Strongest nonlocal sets with minimum cardinality in tripartite systems

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Strong nonlocality, proposed by Halder *et al.* [Phys. Rev. Lett. **122**, 040403 (2019)], is a stronger manifestation than quantum nonlocality. Subsequently, Shi *et al.* presented the concept of the strongest nonlocality [Quantum **6**, 619 (2022)]. Recently, Li and Wang [Quantum **7**, 1101 (2023)] posed the conjecture about a lower bound to the cardinality of the strongest nonlocal set $\mathcal{S}$ in $\otimes_{i=1}^{n} \mathbb{C}^{d_i}$, i.e., $|\mathcal{S}| \leqslant \max_i\{\prod_{j=1}^{n} d_j/d_i + 1\}$. In this work, we construct the strongest nonlocal set of size $d^2 + 1$ in $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$. Furthermore, we obtain the strongest nonlocal set of size $d_2 d_3 + 1$ in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_3}$. Our construction reaches the lower bound, which provides an affirmative solution to Li and Wang's conjecture. ln particular, the strongest nonlocal sets we present here contain the least number of orthogonal states among the available results.

I. INTRODUCTION

In 1964, Bell derived the famous Bell's inequality [1]. Since then, many experiments have been performed to demonstrate violation of Bell's inequality, indicating that the pure entangled states have Bell nonlocality. Unlike Bell nonlocality, in 1999, Bennett *et al.* [2] found that the set of orthogonal product states, which is not perfectly distinguishable under local operations and classical communication (LOCC), also reflects quantum nonlocality. In Ref. [3], Mal and Sen attempted to unify the concepts of Bell nonlocality and Bennett nonlocality in quantum information theory.

A known set of orthogonal quantum states is said to have the property of quantum nonlocality if it is not possible to distinguish them via LOCC. Quantum nonlocality, i.e., Bennett nonlocality, which is widely used in quantum secret sharing and quantum data hiding [4–9], has given rise to research and fruitful results in the past two decades [10–33].

A stronger manifestation of quantum nonlocality in multipartite quantum systems, known as strong quantum nonlocality, was discovered by Halder *et al.* [34]. A set of orthogonal states is said to be strongly nonlocal if it is locally irreducible in every bipartition. Indeed, a locally irreducible set is locally indistinguishable, but the converse is not true in general. They also constructed orthogonal product bases with strong nonlocality in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ and $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$. Soon after, numerous results have emerged concerning the existence of orthogonal product sets (OPSs) and orthogonal entangled sets (OESs) with strong nonlocality [34–47,49] (see Table I for a summary).

For strongly nonlocal OPSs, Yuan *et al.* [36] presented a strongly nonlocal OPS of $6(d-1)^2$ (with respect to $6d^2 - 8d + 4$) elements in $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$ (with respect to $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^{d+1}$). Meanwhile, they gave some examples of strongly nonlocal OPSs in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ and

$\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$. After that, Zhou *et al.* proposed strongly nonlocal OPSs of smaller size in arbitrary three and four-partite systems [37]. By using a general decomposition of the N-dimensional hypercubes for odd $N \geqslant 3$, He *et al.* [38] showed a strongly nonlocal OPS of size $\prod_{i=1}^{N} d_i - \prod_{i=1}^{N}(d_i - 2)$ in $\otimes_{i=1}^{N} \mathbb{C}^{d_i}$. In general n-partite systems with even n, Zhou *et al.* put forward a construction of strongly nonlocal OPSs [39]. In particular, an unextendible product basis (UPB) can also exhibit strong nonlocality. In Ref. [40], Shi *et al.* showed the existence of UPBs that are locally irreducible in every bipartition in $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$. Moreover, the above construction can be generalized to arbitrary three- and four-partite systems [41]. In 2022, Che *et al.* [42] provided a strongly nonlocal UPB with cardinality $(d-1)^3 + 2d + 5$ in $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$ and generalized this approach to arbitrary tripartite systems.

For strongly nonlocal OESs, Shi *et al.* [43] obtained the first result of strongly nonlocal sets with entanglement. In Ref. [44], Wang *et al.* related orthogonal genuinely entangled sets (OGESs) with strong nonlocality to the connectivities of graphs. They also proposed strongly nonlocal OGESs of size $d^3 - (d-2)^2$ (d is odd) and $d^3 - (d-2)^2 + 2$ (d is even) in $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$. Further, they extended this result to the general case. Several strongly nonlocal OESs with cardinality $d^N - (d-1)^N + 1$ have been presented by Shi *et al.* in $(\mathbb{C}^d)^{\otimes N}$, where $N \geqslant 3$ and $d \geqslant 2$ [45]. When $N = 3$ and 4, the sets they constructed were strongly nonlocal OGESs. In N-qutrit systems, Hu *et al.* [46] constructed a strongly nonlocal OGES of size $2 \times 3^{N-1}$.

Recently, Li and Wang [47] proposed the definition of a locally stable set and gave two conjectures: (a) There exists the smallest set of cardinality $\max_i\{d_i + 1\}$ of orthogonal states that is locally stable in $\otimes_{i=1}^{n} \mathbb{C}^{d_i}$. (b) The smallest set exhibiting the strongest nonlocality in $\otimes_{i=1}^{n} \mathbb{C}^{d_i}$ has a cardinality of $\max_i\{\prod_{j=1}^{n} d_j/d_i + 1\}$. In Ref. [48], Cao *et al.* provided a locally stable set of size $\max_i\{d_i + 1\}$, which showed that the first conjecture holds. Li *et al.* [49] constructed the strongest nonlocal sets of size $d_2 d_3 + d_1 - 1$ (with respect to $d^3 + d - 1$) in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_3}$ (with respect to $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d \otimes$

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TABLE I. Results about the strongest nonlocal sets in $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$ or $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_3}$.

Type	Cardinality	References
OPS	$6(d-1)^2$	[36]
OPS	$2[(d_1 d_2 + d_2 d_3 + d_1 d_3)$ $-3(d_1 + d_2 + d_3) + 12]$	[37]
UPB	$d_1 d_2 d_3 - 8(n+1),\ 0 \leqslant n \leqslant \lfloor \frac{d_1 - 3}{2} \rfloor$	[41]
UPB	$(d-1)^3 + 2d + 5$	[42]
OES	$d^3 - d$ (d is odd) $d^3 - d - 6$ (d is even)	[43]
OGES	$d^3 - (d-2)^2$ (d is odd) $d^3 - (d-2)^2 + 2$ (d is even)	[44]
OGES	$d^3 - (d-1)^3 + 1$	[45]
OGES and $|000\rangle$	$\prod_{i=1}^{3} d_i - \prod_{i=1}^{3} (d_i - 1)$	[47]
OGES and $|000\rangle$	$d_2 d_3 + d_1 - 1$	[49]
OGES and $|S_2\rangle$	$d^2 + 1$ (the lower bound)	Theorem 1
OES and $|S\rangle$	$d_2 d_3 + 1$ (the lower bound)	Theorem 2

$\mathbb{C}^d$). The above constructions reach the lower bound only in $\mathbb{C}^2 \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_3}$ and $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. Up to now, whether the conjecture holds in general tripartite systems remains an open question worth considering.

In this work, we construct the strongest nonlocal sets with minimum cardinality in general tripartite systems, confirming the conjecture raised in Ref. [47]. First, we present the strongest nonlocal set of size $d^2 + 1$ in $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$ for $d \geqslant 3$, which consists of the stopper state and special genuinely entangled states [Greenberger-Horne-Zeilinger (GHZ)-like and W-like states], the corresponding Rubik's cube exhibits the perfect geometric symmetry. Moreover, we prove that the smallest orthogonal set of size 21 is the strongest nonlocal set in $\mathbb{C}^3 \otimes \mathbb{C}^4 \otimes \mathbb{C}^5$ and provides the structure to general tripartite systems $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_3}$.

II. PRELIMINARIES

Throughout this paper, we only consider pure quantum states. For the sake of convenience, the states discussed in this paper are un-normalized quantum states. We define $\mathbb{Z}_d = \{0, 1, \ldots, d-1\}$ and choose the computational basis $\{|i\rangle \mid i \in \mathbb{Z}_{d_k}\}$ for each d_k-dimensional subsystem. For each integer $d \geqslant 2$, $\omega_d = e^{\frac{2\pi \sqrt{-1}}{d}}$, i.e., a primitive dth root of unit.

Consider a positive operator-valued measure (POVM) that performed on subsystems, each POVM element can be represented by a $d \times d$ matrix, denote $E = (m_{i,j})_{i,j \in \mathbb{Z}_d}$, under the computational basis of subsystems. A measurement is *trivial* if all its POVM elements are proportional to the identity operator. Otherwise, the measurement is called nontrivial. A measurement is an *orthogonality-preserving local measurement* (OPLM) if the postmeasurement states are mutually orthogonal.

Definition 1 (The strongest nonlocality) [40]. A set $\mathcal{S}$ of orthogonal multipartite quantum states is said to be of *the strongest nonlocality* if only trivial OPLM can be performed for each bipartition of the subsystems.

In a general tripartite quantum system $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C = \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_3}$ ($d_1 \leqslant d_2 \leqslant d_3$), there are three different bipartitions: $A|BC$, $B|CA$, and $C|AB$. Given a set $\mathcal{S}$ of

orthogonal states in $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$, to demonstrate the set $\mathcal{S}$ is the strongest nonlocal set, it is sufficient to show that each joint party BC, CA, and AB can only start with a trivial OPLM. For example, if $E_{BC} = M_{BC}^\dagger M_{BC}$ is one of BC's measurement elements, we use the orthogonality relations $\{\mathbb{I}_A \otimes M_{BC} |\Psi\rangle\}_{|\Psi\rangle \in \mathcal{S}}$, that is,

$$\langle\Phi|\mathbb{I}_A \otimes E_{BC}|\Psi\rangle = \langle\Phi|\mathbb{I}_A \otimes M_{BC}^\dagger M_{BC}|\Psi\rangle = 0, \quad (1)$$

$\forall |\Phi\rangle \neq |\Psi\rangle \in \mathcal{S}$, to show that $E_{BC} \propto \mathbb{I}_{BC}$ (E_{CA} and E_{AB} are similar). Throughout this paper, we assume $E_{BC} = (m_{ij,kl})_{ij,kl \in \mathbb{Z}_{d_2} \times \mathbb{Z}_{d_3}}$, $E_{CA} = (m_{ij,kl})_{ij,kl \in \mathbb{Z}_{d_3} \times \mathbb{Z}_{d_1}}$, and $E_{AB} = (m_{ij,kl})_{ij,kl \in \mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2}}$.

A state $|\phi\rangle_{ABC}$ is called *a genuinely entangled state* if it is entangled in every bipartition. It is well known that the three-qubit GHZ state $(|000\rangle + |111\rangle)/\sqrt{2}$ and the W state $(|100\rangle + |010\rangle + |001\rangle)/\sqrt{3}$ are genuinely entangled states. The states considered throughout this paper (see Fig. 1 for an example) can be separated into the following three classes:

(i) *the GHZ-like states:* $|i_1\rangle_A |j_1\rangle_B |k_1\rangle_C - |i_2\rangle_A |j_2\rangle_B |k_2\rangle_C$, where $i_1 \neq i_2$, $j_1 \neq j_2$ and $k_1 \neq k_2$;

(ii) *the W-like states:* $|i\rangle_A |j\rangle_B |k\rangle_C + \omega_3^1 |j\rangle_A |k\rangle_B |i\rangle_C + \omega_3^2 |k\rangle_A |i\rangle_B |j\rangle_C$, where i, j, and k are not all the same, i.e., (i, j, k) is not of the form (a, a, a); and

(iii) *the stopper state:*

$$|S\rangle = \left(\sum_{i \in \mathbb{Z}_{d_1}} |i\rangle_A\right)\left(\sum_{j \in \mathbb{Z}_{d_2}} |j\rangle_B\right)\left(\sum_{k \in \mathbb{Z}_{d_3}} |k\rangle_C\right).$$

For a GHZ-like state, $|\Phi\rangle = |i_1\rangle_A |j_1\rangle_B |k_1\rangle_C - |i_2\rangle_A |j_2\rangle_B |k_2\rangle_C$, we refer to $(i_1, j_1 k_1)$ and $(i_2, j_2 k_2)$ as two cells of $|\Phi\rangle$ with respect to the bipartition $A|BC$, where i_1 are i_2 are the row indexes, respectively. For a W-like state, $|\Phi\rangle = |i\rangle_A |j\rangle_B |k\rangle_C + \omega_3^1 |j\rangle_A |k\rangle_B |i\rangle_C + \omega_3^2 |k\rangle_A |i\rangle_B |j\rangle_C$, we refer to (i, jk), (j, ki), and (k, ij) as three cells of $|\Phi\rangle$ with respect to the bipartition $A|BC$, where i, j, and k are the row indexes, respectively. Building on the orthogonality of the postmeasurement states, we present the following observations that may be significant in demonstrating that the joint party BC can only start with trivial measurements (CA and AB can be dealt with similarly).

Observation 1. Let $|\Phi\rangle$ and $|\Psi\rangle$ be orthogonal GHZ-like states or W-like states. If only one pair of cells of $|\Phi\rangle$ and $|\Psi\rangle$ with respect to the bipartition $A|BC$ have the same row index,
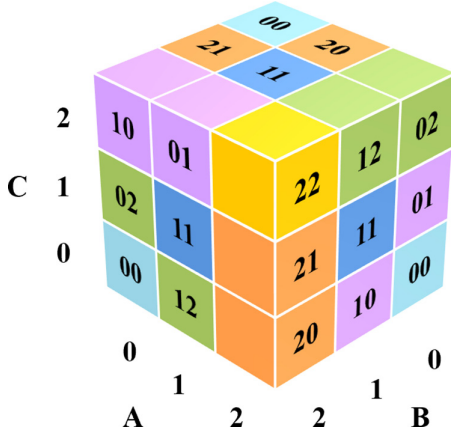
FIG. 1. The Rubik's cube corresponding to the strongest nonlocal set with minimum cardinality given by Eq. (4) in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$. Every cube has an index (i, j, k), where i, j, and k are the indexes in each subsystem A, B, and C. For example, the GHZ-like state $|\phi_{22}\rangle = |2\rangle_A |2\rangle_B |2\rangle_C - |1\rangle_A |1\rangle_B |1\rangle_C$ corresponds to two yellow cubes $(1,1,1)$ and $(2,2,2)$ with label "22." The W-like state $|\phi_{00}\rangle = |2\rangle_A |0\rangle_B |0\rangle_C + \omega_3^1 |0\rangle_A |0\rangle_B |2\rangle_C + \omega_3^2 |0\rangle_A |2\rangle_B |0\rangle_C$ corresponds to three blue cubes $(2,0,0)$, $(0,0,2)$, and $(0,2,0)$ with label "00."

denoted as (i, jk) and $(i, j'k')$, then applying $|\Phi\rangle$ and $|\Psi\rangle$ to Eq. (1) results in

$$m_{jk, j'k'} = 0.$$

For example, consider two orthogonal GHZ-like states $|\phi_{20}\rangle = |2\rangle_A |2\rangle_B |0\rangle_C - |1\rangle_A |0\rangle_B |2\rangle_C$ and $|\phi_{02}\rangle = |2\rangle_A |0\rangle_B |2\rangle_C - |0\rangle_A |2\rangle_B |1\rangle_C$. In Fig. 2, we find that only two cells (2,20) and (2,02) have the unique same row index "2" in the bipartition $A|BC$. Applying Obervation 1 to $|\phi_{20}\rangle$ and $|\phi_{02}\rangle$, we have the equation, that is, $(_A\langle 2|_{BC}\langle 20| -_A\langle 1|_{BC}\langle 02|)\mathbb{I}_A \otimes M_{BC}^\dagger M_{BC}(|2\rangle_A|02\rangle_{BC} - |0\rangle_A|21\rangle_{BC}) = 0$. Hence, we get that $m_{20,02} = 0$. Since $E = E^\dagger$, we have $m_{02,20} = m_{20,02} = 0$.

Observation 2. Let $|\Phi\rangle$ and $|\Psi\rangle$ be orthogonal GHZ-like states or W-like states. If more than one pair of cells of $|\Phi\rangle$ and $|\Psi\rangle$ with respect to the bipartition $A|BC$ have the same row index, denoted as (i, jk) and $(i, j'k')$ and as (i_x, j_xk_x) and $(i_x, j'_xk'_x)$ with $x \in \mathcal{X}$ for some index set $\mathcal{X}$, then applying $|\Phi\rangle$ and $|\Psi\rangle$ to Eq. (1) results in

$$am_{jk, j'k'} + \sum_{x \in \mathcal{X}} a_x m_{j_xk_x, j'_xk'_x} = 0,$$

where $|a| = |a_x| = 1$. If we have known $m_{j_xk_x, j'_xk'_x} = 0$ for $x \in \mathcal{X}$, then we have

$$m_{jk, j'k'} = 0.$$

The cardinality of the index set $\mathcal{X}$ is either 1 or 2 in this paper. For example, given $|\phi_{20}\rangle$ and $|\phi_{12}\rangle = |2\rangle_A |1\rangle_B |2\rangle_C - |1\rangle_A |2\rangle_B |0\rangle_C$, we cannot use Obervation 1 directly. However, we can obtain the equation by Observation 2, i.e., $(_A\langle 2|_{BC}\langle 20| -_A\langle 1|_{BC}\langle 02|)\mathbb{I}_A \otimes M_{BC}^\dagger M_{BC} (|2\rangle_A|12\rangle_{BC} - |1\rangle_A|20\rangle_{BC}) = 0$. It implies that $m_{20,12} + m_{02,20} = 0$. As $m_{02,20} = 0$ can be yielded from Obervation 1; therefore $m_{20,12} = 0$.

Observation 3. Suppose that all off-diagonal entries in the matrix E_{BC} are zeros. Applying the stopper state $|S\rangle$ and a
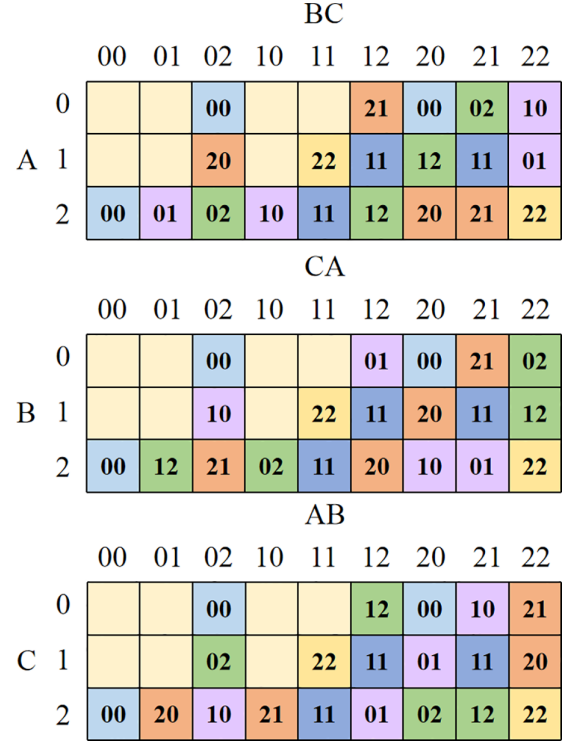

FIG. 2. The 3×9 plane structures of the strongest nonlocal set $\cup_{i,j \in \mathbb{Z}_3}\{|\phi_{ij}\rangle\} \bigcup\{|S_1\rangle\}$ correspond to each bipartition $A|BC$, $B|CA$, and $C|AB$. Every cell has an index (i, jk), where i is the row index in the single subsystem and jk is the column index in the joint subsystem. For example, the GHZ-like state $|\phi_{20}\rangle = |2\rangle_A |20\rangle_{BC} - |1\rangle_A |02\rangle_{BC}$ corresponds to two cells (2,20) and (1,02) with label "20" in the bipartition $A|BC$.

GHZ-like state $|\Phi\rangle = |a\rangle_A |i\rangle_B |j\rangle_C - |b\rangle_A |k\rangle_B |l\rangle_C$ to Eq. (1) yields

$$m_{ij, ij} = m_{kl, kl}.$$

Observation 4. Suppose that all off-diagonal entries in the matrix E_{BC} are zeros. Applying the stopper state $|S\rangle$ and the W-like state $|\Psi\rangle = |i\rangle_A |j\rangle_B |k\rangle_C + \omega_3^1 |j\rangle_A |k\rangle_B |i\rangle_C + \omega_3^2 |k\rangle_A |i\rangle_B |j\rangle_C$ to Eq. (1) yields

$$m_{ij, ij} = m_{jk, jk} = m_{ki, ki}.$$

In fact, under the condition that all off-diagonal entries in the matrix E_{BC} are zeros, Eq. (1) is just

$$m_{jk, jk} + \omega_3^1 m_{ki, ki} + \omega_3^2 m_{ij, ij} = 0, \tag{2}$$

where $m_{jk, jk}$, $m_{ki, ki}$, and $m_{ij, ij}$ are all real numbers. Taking complex conjugation to both sides of Eq. (2), we have

$$m_{jk, jk} + \omega_3^2 m_{ki, ki} + \omega_3^1 m_{ij, ij} = 0. \tag{3}$$

Solving the two linear equations, Eqs. (2) and (3), one has $(m_{jk, jk}, m_{ki, ki}, m_{ij, ij}) = (r, r, r)$ for some real number r.

III. THE STRONGEST NONLOCAL SETS WITH MINIMUM CARDINALITY IN $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$

In this section, we present an orthogonal set of size 10 with the strongest nonlocality in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$, the corresponding

TABLE II. The off-diagonal entries in the matrix $E_{BC} = (m_{ij,kl})_{ij\neq kl\in\mathbb{Z}_3\times\mathbb{Z}_3}$ by Observations 1 and 2.

Observations	Pair of states	Key entries	Pair of states	Remaining entries	Pair of states	Remaining entries
Observation 1	$\lvert\phi_{02}\rangle, \lvert\phi_{11}\rangle$	$m_{02,11} = m_{11,02} = 0$				
	$\lvert\phi_{02}\rangle, \lvert\phi_{12}\rangle$	$m_{02,12} = m_{12,02} = 0$	$\lvert\phi_{00}\rangle, \lvert\phi_{11}\rangle$	$m_{00,11} = m_{11,00} = 0$	$\lvert\phi_{01}\rangle, \lvert\phi_{02}\rangle$	$m_{01,02} = m_{02,01} = 0$
	$\lvert\phi_{02}\rangle, \lvert\phi_{20}\rangle$	$m_{02,20} = m_{20,02} = 0$	$\lvert\phi_{00}\rangle, \lvert\phi_{01}\rangle$	$m_{00,01} = m_{01,00} = 0$	$\lvert\phi_{01}\rangle, \lvert\phi_{10}\rangle$	$m_{01,10} = m_{10,01} = 0$
	$\lvert\phi_{02}\rangle, \lvert\phi_{22}\rangle$	$m_{02,22} = m_{22,02} = 0$	$\lvert\phi_{00}\rangle, \lvert\phi_{12}\rangle$	$m_{00,12} = m_{12,00} = 0$	$\lvert\phi_{01}\rangle, \lvert\phi_{21}\rangle$	$m_{01,21} = m_{21,01} = 0$
	$\lvert\phi_{11}\rangle, \lvert\phi_{21}\rangle$	$m_{11,21} = m_{21,11} = 0$	$\lvert\phi_{00}\rangle, \lvert\phi_{20}\rangle$	$m_{00,20} = m_{20,00} = 0$	$\lvert\phi_{10}\rangle, \lvert\phi_{12}\rangle$	$m_{10,12} = m_{12,10} = 0$
	$\lvert\phi_{12}\rangle, \lvert\phi_{21}\rangle$	$m_{12,21} = m_{21,12} = 0$	$\lvert\phi_{00}\rangle, \lvert\phi_{22}\rangle$	$m_{00,22} = m_{22,00} = 0$	$\lvert\phi_{10}\rangle, \lvert\phi_{20}\rangle$	$m_{10,20} = m_{20,10} = 0$
	$\lvert\phi_{20}\rangle, \lvert\phi_{21}\rangle$	$m_{20,21} = m_{21,20} = 0$	$\lvert\phi_{11}\rangle, \lvert\phi_{10}\rangle$	$m_{11,10} = m_{10,11} = 0$	$\lvert\phi_{10}\rangle, \lvert\phi_{22}\rangle$	$m_{10,22} = m_{22,10} = 0$
	$\lvert\phi_{21}\rangle, \lvert\phi_{22}\rangle$	$m_{21,22} = m_{22,21} = 0$				
Observation 2	$\lvert\phi_{02}\rangle, \lvert\phi_{21}\rangle$	$m_{02,21} = m_{21,02} = 0$				
	$\lvert\phi_{12}\rangle, \lvert\phi_{20}\rangle$	$m_{12,20} = m_{20,12} = 0$	$\lvert\phi_{01}\rangle, \lvert\phi_{12}\rangle$	$m_{01,12} = m_{12,01} = 0$		
	$\lvert\phi_{20}\rangle, \lvert\phi_{22}\rangle$	$m_{20,22} = m_{22,20} = 0$	$\lvert\phi_{01}\rangle, \lvert\phi_{20}\rangle$	$m_{01,20} = m_{20,01} = 0$	$\lvert\phi_{00}\rangle, \lvert\phi_{02}\rangle$	$m_{00,02} = m_{02,00} = 0$
	$\lvert\phi_{11}\rangle, \lvert\phi_{12}\rangle$	$m_{11,12} = m_{12,11} = 0$	$\lvert\phi_{01}\rangle, \lvert\phi_{22}\rangle$	$m_{01,22} = m_{22,01} = 0$	$\lvert\phi_{00}\rangle, \lvert\phi_{10}\rangle$	$m_{00,10} = m_{10,00} = 0$
	$\lvert\phi_{11}\rangle, \lvert\phi_{20}\rangle$	$m_{11,20} = m_{20,11} = 0$	$\lvert\phi_{10}\rangle, \lvert\phi_{02}\rangle$	$m_{10,02} = m_{02,10} = 0$	$\lvert\phi_{00}\rangle, \lvert\phi_{21}\rangle$	$m_{00,21} = m_{21,00} = 0$
	$\lvert\phi_{11}\rangle, \lvert\phi_{22}\rangle$	$m_{11,22} = m_{22,11} = 0$	$\lvert\phi_{10}\rangle, \lvert\phi_{21}\rangle$	$m_{10,21} = m_{21,10} = 0$	$\lvert\phi_{11}\rangle, \lvert\phi_{01}\rangle$	$m_{11,01} = m_{01,11} = 0$
	$\lvert\phi_{12}\rangle, \lvert\phi_{22}\rangle$	$m_{12,22} = m_{22,12} = 0$				

Rubik's cube of which is given by Fig. 1. Subsequently, we propose the strongest nonlocal set with minimum cardinality in $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$, where $d \geqslant 3$.

Lemma 1. The set $\cup_{i,j\in\mathbb{Z}_3}\{\lvert\phi_{ij}\rangle\}\bigcup\{\lvert S_1\rangle\}$ of size 10 given by Eq. (4) is strongest nonlocal in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$:

$$\lvert\phi_{22}\rangle = \lvert 2\rangle_A\lvert 2\rangle_B\lvert 2\rangle_C - \lvert 1\rangle_A\lvert 1\rangle_B\lvert 1\rangle_C,$$

$$\lvert\phi_{20}\rangle = \lvert 2\rangle_A\lvert 2\rangle_B\lvert 0\rangle_C - \lvert 1\rangle_A\lvert 0\rangle_B\lvert 2\rangle_C,$$

$$\lvert\phi_{21}\rangle = \lvert 2\rangle_A\lvert 2\rangle_B\lvert 1\rangle_C - \lvert 0\rangle_A\lvert 1\rangle_B\lvert 2\rangle_C,$$

$$\lvert\phi_{02}\rangle = \lvert 2\rangle_A\lvert 0\rangle_B\lvert 2\rangle_C - \lvert 0\rangle_A\lvert 2\rangle_B\lvert 1\rangle_C,$$

$$\lvert\phi_{12}\rangle = \lvert 2\rangle_A\lvert 1\rangle_B\lvert 2\rangle_C - \lvert 1\rangle_A\lvert 2\rangle_B\lvert 0\rangle_C,$$

$$\lvert\phi_{10}\rangle = \lvert 2\rangle_A\lvert 1\rangle_B\lvert 0\rangle_C - \lvert 0\rangle_A\lvert 2\rangle_B\lvert 2\rangle_C,$$

$$\lvert\phi_{01}\rangle = \lvert 2\rangle_A\lvert 0\rangle_B\lvert 1\rangle_C - \lvert 1\rangle_A\lvert 2\rangle_B\lvert 2\rangle_C,$$

$$\lvert\phi_{00}\rangle = \lvert 2\rangle_A\lvert 0\rangle_B\lvert 0\rangle_C + \omega_3^1\lvert 0\rangle_A\lvert 0\rangle_B\lvert 2\rangle_C + \omega_3^2\lvert 0\rangle_A\lvert 2\rangle_B\lvert 0\rangle_C,$$

$$\lvert\phi_{11}\rangle = \lvert 2\rangle_A\lvert 1\rangle_B\lvert 1\rangle_C + \omega_3^1\lvert 1\rangle_A\lvert 1\rangle_B\lvert 2\rangle_C + \omega_3^2\lvert 1\rangle_A\lvert 2\rangle_B\lvert 1\rangle_C,$$

$$\lvert S_1\rangle = \lvert 0+1+2\rangle_A\lvert 0+1+2\rangle_B\lvert 0+1+2\rangle_C. \tag{4}$$

Proof. Denote that $\mathcal{S}_1 = \cup_{i,j\in\mathbb{Z}_3}\{\lvert\phi_{ij}\rangle\}$. In Fig. 2, the set $\mathcal{S}_1\bigcup\{\lvert S_1\rangle\}$ has a similar plane structure in every bipartition $A\lvert BC$, $B\lvert CA$, and $C\lvert AB$ under the cyclic permutation. Thus, we only need to prove that the measurement applied to the joint subsystem BC is trivial.

Step 1. According to Observations 1 and 2, we obtain that all key off-diagonal entries are zeros, i.e., $m_{ij,kl} = 0$ for $ij \neq kl \in \{02, 11, 12, 20, 21, 22\}$.

Step 2. Consider $\lvert\phi_{pq}\rangle$ and $\lvert\phi_{st}\rangle$ for $pq \neq st$, $pq \in \{00, 01, 10\}$, and $st \in \mathbb{Z}_3 \times \mathbb{Z}_3$, we get that the remaining off-diagonal entries are zeros, that is, $m_{pq,st} = 0$. Thus, all off-diagonal entries in the matrix E_{BC} are zeros, that is, $m_{ij,kl} = m_{kl,ij} = 0$ for $ij \neq kl \in \mathbb{Z}_3 \times \mathbb{Z}_3$ from Table II.

Step 3. Applying Observation 3 to the stopper state $\lvert S_1\rangle$ and $\lvert\phi_{ij}\rangle$ for $ij \in \mathbb{Z}_3 \times \mathbb{Z}_3\backslash\{00, 11\}$, we get $m_{02,02} = m_{12,12} = m_{20,20} = m_{21,21}$ and $m_{01,01} = m_{10,10} = m_{11,11} = m_{22,22}$. Consider the stopper state $\lvert S_1\rangle$ and the W-like states $\lvert\phi_{00}\rangle$ and $\lvert\phi_{11}\rangle$ by Observation 4, we have $m_{00,00} = m_{02,02} = m_{20,20}$ and

$m_{11,11} = m_{12,12} = m_{21,21}$. In short, all diagonal entries are equal. ∎

The proof that E_{BC} is proportional to an identity matrix is shown in Fig. 3. Therefore, the POVM element E_{BC} is trivial. This completes the proof.

The latest result in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ is the strongest nonlocal set of size 11 given by Li *et al.* [49]. Here, we provide the strongest nonlocal set of the smallest size 10, which positively answers an open problem raised by Yuan *et al.* [36]
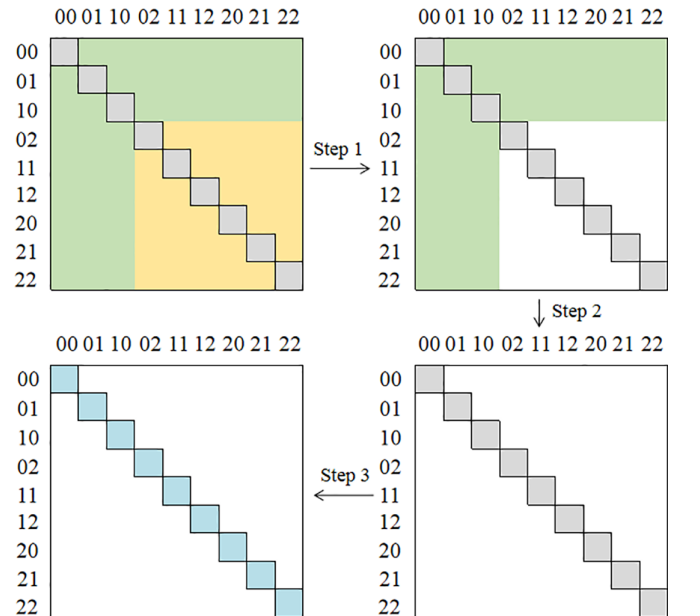

FIG. 3. Applying observations to prove that the set $\cup_{i,j\in\mathbb{Z}_3}\{\lvert\phi_{ij}\rangle\}\bigcup\{\lvert S_1\rangle\}$ is the strongest nonlocal set. The yellow (green) entries correspond to the key (remaining) off-diagonal entries, the gray entries represent that all diagonal entries are unknown. Step 1, we obtain that all key off-diagonal entries are zeros, which correspond to the white entries. Step 2, all remaining off-diagonal entries are zeros. Step 3, all diagonal entries are equal by Observations 3 and 4, which are shown by the blue entries.
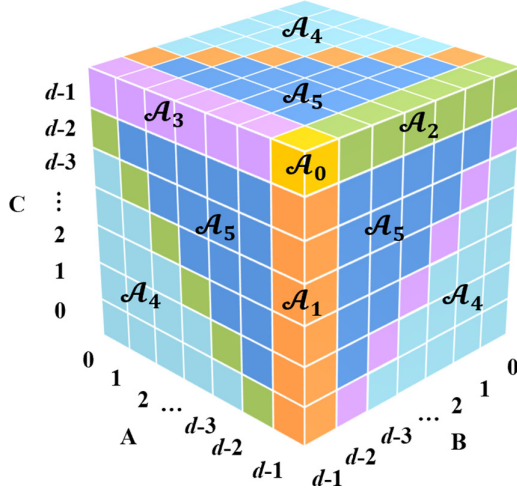
FIG. 4. The Rubik's cube corresponding to the strongest nonlocal set with minimum cardinality given by Eq. (5) in $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$ for $d \geqslant 3$.

and reaches the lower bound on the strongest nonlocal set in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ [47].

Now, we put forward the strongest nonlocal set with minimum cardinality in three-qudit systems, the corresponding Rubik's cube of which is shown in Fig. 4.

Theorem 1. The set $\cup_{i=0}^5 \mathcal{A}_i \bigcup \{|S_2\rangle\}$ of size $d^2 + 1$ given by Eq. (5) is strongest nonlocal in $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$ for $d \geqslant 3$:

$$\mathcal{A}_0 = \{|\phi_{\hat{d}\hat{d}}\rangle = |\hat{d}\rangle_A |\hat{d}\rangle_B |\hat{d}\rangle_C - |d^*\rangle_A |d^*\rangle_B |d^*\rangle_C\},$$

$$\mathcal{A}_1 = \{|\phi_{\hat{d}i}\rangle = |\hat{d}\rangle_A |\hat{d}\rangle_B |i\rangle_C - |d^* - i\rangle_A |i\rangle_B |\hat{d}\rangle_C \,|\, i \in \mathbb{Z}_{\hat{d}}\},$$

$$\mathcal{A}_2 = \{|\phi_{i\hat{d}}\rangle = |\hat{d}\rangle_A |i\rangle_B |\hat{d}\rangle_C - |i\rangle_A |\hat{d}\rangle_B |d^* - i\rangle_C \,|\, i \in \mathbb{Z}_{\hat{d}}\},$$

$$\mathcal{A}_3 = \{|\phi_{(d^*-i)i}\rangle = |\hat{d}\rangle_A |d^* - i\rangle_B |i\rangle_C$$
$$- |i\rangle_A |\hat{d}\rangle_B |\hat{d}\rangle_C \,|\, i \in \mathbb{Z}_{\hat{d}}\},$$

$$\mathcal{A}_4 = \{|\phi_{kl}\rangle = |\hat{d}\rangle_A |k\rangle_B |l\rangle_C + \omega_3^1 |k\rangle_A |l\rangle_B |\hat{d}\rangle_C$$
$$+ \omega_3^2 |l\rangle_A |\hat{d}\rangle_B |k\rangle_C \,|\, k, l \in \mathbb{Z}_{\hat{d}}, k + l \geqslant d - 1\},$$

$$\mathcal{A}_5 = \{|\phi_{st}\rangle = |\hat{d}\rangle_A |s\rangle_B |t\rangle_C + \omega_3^1 |s\rangle_A |t\rangle_B |\hat{d}\rangle_C$$
$$+ \omega_3^2 |t\rangle_A |\hat{d}\rangle_B |s\rangle_C \,|\, s, t \in \mathbb{Z}_{\hat{d}}, 0 \leqslant s + t \leqslant d - 3\},$$

$$|S_2\rangle = \left(\sum_{i \in \mathbb{Z}_d} |i\rangle_A\right)\left(\sum_{j \in \mathbb{Z}_d} |j\rangle_B\right)\left(\sum_{k \in \mathbb{Z}_d} |k\rangle_C\right), \quad (5)$$

where $\hat{d} = d - 1$ and $d^* = d - 2$.

Proof. The set $\cup_{i=0}^5 \mathcal{A}_i \bigcup \{|S_2\rangle\}$ has a similar plane structure in every bipartition under the cyclic permutation. Therefore, it is sufficient to show that the matrix $E_{BC} \propto \mathbb{I}_{BC}$.

Since $E_{BC} = E_{BC}^\dagger$, if $m_{ij,kl} = 0$, then $m_{kl,ij} = 0$ for $ij \neq kl \in \mathbb{Z}_d \times \mathbb{Z}_d$. For convenience, we divide all off-diagonal entries into two parts, called the key and the remaining off-diagonal entries. The key off-diagonal entries are $m_{ij,kl}$ for $ij \neq kl \in \mathcal{S}$, where $\mathcal{S} = \{s\hat{d}, \hat{d}s, d^*d^*, \hat{d}\hat{d} \,|\, s \in \mathbb{Z}_{\hat{d}}\}$. Naturally, the remaining off-diagonal entries are $m_{pq,st}$ for $pq \neq st$, $pq \in \mathbb{Z}_d \times \mathbb{Z}_d \backslash \mathcal{S}$, and $st \in \mathbb{Z}_d \times \mathbb{Z}_d$.

Consider $\mathcal{A}_0$, $\mathcal{A}_1$, $\mathcal{A}_2$, and $|\phi_{d^*d^*}\rangle$ of $\mathcal{A}_5$, we get all key off-diagonal entries are zeros from Table III. However, only two off-diagonal entries $m_{\frac{d^*}{2}\hat{d},\hat{d}\frac{d^*}{2}}$ and $m_{\hat{d}\frac{d^*}{2},\frac{d^*}{2}\hat{d}}$ are not available from Table III when $\hat{d}$ is even. In order to show that $m_{\frac{d^*}{2}\hat{d},\hat{d}\frac{d^*}{2}} = m_{\hat{d}\frac{d^*}{2},\frac{d^*}{2}\hat{d}} = 0$, we first prove that $m_{\frac{d^*}{2}\hat{d},ij} = m_{ij,\frac{d^*}{2}\hat{d}} = 0$ for $ij \in \mathbb{Z}_d \times \mathbb{Z}_d \backslash \{\mathcal{S}, \frac{d^*}{2}\hat{d}, \hat{d}\frac{d^*}{2}\}$. We separate the argument into three cases.

(i) Neither i nor j is $\frac{d^*}{2}$. Consider $|\phi_{\frac{d^*}{2}\hat{d}}\rangle$ and $|\phi_{ij}\rangle$ by Observation 1, then one obtains $m_{\frac{d^*}{2}\hat{d},ij} = 0$.

(ii) Only one of i or j is $\frac{d^*}{2}$, then $|\phi_{ij}\rangle$ belongs to $\mathcal{A}_4$ or $\mathcal{A}_5$. As $m_{\hat{d}\frac{d^*}{2},st} = 0$ for $st \in \mathcal{S} \backslash \{\frac{d^*}{2}\hat{d}\}$, consider $|\phi_{\frac{d^*}{2}\hat{d}}\rangle$ and $|\phi_{ij}\rangle$ by Observation 2, then one obtains $m_{\frac{d^*}{2}\hat{d},ij} = 0$.

(iii) Both of i and j are $\frac{d^*}{2}$, then $|\phi_{\frac{d^*}{2}\frac{d^*}{2}}\rangle$ belongs to $\mathcal{A}_3$. As $m_{\hat{d}\frac{d^*}{2},\hat{d}\hat{d}} = 0$, consider $|\phi_{\frac{d^*}{2}\hat{d}}\rangle$ and $|\phi_{\frac{d^*}{2}\frac{d^*}{2}}\rangle$ by Observation 2, then one obtains $m_{\frac{d^*}{2}\hat{d},\frac{d^*}{2}\frac{d^*}{2}} = 0$.

Next, we show that $m_{\frac{d^*}{2}\hat{d},\hat{d}\frac{d^*}{2}} = m_{\hat{d}\frac{d^*}{2},\frac{d^*}{2}\hat{d}} = 0$. Applying $|\phi_{\hat{d}\frac{d^*}{2}}\rangle$ of $\mathcal{A}_1$ and $|\phi_{\frac{d^*}{2}\hat{d}}\rangle$ of $\mathcal{A}_2$ to Eq. (1) results in

$$0 = \langle \phi_{\hat{d}\frac{d^*}{2}}| \mathbb{I}_A \otimes E_{BC} |\phi_{\frac{d^*}{2}\hat{d}}\rangle = m_{\frac{d^*}{2}\hat{d},\hat{d}\frac{d^*}{2}} + m_{\hat{d}\frac{d^*}{2},\frac{d^*}{2}\hat{d}}. \quad (6)$$

As $m_{\frac{d^*}{2}\hat{d},\hat{d}\frac{d^*}{2}}$ is the complex conjugation of $m_{\hat{d}\frac{d^*}{2},\frac{d^*}{2}\hat{d}}$, $m_{\frac{d^*}{2}\hat{d},\hat{d}\frac{d^*}{2}}$ must be of the form $r\sqrt{-1}$ for some real number r. Then, applying $|\phi_{\frac{d^*}{2}\hat{d}}\rangle$ of $\mathcal{A}_2$ and the stopper state $|S_2\rangle$ to Eq. (1) gives the following equation:

$$0 = \langle \phi_{\frac{d^*}{2}\hat{d}}| \mathbb{I}_A \otimes E_{BC} |S_2\rangle,$$

that is,

$$0 = m_{\frac{d^*}{2}\hat{d},\frac{d^*}{2}\hat{d}} - m_{\hat{d}\frac{d^*}{2},\hat{d}\frac{d^*}{2}} + m_{\frac{d^*}{2}\hat{d},\hat{d}\frac{d^*}{2}} - m_{\hat{d}\frac{d^*}{2},\frac{d^*}{2}\hat{d}}. \quad (7)$$

As $m_{\frac{d^*}{2}\hat{d},\frac{d^*}{2}\hat{d}}$ and $m_{\hat{d}\frac{d^*}{2},\hat{d}\frac{d^*}{2}}$ are real numbers and $m_{\hat{d}\frac{d^*}{2},\frac{d^*}{2}\hat{d}} = -r\sqrt{-1}$, Eq. (7) implies that both the real and imaginary part are zeros. Therefore, $r = 0$. Hence, $m_{\frac{d^*}{2}\hat{d},\hat{d}\frac{d^*}{2}} = m_{\hat{d}\frac{d^*}{2},\frac{d^*}{2}\hat{d}} = 0$. Thus, all key off-diagonal entries $m_{ij,kl}$ are zeros, where $ij \neq kl \in \mathcal{S}$.

Let us show that all remaining off-diagonal entries $m_{pq,st}$ are zeros for $pq \neq st$, $pq \in \mathbb{Z}_d \times \mathbb{Z}_d \backslash \mathcal{S}$, and $st \in \mathbb{Z}_d \times \mathbb{Z}_d$. Consider $|\phi_{pq}\rangle$ and $|\phi_{st}\rangle$ by Eq. (1), then we obtain the following equation:

$$0 = \langle \phi_{pq}| \mathbb{I}_A \otimes E_{BC} |\phi_{st}\rangle = m_{pq,st} + am_{p_a q_a, s_a t_a} + bm_{p_b q_b, s_b t_b}.$$

Obviously, most of them are only composed of two terms. For $m_{p_x q_x, s_x t_x}$ ($x \in \{a, b\}$), the terms come from the key off-diagonal entries that are zeros. Therefore, it is not difficult to show that we can apply Observation 1 or Observation 2 to $|\phi_{pq}\rangle$ and $|\phi_{st}\rangle$, which yields $m_{pq,st} = 0$.

Eventually, applying Observation 3 to the stopper state $|S_2\rangle$ and $\mathcal{A}_i$ ($i \in \mathbb{Z}_4$), some diagonal entries are equal, i.e., $m_{\hat{d}i,\hat{d}i} = m_{\hat{d}(d^*-i),\hat{d}(d^*-i)} = m_{i\hat{d},i\hat{d}} = m_{(d^*-i)\hat{d},(d^*-i)\hat{d}}$ for $i \in \mathbb{Z}_{\lceil \frac{d-1}{2} \rceil}$. Consider the stopper state $|S_2\rangle$ and $\mathcal{A}_4$ and $\mathcal{A}_5$ by Observation 4; we deduce that $m_{kl,kl} = m_{l\hat{d},l\hat{d}} = m_{\hat{d}k,\hat{d}k}$ for $k, l \in \mathbb{Z}_{\hat{d}}, k + l \neq d^*$. Therefore, the matrix $E_{BC} \propto \mathbb{I}_{BC}$. ∎

TABLE III. The key off-diagonal entries with respect to coordinates of $\mathcal{S} = \{\hat{d}i, j\hat{d}, d^*d^*, \hat{d}\hat{d} \mid i, j \in \mathbb{Z}_{\hat{d}}\}$ in the matrix $E_{BC} = (m_{ij,kl})_{ij \neq kl \in \mathbb{Z}_d \times \mathbb{Z}_d}$ by Observations 1 and 2.

Observations	Sets	Pair of states	Key entries	Value range	Precondition
Observation 1	$\mathcal{A}_1$	$|\phi_{\hat{d}i}\rangle, |\phi_{\hat{d}j}\rangle$	$m_{\hat{d}i,\hat{d}j} = m_{\hat{d}j,\hat{d}i} = 0$	$i \neq j$	Null
	$\mathcal{A}_2$	$|\phi_{i\hat{d}}\rangle, |\phi_{j\hat{d}}\rangle$	$m_{i\hat{d},j\hat{d}} = m_{j\hat{d},i\hat{d}} = 0$	$i \neq j$	Null
	$\mathcal{A}_1, \mathcal{A}_2$	$|\phi_{\hat{d}i}\rangle, |\phi_{j\hat{d}}\rangle$	$m_{\hat{d}i,j\hat{d}} = m_{j\hat{d},\hat{d}i} = 0$	$i + j \neq d^*$	Null
	$\mathcal{A}_1, \mathcal{A}_5$	$|\phi_{\hat{d}i}\rangle, |\phi_{d^*d^*}\rangle$	$m_{\hat{d}i,d^*d^*} = m_{d^*d^*,\hat{d}i} = 0$	$i \neq 0$	Null
	$\mathcal{A}_2, \mathcal{A}_5$	$|\phi_{i\hat{d}}\rangle, |\phi_{d^*d^*}\rangle$	$m_{i\hat{d},d^*d^*} = m_{d^*d^*,i\hat{d}} = 0$	$i \neq d^*$	Null
	$\mathcal{A}_0, \mathcal{A}_1$	$|\phi_{\hat{d}\hat{d}}\rangle, |\phi_{\hat{d}i}\rangle$	$m_{\hat{d}i,\hat{d}\hat{d}} = m_{\hat{d}\hat{d},\hat{d}i} = 0$	$i \neq 0$	Null
	$\mathcal{A}_0, \mathcal{A}_2$	$|\phi_{\hat{d}\hat{d}}\rangle, |\phi_{i\hat{d}}\rangle$	$m_{i\hat{d},\hat{d}\hat{d}} = m_{\hat{d}\hat{d},i\hat{d}} = 0$	$i \neq d^*$	Null
Observation 2	$\mathcal{A}_1, \mathcal{A}_2$	$|\phi_{i\hat{d}}\rangle, |\phi_{\hat{d}(d^*-i)}\rangle$	$m_{\hat{d}(d^*-i),i\hat{d}} = m_{i\hat{d},\hat{d}(d^*-i)} = 0$		$m_{\hat{d}i,i\hat{d}} = 0$
	$\mathcal{A}_1, \mathcal{A}_5$	$|\phi_{\hat{d}0}\rangle, |\phi_{d^*d^*}\rangle$	$m_{\hat{d}0,d^*d^*} = m_{d^*d^*,\hat{d}0} = 0$		$m_{0\hat{d},d^*\hat{d}} = m_{0\hat{d},\hat{d}d^*} = 0$
	$\mathcal{A}_2, \mathcal{A}_5$	$|\phi_{d^*\hat{d}}\rangle, |\phi_{d^*d^*}\rangle$	$m_{d^*\hat{d},d^*d^*} = m_{d^*d^*,d^*\hat{d}} = 0$	$i \neq \frac{d^*}{2}$	$m_{\hat{d}0,d^*\hat{d}} = m_{\hat{d}0,\hat{d}d^*} = 0$
	$\mathcal{A}_0, \mathcal{A}_1$	$|\phi_{\hat{d}\hat{d}}\rangle, |\phi_{\hat{d}0}\rangle$	$m_{\hat{d}0,\hat{d}\hat{d}} = m_{\hat{d}\hat{d},\hat{d}0} = 0$		$m_{d^*d^*,0\hat{d}} = 0$
	$\mathcal{A}_0, \mathcal{A}_2$	$|\phi_{\hat{d}\hat{d}}\rangle, |\phi_{d^*\hat{d}}\rangle$	$m_{d^*\hat{d},\hat{d}\hat{d}} = m_{\hat{d}\hat{d},d^*\hat{d}} = 0$		$m_{\hat{d}0,d^*d^*} = 0$
	$\mathcal{A}_0, \mathcal{A}_5$	$|\phi_{\hat{d}\hat{d}}\rangle, |\phi_{d^*d^*}\rangle$	$m_{d^*d^*,\hat{d}\hat{d}} = m_{\hat{d}\hat{d},d^*d^*} = 0$		$m_{\hat{d}d^*,d^*d^*} = m_{d^*\hat{d},d^*d^*} = 0$

IV. THE STRONGEST NONLOCAL SETS WITH MINIMUM CARDINALITY IN $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_3}$

First of all, we construct the strongest nonlocal set of the smallest size in $\mathbb{C}^3 \otimes \mathbb{C}^4 \otimes \mathbb{C}^5$ based on the set given by Lemma 1.

Lemma 2. The set $\cup_{ij \in \mathbb{Z}_4 \times \mathbb{Z}_5} \{|\phi_{ij}\rangle\} \bigcup \{|S_3\rangle\}$ of size 21 given by Eqs. (4) and (8) is strongest nonlocal in $\mathbb{C}^3 \otimes \mathbb{C}^4 \otimes \mathbb{C}^5$:

$$|\phi_{03}\rangle = |2\rangle_A |0\rangle_B |3\rangle_C - |0\rangle_A |2\rangle_B |3\rangle_C,$$

$$|\phi_{13}\rangle = |2\rangle_A |1\rangle_B |3\rangle_C - |1\rangle_A |2\rangle_B |3\rangle_C,$$

$$|\phi_{04}\rangle = |2\rangle_A |0\rangle_B |4\rangle_C - |0\rangle_A |2\rangle_B |4\rangle_C,$$

$$|\phi_{14}\rangle = |2\rangle_A |1\rangle_B |4\rangle_C - |1\rangle_A |2\rangle_B |4\rangle_C,$$

$$|\phi_{23}\rangle = |2\rangle_A |2\rangle_B |3\rangle_C - |0\rangle_A |0\rangle_B |1\rangle_C,$$

$$|\phi_{24}\rangle = |2\rangle_A |2\rangle_B |4\rangle_C - |0\rangle_A |0\rangle_B |3\rangle_C,$$

$$|\phi_{30}\rangle = |2\rangle_A |3\rangle_B |0\rangle_C - |0\rangle_A |3\rangle_B |2\rangle_C,$$

$$|\phi_{31}\rangle = |2\rangle_A |3\rangle_B |1\rangle_C - |1\rangle_A |3\rangle_B |2\rangle_C,$$

$$|\phi_{32}\rangle = |2\rangle_A |3\rangle_B |2\rangle_C - |0\rangle_A |1\rangle_B |0\rangle_C,$$

$$|\phi_{33}\rangle = |2\rangle_A |3\rangle_B |3\rangle_C - |0\rangle_A |1\rangle_B |3\rangle_C,$$

$$|\phi_{34}\rangle = |2\rangle_A |3\rangle_B |4\rangle_C - |0\rangle_A |1\rangle_B |4\rangle_C,$$

$$|S_3\rangle = |0+1+2\rangle_A |0+1+2+3\rangle_B |0+1+2+3+4\rangle_C, \quad (8)$$

where $\cup_{i,j \in \mathbb{Z}_3} \{|\phi_{ij}\rangle\}$ are the same as the states given by Eq. (4).

Proof. Denote $\mathcal{S}_1 = \cup_{i,j \in \mathbb{Z}_3} \{|\phi_{ij}\rangle\}$, $\mathcal{S}_2 = \cup_{i \in \mathbb{Z}_3, j \in \{3,4\}} \{|\phi_{ij}\rangle\}$, and $\mathcal{S}_3 = \cup_{i \in \mathbb{Z}_5} \{|\phi_{3i}\rangle\}$. In Figs. 5 and 6, we show the plane structures of $\cup_{i=1}^3 \mathcal{S}_i \bigcup \{|S_3\rangle\}$ in every bipartition.

In Lemma 1, $\mathcal{S}_1 \bigcup \{|S_1\rangle\}$ is the strongest nonlocal set in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$, this means that the matrix $E = (m_{ij,kl})_{ij,kl \in \mathbb{Z}_3 \times \mathbb{Z}_3}$ is proportional to an identity matrix. Since $E = E^\dagger$, if $m_{ij,kl} = 0$, there must be $m_{kl,ij} = 0$, where $ij \neq kl \in \mathbb{Z}_3 \times \mathbb{Z}_3$.

For the matrix $E_{BC} = (m_{ij,kl})_{ij,kl \in \mathbb{Z}_4 \times \mathbb{Z}_5}$. Consider $\mathcal{S}_2$ and $\mathcal{S}_1$ by Observations 1 and 2, like $|\phi_{23}\rangle$ and $\mathcal{S}_1$, we get $m_{23,ij} = 0$ for $ij \in \mathbb{Z}_3 \times \mathbb{Z}_3$; from $|\phi_{03}\rangle$, $|\phi_{13}\rangle$, and $\mathcal{S}_1$, we have $m_{03,ij} = 0$ and $m_{13,ij} = 0$, where $ij \in \mathbb{Z}_3 \times \mathbb{Z}_3$; for $|\phi_{24}\rangle$ and $\mathcal{S}_1$, we get $m_{24,ij} = 0$ for $ij \in \mathbb{Z}_3 \times \mathbb{Z}_3$; and from $|\phi_{04}\rangle$, $|\phi_{14}\rangle$, and $\mathcal{S}_1$, we obtain $m_{04,ij} = 0$ and $m_{14,ij} = 0$, where $ij \in \mathbb{Z}_3 \times \mathbb{Z}_3$. Consider $\mathcal{S}_2$, we get $m_{i3,j3} = m_{i4,j4} = m_{k3,l4} = 0$ for $i \neq j \in \mathbb{Z}_3$, $k, l \in \mathbb{Z}_3$. Consider $\mathcal{S}_3$ and $\mathcal{S}_1$, like $|\phi_{32}\rangle$ and $\mathcal{S}_1$, we get $m_{32,ij} = 0$ for $ij \in \mathbb{Z}_3 \times \mathbb{Z}_3$; given $|\phi_{3s}\rangle$ and $\mathcal{S}_1$, we have $m_{3s,ij} = 0$, where $s \in \{0, 1, 3, 4\}$, $ij \in \mathbb{Z}_3 \times \mathbb{Z}_3$. From $\mathcal{S}_3$ and $\mathcal{S}_2$, we get $m_{3i,kl} = 0$ for $i \in \mathbb{Z}_5$, $kl \in \{j3, j4 | j \in \mathbb{Z}_3\}$. Consider $\mathcal{S}_3$, we provide $m_{3i,3j} = 0$ for $i \neq j \in \mathbb{Z}_5$. Hereto we prove that all off-diagonal entries are zeros by Observations 1 and 2, i.e., $m_{ij,kl} = 0$ for $ij \neq kl$, $ij \in \{s3, s4, 3t | s \in \mathbb{Z}_3, t \in \mathbb{Z}_5\}$, $kl \in \mathbb{Z}_4 \times \mathbb{Z}_5$. Applying Observation 3 to $\mathcal{S}_2$, $\mathcal{S}_3$, and $|S\rangle$, we obtain that all diagonal entries are equal, i.e., $m_{ij,ij} = m_{kl,kl}$ for $ij \neq kl \in \mathbb{Z}_4 \times \mathbb{Z}_5$. Thus, the matrix $E_{BC} \propto \mathbb{I}_{BC}$.

Consider the matrix $E_{CA} = (m_{ij,kl})_{ij,kl \in \mathbb{Z}_5 \times \mathbb{Z}_3}$. Applying Observations 1 and 2 to $\mathcal{S}_2$ and $\mathcal{S}_1$, like $|\phi_{23}\rangle$ and $\mathcal{S}_1$, we get $m_{32,ij} = 0$ for $ij \in \mathbb{Z}_3 \times \mathbb{Z}_3$; from $|\phi_{03}\rangle$, $|\phi_{13}\rangle$, and $\mathcal{S}_1$, we have $m_{30,ij} = 0$ and $m_{31,ij} = 0$ for $ij \in \mathbb{Z}_3 \times \mathbb{Z}_3$; given $|\phi_{24}\rangle$
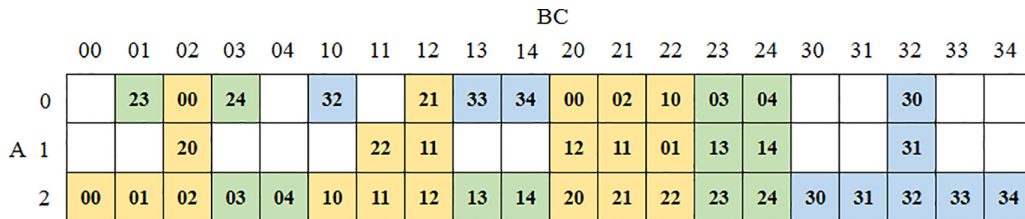

FIG. 5. The 3×20 plane structure of the strongest nonlocal set $\cup_{ij \in \mathbb{Z}_4 \times \mathbb{Z}_5} \{|\phi_{ij}\rangle\} \bigcup \{|S_3\rangle\}$ corresponds to the bipartition $A|BC$. The yellow, green, and blue entries correspond to $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{S}_3$, respectively.

FIG. 6. The 4×15 and 5×12 plane structures of the strongest nonlocal set $\cup_{ij \in \mathbb{Z}_4 \times \mathbb{Z}_5} \{|\phi_{ij}\rangle\} \bigcup \{|S_3\rangle\}$ correspond to the bipartitions $B|CA$ and $C|AB$.

$$
\begin{aligned}
\mathcal{A}_6 &= \{|\phi_{i(d_1+j)}\rangle = |\hat{d}_1\rangle_A |i\rangle_B |d_1+j\rangle_C - |i\rangle_A |\hat{d}_1\rangle_B |d_1+j\rangle_C \mid i \in \mathbb{Z}_{\hat{d}_1}, j \in \mathbb{Z}_{d_3-d_1}\}, \\
\mathcal{A}_7 &= \{|\phi_{\hat{d}_1 d_1}\rangle = |\hat{d}_1\rangle_A |\hat{d}_1\rangle_B |d_1\rangle_C - |0\rangle_A |0\rangle_B |1\rangle_C\}, \\
\mathcal{A}_8 &= \{|\phi_{\hat{d}_1(d_1+1+i)}\rangle = |\hat{d}_1\rangle_A |\hat{d}_1\rangle_B |d_1+1+i\rangle_C - |0\rangle_A |0\rangle_B |d_1+i\rangle_C \mid i \in \mathbb{Z}_{d_3-d_1-1}\}, \\
\mathcal{A}_9 &= \{|\phi_{(d_1+j)i}\rangle = |\hat{d}_1\rangle_A |d_1+j\rangle_B |i\rangle_C - |i\rangle_A |d_1+j\rangle_B |\hat{d}_1\rangle_C \mid i \in \mathbb{Z}_{\hat{d}_1}, j \in \mathbb{Z}_{d_2-d_1}\}, \\
\mathcal{A}_{10} &= \{|\phi_{d_1\hat{d}_1}\rangle = |\hat{d}_1\rangle_A |d_1\rangle_B |\hat{d}_1\rangle_C - |0\rangle_A |1\rangle_B |0\rangle_C\}, \\
\mathcal{A}_{11} &= \{|\phi_{(d_1+1+i)\hat{d}_1}\rangle = |\hat{d}_1\rangle_A |d_1+1+i\rangle_B |\hat{d}_1\rangle_C - |0\rangle_A |d_1+i\rangle_B |0\rangle_C \mid i \in \mathbb{Z}_{d_2-d_1-1}\}, \\
\mathcal{A}_{12} &= \{|\phi_{d_1(d_1+i)}\rangle = |\hat{d}_1\rangle_A |d_1\rangle_B |d_1+i\rangle_C - |0\rangle_A |1\rangle_B |d_1+i\rangle_C \mid i \in \mathbb{Z}_{d_3-d_1}\}, \\
\mathcal{A}_{13} &= \{|\phi_{(d_1+1+i)(d_1+j)}\rangle = |\hat{d}_1\rangle_A |d_1+1+i\rangle_B |d_1+j\rangle_C - |0\rangle_A |d_1+i\rangle_B |d_1+j\rangle_C \mid i \in \mathbb{Z}_{d_2-d_1-1}, j \in \mathbb{Z}_{d_3-d_1}\}, \\
|S\rangle &= \left(\sum_{i \in \mathbb{Z}_{d_1}} |i\rangle_A\right)\left(\sum_{j \in \mathbb{Z}_{d_2}} |j\rangle_B\right)\left(\sum_{k \in \mathbb{Z}_{d_3}} |k\rangle_C\right),
\end{aligned}
\tag{9}
$$

and $\mathcal{S}_1$, we get $m_{42,ij} = 0$ for $ij \in \mathbb{Z}_3 \times \mathbb{Z}_3$; and from $|\phi_{04}\rangle$, $|\phi_{14}\rangle$, and $\mathcal{S}_1$, we obtain $m_{40,ij} = 0$ and $m_{41,ij} = 0$ for $ij \in \mathbb{Z}_3 \times \mathbb{Z}_3$. Consider $\mathcal{S}_2$, we have $m_{3i,3j} = m_{4i,4j} = m_{3k,4l} = 0$ for $i \neq j \in \mathbb{Z}_3$, $k, l \in \mathbb{Z}_3$. Applying Observation 3 to $\mathcal{S}_2$ and $|S_3\rangle$, we present that $m_{3i,3i} = m_{4i,4i} = m_{10,10}$ for $i \in \mathbb{Z}_3$. So, the matrix E_{CA} is proportional to an identity matrix.

For the matrix $E_{AB} = (m_{ij,kl})_{ij,kl \in \mathbb{Z}_3 \times \mathbb{Z}_4}$. Consider states $|\phi_{30}\rangle$, $|\phi_{31}\rangle$, $|\phi_{32}\rangle$, and $\mathcal{S}_1$, we get $m_{i3,kl} = 0$ for $i \in \mathbb{Z}_3$, $kl \in \mathbb{Z}_3 \times \mathbb{Z}_3$. Applying Observations 1 and 2 to $|\phi_{30}\rangle$, $|\phi_{31}\rangle$, and $|\phi_{32}\rangle$, we yield $m_{i3,j3} = 0$ for $i \neq j \in \mathbb{Z}_3$. From $|\phi_{30}\rangle$, $|\phi_{31}\rangle$, $|\phi_{32}\rangle$, and $|S_3\rangle$ by Observation 3, we have $m_{03,03} = m_{13,13} = m_{23,23} = m_{01,01}$. Thus the POVM element E_{AB} is trivial, which completes the proof.

In Ref. [49], Li *et al.* proposed the strongest nonlocal set of size 22 in $\mathbb{C}^3 \otimes \mathbb{C}^4 \otimes \mathbb{C}^5$. Here, we construct the strongest nonlocal set of the smallest size 21, which reaches the low bound on the strongest nonlocal sets in $\mathbb{C}^3 \otimes \mathbb{C}^4 \otimes \mathbb{C}^5$.

Theorem 2. The set $\cup_{i=0}^{13} \mathcal{A}_i \bigcup \{|S\rangle\}$ of size $d_2 d_3 + 1$ given by Eqs. (5) and (9) is strongest nonlocal in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_3}$ for $3 \leqslant d_1 \leqslant d_2 \leqslant d_3$:

where $\hat{d}_1 = d_1 - 1$, and $\mathcal{A}_i$ ($i \in \mathbb{Z}_6$) are similar to the sets given by Eq. (5), except that d is replaced by d_1.

Proof. Denote that $\mathcal{B}_1 = \cup_{i=0}^5 \mathcal{A}_i$, $\mathcal{B}_2 = \cup_{i=6}^8 \mathcal{A}_i$, and $\mathcal{B}_3 = \cup_{i=9}^{13} \mathcal{A}_i$. In Theorem 1, $\mathcal{B}_1 \bigcup \{|S\rangle\}$ is the strongest nonlocal set in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_1}$, it implies that $m_{ij,kl} = 0$ and $m_{ij,ij} = m_{kl,kl}$ for $ij \neq kl \in \mathbb{Z}_{d_1} \times \mathbb{Z}_{d_1}$. In order to show that the matrices E_{BC}, E_{CA}, and E_{AB} are trivial, we divide all off-diagonal entries into two parts in every bipartition, i.e., the key and the remaining off-diagonal entries.

In the bipartition $A|BC$, the key off-diagonal entries are $m_{ij,kl}$ for $ij \neq kl \in \mathcal{K}_1$, where $\mathcal{K}_1 = \{d_1^* d_1^*, i\hat{d}_1, \hat{d}_1 j \mid d_1^* = d_1 - 2, i \in \mathbb{Z}_{d_2}, j \in \mathbb{Z}_{d_3}\}$; the remaining off-diagonal entries in the matrix E_{BC} are $m_{pq,st}$ for $pq \neq st$, $pq \in \mathbb{Z}_{d_2} \times \mathbb{Z}_{d_3} \backslash \mathcal{K}_1$, and $st \in \mathbb{Z}_{d_2} \times \mathbb{Z}_{d_3}$. Applying Observations 1 and 2 to the sets $\mathcal{B}_1$, $\mathcal{A}_7$, $\mathcal{A}_8$, $\mathcal{A}_{10}$, and $\mathcal{A}_{11}$, we get that all key off-diagonal entries are zeros. Since $m_{ij,kl} = m_{kl,ij} = 0$ for $ij \neq kl \in \mathcal{K}_1$, applying Observations 1 and 2 to $\mathcal{B}_1$, $\mathcal{B}_2$, and $\mathcal{B}_3$, we obtain that all off-diagonal entries are zeros. Consider $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$, and $|S\rangle$ by Observations 3 and 4, and we have

$m_{ij,ij} = m_{kl,kl}$ for $ij \neq kl \in \mathbb{Z}_{d_2} \times \mathbb{Z}_{d_3}$. Thus, the matrix E_{BC} is trivial.

Consider the bipartition $B|CA$, the key off-diagonal entries are $m_{ij,kl}$ for $ij \neq kl \in \mathcal{K}_2$, where $\mathcal{K}_2 = \{d_1^* d_1^*, j\hat{d}_1 \mid d_1^* = d_1 - 2, j \in \mathbb{Z}_{d_3}\}$; the remaining off-diagonal entries in the matrix E_{CA} are $m_{pq,st}$ for $pq \neq st$, $pq \in \mathbb{Z}_{d_3} \times \mathbb{Z}_{d_1} \backslash \mathcal{K}_2$, and $st \in \mathbb{Z}_{d_3} \times \mathbb{Z}_{d_1}$. Applying Observations 1 and 2 to $\mathcal{B}_1$, $\mathcal{A}_7$, and $\mathcal{A}_8$, we obtain that $m_{ij,kl} = m_{kl,ij} = 0$ for $ij \neq kl \in \mathcal{K}_2$. Based on the fact that all key off-diagonal entries are zeros, applying Observations 1 and 2 to $\mathcal{B}_1$ and $\mathcal{B}_2$, we have $m_{ij,kl} = m_{kl,ij} = 0$ for $ij \neq kl \in \mathbb{Z}_{d_3} \times \mathbb{Z}_{d_1}$. From $\mathcal{B}_1$, $\mathcal{B}_2$, and $|S\rangle$ by applying Observations 3 and 4, it implies that $m_{ij,ij} = m_{kl,kl}$ for $ij \neq kl \in \mathbb{Z}_{d_3} \times \mathbb{Z}_{d_1}$. Therefore, the matrix $E_{CA} \propto \mathbb{I}_{CA}$.

For the bipartition $C|AB$, the key off-diagonal entries are $m_{ij,kl}$ for $ij \neq kl \in \mathcal{K}_3$, where $\mathcal{K}_3 = \{d_1^* d_1^*, \hat{d}_1 i \mid d_1^* = d_1 - 2, i \in \mathbb{Z}_{d_2}\}$; the remaining off-diagonal entries in the matrix E_{AB} are $m_{pq,st}$ for $pq \neq st$, $pq \in \mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2} \backslash \mathcal{K}_3$, and $st \in \mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2}$. According to $\mathcal{B}_1$, $\mathcal{A}_{10}$, and $\mathcal{A}_{11}$ by applying Observations 1 and 2, we obtain that all key off-diagonal entries

are zeros. Since $m_{ij,kl} = m_{kl,ij} = 0$ for $ij \neq kl \in \mathcal{K}_3$, applying Observations 1 and 2 to $\mathcal{B}_1$ and $\mathcal{B}_3$, we get that all off-diagonal entries are zeros. Considering $\mathcal{B}_1$, $\mathcal{B}_3$ and $|S\rangle$ by Observations 3 and 4, we deduce $m_{ij,ij} = m_{kl,kl}$ for $ij \neq kl \in \mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2}$. Hence, the matrix $E_{AB} \propto \mathbb{I}_{AB}$. To sum up, we successfully show that the set $\cup_{i=1}^{3} \mathcal{B}_i \bigcup \{|S\rangle\}$ is the strongest nonlocal set in general tripartite systems. ∎

V. CONCLUSION

In this paper, we constructed the strongest nonlocal sets with minimum cardinality in general tripartite systems. Our result positively answers an open conjecture proposed in Ref. [47]. From Table I, the size of the strongest nonlocal sets we constructed is the smallest of all previous results in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_3}$. In particular, the strongest nonlocal set given by Theorem 1 consists of d^2 orthogonal genuine

entangled states except the stopper state, and the strongest nonlocal set given by Theorem 2 contains $d_2 d_3$ orthogonal entangled states except the stopper state.

There are still some open questions left to be solved. In general N-partite systems $\otimes_{i=1}^{N} \mathbb{C}^{d_i}$, can we construct the strongest nonlocal set of the smallest size $\max_i \{\prod_{j=1}^{n} d_j / d_i + 1\}$? Can we further improve the lower bound on the strongest nonlocal sets?

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