Epsilon measures of state-based quantum resource theory

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The quantification of state-based quantum "resources" such as entanglement, coherence, and nonstabilizer states lies at the heart of quantum science and technology, providing potential advantages over classical methods. In a realistic scenario, due to the imperfections and uncertainties in physical devices, we are unable to perfectly prepare or detect the true quantum states. Consequently, it is necessary to study the quantification of quantum resources under such circumstances. In this work, by focusing on the state-based quantum resource theory, we introduce a family of resource measures called ϵ measure that relies on a precision parameter to address this issue. This family of resource measures inherits the fundamental properties of the original resource measure, such as weak monotonicity, convexity, monogamy, and so forth. Furthermore, the ϵ measure remains continuous irrespective of whether the original measure is continuous or not. We also investigate the ϵ measure of distance-based resource quantifiers, and some interesting properties are presented. As part of the applications, we derived several formulas for the ϵ measure of coherence, nonstabilizerness, asymmetry, nonuniformity, and imaginarity. Additionally, we offered an upper bound for the ϵ measure of the resource rank. Finally, we outline how this work can be extended to channel-based resources among others.

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I. INTRODUCTION

Quantum resource theories (QRTs) provide a highly versatile and powerful framework for exploring various phenomena in quantum physics [1–15]. They involve the examination and analysis of the available resources in quantum systems and how these resources can be harnessed for specific quantum tasks. Quantum resources encompass various aspects, including entanglement [7,11,16], coherence [8], and nonstabilizer states [10,17–19], among others. These resources play a pivotal role in enabling quantum computation [10,20–22], quantum communication [23], and quantum information processing [5,24,25]. In general, depending on the nature of the research focus, a QRT can be divided into two categories: static QRT and dynamical QRT. The study object of static QRT is quantum states, hence, it is also referred to as state-based QRT. On the other hand, dynamical QRT covers various scenarios, including quantum channel [26,27], quantum incompatibility [28,29], and measurement sharpness [30,31], and more [32]. This article primarily investigates state-based QRT, and unless explicitly stated otherwise, any reference to QRT in the text refers to state-based QRT. A QRT is characterized by a set of free states and a corresponding set of free operations that preserve the free states. States that do not belong to the set of free states are considered to possess resources [2]. For example, in the QRT of entanglement, the free states can be considered as separable states, and the free operations are local operation and classical communication (LOCC). This leads to two fundamental problems in QRTs: quantification and conversion of resources [33,34]. The goal of resource quantification is to quantify the amount of the resource in a quantum state. The conversion of resources is asking whether one resource state can be converted into the other state via a free operation. There is also a profound connection between these two problems: Intuitively, a quantum state with a larger "resource quantity" is more valuable and can be transformed into a greater variety of other quantum states via free operations. For example, in the QRTs of entanglement, accessible entanglement characterizes the proficiency of a state to generate other states via LOCC, whereas the source entanglement characterizes the set of states that can be reached via LOCC acting on the given state of interest [35]. This type of quantifier has also been studied in the context of QRT of coherence [36]. In this paper, we propose ϵ measure of QRT which depends on a precision parameter ϵ, following the logistics given in Refs. [37,38]. Differing from the one presented in Refs. [37,38] solely on specific quantum resources like entanglement and coherence, this work explores ϵ measure within a broader framework of general QRTs: any QRT admitting the tensor-product structure assumption, as elaborated later. Within this framework, we investigate some fundamental properties and applications of the ϵ measure. It is worth noting that this framework not only encompasses quantum resources like entanglement and coherence but also extends to other quantum resources such as asymmetry [39,40], nonstabilizerness, imaginarity [14,41,42], nonuniformity [43,44], and more. Additionally, there are three

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reasons we introduce ϵ measure of QRT. First, the most fundamental state transformation in a QRT is the one-shot convertibility, which involves converting one resource state to another using the free operations of the theory [33,34]. However, in the most QRTs, it is generally not possible to achieve a perfect transformation of one specific state to another using only the free operations, or vice versa. Therefore, it becomes crucial to characterize the ability to efficiently distill resource states from prepared states or dilute a unit resource state to the target state. As demonstrated in the main results, we show that the ϵ measure of QRT can be utilized in the one-shot scenario to provide a lower bound on the one-shot dilution cost of QRT. The second reason is that, realistically, any physical apparatus for quantum information processing can only achieve a certain degree of precision and reliability. In this realistic scenario, we need to estimate the amount of resource in a state that is only partially known [45,46]. To address this issue, when considering a resource measure R, we can assume that the amount of resource of the output state $\tilde{\rho}$ is related to the true state ρ through a parameter ϵ, which depends on the realistic processing. Thus, in order to estimate the true amount $R(\rho)$, we utilize the ϵ measure R_ϵ, which is guaranteed to be present in the system, given that the output state $\tilde{\rho}$ has some approximation ϵ. Mathematically, R_ϵ quantifies the minimum guaranteed amount of resource, under the condition that the state $\tilde{\rho}$ which has been prepared is within a distance ϵ from some state ρ. The third reason for introducing the ϵ measure of QRT is that resource measures do not necessarily require smoothness. This can be exemplified by the resource measure called logarithmic robustness $R_{\rm lr}$, which can be defined as [47]

$$R_{\rm lr}(\rho) = \inf_{\sigma\in\mathcal{F}} D_{\max}(\rho||\sigma), \tag{1}$$

where $\mathcal{F}$ is the set of free states and $D_{\max}(\rho||\sigma) = \log_2 \min\{\lambda|\rho \leqslant \lambda\sigma\}$ is the max-relative entropy [48]. The lack of smoothness in logarithmic robustness primarily stems from the inherent discontinuity of the max-relative entropy. For instance, consider two quantum states [49]:

$$\rho = \left(\tfrac{1}{2} - \epsilon\right)|0\rangle\langle 0| + \tfrac{1}{2}|1\rangle\langle 1| + \epsilon|2\rangle\langle 2|,$$
$$\sigma = \tfrac{1}{2}|0\rangle\langle 0| + \tfrac{1}{2}|1\rangle\langle 1|.$$

It is easy to see from the definition of max-relative entropy that when $\epsilon \in (0, \frac{1}{2}]$, $D_{\max}(\rho||\sigma) = +\infty$, while for $\epsilon = 0$, $D_{\max}(\rho||\sigma) = 0$. As discussed before, in a realistic scenario, preparing a physical system in a state ρ inevitably introduces some error, causing a small discrepancy (in terms of some distance) between the expected state ρ and the actually prepared state with some small $\epsilon > 0$. Therefore, discontinuous resource measures may lack practical physical significance without undergoing a smoothing process. Hence, the introduction of the ϵ measure R_ϵ becomes necessary to overcome these issues. Moreover, the ϵ measure R_ϵ inherits the fundamental properties of the original resource measure R, such as weak monotonicity, convexity, monogamy, and so forth. This paper is organized as follows. In Sec. II, we first introduce the necessary notation and definitions we need. In Sec. III, we provide our main results. In Sec. IV, we present some examples as applications. Specifically, we derive several formulas for the ϵ measure of QRT and provide an upper bound for the ϵ measure of the resource rank. We outline how the our results can be extended to the channel-based resource in Sec. V. We summarize our results in Sec. VI.

II. ϵ MEASURE OF QRT

We begin by introducing the framework of QRT of quantum states, in which quantum states are the central objects under study. We consider a Hilbert space $\mathcal{H}$ of finite dimension d. The density matrix space in $\mathcal{H}$ is denoted as $\mathcal{D}(\mathcal{H})$. We use the 2-tuple $\mathcal{R} = (\mathcal{F}, \mathcal{O})$ to denote a QRT, the set of free states is denoted by $\mathcal{F}$, and the set of free operations is denoted by $\mathcal{O}$. They are subsets of quantum states and quantum channels [completely positive trace-preserving (CPTP) maps], respectively, and are considered to be accessible for free within a given framework. The basic two requirements for a functional $R : \mathcal{D}(\mathcal{H}) \to \mathbb{R}^+$ being a resource measure for $(\mathcal{F}, \mathcal{O})$ are [1,2] as follows:

[A1] (Non-negativity): $R(\rho) \geqslant 0$ and the equality holds if and only if $\rho \in \mathcal{F}$. [A2] (Weak monotonicity): $R(\rho) \geqslant R(\Lambda(\rho))$, where $\Lambda \in \mathcal{O}$ is a CPTP map.

[A1] and [A2] are the minimal requirements of a QRT, in practice there are other natural properties that one might desire in a QRT:

[A3] (Strong monotonicity): $R(\rho) \geqslant \sum_k p_k R(\Lambda_k(\rho)/p_k)$, where $\Lambda = \sum_k \Lambda_k \in \mathcal{O}$ with Λ_ks are CP trace nonincreasing maps and $p_k = \mathrm{Tr}[\Lambda_k(\rho)]$.

[A4] (Convexity): $\sum_k p_k R(\rho_k) \geqslant R(\sum_k p_k \rho_k)$ for any ensemble $\{p_k, \rho_k\}$.

In practical terms, one may have additional desirable properties in a QRT. These can be most evidently grouped together under what we will call a *tensor-product structure*. A QRT admits a tensor-product structure if the following two free operations are allowed [1,2]:

[B1] (Appending free states): Appending a free state is a free operation, i.e., for any state $\sigma \in \mathcal{F}$, the operation $\Lambda(\rho) = \rho \otimes \sigma$ is a free operation.

[B2] (Discarding a system): Discarding a system is a free operation, i.e., the partial trace $\mathrm{Tr}_B(\rho^{AB}) = \rho^A$ is a free operation from $\mathcal{H}_{AB}$ to $\mathcal{H}_A$.

Remark. Most of the physically motivated and previously studied QRTs admit a tensor-product structure, such as the QRTs of entanglement [11], coherence [9], asymmetry [50], nonstabilizerness [10], imaginarity [14,41,42], and athermality [13,51]. However, there are less intuitive but still important QRTs that do not possess a tensor-product structure, such as superactivation of quantum nonlocality [52]. The distance function D serves as a crucial tool in quantum information theory [53]. The distance must fulfill non-negativity, symmetry, and the triangle inequality. While certain functions may not meet these fundamental distance conditions, they are extensively utilized in information theory, such as relative entropy $S(\rho||\sigma)$. In this paper, we impose an additional condition on the distance D-convexity or joint convexity, i.e.,

$$D\left(\sum_i p_i\rho_i, \sum_i p_i\sigma_i\right) \leqslant \sum_i p_i D(\rho_i, \sigma_i), \tag{2}$$

where $p_i \geqslant 0$ and $\sum_i p_i \geqslant 0$. In quantum information theory, it is crucial for the distance D to be contractive under CPTP maps [53]:

$$D(\Lambda(\rho), \Lambda(\sigma)) \leqslant D(\rho, \sigma). \tag{3}$$

Now, we will introduce the ϵ measure of QRT:

Definition 1 (ϵ measure of QRT). For any resource measure R, the ϵ measure of QRT of $\rho \in \mathcal{D}(\mathcal{H})$ is defined as

$$R_\epsilon^{(D)}(\rho) = \inf\left\{R(\sigma)|\sigma \in B_\epsilon^{(D)}(\rho)\right\}, \tag{4}$$

where $B_\epsilon^{(D)}(\rho) = \{\sigma \in \mathcal{D}(\mathcal{H})|D(\rho, \sigma) \leqslant \epsilon\}$ is the ϵ-Ball of ρ, and D is the distance between two quantum states ρ and σ.

Note that the distance and the resource measure function may coincide, or they may diverge. Consequently, we will refrain from using the superscript D to prevent potential confusion. In essence, the ϵ measure of the resource, denoted as R_ϵ, characterizes the smallest quantity of resource ensured to exist within an ϵ-Ball centered on the specified quantum state ρ. This approach furnishes a method to acquire a continuous function, even in cases where the original measure R lacks this property.

III. MAIN RESULTS

A. Properties

In this section, we will study some properties of ϵ measure of QRT.

Proposition 1. For any resource measure R, the ϵ measure R_ϵ is also a resource measure.

Proof. First, we observe that for any $\rho \in \mathcal{F}$, $0 \leqslant R_\epsilon(\rho) \leqslant R(\rho) = 0$. Second, σ^* is the optimal solution of Eq. (4), as $R(\sigma^*) \geqslant R[\Lambda(\sigma^*)]$ which implies that

$$\begin{aligned} R_\epsilon(\rho) &= R(\sigma^*) \\ &\geqslant R[\Lambda(\sigma^*)] \\ &\geqslant \inf\left\{R(\sigma)|\sigma \in B_\epsilon^{(D)}[\Lambda(\rho)]\right\} \\ &= R_\epsilon[\Lambda(\rho)]. \end{aligned}$$

The second inequality hold is due to the contractive of distance D: $\epsilon \geqslant D(\rho, \sigma) \geqslant D(\Lambda(\rho), \Lambda(\sigma)) \Rightarrow \Lambda(\sigma) \in B_\epsilon^{(D)}[\Lambda(\rho)]$. ■

The requirement for the distance D to be contractive under CPTP maps is a crucial property for R_ϵ satisfying the minimal requirements ([A1] and [A2]) to be a ϵ measure, as demonstrated in Refs. [37,38]. Convexity is another essential property for any resource measure. The following proposition demonstrates that the requirement for the distance D to be jointly convex is crucial for R_ϵ to satisfy condition [A4]:

Proposition 2. For any convex resource measure $R(\rho)$, the ϵ measure $R_\epsilon(\rho)$ is also a convex measure for any jointly convex distance D.

Proof. Let $\sigma_1^* \in B_\epsilon^{(D)}(\rho_1)$ and $\sigma_2^* \in B_\epsilon^{(D)}(\rho_2)$ be the optimal solution of $R_\epsilon(\rho_1)$ and $R_\epsilon(\rho_2)$, respectively. Then the following equality holds:

$$\begin{aligned} pR_\epsilon(\rho_1) + (1-p)R_\epsilon(\rho_2) &= pR(\sigma_1^*) + (1-p)R(\sigma_2^*) \\ &\overset{\text{(a)}}{\geqslant} R[p\sigma_1^* + (1-p)\sigma_2^*] \\ &\overset{\text{(b)}}{\geqslant} R_\epsilon[p\rho_1 + (1-p)\rho_2]. \end{aligned}$$

Here, (a) is a result of the convexity of R, and (b) is due to the following equality: $D(p\rho_1 + (1-p)\rho_2, p\sigma_1^* + (1-p)\sigma_2^*) \leqslant pD(\rho_1, \sigma_1^*) + (1-p)D(\rho_2, \sigma_2^*) \leqslant \epsilon$. ■

Common distance functions satisfying jointly convexity include trace distance [54], squared Bures distance [55], squared Hellinger distance [55], etc. Note that the relative entropy is also jointly convex. Strong monotonicity is another desired property for any resource measure. The following result shows that even if R satisfies strong monotonicity, R_ϵ may not necessarily satisfy strong monotonicity.

Proposition 3. R satisfies the [A3] $\not\Rightarrow$ R_ϵ satisfies the [A3].

Proof. Let us consider a state $\rho = p\sigma \otimes |0\rangle\langle 0| + (1-p)\tau \otimes |1\rangle\langle 1|$, where $R_\epsilon(\sigma) > 0$ and $\tau \in \mathcal{F}$. Without loss of generality, we assume that D is convex. For $p \leqslant \frac{\epsilon}{D(\sigma,\tau)}$, we have

$$\begin{aligned} D(\rho, \tau \otimes |1\rangle\langle 1|) &\overset{\text{(a)}}{\leqslant} pD(\sigma \otimes |0\rangle\langle 0|, \tau \otimes |1\rangle\langle 1|) \\ &\overset{\text{(b)}}{\leqslant} pD(\sigma, \tau) \\ &\overset{\text{(c)}}{\leqslant} \epsilon. \end{aligned}$$

Here, (a) is due to the convexity of D, (b) is due to the condition [B2], and (c) follows from $p \leqslant \frac{\epsilon}{D(\sigma,\tau)}$. This implies $R_\epsilon(\rho) = 0$ [since $0 \leqslant R_\epsilon(\rho) \leqslant R_\epsilon(\tau \otimes |1\rangle\langle 1|) \leqslant R_\epsilon(\tau) = 0$]. Therefore, for the state ρ, we have $pR_\epsilon(\sigma \otimes |0\rangle\langle 0|) + (1-p)R_\epsilon(\tau \otimes |1\rangle\langle 1|) = pR_\epsilon(\sigma) > 0 = R_\epsilon(\rho)$. ■

In general, it is challenging to verify whether a resource measure satisfies strong monotonicity. However, a property called flag additivity can simplify this problem. For example, in the QRT of coherence, directly verifying coherence based on trace distance is difficult [56]. Nonetheless, leveraging flag additivity allows for a direct demonstration that coherence based on trace distance does not satisfy strong monotonicity [15,57,58]. Therefore, it is necessary to investigate whether R_ϵ satisfies flag additivity. The flag additivity can be defined as

$$R\left(\sum_{i=1}^d p_i\rho_i^A \otimes |\psi_i\rangle\langle\psi_i|^B\right) = \sum_{i=1}^d p_i R\left(\rho_i^A\right). \tag{5}$$

The equation above holds for the "flag basis" $\{|\psi_i\rangle\langle\psi_i|\}$ which satisfies (i) $|\psi_i\rangle \in \mathcal{F}$ for $i \in [d]$, and (ii) the projective measurement $\{|\psi_i\rangle\langle\psi_i|\} \in \mathcal{O}$ for $i \in [d]$. In Ref. [15], the authors have demonstrated that for any resource measure R, the flag additivity holds if and only if [A3] and [A4] hold. Thus, as a direct corollary of Proposition 3, even if R satisfies flag additivity, R_ϵ may not necessarily satisfy flag additivity.

Remark. As demonstrated in Appendix A, any QRT admits tensor-product structure, the resource measure of QRT which is based on quantum relative Rényi entropy D_α, sandwiched Rényi entropy $\tilde{D}_\alpha$, and relative entropy $S(\rho||\sigma)$ satisfies the flag additivity, where $\alpha \in [1, 2]$. Furthermore, we provide an illustrative example to showcase the practical application of flag additivity in Appendix A. While R_ϵ does not satisfy flag additivity, it adheres to the following equation as a resource measure:

$$R_\epsilon(\rho_A \otimes \sigma_B) = R_\epsilon(\rho_A), \tag{6}$$

where $\sigma_B \in \mathcal{F}$. This is a result of $R_\epsilon(\rho_A) \overset{\text{[B1]}}{\geqslant} R_\epsilon(\rho_A \otimes \sigma_B) \overset{\text{[B2]}}{\geqslant} R_\epsilon(\rho_A)$. Therefore, Eq. (6) can be interpreted as a "weak" form of flag additivity that holds for any resource measure. Subadditivity is an important property for resource measures in quantum information theory, playing a crucial role in the analysis of many information processing processes [59–62]. A resource measure R is subadditive if $R(\rho) + R(\sigma) \geqslant R(\rho \otimes \sigma)$ holds for any $\rho, \sigma \in \mathcal{D}(\mathcal{H})$. For any convex QRT [a QRT $\mathcal{R} = (\mathcal{F}, \mathcal{O})$ is convex if $\mathcal{O}$ is convex. The QRT of entanglement, coherence, asymmetry, and athermality are all convex], which admits a tensor-product structure, the resource measures of QRT based on quantum relative Rényi entropy D_α, sandwiched Rényi entropy $\tilde{D}_\alpha$, and relative entropy $S(\rho||\sigma)$ satisfy subadditivity, where $\alpha \in [1, 2]$ [1]. In general, the subadditivity of R does not imply the subadditivity of R_ϵ. However, the following result shows that R_ϵ exhibits a "weak" subadditivity, where subadditivity of R can be obtained by setting ϵ to 0.

Proposition 4. If R is a subadditive resource measure, then for any $\epsilon = \epsilon_1 + \epsilon_2$ $(\epsilon_1, \epsilon_2 \geqslant 0)$, R_ϵ satisfies the following:

$$R_{\epsilon_1}(\rho_1) + R_{\epsilon_2}(\rho_2) \geqslant R_\epsilon(\rho_1 \otimes \rho_2). \tag{7}$$

Here, the distance D considered for $B_\epsilon^{(D)}(\rho)$ is the purified distance $D(\rho, \sigma) = P(\rho, \sigma) := \sqrt{1 - F^2(\rho, \sigma)}$ with $F(\rho, \sigma) := \mathrm{Tr}|\sqrt{\rho}\sqrt{\sigma}|$ being fidelity [63].

Proof. First, note that $D(\rho, \sigma) = \sqrt{1 - F^2(\rho, \sigma)} \leqslant \epsilon \iff F^2(\rho, \sigma) \geqslant 1 - \epsilon^2$. Then, we have

$$\begin{aligned} F^2(\rho_1 \otimes \rho_2, \sigma_1 \otimes \sigma_2) &= F^2(\rho_1, \sigma_1) F^2(\rho_2, \sigma_2) \\ &\geqslant \left(1 - \epsilon_1^2\right)\left(1 - \epsilon_2^2\right) \\ &= 1 - \left(\epsilon_1^2 + \epsilon_2^2\right) + \epsilon_1^2 \epsilon_2^2 \\ &\geqslant 1 - (\epsilon_1 + \epsilon_2)^2, \end{aligned}$$

which implies $D(\rho_1 \otimes \rho_2, \sigma_1 \otimes \sigma_2) \leqslant \epsilon = \epsilon_1 + \epsilon_2$. Then, let $\sigma_1^* \in B_{\epsilon_1}^{(D)}(\rho_1)$ and $\sigma_2^* \in B_{\epsilon_2}^{(D)}(\rho_2)$ be the optimal solutions of $R_{\epsilon_1}(\rho_1)$ and $R_{\epsilon_2}(\rho_2)$, respectively. The following inequality holds:

$$\begin{aligned} R_{\epsilon_1}(\rho_1) + R_{\epsilon_2}(\rho_2) &= R(\sigma_1^*) + R(\sigma_2^*) \\ &\overset{\text{(a)}}{\geqslant} R(\sigma_1^* \otimes \sigma_2^*) \\ &\overset{\text{(b)}}{\geqslant} \inf_{\sigma \in B_\epsilon^{(D)}(\rho_1 \otimes \rho_2)} R(\sigma) \\ &= R_\epsilon(\rho_1 \otimes \rho_2), \end{aligned}$$

where (a) is the result of the subadditivity of R, and (b) is due to $D(\rho_1 \otimes \rho_2, \sigma_1^* \otimes \sigma_2^*) \leqslant \epsilon = \epsilon_1 + \epsilon_2$. ■

Next, we will investigate the continuity of the resource measure. First, we present the following proposition:

Proposition 5. Let D be a convex distance and R be a convex and bounded functional. For any two quantum states ρ_1 and ρ_2 with distance $D(\rho_1, \rho_2) = \eta$. Let $R_{\epsilon_i}(\rho_i) = R(\sigma_i)$ for $i = 1, 2$, and consider the state $\tau_\lambda = (1 - \lambda)\rho_1 + \lambda\sigma_2$ with $0 \leqslant \lambda \leqslant \frac{\epsilon_1}{\epsilon_1 + \eta}$. We have the following relation:

$$R_{\epsilon_1}(\rho_1) - R_{\epsilon_2}(\rho_2) \leqslant (1 - \lambda)[R(\rho_1) - R_{\epsilon_2}(\rho_2)]. \tag{8}$$

Proof. Since D is a convex distance, for the state τ_λ, we have

$$\begin{aligned} D(\rho_1, \tau_\lambda) &= D(\rho_1, (1 - \lambda)\rho_1 + \lambda\sigma_2) \\ &\leqslant \lambda D(\rho_1, \sigma_2) \\ &\leqslant \lambda D(\rho_1, \rho_2) + D(\rho_2, \sigma_2) \\ &\leqslant \lambda(\eta + \epsilon_2) \\ &\leqslant \epsilon_1, \end{aligned}$$

where we have used the triangle inequality in the second inequality and in the final equality, we have the condition $0 \leqslant \lambda \leqslant \frac{\epsilon_1}{\epsilon_1 + \eta}$. Consequently, the following inequality holds:

$$\begin{aligned} R_{\epsilon_1}(\rho_1) &\leqslant R(\tau_\lambda) \\ &\leqslant (1 - \lambda)R(\rho_1) + \lambda R(\sigma_2) \\ &= (1 - \lambda)R(\rho_1) + \lambda R_{\epsilon_2}(\rho_2). \end{aligned} \tag{9}$$

This implies

$$R_{\epsilon_1}(\rho_1) - R_{\epsilon_2}(\rho_2) \leqslant (1 - \lambda)[R(\rho_1) - R_{\epsilon_2}(\rho_2)]. \tag{10}$$

■

As a by-product of the above proposition, we obtain the following two corollaries.

Corollary 1. If $\epsilon_1 = \epsilon_2 = \epsilon$ and $\rho_1 \neq \rho_2$, let $\lambda = \frac{\epsilon}{\epsilon + \eta}$. Then

$$R_\epsilon(\rho_1) - R_\epsilon(\rho_2) \leqslant \frac{\eta}{\eta + \epsilon}[R(\rho_1) - R_\epsilon(\rho_2)]. \tag{11}$$

Furthermore, if $D(\rho_1, \rho_2) = \eta \to 0$, $|R_\epsilon(\rho_1) - R_\epsilon(\rho_2)| \to 0$.

Proof. In order to complete our proof, we only need to show the lower bound of Eq. (11) tends to 0 as $\eta \to 0$. Note that we can exchange the role of ρ_1 and ρ_2 in Eq. (11). Then, we have

$$R_\epsilon(\rho_1) - R_\epsilon(\rho_2) \geqslant -\frac{\eta}{\eta + \epsilon}[R(\rho_2) - R_\epsilon(\rho_1)]. \tag{12}$$

Let $M = \max\{R(\rho_1) - R_\epsilon(\rho_2), R(\rho_2) - R_\epsilon(\rho_1)\}$, then $|R_\epsilon(\rho_1) - R_\epsilon(\rho_2)| \leqslant \frac{\eta}{\eta + \epsilon} M \to 0$ as $\eta \to 0$. ■

This indicates that the ϵ measure of QRT is always continuous regardless of whether the original convex resource measure function is continuous with respect to the density matrix ρ.

Corollary 2. If $\rho_1 = \rho_2 = \rho$ and $\epsilon_2 \geqslant \epsilon_1 > 0$, let $\lambda = \frac{\epsilon_1}{\epsilon_2}$. Then

$$R_{\epsilon_1}(\rho) - R_{\epsilon_2}(\rho) \leqslant \frac{\epsilon_2 - \epsilon_1}{\epsilon_2}[R(\rho) - R_{\epsilon_2}(\rho)]. \tag{13}$$

Furthermore, if $\epsilon_2 - \epsilon_1 \to 0$, then $|R_{\epsilon_1}(\rho) - R_{\epsilon_2}(\rho)| \to 0$.

It shows that ϵ measure of QRT is continuous of ϵ.

B. Monogamy of correlated resource

Similar to subadditivity, strong monotonicity, and convexity in QRTs, monogamy stands out as a pivotal property of correlated resources, such as entanglement [64–68] and discord [69,70]. This principle asserts that correlated resources cannot be freely distributed within a multipartite quantum system. While not always considered essential in quantitative analyses, monogamy has proven its significance in various quantum information tasks and other domains of physics.

Applications include quantum key distribution [71], the classification of quantum states [72], condensed-matter physics [73], frustrated spin systems [74], and even black-hole physics [75].

The correlated resource is referred to the resource measure R of bipartite states $\rho_{AB} \in \mathcal{D}(\mathcal{H}_{AB})$, which satisfies the following conditions [70]:

(1) $R(\rho_{AB}) \geqslant 0$;

(2) $R(\rho_{AB}) = R(U_A \otimes U_B \rho_{AB} U_A \otimes U_B)$ where U_A, U_B are local unitary matrixes;

(3) $R(\rho_{AB}) \geqslant R_{A|BC}(\rho_{AB} \otimes |0\rangle\langle 0|_C)$. In this paper, we adopt the definition of monogamy as stated in Ref. [64]. If there exists a nontrivial function $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, such that

$$R(\rho_{A|BC}) \geqslant f(R(\rho_{AB}), R(\rho_{AC})) \tag{14}$$

holds for any correlated resource measure R [64–68,70]. We refer to this property as monogamy. It is often assumed that $f(x, y)$ is monotonically increasing and continuous. For example, a common choice is $f(x, y) = (x^\alpha + y^\alpha)^{\frac{1}{\alpha}}$ with $\alpha > 0$, which has been explored in the context of various correlated measures [65–68,76,77]. It is worth noting that the CKW-type inequality can be derived when $\alpha = 1$ [65,66]. Based on the above assumptions, we obtain the following result:

Proposition 6. For any correlated resource measure R, R is monogamous [i.e., R satisfies Eq. (14)] implies that R_ϵ is monogamous.

Proof. Suppose $\tau^*_{ABC} \in B^{(D)}_\epsilon(\rho_{ABC})$ reaches the optimal solution of $R_\epsilon(\rho_{A|BC})$, and we obtain

$$\begin{aligned} R_\epsilon(\rho_{A|BC}) &= \min\left\{R(\tau_{A|BC}) | \tau_{ABC} \in B^{(D)}_\epsilon(\rho_{ABC})\right\} \\ &= R(\tau^*_{A|BC}) \\ &\geqslant f(R(\tau^*_{AB}), R(\tau^*_{AC})) \\ &\geqslant f(R_\epsilon(\rho_{AB}), R_\epsilon(\rho_{AC})). \end{aligned}$$

where $\tau^*_{AB} \in B^{(D)}_\epsilon(\rho_{AB})$ and $\tau^*_{AC} \in B^{(D)}_\epsilon(\rho_{AC})$ in the first inequality, and we have used monotonicity of f in the last inequality. ■

This result shows that if the original correlated measure R satisfies monogamy, then the correlated measure R_ϵ with smooth parameter ϵ will also satisfy the corresponding monogamy relation. In practical experiments, we need to estimate the amount of correlated resource in a state that is only partially known [45,46]. Proposition 13 has practical significance for experimentally verifying whether an correlated measure satisfies monogamy properties. As an application, we have provided an example in Appendix B demonstrating how Proposition 13 can be employed to differentiate between Greenberger-Horne-Zeilinger (GHZ) states and W states within the context of the QRT of discord [1,72].

C. ϵ measure based on the distance measures

In this section, we will consider following distance-based resource quantifier:

$$R_D(\rho) = \min_{\sigma\in\mathcal{F}} D(\rho, \sigma). \tag{15}$$

It is easy to show that distance-based resource quantifier is a resource measure

$$\begin{aligned} R_D[\Lambda(\rho)] &= \min_{\sigma\in\mathcal{F}} D(\Lambda(\rho), \sigma) \\ &\leqslant \min_{\Lambda(\sigma)\in\mathcal{F}} D(\Lambda(\rho), \Lambda(\sigma)) \\ &\leqslant \min_{\sigma\in\mathcal{F}} D(\rho, \sigma) \\ &= R_D(\rho). \end{aligned}$$

We obtain the following simplification of the evaluation of distance-based resource measures:

Proposition 7. Suppose D is a convex and contractive distance. Let $\rho^\sigma_p = (1-p)\rho + p\sigma$ and $R_D(\rho) = D(\rho, \sigma^*)$. Then, we have

$$R_D\big(\rho^{\sigma^*}_p\big) = (1-p)R_D(\rho). \tag{16}$$

Proof. On the one hand, we have

$$\begin{aligned} R_D\big(\rho^{\sigma^*}_p\big) &= \min_{\sigma\in\mathcal{F}} D\big(\rho^{\sigma^*}_p, \sigma\big) \\ &\leqslant \min_{\sigma\in\mathcal{F}}[(1-p)D(\rho, \sigma) + pD(\sigma^*, \sigma)] \\ &\leqslant (1-p)R_D(\rho). \end{aligned}$$

On the other hand, by using triangular inequality, we have

$$\begin{aligned} R_D\big(\rho^{\sigma^*}_p\big) &= \min_{\sigma\in\mathcal{F}} D\big(\rho^{\sigma^*}_p, \sigma\big) \\ &\geqslant \min_{\sigma\in\mathcal{F}} \left[D(\rho, \sigma) - D\big(\rho, \rho^{\sigma^*}_p\big)\right] \\ &\geqslant \min_{\sigma\in\mathcal{F}}[D(\rho, \sigma) - pD(\rho, \sigma^*)] \\ &= (1-p)R_D(\rho). \end{aligned}$$

■

Lemma 1. For any $\rho \in \mathcal{D}(\mathcal{H})$ and $\sigma \in \mathcal{F}$, the map

$$\Lambda^\sigma_p(\rho) = (1-p)\rho + p\sigma \tag{17}$$

is a free operation.

Proof. First, note that the replacement map $\Lambda'(\rho) = \sigma$ is a free operation, which can be obtained via conditions [B1] and [B2]. Then, the map $\Lambda^\sigma_p(\rho) = (1-p)\rho + p\sigma$ can be regarded as a mixture of the replacement map and the identity map. Thus, the map Λ^σ_p is a free map. ■

Remark. The depolarizing channel, defined as

$$\Lambda(\rho) = (1-p)\rho + p\frac{I}{d}, \tag{18}$$

can be considered as a free operation for any QRTs that adhere to the tensor-product structure assumption, with the maximally mixed state $\frac{I}{d}$ serving as the free state. The maximally mixed state $\frac{I}{d}$ serves as a free state in various QRTs, including but not limited to entanglement [7], coherence [8,9], asymmetry [39,40], nonstabilizerness [10], thermodynamics [13], nonuniformity [43,44], purity [78,79], imaginarity [14,41], and more. We are now prepared to introduce lower and upper bounds for the ϵ measure R_ϵ using both the original measure R and the distance-based measure R_D.

Proposition 8. Let $R(\rho)$ be a convex resource measure, and D is a convex contractive distance. Then $R_\epsilon^{(D)}(\rho)$ satisfies

$$\min_{\text{so that } R_D(\tau)=R_D(\rho)-\epsilon} R(\tau) \leqslant R_\epsilon^{(D)}(\rho) \leqslant \left(1-\frac{\epsilon}{R_D(\rho)}\right)R(\rho). \tag{19}$$

Proof. Since $D(\Lambda(\rho),\rho)\leqslant\epsilon$ implies $R_\epsilon(\rho)\leqslant R(\rho)$. Let us consider the operation $\Lambda_p^\sigma(\rho)=\rho_p^\sigma=(1-p)\rho+p\sigma$ with $\sigma\in\mathcal{F}$, which is a free operation from the Lemma 1. From the convexity of both R and D, we have $R(\rho_p^\sigma)\leqslant(1-p)R(\rho)$ and $D(\rho_p^\sigma,\rho)\leqslant pD(\sigma,\rho)$, which implies when $p\leqslant\frac{\epsilon}{D(\sigma,\rho)}$, we have $D(\rho_p^\sigma,\rho)\leqslant\epsilon$. Then, we obtain

$$\begin{aligned} R_\epsilon^{(D)}(\rho) &\leqslant \min\left\{R\left(\rho_p^\sigma\right)\middle|D\left(\rho_p^{\sigma^*},\rho\right)\leqslant\epsilon\right\} \\ &\leqslant \min\{(1-p)R(\rho)|D(\rho_p^{\sigma^*},\rho)\leqslant\epsilon\} \\ &\overset{(a)}{\leqslant} \min\left\{\left(1-\frac{\epsilon}{D(\rho,\sigma)}\right)R(\rho)|\sigma\in\mathcal{F}\right\} \\ &\overset{(b)}{\leqslant} \left(1-\frac{\epsilon}{R_D(\rho)}\right)R(\rho), \end{aligned}$$

where (a) holds is due to for a fixed value $p=\frac{\epsilon}{D(\rho,\sigma)}$, and (b) holds is due to $R_D(\rho)=\min_{\sigma\in\mathcal{F}}D(\rho,\sigma)$. On the other hand, we have $R_\epsilon(\rho)=R(\rho^*)\geqslant R[\Lambda(\rho^*)]$, where $\rho^*\in B_\epsilon(\rho)$ is the optimal solution of $R_\epsilon(\rho)$. Consider a free state σ^* which is the optimal solution of $R_D(\rho^*)$. From the triangle inequality

$$\begin{aligned} R_\epsilon^{(D)}(\rho^*) &= D(\rho^*,\sigma^*) \\ &\geqslant D(\rho,\sigma^*)-D(\rho^*,\rho) \\ &\overset{(a)}{\geqslant} R_D(\rho)-\epsilon, \end{aligned}$$

where (a) is due to σ^* is not the optimal solution of $R_D(\rho)$ in general. Now, let $s=1-\frac{R_D(\rho)-\epsilon}{R_D(\rho^*)}$ and note that $0\leqslant s\leqslant 1$. Then, from Proposition 7, we have $R_D(\Lambda_s^{\sigma^*}(\rho^*))=R_D(\rho)-\epsilon$. Therefore,

$$\begin{aligned} R_\epsilon(\rho) &= R(\rho^*) \\ &\geqslant R_D\left[\Lambda_s^{\sigma^*}(\rho^*)\right] \\ &\geqslant \min\{R(\tau)|R_D(\tau)=R_D(\rho)-\epsilon\}, \end{aligned}$$

where $\Lambda_s^{\sigma^*}$ is a free operation. ■

Although, the relative entropy $S(\rho||\sigma)$ is not a distance since it is not symmetric and it does not satisfy the triangle inequality. Nevertheless, it possesses three crucial properties shared by the aforementioned distances: it exhibits contractivity under CPTP maps and joint convexity, $S(\rho||\sigma)=0$ if and only if $\rho=\sigma$ [53]. It is thus clear that most of the proofs given in the case of $R_\epsilon^{(D)}$ hold also for

$$R_{\text{rel},\epsilon}^{(D)}(\rho)=\inf_{\sigma\in B_\epsilon^{(D)}(\rho)} R(\sigma). \tag{20}$$

For example, $R_{\text{rel},\epsilon}^{(D)}(\rho)$ exhibits continuity with respect to ρ, and it satisfies $R_{\text{rel},\epsilon}^{(D)}(\rho)\leqslant R_{\text{rel}}(\rho)-\epsilon$, where D is the relative entropy. In a general QRT, it is not always true that the resource measures keep the ordered [42], i.e., it is not always true that, for any states ρ and σ,

$$R_1(\rho)\geqslant R_1(\sigma) \iff R_2(\rho)\geqslant R_2(\sigma) \tag{21}$$

holds for two resource measures R_1 and R_2. By using ϵ measure of QRT, we can construct a class of resource measures which keep the ordered whenever it is a complete order resource theory:

Corollary 3. Consider a distance-based resource measure R_D defined in Eq. (15), and let us take its ϵ generation to be

$$R_{D,\epsilon}(\rho):=\min_{\sigma\in B_\epsilon^{(D)}(\rho)} R_D(\sigma). \tag{22}$$

Then, $R_D(\rho)\geqslant R_D(\sigma) \iff R_{D,\epsilon}(\rho)\geqslant R_{D,\epsilon}(\sigma)$.

Proof. Proposition 8 implies $R_{D,\epsilon}(\rho)=R_D(\rho)-\epsilon$ when we set $R_\epsilon^{(D)}(\rho)=R_{D,\epsilon}(\rho)$ in Eq. (19). Thus, $R_D(\rho)\geqslant R_D(\sigma)$ holds if and only if $R_{D,\epsilon}(\rho)\geqslant R_{D,\epsilon}(\sigma)$. ■

D. One-shot dilution cost and smooth asymptotic resource measures

Resource dilution stands out as important subclasses of resource manipulation tasks. Resource dilution is a protocol to transform a state ϕ_m in the family $\mathbb{T}$ of reference states to a given state ρ using free operations, in which a reference state is to be transformed to the desired state. The optimal performance of such task is characterized by the one-shot dilution cost of QRT, defined as [33]

$$R_{c,\epsilon}^{(1)}(\rho)=\inf_{\Lambda\in\mathcal{O}}\{c|D(\rho,\Lambda(\phi))\leqslant\epsilon,\phi\in\mathbb{T}\}, \tag{23}$$

where $c=R(\phi)$.

Proposition 9. The ϵ measure of QRT is a lower bound of its resource dilution, i.e., for any resource measure R, we have

$$R_{c,\epsilon}^{(1)}(\rho)\geqslant R_\epsilon(\rho). \tag{24}$$

Proof. Suppose Λ^* is the optimal free operation for $R_{c,\epsilon}^{(1)}(\rho)$ and we have

$$\begin{aligned} R_{c,\epsilon}^{(1)}(\rho) &= R(\phi) \\ &\geqslant \inf_{\sigma\in B_\epsilon(\rho)} R_\epsilon(\sigma) \\ &= R_\epsilon(\rho), \end{aligned}$$

where the inequality holds is due to $D(\rho,\Lambda^*(\phi))\leqslant\epsilon$, or equivalently, $\Lambda^*(\phi)\in B_\epsilon(\rho)$. ■

This observation underscores that the ϵ measure of QRT denoted as R_ϵ furnishes a lower bound for the requisite minimum number of reference states from $\mathbb{T}$ essential for reliable resource dilution. Meanwhile, we can define one-shot dilution cost under free catalysts as

$$R_{c,\text{cat}}^{(1),\epsilon}(\rho)=\inf_{\Lambda\in\mathcal{O}}\inf_{\sigma\in\mathcal{F}}\{c|D(\rho\otimes\sigma,\Lambda(\phi\otimes\sigma))\leqslant\epsilon,\phi\in\mathbb{T}\}, \tag{25}$$

where $c=R(\phi)$. From one-shot dilution cost under free catalysts, we have the following result:

Proposition 10. The ϵ measure of QRT is a lower bound of its resource dilution cost under free catalysts, i.e., for any resource measure R, we have

$$R_{c,\text{cat}}^{(1),\epsilon}(\rho)\geqslant R_\epsilon(\rho). \tag{26}$$

Proof. Suppose Λ^* is the optimal free operation for $R^{(1),\epsilon}_{c,\text{cat}}(\rho)$ and we have

$$\begin{aligned}R^{(1),\epsilon}_{c,\text{cat}}(\rho) &= R(\phi)\\ &= R(\phi\otimes\sigma)\\ &\geqslant R[\Lambda^*(\phi\otimes\sigma)]\\ &\geqslant \inf_{\tau\in B_\epsilon(\rho\otimes\sigma)} R_\epsilon(\tau)\\ &= R_\epsilon(\rho\otimes\sigma)\\ &= R_\epsilon(\rho),\end{aligned}$$

where the second equality hold is due to the condition [B1], and the second inequality hold is due to $D(\rho\otimes\sigma, \Lambda^*(\phi\otimes\sigma)) \leqslant \epsilon$ or, equivalently, $\Lambda^*(\phi\otimes\sigma)\in B_\epsilon(\rho\otimes\sigma)$. ■

The asymptotic resource measures play a crucial role in characterizing the long-term behavior of quantum systems and their operations in QRTs. These measures are particularly relevant when dealing with the asymptotic limit of many identical or independent copies of quantum states [1–3,8]. For a ϵ measure of QRT R_ϵ, the lower and upper regularizations of resource measure can be defined as follows:

$$R^{\inf,\infty}(\rho) = \lim_{\epsilon\to 0^+}\liminf_{n\to\infty}\frac{1}{n}R_\epsilon(\rho^{\otimes n}), \tag{27}$$

$$R^{\sup,\infty}(\rho) = \lim_{\epsilon\to 0^+}\limsup_{n\to\infty}\frac{1}{n}R_\epsilon(\rho^{\otimes n}). \tag{28}$$

In a more concise manner, we refer to these as the smooth regularizations of the resource measures.

Proposition 11. The smooth regularizations of resource measures in Eqs. (27) and (28) are resource measures. That is,

$$R^{\inf,\infty}(\rho) \geqslant R^{\inf,\infty}[\Lambda(\rho)], \tag{29}$$

$$R^{\sup,\infty}(\rho) \geqslant R^{\sup,\infty}[\Lambda(\rho)] \tag{30}$$

hold for any free operations $\Lambda\in\mathcal{O}$, and $R^{\inf,\infty}(\rho) = R^{\sup,\infty}(\rho) = 0$ hold for any free states $\rho\in\mathcal{F}$.

Proof. We define the two quantities as follows:

$$R^{\inf,\infty}_\epsilon(\rho) = \liminf_{n\to\infty}\frac{1}{n}R_\epsilon(\rho^{\otimes n}), \tag{31}$$

$$R^{\sup,\infty}_\epsilon(\rho) = \limsup_{n\to\infty}\frac{1}{n}R_\epsilon(\rho^{\otimes n}). \tag{32}$$

For $R^{\inf,\infty}_\epsilon(\rho)$, we have

$$\begin{aligned}R^{\inf,\infty}_\epsilon(\rho) &= \liminf_{n\to\infty}\frac{1}{n}R_\epsilon(\rho^{\otimes n})\\ &\geqslant \liminf_{n\to\infty}\frac{1}{n}R_\epsilon[\Lambda^{\otimes n}(\rho^{\otimes n})]\\ &= \liminf_{n\to\infty}\frac{1}{n}R_\epsilon[\Lambda(\rho)^{\otimes n}]\\ &= R^{\inf,\infty}_\epsilon[\Lambda(\rho)],\end{aligned}$$

where the equality holds due to $\Lambda\in\mathcal{O}\Rightarrow\Lambda^{\otimes n}\in\mathcal{O}$. Therefore, we get

$$\begin{aligned}R^{\inf,\infty}(\rho) &= \lim_{\epsilon\to 0^+} R^{\inf,\infty}_\epsilon(\rho)\\ &\geqslant \lim_{\epsilon\to 0^+} R^{\inf,\infty}_\epsilon[\Lambda(\rho)]\\ &= R^{\inf,\infty}[\Lambda(\rho)].\end{aligned}$$

The same reasoning above also holds for Eq. (30). It is easy to see that $R^{\inf,\infty}(\rho) = R^{\sup,\infty}(\rho) = 0$ for any free states ρ. ■

This demonstrates that the smooth regularizations of resource measures are valid asymptotic resource measures in QRTs. Furthermore, these smooth regularizations of resource measures can play a crucial role in investigations within quantum information theory in the asymptotic regime, a domain extensively utilized to scrutinize the interplay between resource distillation and resource dilution [33]. For example, in the QRT of entanglement, the logarithmic robustness R_{lr} denoted as Eq. (1) lacks asymptotic continuity, and it remains uncertain whether it exhibits weak additivity. However, the smooth regularizations of R_{lg}, which read as

$$R^{\sup,\infty}_{\text{lr}}(\rho) = \lim_{\epsilon\to 0^+}\limsup_{n\to\infty}\frac{1}{n}\inf_{\sigma\in B^{(D)}_\epsilon(\rho^{\otimes n})} R_{\text{lr}}(\sigma), \tag{33}$$

demonstrate both asymptotic continuity and weak additivity [80]. Meanwhile, it was shown in Ref. [81] that the entanglement cost E_c under asymptotically nonentangling maps is equal to $R^{\sup,\infty}_{\text{lr}}(\rho)$, i.e., $E_c(\rho) = R^{\sup,\infty}_{\text{lr}}(\rho)$. We refer readers to Refs. [81,82] for the details of E_c.

IV. APPLICATION OF ϵ MEASURE OF QRT

In this section, we will provide six examples of applications of ϵ measure of QRT.

Example 1: ϵ robustness of coherence. Consider a single-qubit state ρ with the Bloch vector $\vec{r} = (r_x, r_y, r_z)$, its robustness of coherence is $C_R(\rho) = \sqrt{r_x^2 + r_y^2}$. The ϵ robustness of coherence can be defined as

$$C_{R,\epsilon}(\rho) = \min_{\sigma\in B^{(D)}_\epsilon(\rho)} C_R(\sigma), \tag{34}$$

where we use trace distance $D(\rho,\sigma) = \frac{1}{2}|\vec{r}-\vec{s}|$ to denote D and $\vec{s} = (s_x, s_y, s_z)$ is the Bloch vector of σ. Thus, the ϵ robustness of coherence for single-qubit state ρ can be written as

$$\min\ C_R(\sigma) = \sqrt{s_x^2 + s_y^2} \tag{35}$$

$$\text{so that } |\vec{r}-\vec{s}| \leqslant 2\epsilon. \tag{36}$$

This optimization problem is equivalent to

$$\min\ C_R(\sigma)^2 = s_x^2 + s_y^2 \tag{37}$$

$$\text{so that } (s_x - r_x)^2 + (s_y - r_y)^2 + (s_z - r_z)^2 \leqslant 4\epsilon^2. \tag{38}$$

By using the Lagrange multiplier method, we obtain the ϵ robustness of coherence as

$$C_{R,\epsilon}(\rho) = \sqrt{r_x^2 + r_y^2} - 2\epsilon. \tag{39}$$

■

Example 2: *ϵ robustness of nonstabilizerness.* Consider a single-qubit state ρ with Bloch vector $\vec{r} = (r_x, r_y, r_z)$. The robustness of nonstabilizerness (or the robustness of magic states) M_R for a single-qubit state ρ with Bloch vector $\vec{r} = (r_x, r_y, r_z)$ is defined as [21,83]

$$M_R(\rho) = \begin{cases} |r_x| + |r_y| + |r_z|, & \text{if } \vec{r} \notin \text{STAB}, \\ 0, & \text{if } \vec{r} \in \text{STAB}. \end{cases}$$

Here, STAB represents the set of single-qubit stabilizer states. In the Bloch sphere representation, STAB is characterized by six pure stabilizer states $[|0\rangle, |1\rangle, \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$, and $\frac{1}{\sqrt{2}}(|0\rangle \pm i\,|1\rangle)]$, interconnected to form an octahedron. The ϵ robustness of nonstabilizerness can be defined as

$$M_{R,\epsilon}(\rho) = \min_{\sigma \in B_\epsilon^{(D)}(\rho)} M_R(\sigma), \tag{40}$$

where we use trace distance $D(\rho, \sigma) = \frac{1}{2}|\vec{r} - \vec{s}|$ to denote D and $\vec{s} = (s_x, s_y, s_z)$ is the Bloch vector of σ. If $B_\epsilon^{(D)}(\rho) \cap \text{STAB} \neq \emptyset$, then $M_R(\rho) = 0$. Thus, for the state ρ satisfies $B_\epsilon^{(D)}(\rho) \cap \text{STAB} = \emptyset$, the ϵ robustness of nonstabilizerness for a single-qubit state ρ can be written as

$$\min \; M_R(\sigma) = |s_x| + |s_y| + |s_z| \tag{41}$$

$$\text{so that } |\vec{r} - \vec{s}| \leqslant 2\epsilon. \tag{42}$$

This optimization problem is equivalent to

$$\min \; M_R(\sigma)^2 = s_x^2 + s_y^2 \tag{43}$$

$$\text{so that } (s_x - r_x)^2 + (s_y - r_y)^2 + (s_z - r_z)^2 \leqslant 4\epsilon^2. \tag{44}$$

By using the Lagrange multiplier method, we obtain the ϵ robustness of nonstabilizerness as

$$M_{R,\epsilon}(\rho) = \begin{cases} A_x + B_y + C_z, & \text{if } B_\epsilon^{(D)}(\rho) \cap \text{STAB} \neq \emptyset, \\ 0, & \text{if } B_\epsilon^{(D)}(\rho) \cap \text{STAB} = \emptyset. \end{cases}$$

Here,

$$A_x = \begin{cases} |r_x - \frac{2}{\sqrt{3}}\epsilon|, & \text{if } r_x > 0 \\ |r_x + \frac{2}{\sqrt{3}}\epsilon|, & \text{if } r_x < 0. \end{cases}$$

The definitions of B and C follow a similar structure. In general, ϵ is a very small constant, so we impose $\frac{2}{\sqrt{3}}\epsilon \leqslant \max\{|r_x|, |r_y|, |r_z|\}$ to ensure its smallness. ■

Example 3: *Geometric ϵ measure of asymmetry for two-qubit pure states.* Suppose a bipartite two-qubit system is equipped with the global Hamiltonian $H_{12} = H \otimes \mathbb{1} + \mathbb{1} \otimes H$ with the local Hamiltonian $H = |1\rangle\langle 1|$. Let us consider the QRT of asymmetry with the U(1) group defined by the unitary representation $\mathcal{U}_t : t \to e^{-iH_{12}t} e^{iH_{12}t}$. The free states for this theory are the set of states that are invariant under such symmetric transformations

$$\mathcal{F} = \{\sigma \mid \sigma = \mathcal{U}_t(\sigma), \; \forall\, t\} \tag{45}$$

$$= \{\sigma \mid [\sigma, H_{12}] = 0\} \tag{46}$$

$$= \text{conv}\{|0\rangle\langle 0|, |1\rangle\langle 1|, \{|\psi_{\alpha,\beta}\rangle\langle\psi_{\alpha,\beta}|\}_{\alpha,\beta}\}, \tag{47}$$

where $\{|\psi_{\alpha,\beta}\rangle\langle\psi_{\alpha,\beta}|\}$ denotes the set of pure states parametrized by $\alpha, \beta \in \mathbb{C}$ as $|\psi_{\alpha,\beta}\rangle = \alpha\,|01\rangle + \beta\,|10\rangle$. For a two-qubit pure state $|\phi\rangle = a\,|00\rangle + b\,|01\rangle + c\,|10\rangle + d\,|11\rangle$, its geometric measure of asymmetry (which is also known as Bures distance of asymmetry) is defined as

$$A_G(|\phi\rangle) = \min_{\sigma \in \mathcal{F}} \sqrt{2}\sqrt{1 - F(|\phi\rangle\langle\phi|, \sigma)}, \tag{48}$$

where $F(|\phi\rangle\langle\phi|, \sigma) = \langle\phi|\,\sigma\,|\phi\rangle$ is the fidelity. It is shown in Ref. [84] that $\max_{\sigma\in\mathcal{F}} \langle\phi|\,\sigma\,|\phi\rangle = \max\{|a|^2, |d|^2, 1 - |a|^2 - |d|^2\}$. We focus on the geometric ϵ measure of asymmetry for pure states and the quantity can be written as

$$A_{G,\epsilon}(|\phi\rangle) = \inf_{|\psi\rangle \in B_\epsilon^{(D)}(|\phi\rangle)} A_G(|\psi\rangle), \tag{49}$$

where $|\psi\rangle = a'\,|00\rangle + b'\,|01\rangle + c'\,|10\rangle + d'\,|11\rangle$ and $B_\epsilon^{(D)}(|\phi\rangle) \cap \mathcal{F} = \emptyset$. If the distance is the norm distance $D(|\psi\rangle, |\phi\rangle) = ||\,|\psi\rangle - |\phi\rangle\,||$ with $||\,|\phi\rangle\,||^2 = \langle\phi|\,\phi\rangle$, this optimization problem is equivalent to

$$\min \; A_G(|\psi\rangle)^2 = 2(1 - \max\{|a'|^2, |d'|^2, 1 - |a'|^2 - |d'|^2\}) \tag{50}$$

$$\text{so that } (a - a')^2 + (b - b')^2 + (c - c')^2 + (d - d')^2 \leqslant \epsilon^2. \tag{51}$$

By using the Lagrange multiplier method, we obtain the geometric ϵ measure of asymmetry as

$$A_{G,\epsilon}(|\phi\rangle) = \sqrt{2}\sqrt{1 - \max\{A, B, C, D, E, 1\}}, \tag{52}$$

where

$$A = (\epsilon + |a|)^2,$$

$$B = (\epsilon + |d|)^2,$$

$$C = 1 - [(\epsilon + |a|)^2] - |d|^2,$$

$$D = 1 - |a|^2 - (\epsilon + |d|)^2,$$

$$E = 1 - \left(1 - \frac{\epsilon}{\sqrt{|a|^2 + |d|^2}}\right)^2 a^2 - \left(1 - \frac{\epsilon}{\sqrt{|a|^2 + |d|^2}}\right)^2 d^2.$$

■

Example 4: *ϵ measure of nonuniformity.* Consider the following nonuniformity measure [44,85]:

$$\mathcal{T}_q(\rho) = \frac{d^{q-1} - 1}{q - 1} - d^{q-1} T_q(\rho),$$

where $T_q(\rho) = \frac{1 - \text{Tr}(\rho^q)}{q - 1}$ for $q > 0$ is the Tsallis q entropy [86], and d is the dimension of Hilbert space $\mathcal{H}$. Here, we consider the case of $q = d = 2$, then

$$\mathcal{T}_2(\rho) = 2\,\text{Tr}(\rho^2) - 1 = |\vec{r}|, \tag{53}$$

where $\vec{r} = (r_x, r_y, r_z)$ is the Bloch vector of ρ. Note that this quantity has a nice geometric meaning for a single-qubit: that is, the farther away from the center of the Bloch ball, the more resourceful it is. The ϵ measure of nonuniformity based on the Tsallis 2-entropy can be defined as

$$\mathcal{T}_{2,\epsilon}(\rho) = \min_{\sigma \in B_\epsilon^{(D)}(\rho)} \mathcal{T}_2(\sigma), \tag{54}$$

where D is the trace distance. From the geometric meaning of Eq. (53), we get

$$\mathcal{T}_{2,\epsilon}(\rho) = |\vec{r}| - 2\epsilon. \tag{55}$$

■

Example 5: ϵ *measure of imaginarity*. In the QRT of imaginarity, the robustness of imaginarity, a single-qubit state ρ with the Bloch vector $\vec{r} = (r_x, r_y, r_z)$, is [14,41,42]

$$I(\rho) = |r_y|. \tag{56}$$

The ϵ measure of imaginarity can be defined as

$$I_\epsilon(\rho) = \min_{\sigma \in B_\epsilon^{(D)}(\rho)} I(\sigma), \tag{57}$$

where D is the trace distance. By using the Lagrange multiplier method, we obtain the ϵ measure of imaginarity as

$$I_\epsilon(\rho) = |r_y| - 2\epsilon. \tag{58}$$

■

Example 6: *Upper bound of* ϵ *measure of resource rank*. Resource rank is an important measure, which has been studied in many resource theories, such as resource theory of entanglement [87–89], coherence [90], and nonstabilizerness [20,21,91]. Suppose there is a collection of rank-one free states $\{|\phi_i\rangle\langle\phi_i|\}$ such that $\{|\phi_i\rangle\}$ form a basis of the density matrix space $\mathcal{H}$. Then one can define the resource rank of an arbitrary pure state $|\phi\rangle$ as [1]

$$R_{rk}(|\phi\rangle) = \inf\left\{k \,|\, |\phi\rangle = \sum_{i=1}^{k} c_i\,|\phi_i\rangle\,,\,|\phi_i\rangle \in \mathcal{F}, c_i \in \mathbb{C}\right\}. \tag{59}$$

For a mixed state ρ, its resource rank can be defined by the convex-roof extended method:

$$R_{rk}(\rho) = \inf_{\{p_i,|\phi_i\rangle\}} \sum_i p_i R_{rk}(|\phi_i\rangle), \tag{60}$$

where the infimum is taken over all pure-state decompositions of ρ. We focus on the ϵ measure of resource rank of pure states, and we define the quantity as

$$R_{rk,\epsilon}(|\phi\rangle) = \inf_{|\psi\rangle \in B_\epsilon^{(D)}(|\phi\rangle)} R_{rk}(|\psi\rangle). \tag{61}$$

If the distance is the norm distance $D(|\psi\rangle, |\phi\rangle) = ||\,|\psi\rangle - |\phi\rangle\,||$ with $||\,|\phi\rangle\,||^2 = \langle\phi|\,\phi\rangle$. We have following result:

Proposition 12. Let $|\psi\rangle$ be a normalized n-qubit state with a rank-one free state decomposition $|\phi\rangle = \sum_{i=1}^{k} c_i\,|\phi_i\rangle$, $|\phi_i\rangle \in \mathcal{F}$ are normalized basis and $c_i \in \mathbb{C}$. Then,

$$R_{rk,\epsilon}(|\phi\rangle) \leqslant 1 + \frac{||c||_1^2}{\epsilon}, \tag{62}$$

where $||c||_1 = \sum_i |c_i|$.

The proof of the proposition can be found in Appendix C. ■

V. REMARK ON THE EXTENSION OF THE ϵ MEASURE TO DYNAMICAL QRT

As potential next steps, our findings could be generalized to the dynamical QRT that does not rely exclusively on states. This expansion might include domains like quantum channel [26,27], quantum incompatibility [28,29], and measurement sharpness [30,31]. For instance, let us consider ϵ measures of the channel-based QRT. In a channel-based QRT, denoted as a 2-tuple $\hat{\mathcal{R}} = (\hat{\mathcal{F}}, \hat{\mathcal{O}})$, the set of free channels is denoted by $\hat{\mathcal{F}}$, and the set of free superoperations is denoted by $\hat{\mathcal{O}}$. Free channels $\hat{\mathcal{F}}$ are those quantum channels that do not possess any resource, and free superoperations $\hat{\mathcal{O}}$ are a subset of superchannels that transform free channels into free channels [27]. In this context, superchannels transform a quantum channel $\Lambda^{A\to B}$ into another channel $\Phi^{C\to D}$ through the expression $\Phi^{C\to D} = \Psi_{\text{post}}^{BE\to D} \circ (\Lambda^{A\to B} \otimes \text{id}_E) \circ \Psi_{\text{pre}}^{C\to AE}$, where the superscripts denote input and output systems, id is the identity map, and $\Psi_{\text{post}}, \Psi_{\text{pre}}$ are also quantum channels [92,93]. Let $L(\mathcal{H})$ be the set of quantum channels, the basic two requirements for a functional $\hat{R} : L(\mathcal{H}) \to \mathbb{R}^+$ being a channel resource measure for $(\hat{\mathcal{F}}, \hat{\mathcal{O}})$ are as follows [27]:

[A′1] (Non-negativity): $\hat{R}(\Lambda) \geqslant 0$ and the equality holds if and only if $\Lambda \in \hat{\mathcal{F}}$.

[A′2] (Weak monotonicity): $\hat{R}(\Lambda) \geqslant \hat{R}(\Theta[\Lambda])$, where $\Theta \in \hat{\mathcal{O}}$ is a superoperation.

We can further define their ϵ-measure counterparts as

$$\hat{R}_\epsilon(\Lambda) = \inf_{\Lambda' \in B_\epsilon(\Lambda)} \hat{R}(\Lambda'), \tag{63}$$

where $\Lambda' \in B_\epsilon(\Lambda) \iff \frac{1}{2}||\Lambda - \Lambda'||_\diamond \leqslant \epsilon$ and $||X_A||_\diamond = \max_{\rho_{AE}} \text{Tr}|X_A \otimes \mathbb{1}_E(\rho_{AE})|$ is the diamond norm [27]. The inequality $||\Theta[\Lambda_1] - \Theta[\Lambda_2]||_\diamond \leqslant ||\Lambda_1 - \Lambda_2||_\diamond$, valid for any superchannel Θ and quantum channels Λ_1 and Λ_2 [93], makes it straightforward to deduce that the channel ϵ measure $\hat{R}_\epsilon(\Lambda)$, using the approach outlined in Proposition 1, also qualifies as a channel resource measure. Unlike the state-based QRT, measuring the distance between two channels (e.g., diamond norm) always involves an optimization problem over all quantum states. Therefore, calculating an analytical expression for the channel resource measure ϵ measure $\hat{R}_\epsilon$ is generally challenging. However, one may calculate the distance of two channels by evaluating the (trace) distance of their Choi-Jamiolkowski representations [53]. In essence, $||J_{\Lambda_1} - J_{\Lambda_2}||_1$ can serve as a substitute for calculating $||\Lambda_1 - \Lambda_2||_\diamond$, where $J_\Lambda = \frac{1}{d}(id \otimes \Lambda)(\sum_i |ii\rangle\langle ii|)$ denotes the Choi-Jamiolkowski representation of the channel Λ. In fact, this approach has been studied in the QRT of channel coherence [94], which can simplify the study of channel-based resource to state-based resource. These methods above can also be applied to measure-based QRTs [95], such as quantum incompatibility and measurement sharpness. In general, computing $\hat{R}_\epsilon$ is a constrained optimization problem that can be solved using the method of Lagrange multipliers. For a qubit channel Λ, its Choi-Jamiolkowski representation corresponds to a two-qubit state J_Λ. Computing Eq. (63) becomes the task of finding a two-qubit state $J_{\Lambda'}$ under the constraint $||J_\Lambda - J_{\Lambda'}||_1 \leqslant \epsilon$ that minimizes $\hat{R}_\epsilon(\Lambda')$. In this scenario, solving the system of equations involving $2^2 \times 2^2 - 1 = 15$ parameters is highly challenging. Therefore, we defer the consideration of examples involving the computation of the ϵ measure for channels to future investigations. Further research in these directions is left for future work since it is out of the scope of this paper.

VI. CONCLUSION

In this paper, we have introduced a family of smoothed measures for any QRTs. The properties of ϵ measures are given, ϵ measure R_ϵ inherits the fundamental properties of the original resource measure R, such as weak monotonicity, convexity, monogamy, and so forth. Moreover, the ϵ measure remains continuous regardless of whether the original measure is continuous or not. We have also demonstrated that the ϵ measure of a QRT satisfies a "weak" subadditivity, with the subadditivity of R being a special case when $\epsilon = 0$. In practical scenarios, ϵ measures can serve to estimate the minimum guaranteed amount of prepared resources. These measures find application in one-shot convertibility, particularly by providing a lower bound on the one-shot dilution cost of a QRT. We also found that the smooth regularizations of resource measures are resource measures. As part of our applications, we derived several formulas for the ϵ measures of coherence, nonstabilizerness, asymmetry, nonuniformity, and imaginarity. Additionally, we offered an upper bound for the ϵ measure of the resource rank. This work investigates the properties, applications, and computational examples of the ϵ measure in the QRT based on the tensor-product structure assumption. This general framework not only encompasses the QRTs of entanglement [37] and coherence [38] but also extends to other QRTs such as asymmetry, nonstabilizerness, imaginarity, nonuniformity, and more. Additionally, we present properties of the ϵ measure (e.g., Propositions 4, 13, 10, and 11) and computational examples (e.g., the examples in Sec. IV) that have not been previously addressed in Refs. [37,38]. Finally, we hope that the results drawn in this paper will contribute to a deeper understanding of the role of quantum resources in quantum information processing.

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APPENDIX A: FLAG ADDITIVITY OF QUANTUM RELATIVE RÉNYI ENTROPY, SANDWICHED RÉNYI ENTROPY, AND RELATIVE ENTROPY

In this Appendix, we will show that the resource measure of QRT which is based on quantum relative Rényi entropy $D_\alpha(\rho||\sigma)$, sandwiched Rényi entropy $\tilde{D}_\alpha(\rho||\sigma)$, and relative entropy $S(\rho||\sigma)$ satisfies condition [FA], where $\alpha \in [1, 2]$. Meanwhile, we will give an illustrative example showcasing the application of flag additivity. Recall the quantum relative Rényi entropy D_α is defined as [96]

$$D_\alpha(\rho||\sigma) = \begin{cases} \frac{1}{\alpha-1}\log_2(\mathrm{Tr}\rho^\alpha\sigma^{1-\alpha}), & \text{otherwise} \\ +\infty, & if \alpha \notin (0,1) \wedge \mathrm{supp}(\rho) \not\subset \mathrm{supp}(\sigma) \end{cases} \tag{A1}$$

where $\alpha \in [0, 1) \cup (1, +\infty)$ and the sandwiched Rényi entropy $\tilde{D}_\alpha$ is defined as [97]

$$\tilde{D}_\alpha(\rho||\sigma) = \begin{cases} \frac{1}{\alpha-1}\log_2\{\mathrm{Tr}[(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\sigma^{\frac{1-\alpha}{2\alpha}})^\alpha]\}, & \text{otherwise} \\ +\infty, & if \alpha \notin (0,1) \wedge \mathrm{supp}(\rho) \not\subset \mathrm{supp}(\sigma) \end{cases} \tag{A2}$$

where $\alpha \in (0, 1) \cup (1, +\infty)$ and the relative entropy is defined as

$$S(\rho||\sigma) = \mathrm{Tr}\rho\,\log_2\rho - \mathrm{Tr}\rho\,\log_2\sigma, \tag{A3}$$

with $\mathrm{supp}\rho \subset \mathrm{supp}\sigma$. Thus, one could define a whole family of resource measures for any QRT $\mathcal{R} = (\mathcal{F}, \mathcal{O})$ as follows:

$$R_\alpha(\rho) = \inf_{\sigma\in\mathcal{F}} D_\alpha(\rho||\sigma) \tag{A4}$$

with $\alpha \in [0, 2]$ and

$$\tilde{R}_\alpha(\rho) = \inf_{\sigma\in\mathcal{F}} \tilde{D}_\alpha(\rho||\sigma) \tag{A5}$$

with $\alpha \in [1/2, \infty)$ and

$$R_{\mathrm{rel}}(\rho) = \inf_{\sigma\in\mathcal{F}} S(\rho||\sigma). \tag{A6}$$

Now, we will show the following result:

Proposition 13. For any convex QRT admits tensor-product structure, $R_\alpha(\rho||\sigma)$, $\tilde{R}_\alpha(\rho||\sigma)$, and $R_{rel}(\rho||\sigma)$ satisfy flag additivity for $\alpha \in [1, 2]$.

Proof. First, note that for any resource monotone R, since the flag additivity is equivalent to [A3] and [A4] [15], we only prove [A3] and [A4] hold for $D_\alpha(\rho||\sigma)$, $\tilde{D}_\alpha(\rho||\sigma)$, and $S(\rho||\sigma)$ with $\alpha \in [1, 2]$. We first prove the sandwiched Rényi entropy based measure $\tilde{R}(\rho||\sigma)$ satisfies conditions [A3] and [A4]. The authors in Ref. [1] have shown $D_\alpha(\rho||\sigma)$, $\tilde{D}_\alpha(\rho||\sigma)$, and $S(\rho||\sigma)$ satisfy conditions [A3] and [A4]. ■

In general, the computation of entropic measures is often challenging. The introduction of flag additivity, however, can simplify certain calculations. For instance, consider the relative entropy of entanglement given by

$$R_{\mathrm{rel}}(\rho_{AB}) = \inf_{\sigma_{AB}\in\mathrm{Sep}} S(\rho_{AB}||\sigma_{AB}), \tag{A7}$$

where the set of free states, Sep, comprises separable states. Now, we will calculate the relative entropy of entanglement for the following state using flag addtivity:

$$\rho_{AA'|BB'} = \tfrac{1}{3}\rho_1^{AB} \otimes |\Psi^+\rangle\langle\Psi^+|^{A'B'} + \tfrac{2}{3}\rho_2^{AB} \otimes |\Psi^-\rangle\langle\Psi^-|^{A'B'}, \tag{A8}$$

where

$$\rho_1^{AB} = \lambda|\Psi^+\rangle\langle\Psi^+| + (1-\lambda)|01\rangle\langle01|,$$

$$\rho_2^{AB} = a|00\rangle\langle00| + b|00\rangle\langle11| + b^* |11\rangle 00 + (1-a)|11\rangle\langle11|,$$

$$|\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle).$$

In Ref. [16], Vedral and Plenio have shown that

$$E\left(\rho_1^{AB}\right) = (\lambda-2)\log_2\left(1-\frac{\lambda}{2}\right) + (1-\lambda)\log_2(1-\lambda),$$

$$E\left(\rho_2^{A'B'}\right) = e_+\log_2 e_+ + e_-\log_2 e_- - a\,\log_2 a - (1-a)\log_2(1-a),$$

where

$$e_\pm = \frac{1 \pm \sqrt{1-4a(1-a)-|b|^2}}{2}.$$

By using flag additivity, we obtain that $E(\rho_{AA'|BB'}) = \frac{1}{3}E(\rho_1^{AB}) + \frac{2}{3}E(\rho_2^{AB})$. In essence, flag additivity simplifies the computation of entropic measures, making tasks like calculating the relative entropy of entanglement more straightforward.

APPENDIX B: CLASSIFICATION OF QUANTUM STATES VIA MONOGAMY INEQUALITY OF QRT OF DISCORD

In the QRT of discord, the CKW-type monogamy inequality

$$R(\rho_{A|BC}) \geqslant R(\rho_{AB}) + R(\rho_{AC}) \tag{B1}$$

can be employed to distinguish between the GHZ state and the W state [72]. This involves using discord as the chosen resource measure [98]. Essentially, the three-qubit GHZ state follows monogamy, and the W state does not [72]. Assuming a factory produces GHZ and W states, practical quantum states σ_{ABC} may deviate slightly from the ideal quantum states ρ_{ABC} (GHZ or W states). This discrepancy can occur due to factors such as noise introduced during the preparation of quantum states. The term "noise" in this context can be considered as a depolarizing channel in Eq. (18), leading to a partial loss of polarization information in the quantum states. This deviation can be quantified by a known small constant $\epsilon > 0$, defined as the (trace) distance between the actual quantum states and the ideal quantum states, i.e., $D(\rho_{ABC}, \sigma_{ABC}) \leqslant \epsilon$. The monogamy inequality

$$R_\epsilon(\rho_{A|BC}) \geqslant R_\epsilon(\rho_{AB}) + R_\epsilon(\rho_{AC}) \tag{B2}$$

of Proposition 13 ensures that, even in the presence of slight noise in practical scenarios, it is still possible to distinguish whether the actual quantum state is a GHZ state or a W state.

APPENDIX C: PROOF OF PROPOSITION 12

This approach is based on Ref. [20] and we will first prove the following lemma:

Lemma 2 (Sparsification of QRT). Consider a normalized state $|\phi\rangle = \sum_j c_j\,|\phi_j\rangle$ with $c_j \in \mathbb{C}$ and $|\phi_j\rangle \in \mathcal{F}$. We can construct a random state $|\Omega\rangle = \frac{||c||_1}{k}\sum_{\alpha=1}^k |\omega_\alpha\rangle$ for some integer k, where $|\omega_\alpha\rangle$ $(\alpha \in [k])$ are independent and identically distributed random copies of random variable $|\omega\rangle$ and $|\omega\rangle$ is sampled from the set $\{|W_j\rangle \mid |W_j\rangle = \frac{c_j}{|c_j|}\,|\phi_j\rangle\}$ with probability $p_j = \frac{|c_j|}{||c||_1}$. Then, we have

$$\mathbb{E}(||\,|\phi\rangle - |\Omega\rangle\,||^2) = \frac{||c||_1^2}{k}, \tag{C1}$$

with $||c||_1 = \sum_j |c_j|$ and $||\,|\phi\rangle\,||^2 = \langle\phi|\,\phi\rangle$.

Proof. The state $|\phi\rangle$ can be written as

$$\begin{aligned}|\phi\rangle &= \sum_j c_j\,|\phi_j\rangle\\ &= ||c||_1\sum_j p_j\frac{c_j}{|c_j|}\,|\phi_j\rangle\\ &= ||c||_1\sum_j p_j\,|W_j\rangle\\ &= ||c||_1\mathbb{E}[|\omega\rangle].\end{aligned}$$

The last equality hold is due to the random variable $|\omega\rangle$ sampling from the set $\{|W_j\rangle\}$ with probability p_j. Because we will calculate the expectation

$$\mathbb{E}[||\,|\phi\rangle - |\Omega\rangle\,||^2] = \mathbb{E}[\langle\Omega|\,\Omega\rangle] - \mathbb{E}[\langle\phi|\,\Omega\rangle] - \mathbb{E}[\langle\Omega|\,\phi\rangle] + \mathbb{E}[\langle\phi|\,\phi\rangle]. \tag{C2}$$

It is only needed to calculate the four expectations in the right-hand side of Eq. (C2). First, note that $|\Omega\rangle$ may not be correctly normalized. However, we can bound the expectation $\mathbb{E}[\langle\Omega|\,\Omega\rangle]$ as follows:

$$\begin{aligned}\mathbb{E}[\langle\Omega|\,\Omega\rangle] &= \frac{||c||_1^2}{k^2}\sum_{\alpha=1}^k\mathbb{E}[\langle\omega_\alpha|\,\omega_\alpha\rangle] + \frac{||c||_1^2}{k^2}\sum_{\alpha\neq\beta}\mathbb{E}[\langle\omega_\alpha|\,\omega_\beta\rangle]\\ &= \frac{||c||_1^2}{k}E[\langle\omega|\,\omega\rangle] + \frac{k(k-1)}{k^2}\\ &\leqslant 1 + \frac{||c||_1^2}{k}.\end{aligned}$$

Second, since $|\Omega\rangle = \frac{||c||_1}{k}\sum_{\alpha=1}^k|\omega_\alpha\rangle$ be considered as the average of the k independent and identically distributed random copies $|\omega\rangle$, we have

$$\mathbb{E}[\langle\phi|\,\Omega\rangle] = \frac{||c||_1^2}{k}\mathbb{E}[\langle\omega|\,\omega_\alpha\rangle] = 1 \tag{C3}$$

and

$$\mathbb{E}[\langle\Omega|\,\phi\rangle] = 1. \tag{C4}$$

By substituting the inequalities above into Eq (C2), we obtain

$$\mathbb{E}(||\,|\phi\rangle - |\Omega\rangle\,||^2) = \frac{||c||_1^2}{k}. \tag{C5}$$

■

Now, we will prove the Proposition 12:

Proof. From Lemma 2, when we choose $k \geqslant \frac{||c||_1^2}{\epsilon^2}$, then $\mathbb{E}(|| \, |\phi\rangle - |\Omega\rangle \, ||^2) = \frac{||c||_1^2}{k} \leqslant \epsilon^2$. Since the random state $|\Omega\rangle$ is a sum of k free states, there must exist one $|\Omega\rangle$ that ϵ approximates $|\phi\rangle$. Consequently, from the definition of ϵ measure of resource rank, we have

$$R_{rk,\epsilon}(|\phi\rangle) \leqslant 1 + \frac{||c||_1^2}{\epsilon}, \tag{C6}$$

which completes the proof. ■

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