Galilean relativity and wave-particle duality imply the Schrödinger equation

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(Received 13 March 2024; accepted 26 April 2024; published 9 May 2024)

We show that the Schrödinger equation can be derived assuming the Galilean covariance of a generic wave equation and the validity of the de Broglie's wave-particle duality hypothesis. We also obtain from this set of assumptions the transformation law for the wave function under a Galilean boost and prove that complex wave functions are unavoidable for a consistent description of a physical system. The extension to the relativistic domain of the above analysis is also provided. We show that Lorentz covariance and wave-particle duality are consistent with two different transformation laws for the wave function under a Lorentz boost. This leads to two different wave equations, namely, the Klein-Gordon equation and the Lorentz covariant Schrödinger equation.

DOI: 10.1103/PhysRevA.109.052209

I. INTRODUCTION

Schrödinger arrived at his eponymous equation inspired by Hamilton's analogy between ordinary mechanics and geometrical optics. Instead of geometrical optics, Schrödinger worked with physical (wave) optics and searched for its "mechanical" analog. This search, guided by the de Broglie's wave-particle duality hypothesis [1], eventually led to what we know today as the Schrödinger equation [2].[1]

Our main goal in this work is to derive the Schrödinger equation assuming the de Broglie's wave-particle duality hypothesis and the Galilean covariance of the wave equation alone. Furthermore, we show that this connection between the Schrödinger equation and Galilean covariance is unique, namely, the Schrödinger equation is the only linear wave equation satisfying both de Broglie's hypothesis and Galilean covariance. We also obtain from this set of assumptions the transformation rule for the wave function after a Galilean boost [6–9] and an alternative proof that complex wave functions are unavoidable [10–12].

The last part of this work deals with the relativistic extension of the previous analysis. We show that two different classes of transformation laws for the wave function are compatible with Lorentz covariance. One transformation law leads to the Klein-Gordon equation [13–16] and the other one to the Lorentz covariant Schrödinger equation [17].

Before we move on, we should mention that the Schrödinger equation can be obtained by introducing, in very specific ways, stochastic fields or probabilistic arguments into classical physics [18–23], and most of the time by invoking the classical Hamilton-Jacobi equation [19,21–23]. Of particular notice is Ref. [24], where the Schrödinger equation was obtained by assuming the existence of a complex wave function that satisfied an arbitrary linear wave equation with, at most, a first-order time derivative.

II. ASSUMPTIONS

A. de Broglie's hypothesis

The wave-particle duality postulated by de Broglie [1] dictates that any massive particle also has a wave-like character, quantitatively expressed by the following relations: $\lambda = h/p$ and $\nu = E/h$, where λ is the particle's "wavelength," ν is its "frequency," h is Planck's constant, p is the magnitude of the particle's momentum $\mathbf{p}$, and E its energy.

In modern notation, de Broglie's hypothesis means that a particle's energy E and momentum $\mathbf{p}$ are given by

$$E = \hbar\omega, \tag{1}$$

$$\mathbf{p} = \hbar\mathbf{k}, \tag{2}$$

where $\hbar = h/(2\pi)$, the angular frequency $\omega = 2\pi\nu$, the wave number $k = |\mathbf{k}| = 2\pi/\lambda$, and $\mathbf{k}$ is the wave vector associated with the particle's wave function.

Throughout de Broglie's Ph.D. thesis [1] it is implicit that one should look for a wave equation governing the particle's dynamics. This is what Schrödinger accomplished three years after de Broglie presented his Ph.D. thesis [2]. One hundred years later, our main goal here is to use de Broglie's wave-particle duality, Eqs. (1) and (2), plus the following assumption to derive the Schrödinger equation.

B. Galilean covariance

In this work, Galilean covariance refers to mathematical expressions or physical laws that does not change under spatial rotations and Galilean boosts. We will not be dealing with space or time translations. These latter two operations plus spatial rotations and Galilean boosts constitute the inhomogeneous Galilean group. If we exclude space and time translations, we have the homogeneous Galilean

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[1]We should mention that Schrödinger was also influenced by Einstein's second paper on the Bose-Einstein condensate [3], which is acknowledged by Schrödinger himself [4,5].

group. Note that Galilean boosts are also known as Galilean transformations.

If S and S' are two inertial reference frames, with S' moving away from S with velocity $\mathbf{v}$, the Cartesian coordinates locating a given event and the time of its occurrence in both frames are related by the following rules according to the Galilean relativity:

$$t = t', \tag{3}$$

$$\mathbf{r} = \mathbf{r}' + \mathbf{v}t'. \tag{4}$$

For simplicity and without losing generality, the above relations assume that the origin of time and space coincide in both inertial frames. We also have, in an obvious notation, that $\mathbf{r} = (x, y, z)$ and $\mathbf{r}' = (x', y', z')$ are the space coordinates of the event in S and S', respectively, and t and t' are the corresponding time of occurrence of the event. Equations (3) and (4) are what we call a Galilean boost or transformation.

Using Einstein's summation notation, the most general way of writing a wave equation is

$$a^{\mu\nu}\partial_\mu\partial_\nu\Psi(\mathbf{r},t) + b^\mu\partial_\mu\Psi(\mathbf{r},t) + f(\mathbf{r},t)\Psi(\mathbf{r},t) = 0. \tag{5}$$

Note that by wave equation we mean the most general homogeneous linear partial differential equation in the variables x, y, z, and t of order less than or equal to 2, and with constant coefficients multiplying the derivatives. We also assume that a wave equation has to provide solutions that propagate the waves along the spacetime, especially for free particles with nonzero momentum.

In Eq. (5), $a^{\mu\nu}$ and b^μ are constants (Galilean invariants) and $f(\mathbf{r},t)$ is an arbitrary function but a strict Galilean scalar, namely, $f(\mathbf{r},t) = f'(\mathbf{r}',t')$ after a spatial rotation or a Galilean boost. Also, $\partial_\mu = \frac{\partial}{\partial x^\mu}$ and $\mu = 0, 1, 2, 3$, where $x^0 = ct$, $x^1 = x$, $x^2 = y$, and $x^3 = z$, with c being the speed of light in vacuum.

With the previous notation, we can state the principle of Galilean covariance as follows. If inertial frame S' is connected to S by a spatial rotation or a Galilean boost, then if in S the wave equation is given by Eq. (5), in S' it should look, up to an overall nonnull multiplicative factor

$$a^{\mu\nu}\partial_{\mu'}\partial_{\nu'}\Psi'(\mathbf{r}',t') + b^\mu\partial_{\mu'}\Psi'(\mathbf{r}',t') + f'(\mathbf{r}',t')\Psi'(\mathbf{r}',t') = 0. \tag{6}$$

III. OBTAINING THE SCHRÖDINGER EQUATION

A. Covariance under spatial rotations

According to de Broglie's hypothesis, a particle with mass m has an associated wave function describing its dynamics. In the nonrelativistic domain and for a free particle with nonnull momentum, this wave function must describe the fact that this particle moves away with speed $v = p/m$. Whether or not we deal with a localized particle, or with a real or complex wave function, variables $\mathbf{r}$ and t are constrained by the following relation if we want propagating waves,

$$\mathbf{k}\cdot\mathbf{r} - \omega t. \tag{7}$$

Since the scalar product $\mathbf{k}\cdot\mathbf{r}$ is invariant under spatial rotations, and by using that the vacuum is isotropic, Galilean covariance implies that under any spatial rotation the wave functions in frames S and S' are related by the following rule:[2]

$$\Psi(\mathbf{r},t) = \Psi'(\mathbf{r}',t). \tag{8}$$

Note that any Euclidean vector in S' is connected to its corresponding representation in S by an orthogonal transformation belonging to the SO(3) group, i.e., $\mathbf{r}' = M\mathbf{r}$, with $M \in$ SO(3). Moreover, if we assume $\Psi(\mathbf{r},t) = \alpha\Psi'(\mathbf{r}',t)$, where α is a constant, it is easy to see that $\alpha = 1$. Indeed, if we rotate from S to S' and then to S'' we get $\Psi(\mathbf{r},t) = \alpha^2\Psi''(\mathbf{r}'',t)$. If we rotate directly from S to S'' we have $\Psi(\mathbf{r},t) = \alpha\Psi''(\mathbf{r}'',t)$. Comparing both expressions for $\Psi(\mathbf{r},t)$ we obtain that $\alpha^2 = \alpha$. Since α cannot be zero, the only valid solution is $\alpha = 1$.

Using the transformation law given by Eq. (8), it can be shown that we guarantee covariance under spatial rotations if, and only if, for $j, k = 1, 2, 3$ [17,25,26],

$$b^j = 0, \tag{9}$$

$$a^{j0} + a^{0j} = 0, \tag{10}$$

$$a^{jk} + a^{kj} = 0, \quad \text{for } j \neq k, \tag{11}$$

$$a^{jj} = a^{kk}. \tag{12}$$

Using Eqs. (9) to (12) and renaming the constant parameters multiplying the derivatives, Eq. (5) becomes

$$\overline{A}\partial_t^2\Psi(\mathbf{r},t) + \overline{B}\nabla^2\Psi(\mathbf{r},t) + \overline{C}\partial_t\Psi(\mathbf{r},t) + f(\mathbf{r},t)\Psi(\mathbf{r},t) = 0. \tag{13}$$

Equation (13) is the most general linear partial differential equation of order two compatible with covariance under any spatial rotation. To arrive at Eq. (13), we use that $\partial_0 = \partial^0$, $\partial_j = -\partial^j$, for $j = 1, 2, 3$, $\partial_t^2 = \partial^2/\partial t^2$, and $-\partial_j\partial^j = \nabla^2$, where the latter is the Laplacian, an invariant under spatial rotations.

B. Covariance under Galilean boosts

Since Eq. (13) is covariant under spatial rotations, we can work without losing in generality with a Galilean boost along the x axis, where $\mathbf{v} = (v, 0, 0)$,

$$t = t', \tag{14}$$

$$x = x' + vt', \tag{15}$$

$$y = y', \tag{16}$$

$$z = x'. \tag{17}$$

Using the chain rule and Eqs. (14) to (17), the first-order derivatives change as follows:

$$\partial_t = \partial_{t'} - v\partial_{x'}, \tag{18}$$

$$\partial_j = \partial_{j'}, \ \text{for } j = x, y, z. \tag{19}$$

[2]If $\mathbf{k}$ and $\mathbf{r}$ are not related by $\mathbf{k}\cdot\mathbf{r}$, rotational covariance can be obtained assuming $\Psi(\mathbf{r},t) = g(\mathbf{r}',t)\Psi'(\mathbf{r}',t)$ [17].

We also assume that the wave function changes according to the following rule after a Galilean boost:

$$\Psi(\mathbf{r},t) = g(\mathbf{r}',t')\Psi'(\mathbf{r}',t'), \tag{20}$$

with $g(\mathbf{r}',t')$ representing an arbitrary function of $\mathbf{r}'$ and t'. Equation (20) is the most general way of representing how a wave function changes after an arbitrary symmetry operation [17,25,26].

Using Eqs. (18) to (20), we obtain that the term multiplying $\overline{A}$ in Eq. (13) is transformed to several different terms containing pure or mixed derivatives. One of these terms is the following mixed derivative:

$$-2\overline{A}vg(\mathbf{r}',t')\frac{\partial^2\Psi'(\mathbf{r}',t')}{\partial t'\partial x'}. \tag{21}$$

Another term proportional to $\partial_{t'}\partial_{x'}\Psi'(\mathbf{r}',t')$ cannot be found anywhere else in the transformed equation after the Galilean boost. The derivatives multiplying the other constants in Eq. (13) cannot provide a mixed derivative in the t' and x' variables if we use Eqs. (18) to (20). Therefore, since $g(\mathbf{r}',t') \neq 0$ and the transformed term given by Eq. (21) is not in Eq. (13) before the transformation, Galilean covariance implies that $\overline{A} = 0$. Note that since the boost is along the x axis, $g(\mathbf{r}',t')$ depends only on x and t.

This result is interesting by its own since it implies that we cannot have a covariant equation with a second-order derivative in time. In other words, we proved the following result.

Theorem 1. If the order of the differential equation is at most two, Galilean covariance implies that the wave equation cannot have a second-order time derivative.

This is another way of understanding why the Schrödinger equation has only a first-order time derivative. If higher than second-order derivatives are allowed, we show in the Appendix that when fourth-order derivatives are present we can have a second-order time derivative and a covariant differential equation. However, for the free-particle case, this equation is essentially the squared Schrödinger operator acting on Ψ.

Setting $\overline{A} = 0$ in Eq. (13), Eqs. (18) to (20) lead to the following transformed wave equation if we drop the primes and factor out the common term $g(\mathbf{r}',t')$:

$$\overline{B}\nabla^2\Psi + \overline{C}\partial_t\Psi + f\Psi + \left[\frac{2\overline{B}}{g}\frac{\partial g}{\partial x} - \overline{C}v\right]_1 \partial_x\Psi + \left[\frac{\overline{B}}{g}\frac{\partial^2 g}{\partial x^2} + \frac{\overline{C}}{g}\frac{\partial g}{\partial t} - \frac{\overline{C}v}{g}\frac{\partial g}{\partial x}\right]_2 \Psi = 0. \tag{22}$$

To obtain covariance, the two brackets above should be zero, i.e., $[\ \ ]_1 = [\ \ ]_2 = 0$. By demanding that $[\ \ ]_1 = 0$ and taking the spatial derivative we obtain

$$\frac{\overline{B}}{g}\frac{\partial^2 g}{\partial x^2} = \frac{\overline{C}v}{2g}\frac{\partial g}{\partial x}. \tag{23}$$

Using Eq. (23), the second condition for covariance, namely, $[\ \ ]_2 = 0$, can be written as

$$\frac{\partial g}{\partial t} = \frac{v}{2}\frac{\partial g}{\partial x}. \tag{24}$$

This is the one-way wave equation whose general solution is

$$g(x,t) = h\left(t + \frac{2}{v}x\right). \tag{25}$$

Inserting $g(x,t)$ back into $[\ \ ]_1 = 0$ gives

$$\frac{dh(u)}{du} = \frac{\overline{C}v^2}{4\overline{B}}h(u), \tag{26}$$

where $u = t + 2x/v$. The general solution to the above equation is

$$h(u) = g_0 \exp\left[\frac{\overline{C}v^2}{4\overline{B}}u\right], \tag{27}$$

with g_0 being an arbitrary constant. Finally, returning to g and using the definition of u we obtain

$$g(x,t) = g_0 \exp\left[\frac{\overline{C}}{4\overline{B}}v^2 t + \frac{\overline{C}}{2\overline{B}}vx\right]. \tag{28}$$

Putting back the primes, dividing and multiplying the exponent by the particle's mass m, we arrive at the following solution for a boost in an arbitrary direction:

$$g(\mathbf{r}',t') = g_0 \exp\left[\frac{\overline{C}}{2m\overline{B}}\left(\frac{mv^2}{2}t' + m\mathbf{v}\cdot\mathbf{r}'\right)\right]. \tag{29}$$

Note that if we go directly from frame S to S'' or from S to S' and then to S'', we get that $g_0 = 1$.

Equation (29) is the most general function $g(\mathbf{r}',t')$ that under a general Galilean boost [cf. Eqs. (3) and (4)] guarantees the covariance of the wave equation (13) if its wave function transforms according to Eq. (20).

If $g(\mathbf{r}',t')$ is a constant, we must have $\overline{C} = 0$. This implies the following differential equation after setting $\overline{A} = \overline{C} = 0$ in Eq. (13):

$$\overline{B}\nabla^2\Psi(x) + f(x)\Psi(x) = 0. \tag{30}$$

Note that Eq. (30) is the Helmholtz equation with nonconstant eigenvalues $f(x)/\overline{B}$. But the most important point is that the above equation has no time derivatives. This means that it is not a wave equation at all. Putting it differently, we proved the following result.

Theorem 2. Galilean covariance implies that the wave function cannot be a strict scalar under a Galilean boost, i.e., it is impossible to have a covariant wave equation such that, up to an overall constant phase, $\Psi(\mathbf{r},t) = \Psi'(\mathbf{r}',t')$ after a Galilean boost.

C. de Broglie's wave-particle duality

If we now set $\overline{C} \neq 0$, we have that, in general, $g(\mathbf{r}',t')$ is a function of $\mathbf{r}'$ and t'. Moreover, whenever $v \neq 0$, looking at Eq. (29) we note that if $\overline{C}/\overline{B}$ is a positive real number, $g(\mathbf{r}',t')$ diverges as a function of the time. In addition, if $\overline{C}/\overline{B}$ is a negative real number, it tends to zero as the time increases. Since $\Psi(\mathbf{r},t) = g(\mathbf{r}',t')\Psi'(\mathbf{r}',t')$, this implies that if $\overline{C}/\overline{B}$ is real and $|\Psi'(\mathbf{r}',t')|$ is a nonzero bounded function for all t', we have that $|\Psi(\mathbf{r},t)|$ is either zero or infinity after a sufficiently long time.

However, according to de Broglie's hypothesis, there must be a wave function with wave number $\mathbf{k}$ and frequency ω associated to a particle of mass m. For a free particle with nonzero velocity, the magnitude of this wave function cannot be infinity or zero. Otherwise no physical meaning could be attributed to it and we could not extract from it a meaningful wave number and frequency that must be associated to the moving free particle according to de Broglie's hypothesis. Therefore, we can only satisfy both Galilean covariance and de Broglie's wave-particle duality if $\overline{C}/\overline{B}$ is a pure imaginary number. In other words, the wave-particle duality should be valid in any inertial reference frame and this would not be the case if we had a zero or divergent wave function.

The fact that $g(\mathbf{r}', t')$ must be a complex number implies that we cannot have a real quantum mechanics. Complex numbers, or equivalently complex wave functions, are mandatory when both Galilean covariance and de Broglie's hypothesis are assumed. This can be proved by the following simple argument. Assume that in the inertial reference frame S', moving away from S with velocity $\mathbf{v}$, we are able to somehow completely describe a particle of mass m using a purely real wave function. If we now go to frame S, the particle's wave function is given by Eq. (20). Since $g(\mathbf{r}', t')$ is necessarily a complex number, in frame S the wave function will thus be necessarily a complex number also. Therefore, we proved the following interesting result.

Theorem 3. Galilean covariance and the de Broglie's wave-particle duality imply that complex wave functions are unavoidable to properly describe a physical system.

To finally arrive at the Schrödinger equation, we need to determine the value of $\overline{C}/\overline{B}$. This is accomplished using de Broglie's relations, Eqs. (1) and (2), which lead to a specific dispersion relation for free particles. We then require that the wave equation that we have so far, Eq. (32), must yield the same dispersion relation for its plane wave solution. This will fix the value of $\overline{C}/\overline{B}$.

For a nonrelativistic free particle of mass m moving with constant velocity $\mathbf{v}$, we have that its momentum and energy are $\mathbf{p} = m\mathbf{v}$ and $E = p^2/(2m)$. The second equation together with Eqs. (1) and (2) lead to the following dispersion relation:

$$\omega = \frac{\hbar k^2}{2m}. \tag{31}$$

The wave equation we have so far can be written as follows:

$$\overline{B}\nabla^2\Psi(\mathbf{r}, t) + \overline{C}\partial_t\Psi(\mathbf{r}, t) + f(\mathbf{r}, t)\Psi(\mathbf{r}, t) = 0. \tag{32}$$

If we insert the ansatz

$$\Psi(\mathbf{r}, t) = \Psi_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}, \tag{33}$$

where Ψ_0 is an arbitrary constant, into Eq. (32) we get

$$\overline{B}k^2 + i\overline{C}\omega - f(\mathbf{r}, t) = 0. \tag{34}$$

If we now demand that ω and k are related by Eq. (31), we have that Eq. (34) becomes

$$k^2\left(\overline{B} + \frac{i\hbar}{2m}\overline{C}\right) - f(\mathbf{r}, t) = 0. \tag{35}$$

Since $f(\mathbf{r}, t)$ cannot depend on k, the only solution to Eq. (35) is

$$\frac{\overline{C}}{\overline{B}} = \frac{i2m}{\hbar}, \tag{36}$$

$$f(\mathbf{r}, t) = 0. \tag{37}$$

We thus fixed the value of $\overline{C}/\overline{B}$ and discovered that for a free particle $f(\mathbf{r}, t) = 0$. With Eq. (36) we can write the final expression for the transformation rule for the wave equation after a Galilean boost [cf. Eq. (29)],

$$g(\mathbf{r}', t') = \exp\left[\frac{i}{\hbar}\left(\frac{mv^2}{2}t' + m\mathbf{v}\cdot\mathbf{r}'\right)\right]. \tag{38}$$

Equations (36) and (37) when inserted into Eq. (32) give the Schrödinger equation for a free particle. To write Eq. (32) exactly as we write today the Schrödinger equation, we express $\overline{B}$ and $\overline{C}$ as follows:

$$\overline{B} = \frac{\hbar^2}{2m}D, \tag{39}$$

$$\overline{C} = i\hbar D. \tag{40}$$

Equations (39) and (40) satisfy Eq. (36) and D is an arbitrary constant. Inserting Eqs. (39) and (40) into Eq. (32) and using Eq. (37), we obtain the free-particle Schrödinger equation after dropping the overall constant D,

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi. \tag{41}$$

We can also fix the value of f in Eq. (32) by studying the case of a particle in a constant potential V and then generalizing the obtained equation to a position and time-dependent potential. For a constant potential, the particle's energy is $E = p^2/(2m) + V = \hbar^2k^2/(2m) + V$. The dispersion relation now is

$$\omega = \frac{\hbar k^2}{2m} + \frac{V}{\hbar}. \tag{42}$$

Inserting Eqs. (33) and (42) into Eq. (32), and using Eqs. (39) and (40), we get

$$f(\mathbf{r}, t) = -DV. \tag{43}$$

Using Eqs. (39), (40), and (43), we can write Eq. (32) after dropping the overall constant D as follows:

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi. \tag{44}$$

Equation (44) is the celebrated Schrödinger equation, derived here using only two assumptions, namely, Galilean covariance and de Broglie's wave-particle duality relations.

IV. OBTAINING THE RELATIVISTIC EQUATIONS

Since we will be dealing with relativistic wave equations, it is convenient to use the four-vector notation. The four-vector notation can be summarized as follows [16]. The contravariant four-vector x^μ is defined as $(x^0, x^1, x^2, x^3) = (ct, x, y, z)$, where c is the speed of light in vacuum. Using the metric where $g_{00} = 1, g_{11} = g_{22} = g_{33} = -1$, with $g_{\mu\nu} = 0$ otherwise, the covariant four-vector is $x_\mu = g_{\mu\nu}x^\nu$, i.e., $(x_0, x_1, x_2, x_3) = (ct, -x, -y, -z)$. The Einstein summation convention is assumed, where the Greek indexes run from 0 to 3 while the Latin ones go from 1 to 3. The scalar product between two four-vectors is $x^\mu y_\mu$ and between two spatial vectors is $\mathbf{x}\cdot\mathbf{y} = -x^j y_j$. The covariant four-gradient is defined as $\partial_\mu = (\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3})$ and the contravariant four-gradient by $\partial^\mu = g^{\mu\nu}\partial_\nu$, where $g^{\mu\nu} = g_{\mu\nu}$.

In the four-vector notation, for instance, the Galilean boosts (3) and (4) can be written as

$$x^0 = x^{0'}, \tag{45}$$

$$x^j = x^{j'} + \beta^j x^{0'}, \tag{46}$$

where $\beta^j = v^j/c$. Note that $\beta = |\beta| = |\mathbf{v}|/c = v/c$.

The analysis carried out up to Eq. (13) applies here as well since we assume for the wave function the same transformation law under spatial rotations of the nonrelativistic case [cf. Eq. (8)]. It is convenient to rewrite Eq. (13) as

$$A\partial_0^2\Psi(x) - B\partial_j\partial^j\Psi(x) + C\partial_0\Psi(x) + f(x)\Psi(x) = 0, \tag{47}$$

where $A = c^2\overline{A}$, $B = \overline{B}$, $C = c\overline{C}$, and $x = (x^0, x^1, x^2, x^3)$.

As before, due to the covariance of the wave equation (47) under spatial rotations, we employ a boost along the x^1 axis without losing in generality. However, we must work now with a Lorentz instead of a Galilean boost

$$x^0 = \gamma(x^{0'} + \beta x^{1'}), \tag{48}$$

$$x^1 = \gamma(x^{1'} + \beta x^{0'}), \tag{49}$$

$$x^2 = x^{2'}, \tag{50}$$

$$x^3 = x^{3'}, \tag{51}$$

where

$$\beta = \frac{v}{c} \text{ and } \gamma = \frac{1}{\sqrt{1-\beta^2}}. \tag{52}$$

The analogs of Eqs. (18) and (19) are

$$\partial_0 = \gamma(\partial_{0'} - \beta\partial_{1'}), \tag{53}$$

$$\partial_1 = \gamma(\partial_{1'} - \beta\partial_{0'}), \tag{54}$$

$$\partial_2 = \partial_{2'}, \tag{55}$$

$$\partial_3 = \partial_{3'}. \tag{56}$$

Inserting Eqs. (20) and (53) to (56) into Eq. (47), and assuming $f(x)$ to be a strict Lorentz scalar, $f(x) = f'(x')$, we obtain after factoring out the common $g(x')$ term

$$\begin{aligned} &A'\partial_{0'}^2\Psi'(x') + B'\partial_{1'}^2\Psi'(x') + B\partial_{2'}^2\Psi'(x') + B\partial_{3'}^2\Psi'(x') \\ &+ C'\partial_{0'}\Psi'(x') + D'\partial_{0'}\partial_{1'}\Psi'(x') + E'\partial_{1'}\Psi'(x') \\ &+ F'\Psi'(x') + f'(x')\Psi'(x') = 0, \end{aligned} \tag{57}$$

where

$$A' = \gamma^2(A + \beta^2 B), \tag{58}$$

$$B' = \gamma^2(\beta^2 A + B), \tag{59}$$

$$\begin{aligned} C' = \gamma\bigg[&C + 2(A+\beta^2 B)\gamma\frac{\partial_{0'}g(x')}{g(x')} \\ &- 2(A+B)\beta\gamma\frac{\partial_{1'}g(x')}{g(x')}\bigg], \end{aligned} \tag{60}$$

$$D' = -2\gamma^2\beta(A+B), \tag{61}$$

$$\begin{aligned} E' = \gamma\bigg[&-C\beta - 2(A+B)\beta\gamma\frac{\partial_{0'}g(x')}{g(x')} \\ &+ 2(B+\beta^2 A)\gamma\frac{\partial_{1'}g(x')}{g(x')}\bigg], \end{aligned} \tag{62}$$

$$\begin{aligned} F' = \gamma\bigg[&C\frac{\partial_{0'}g(x')}{g(x')} + (A+\beta^2 B)\gamma\frac{\partial_{0'}^2 g(x')}{g(x')} \\ &- C\beta\frac{\partial_{1'}g(x')}{g(x')} - 2(A+B)\beta\gamma\frac{\partial_{0'}\partial_{1'}g(x')}{g(x')} \\ &+ (B+\beta^2 A)\gamma\frac{\partial_{1'}^2 g(x')}{g(x')}\bigg]. \end{aligned} \tag{63}$$

We first note that in Eq. (57), the mixed derivative term, $D'\partial_{0'}\partial_{1'}\Psi'(x')$, should be zero since it is not present in the wave equation before the Lorentz boost. Therefore, we must have $D' = 0$, which, according to Eq. (61), implies

$$A = -B. \tag{64}$$

If we now use Eq. (64) and the mathematical identity $(1-\beta^2)\gamma^2 = 1$, Eq. (57) can be written as

$$\begin{aligned} &B\partial_{\mu'}\partial^{\mu'}\Psi'(x') - C'\partial_{0'}\Psi'(x') - f'(x')\Psi'(x') \\ &- E'\partial_{1'}\Psi'(x') - F'\Psi'(x') = 0, \end{aligned} \tag{65}$$

where

$$C' = C\gamma - 2B\frac{\partial_{0'}g(x')}{g(x')}, \tag{66}$$

$$E' = -C\beta\gamma + 2B\frac{\partial_{1'}g(x')}{g(x')}, \tag{67}$$

$$\begin{aligned} F' = &C\gamma\left(\frac{\partial_{0'}g(x')}{g(x')} - \beta\frac{\partial_{1'}g(x')}{g(x')}\right) \\ &+ B\left(\frac{\partial_{1'}^2 g(x')}{g(x')} - \frac{\partial_{0'}^2 g(x')}{g(x')}\right). \end{aligned} \tag{68}$$

Comparing Eq. (65) with the original differential equation before the Lorentz boost, i.e., comparing it with the equation below, which is Eq. (47) when $A = -B$,

$$B\partial_\mu\partial^\mu\Psi(x) - C\partial_0\Psi(x) - f(x)\Psi(x) = 0, \tag{69}$$

we realize that to guarantee covariance three conditions must be met,

$$C' = C, \tag{70}$$

$$E' = 0, \tag{71}$$

$$F' = 0. \tag{72}$$

There are two distinct classes of solutions that satisfy Eqs. (70) to (72).

A. Klein-Gordon equation

The first class of solutions assumes that it is possible to obtain a relativistic wave equation such that the wave function is a strict scalar under a Lorentz boost, i.e., $\Psi(x) = \Psi'(x')$. This is achieved if we impose that $g(x') = 1$. Note that the results below are also true for any nonnull constant $g(x')$.

For a constant g we immediately see that $F' = 0$ since it only depends on derivatives of g. Furthermore, Eqs. (66) and (67) become, respectively, $C' = C\gamma$ and $E' = -C\beta\gamma$. We can only have $C' = C$ and $E' = 0$, and thus satisfy Eqs. (70) and (71), if $C = 0$.

With $C = 0$ the wave equation becomes

$$B\partial_\mu\partial^\mu\Psi(x) - f(x)\Psi(x) = 0, \tag{73}$$

which is the Klein-Gordon equation if we set $f(x) = -Bm^2c^2/\hbar^2$. We can arrive at the previous value for $f(x)$ by demanding the validity of the Einstein energy-momentum relation, namely, $E^2 = m^2c^4 + p^2c^2$, and by applying de Broglie's hypothesis to obtain from the Einstein energy-momentum relation the corresponding dispersion relation that the plane wave solution to Eq. (73) must satisfy.

The present analysis can be summarized in the following theorem.

Theorem 4. Lorentz covariance is compatible with the wave function being a strict scalar under a Lorentz boost, i.e., $\Psi(x) = \Psi'(x')$ after a Lorentz boost. In addition, together with de Broglie's hypothesis and Einstein energy-momentum relation, they lead to the Klein-Gordon equation.

It is not difficult to see that the logic that led to the above theorem can be easily reverted, leading to the following result.

Theorem 5. Lorentz covariance, de Broglie's hypothesis, and the Einstein energy-momentum relation imply the Klein-Gordon equation and that the wave function is, up to an overall constant, a strict scalar under a Lorentz boost, i.e., $\Psi(x) = \Psi'(x')$ after a Lorentz boost.

The proof of the above theorem is as follows. Lorentz covariance, i.e., covariance under spatial rotations and relativistic boosts, leads to the wave equation (69) and the auxiliary conditions (70) to (72) that will allow us to obtain the transformation law for the wave function under a Lorentz boost. If we now use the Einstein energy-momentum relation and de Broglie's wave-particle hypothesis, we obtain the following dispersion relation:

$$\hbar^2\omega^2 = m^2c^4 + c^2\hbar^2k^2. \tag{74}$$

If we use Eq. (74) and the plane wave ansatz (33), the wave equation (69) becomes

$$\frac{m^2c^2}{\hbar^2}B - \frac{i\omega}{c}C + f(x) = 0. \tag{75}$$

Since B, C, and f are independent of k and ω, the only solution to Eq. (75) compatible with this constraint is

$$C = 0, \tag{76}$$

$$f(x) = -\frac{m^2c^2}{\hbar^2}B. \tag{77}$$

However, if $C = 0$, Eq. (70) implies that $C' = 0$. Looking at the definition of C', Eq. (66), this implies that $g(x')$ should not depend on $x^{0'}$. Similarly, Eqs. (76), (71), and (67) imply that $g(x')$ should not depend on $x^{1'}$. In other words, $g(x')$ is a constant, proving that $\Psi(x)$ is a strict scalar under a Lorentz boost. Note that if $g(x)$ is a constant, the remaining constraint, Eq. (72), is automatically satisfied. Finally, if we insert $f(x)$ as given in Eq. (77) into the wave equation (69) and use the fact that $C = 0$, we obtain after dropping the common factor B the Klein-Gordon equation.

B. Lorentz covariant Schrödinger equation

The second class of solutions to Eqs. (70) to (72) no longer assumes a constant $g(x')$. We start by solving Eq. (71). Using Eq. (67), $E' = 0$ leads to the following partial differential equation:

$$\frac{\partial g(x^{0'}, x^{1'})}{\partial x^{1'}} = \frac{C\gamma\beta}{2B} g(x^{0'}, x^{1'}), \tag{78}$$

whose general solution is

$$g(x^{0'}, x^{1'}) = h(x^{0'}) \exp\left(\frac{C\gamma\beta}{2B}x^{1'}\right). \tag{79}$$

Using Eq. (70), we have that Eq. (66) gives the following partial differential equation:

$$\frac{\partial g(x^{0'}, x^{1'})}{\partial x^{0'}} = \frac{C(\gamma - 1)}{2B} g(x^{0'}, x^{1'}). \tag{80}$$

Inserting Eq. (79) into Eq. (80) we get

$$\frac{dh(x^{0'})}{dx^{0'}} = \frac{C(\gamma - 1)}{2B} h(x^{0'}), \tag{81}$$

the solution of which is

$$h(x^{0'}) = g_0 \exp\left[\frac{C(\gamma - 1)}{2B}x^{0'}\right], \tag{82}$$

where $g_0 = 1$ (see Sec. III B).

Therefore, using Eqs. (79) and (82) we obtain

$$g(x^{0'}, x^{1'}) = \exp\left\{\frac{C}{2B}[(\gamma - 1)x^{0'} + \gamma\beta x^{1'}]\right\}. \tag{83}$$

The remaining constraint, Eq. (72), is automatically satisfied if we use the solution above, Eqs. (83) and (68). It is worth mentioning that we recover the case of a constant $g(x')$ if $C = 0$, which we proved to be the case by a different route in Sec. IV A.

Looking at Eq. (83), we realize that the same analysis carried out for the nonrelativistic case concerning the complex nature of $g(x')$ applies here if $g(x')$ is not a constant ($C \neq 0$). This implies that C/B must be a pure imaginary number if $C \neq 0$ and that complex wave functions are unavoidable (cf. Sec. III C). Note that if $g(x')$ is a constant, the previous analysis does not apply. This implies that real wave functions are compatible with Lorentz covariance and the Klein-Gordon equation.

Without further input, C/B is an arbitrary pure imaginary number. We can fix its value by demanding that the nonrelativistic limit ($\beta = v/c \ll 1$) of Eq. (83) tends to Eq. (38), its nonrelativistic analog.

The nonrelativistic limit of Eq. (83) is obtained by expanding its exponent in powers of v/c and keeping only the dominant terms. Since $\gamma - 1 \approx v^2/(2c^2)$ and $\gamma\beta \approx v/c$, Eq. (83) becomes

$$g(x', t') \approx \exp\left[\frac{C}{2Bc}\left(\frac{v^2}{2}t' + vx'\right)\right]. \tag{84}$$

We used that $x^{0'} = ct'$ and $x^{1'} = x'$ to arrive at Eq. (84). Note that x can be the shorthand notation for the four-vector (x^0, x^1, x^2, x^3) or simply the variable associated with the x axis. The context makes it clear which meaning one should attribute to x. Comparing Eqs. (84) and (38), where we set $\mathbf{v} = (v, 0, 0)$ in the later equation, we realize that they are equal if

$$\frac{C}{B} = \frac{i2mc}{\hbar}. \tag{85}$$

Inserting Eq. (85) into Eq. (83) we get

$$g(x^{0'}, x^{1'}) = \exp\left\{\frac{i}{\hbar}[(\gamma - 1)mcx^{0'} + \gamma mvx^{1'}]\right\}. \tag{86}$$

If we use that $x^{0'} = ct'$, $x^{1'} = x$, and note that $vx = \mathbf{v}\cdot\mathbf{r}$, we obtain from Eq. (86) the transformation law for a Lorentz boost in an arbitrary direction (along the direction of $\mathbf{v}$)

$$g(\mathbf{r}', t') = \exp\left\{\frac{i}{\hbar}[(\gamma - 1)mc^2t' + \gamma m\mathbf{v}\cdot\mathbf{r}']\right\}. \tag{87}$$

We are still left with one free parameter to fix the values of C and B, provided we respect Eq. (85). Employing the same convention of the non-relativistic case, namely, Eqs. (39) and (40), we obtain the following wave equation from Eq. (69)

$$\partial_\mu\partial^\mu\Psi - i\frac{2mc}{\hbar}\partial_0\Psi + \frac{2mV}{\hbar^2}\Psi = 0. \tag{88}$$

Note that we also used the nonrelativistic convention, Eq. (43), to rename $f(x)$. In the present case, V should be interpreted as a relativistic invariant or a relativistic scalar under proper Lorentz transformations whose nonrelativistic limit tends to the Newtonian potential energy [17].

Equation (88) is the Lorentz covariant Schrödinger equation obtained in Ref. [17] by a different set of assumptions. It can also be written as follows, akin to the way we write the nonrelativistic Schrödinger equation

$$-\frac{\hbar^2}{2mc^2}\frac{\partial^2\Psi}{\partial t^2} + i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi. \tag{89}$$

Looking at Eq. (89) we realize that it differs from the nonrelativistic Schrödinger equation by its first term. It is also clear that when $c \to \infty$, another way to obtain the nonrelativistic limit, this term goes to zero and we recover the Schrödinger equation exactly.

Theorems 4 and 5 indicate that the plane wave solution to Eq. (88) does not satisfy the dispersion relation (74), unless $m = 0$. In this particular case $g(x')$ becomes a constant according to Eq. (87) and the assumptions that led to those theorems are fulfilled.

To better understand this point, let S' be the rest frame of a particle with mass m that is moving with constant speed $\mathbf{v}$ with respect to an inertial frame S. For the free-particle case ($V = 0$), we have that in S' the wave function $\Psi'(\mathbf{r}', t') = \Psi_0$, with Ψ_0 being a constant, is a solution to Eq. (88) that has a clear physical meaning. Note that this is not the case for the Klein-Gordon equation, which does not accept a constant solution if the mass is not zero.

A constant solution in S' means a plane wave with a null wave vector and zero frequency. As such, the same interpretation of the nonrelativistic case applies here, where we should understand $\hbar\omega$ as the kinetic energy of the particle. This becomes even clearer if we use the transformation rule (20) and Eq. (87) to obtain the wave function in S from the one in S', namely,

$$\begin{aligned}\Psi(\mathbf{r}, t) &= \Psi_0 \exp\left\{\frac{i}{\hbar}[(\gamma - 1)mc^2t' + \gamma m\mathbf{v}\cdot\mathbf{r}']\right\},\\ &= \Psi_0 \exp\left\{\frac{i}{\hbar}[-(\gamma - 1)mc^2t + \gamma m\mathbf{v}\cdot\mathbf{r}]\right\}.\end{aligned} \tag{90}$$

To obtain the last line, we used the inverse of the Lorentz boost given by Eqs. (48) to (51) to express the primed variables as functions of the unprimed ones.

Comparing Eq. (90) with the standard way of writing a plane wave, Eq. (33), we recognize that

$$\hbar\omega = (\gamma - 1)mc^2, \tag{91}$$

$$\hbar\mathbf{k} = \gamma m\mathbf{v}. \tag{92}$$

Note that $(\gamma - 1)mc^2$ is the usual relativistic kinetic energy of the particle and $\gamma m\mathbf{v}$ its relativistic momentum from the point of view of S. This is consistent with the interpretation we gave above about the constant solution in the particle's rest frame, where both the kinetic energy and momentum are zero.

A detailed analysis of Eq. (89) in several different scenarios and external potentials is given in Ref. [17] as well as its connection with the Klein-Gordon equation. The second quantization of Eq. (89) is also provided in Ref. [17] as well as a generalized Lorentz covariant Schrödinger equation in which the transformation law for the wave function under a boost and under spatial rotations are given by a nonconstant $g(x')$. The extension of the present ideas for a spin-1/2 elementary particle is given in Refs. [25,26], where the first and second quantized theories are developed. The common feature of the quantum field theories built on Eq. (89) [17] and on its spin-1/2 extension [25,26] can be summarized in the fact that particles and antiparticles with the same mass do not have the same dispersion relation anymore. This points to a fully relativistic way of understanding the asymmetry between matter and antimatter in the present day universe [17,25,26].

V. CONCLUSION

We showed that it is possible to derive the Schrödinger equation using only two assumptions, namely, de Broglie's wave-particle duality hypothesis and the Galilean covariance of the wave equation. The first assumption means that there is a wave function associated with a massive particle and that its energy and momentum are connected to the frequency and wave vector of that wave as prescribed by de Broglie. The second assumption is the Galilean relativity principle, which, in the sense employed in this work, postulates that the wave equation (laws of physics) should look the same after either a spatial rotation or a Galilean transformation (Galilean boost).

The above analysis not only lead unambiguously to the nonrelativistic Schrödinger equation, but also to the transformation law for the wave function after a Galilean boost. It also lead to the proof that the wave function cannot be a strict scalar under a Galilean transformation, namely, it is not possible to have a nonrelativistic wave equation satisfying the two assumptions outlined above such that $\Psi(\mathbf{r}, t) \rightarrow \Psi(\mathbf{r}, t)$ after a Galilean boost.

Furthermore, we showed that Galilean covariance and de Broglie's wave-particle duality also imply that complex wave functions are unavoidable for a consistent description of a physical system in all inertial frames. We also showed that any wave equation compatible with those two assumptions cannot have a second-order time derivative, which is an alternative way of understanding why the Schrödinger equation has only a first-order time derivative [24].

The extension of the above results to the relativistic domain was given in the end of this work. We derived the wave equations compatible with de Broglie's wave-particle duality hypothesis and Lorentz covariance, where the latter term means covariance under spatial rotations and Lorentz boosts.

We showed that Lorentz covariance is compatible with a wave function that transforms under a Lorentz boost as a strict scalar, i.e., $\Psi(\mathbf{r}, t) \rightarrow \Psi(\mathbf{r}, t)$ after a Lorentz transformation. Moreover, we showed that Lorentz covariance plus de Broglie's hypothesis and the Einstein energy-momentum relation lead unambiguously to the Klein-Gordon equation if, and only if, $\Psi(\mathbf{r}, t) \rightarrow \Psi(\mathbf{r}, t)$ under proper Lorentz transformations (spatial rotations and boosts).

However, we showed that Lorentz covariance is also compatible with a wave function that is not a strict scalar under a Lorentz boost, namely, with a wave function that transforms after a Lorentz boost as follows, $\Psi(\mathbf{r}, t) \rightarrow g(\mathbf{r}, t)\Psi(\mathbf{r}, t)$. By requiring that the nonrelativistic approximation of $g(\mathbf{r}, t)$ should tend to Schrödinger's wave-function transformation law, we uniquely determined it and also its associated wave equation. This wave equation was shown to be the Lorentz-covariant Schrödinger equation, derived in Ref. [17] by a different set of assumptions.

Also, an interesting extension of the ideas contained in this work would be to search for the most general second-order linear wave equation and the respective wave-function transformation rule by requiring "Einstein's general covariance" [27] for the wave equation. This approach may lead to a wave equation that incorporates the gravitational field from the start, opening the door to a consistent quantum theory of gravity.

Finally, we would like to call attention to an open problem that we were not able to solve so far, despite several attempts. The solution to this problem might lead to an even deeper understanding of nonrelativistic quantum mechanics, in particular about the origin and meaning of the measurement postulate and Born rule, i.e., the statistical interpretation of the wave function. The problem is the following. Using the two assumptions outlined above, we were not able to derive the Born rule. We believe it is not possible to arrive at it from those two assumptions alone. However, we do not know either what third physical assumption one should bring to the table to arrive at it. In other words, what is the extra ingredient, the extra basic physical law, that we need to prove the Born rule and completely build nonrelativistic quantum mechanics without ad hoc postulates that do not have a clear physical meaning?

ACKNOWLEDGMENTS

G.R. thanks the Brazilian agency CNPq (National Council for Scientific and Technological Development) for funding and the referees for their insightful suggestions.

APPENDIX : HIGHER-ORDER DIFFERENTIAL EQUATIONS

1. Third-order differential equation

The most general linear third-order partial differential equation on the variables x^μ is

$$a^{\mu\nu\kappa}\partial_\mu\partial_\nu\partial_\kappa\Psi(x) + a^{\mu\nu}\partial_\mu\partial_\nu\Psi(x) + b^\mu\partial_\mu\Psi(x) + f(x)\Psi(x) = 0, \tag{A1}$$

where we assume that $f(x) \rightarrow f(x)$ after a spatial rotation or a Galilean boost. In other words, $f(x)$ is a strict scalar under those symmetry operations.

Since the order of the derivatives can be interchanged at will (we are assuming that $\Psi(x)$ is a well-behaved infinitely differentiable function), we slightly modify the Einstein summation convention to simplify the analysis. When two or more indexes are summed, we restrict the sum such that each subsequently index is greater or equal to its predecessor. For instance,

$$a^{\mu\nu}\partial_\mu\partial_\nu = a^{11}\partial_1\partial_1 + a^{12}\partial_1\partial_2 + a^{22}\partial_2\partial_2. \tag{A2}$$

We now start analyzing the constraints on the coefficients $a^{\mu\nu\kappa}$ coming from demanding the covariance of the differential equation (A1) under spatial rotations. We also assume that the wave function is a strict scalar under spatial rotations, namely, $\Psi(x) \rightarrow \Psi(x)$ after an arbitrary spatial rotation.

The analysis is carried out most simply by separating it into two distinct parts. The first one deals with the case where the first index is 0, a temporal derivative. The second case considers the scenario where only pure spatial derivatives are present. These two cases exhaust all possibilities.

When $\mu = 0$ we have the following third-order term:

$$a^{0\nu\kappa}\partial_0\partial_\nu\partial_\kappa\Psi(x). \tag{A3}$$

Since the first index is fixed and equal to zero and under any spatial rotation $\partial_0 \rightarrow \partial_0$, what we have here is an actual two-index term. Therefore, this term and the other lower-order terms in Eq. (A1) can be written as follows after renaming some of the dummy indexes:

$$\overline{a}^{\mu\nu}\partial_\mu\partial_\nu\Psi(x) + b^\mu\partial_\mu\Psi(x) + f(x)\Psi(x) = 0, \tag{A4}$$

where

$$\overline{a}^{\mu\nu} = a^{0\mu\nu}\partial_0 + a^{\mu\nu}. \tag{A5}$$

According to the results of the main text (see Sec. III A), by demanding the covariance of Eq. (A4) under spatial rotations we obtain

$$\overline{a}^{00}\partial_0^2\Psi(x) - \overline{a}^{11}\partial_j\partial^j\Psi(x) + b^0\partial_0\Psi(x) + f(x)\Psi(x) = 0, \tag{A6}$$

where we used that $a^{11} = a^{22} = a^{33}$ and $\partial_j = -\partial^j$. Inserting Eq. (A5) into Eq. (A6) we get

$$a^{000}\partial_0^3\Psi(x) - a^{011}\partial_0\partial_j\partial^j\Psi(x) + a^{00}\partial_0^2\Psi(x) - a^{11}\partial_j\partial^j\Psi(x) + b^0\partial_0\Psi(x) + f(x)\Psi(x) = 0. \quad (A7)$$

When $\mu \neq 0$ we have the following third-order term:

$$a^{ijk}\partial_i\partial_j\partial_k\Psi(x), \quad (A8)$$

where no temporal index is present.

If the three indexes are equal, let us say equal to j, a rotation of π radians about any one of the other two remaining orthogonal axes leads to $\partial_j \to -\partial_j$. This implies that $\partial_j^3 \to -\partial_j^3$ and, thus, that $a^{jjj} = 0$ for $j = 1, 2, 3$. This is true since after the spatial rotation the term $a^{jjj}\partial_j^3\Psi(x)$ changes to $-a^{jjj}\partial_j^3\Psi(x)$. Since the last term was not in the differential equation before the transformation, and there are other terms in the equation that do not change sign under this symmetry operation, we can only guarantee covariance if $a^{jjj} = 0$.

If two indexes are equal, a π radian rotation about the axis labeled by this index will change the sign of the derivative related to the index that is different. In other words, we will have $a^{jjk}\partial_j^2\partial_k\Psi(x) \to -a^{jjk}\partial_j^2\partial_k\Psi(x)$ and the same argument we employed above to prove that $a^{jjj} = 0$ leads to $a^{jjk} = 0$, for $j \neq k$.

Finally, if the three indexes are different, we implement a rotation of $\pi/2$ radians about any one of the three orthogonal axes. For instance, rotating $\pi/2$ radians about the k axis (z-axis), we have that $\partial_k \to \partial_k$, $\partial_i \to \partial_j$ and $\partial_j \to -\partial_i$, where i and j mean the x axis and and y axis, respectively. Therefore, in general, we will have $a^{ijk}\partial_i\partial_j\partial_k\Psi(x) \to -a^{ijk}\partial_i\partial_j\partial_k\Psi(x)$ and thus by similar arguments already employed in the two previous cases we must have $a^{ijk} = 0$, if $i \neq j \neq k$.

The three cases above combined imply that $a^{ijk} = 0$ for all possible values of i, j, k. Therefore, Eq. (A7) is the most general third-order differential equation compatible with covariance under spatial rotations.

We need now to impose covariance under Galilean boosts. In this scenario the wave function changes according to the following rule:

$$\Psi(x) \to g(x)\Psi(x), \quad (A9)$$

where $g(x) \neq 0$. Note that the shorthand notation for $\Psi(x^0, x^1, x^2, x^3)$ is $\Psi(x)$. After a Galilean boost we also have that

$$\partial_0 \to \partial_0 - \beta\partial_x, \quad (A10)$$

where $\beta = v/c \neq 0$, while the spatial derivatives remain unchanged (see Secs. III B and IV).

Using Eqs. (A9) and (A10), we have that $a^{000}\partial_0^3\Psi(x)$ yield after the Galilean boost the following term (among many others, of course):

$$-3\beta g(x)a^{000}\partial_0^2\partial_x\Psi(x). \quad (A11)$$

A term proportional to $\partial_0^2\partial_x\Psi(x)$ will not appear anywhere else after transforming Eq. (A7). This comes about because its other terms are either at most of second order in the derivatives or first order in the time derivative when the order of the derivative is higher than 2. Therefore, since Eq. (A11) was not in Eq. (A7) before the Galilean transformation, we must have that

$$a^{000} = 0 \quad (A12)$$

to guarantee covariance under a Galilean boost.

Moving to the term $-a^{001}\partial_0\partial_j\partial^j\Psi(x)$, we have that after a Galilean boost one of the new terms is

$$\beta a^{001}g(x)\partial_x\partial_j\partial^j\Psi(x). \quad (A13)$$

Since $a^{000} = 0$ and the other terms in Eq. (A7) are, at most, of second order in the derivatives, the fact that a term proportional to $\partial_x\partial_j\partial^j\Psi(x)$ is absent before the Galilean transformation implies that we necessarily have

$$a^{001} = 0 \quad (A14)$$

to preserve the covariance of Eq. (A7) under a Galilean boost.

Combining Eqs. (A12) and (A14), we have that Eq. (A7) becomes

$$a^{00}\partial_0^2\Psi(x) - a^{11}\partial_j\partial^j\Psi(x) + b^0\partial_0\Psi(x) + f(x)\Psi(x) = 0. \quad (A15)$$

In other words, if we restrict the order of the derivatives up to three, there is no third-order derivative term compatible with both covariance under spatial rotations and Galilean boosts. As we show next, we can only have a compatible third-order term if we allow the presence of fourth-order terms as well.

2. Fourth-order differential equation

The same notation, conventions, and assumptions laid out at the previous subsection are valid here. The most general linear fourth order partial differential equation on the variables x^μ can be written as

$$a^{\mu\nu\kappa\epsilon}\partial_\mu\partial_\nu\partial_\kappa\partial_\epsilon\Psi(x) + a^{\mu\nu\kappa}\partial_\mu\partial_\nu\partial_\kappa\Psi(x) + a^{\mu\nu}\partial_\mu\partial_\nu\Psi(x) + b^\mu\partial_\mu\Psi(x) + f(x)\Psi(x) = 0. \quad (A16)$$

The constraints on the coefficients $a^{\mu\nu\kappa\epsilon}$ due to the covariance of the differential equation (A16) under spatial rotations are obtained as follows. Similarly to the way we worked out the third-order differential equation, we start with the case where the first index is 0. The other case, where $\mu \neq 0$, contains only spatial derivatives and these two cases exhaust all possibilities.

When $\mu = 0$ we have

$$a^{0\nu\kappa\epsilon}\partial_0\partial_\nu\partial_\kappa\partial_\epsilon\Psi(x). \quad (A17)$$

Since under any spatial rotation $\partial_0 \to \partial_0$, we are actually working with an effective three-index term. This term and the lower-order terms of Eq. (A16) can be written as follows to highlight this point:

$$\overline{a}^{\mu\nu\kappa}\partial_\mu\partial_\nu\partial_\kappa\Psi(x) + a^{\mu\nu}\partial_\mu\partial_\nu\Psi(x) + b^\mu\partial_\mu\Psi(x) + f(x)\Psi(x) = 0, \quad (A18)$$

where

$$\overline{a}^{\mu\nu\kappa} = a^{0\mu\nu\kappa}\partial_0 + a^{\mu\nu\kappa}. \quad (A19)$$

Following the analysis of the previous subsection, if we require the covariance of Eq. (A18) under spatial rotations we get that it becomes

$$\overline{a}^{000}\partial_0^3\Psi(x) - \overline{a}^{011}\partial_0\partial_j\partial^j\Psi(x) + a^{00}\partial_0^2\Psi(x) - a^{11}\partial_j\partial^j\Psi(x) + b^0\partial_0\Psi(x) + f(x)\Psi(x) = 0. \quad \text{(A20)}$$

Inserting Eq. (A19) into Eq. (A20) we get

$$a^{0000}\partial_0^4\Psi(x) - a^{0011}\partial_0^2\partial_j\partial^j\Psi(x) + a^{000}\partial_0^3\Psi(x) - a^{011}\partial_0\partial_j\partial^j\Psi(x) + a^{00}\partial_0^2\Psi(x) - a^{11}\partial_j\partial^j\Psi(x) + b^0\partial_0\Psi(x) + f(x)\Psi(x) = 0. \quad \text{(A21)}$$

When $\mu \neq 0$ we have the following fourth-order term:

$$a^{ijkl}\partial_i\partial_j\partial_k\partial_l\Psi(x), \quad \text{(A22)}$$

where no temporal index is present.

If the four indexes are equal to j, a rotation of $\pi/2$ radians about any other orthogonal axes not labeled by j leads to $\partial_j^4 \leftrightarrow \partial_k^4$ or $\partial_j^4 \leftrightarrow \partial_l^4$, where $j \neq k \neq l$. This implies that $a^{1111} = a^{2222} = a^{3333}$ to ensure covariance under these particular rotations. For example, a counterclockwise rotation of $\pi/2$ radians about the x^3 axis leads to the following changes, $a^{1111}\partial_1^4\Psi(x) \to a^{1111}\partial_2^4\Psi(x)$ and $a^{2222}\partial_2^4\Psi(x) \to a^{2222}\partial_1^4\Psi(x)$, which implies that $a^{1111} = a^{2222}$ when covariance is required. A similar analysis changing the axis of rotation gives $a^{1111} = a^{3333}$.

If we have three identical indexes, namely, $jjjk$, with $j \neq k$, a π radian rotation about the x^k axis leads to $a^{jjjk}\partial_j^3\partial_k\Psi(x) \to -a^{jjjk}\partial_j^3\partial_k\Psi(x)$ since $\partial_j \to -\partial_j$ and $\partial_k \to \partial_k$ under this rotation. This implies that $a^{jjjk} = 0$ to guarantee covariance.

If two indexes are equal and the other two are different from the latter and from each other, namely, $jjkl$, with $j \neq k \neq l$, a counterclockwise π radian rotation about the x^k axis gives $\partial_k \to \partial_k$, $\partial_j \to -\partial_j$, and $\partial_l \to -\partial_l$. This implies that $a^{jjkl}\partial_j^2\partial_k\partial_l\Psi(x) \to -a^{jjkl}\partial_j^2\partial_k\partial_l\Psi(x)$ and thus we must have $a^{jjkl} = 0$ to obtain covariance.

Now, if two indexes are equal and the other two are also equal but different from the other pair, namely, $jjkk$, with $j \neq k$, a counterclockwise $\pi/2$ radian rotation about the x^j axis gives $\partial_j^2 \to \partial_j^2$ and $\partial_k^2 \leftrightarrow \partial_l^2$, where $l \neq j$ and $l \neq k$. This implies that $a^{jjkk}\partial_j^2\partial_k^2\Psi(x) \to a^{jjkk}\partial_j^2\partial_l^2\Psi(x)$ while $a^{jjll}\partial_j^2\partial_l^2\Psi(x) \to a^{jjll}\partial_j^2\partial_k^2\Psi(x)$ and thus we must have $a^{jjkk} = a^{jjll}$ to have covariance.

Combining the results of the last four paragraphs, we obtain that Eq. (A22) becomes

$$\left[a^{1111}\left(\partial_1^4 + \partial_2^4 + \partial_3^4\right) + a^{1122}\left(\partial_1^2\partial_2^2 + \partial_1^2\partial_3^2 + \partial_2^2\partial_3^2\right)\right]\Psi(x). \quad \text{(A23)}$$

We need one last rotation, namely, a counterclockwise $\pi/4$ radian rotation about the x^3 axis. In this case $\partial_3 \to \partial_3$, $\partial_1 \to (\partial_1 + \partial_2)/\sqrt{2}$ and $\partial_2 \to (\partial_2 - \partial_1)/\sqrt{2}$. Therefore, transforming Eq. (A23) we get

$$a^{1111}\left(\frac{\partial_1^4 + \partial_2^4}{2} + 3\partial_1^2\partial_2^2 + \partial_3^4\right)\Psi(x) + a^{1122}\left(\frac{\partial_1^4 + \partial_2^4}{4} - \frac{\partial_1^2\partial_2^2}{2} + \partial_1^2\partial_3^2 + \partial_2^2\partial_3^2\right)\Psi(x). \quad \text{(A24)}$$

Looking at Eq. (A24) we note that, individually, the operators multiplying a^{1111} and a^{1122} are not covariant under the latter rotation. However, the whole expression can be made covariant if we set

$$a^{1122} = 2a^{1111}. \quad \text{(A25)}$$

Indeed, using Eq. (A25) we have that Eq. (A24) becomes

$$a^{1111}\left(\partial_1^2 + \partial_2^2 + \partial_3^2\right)^2\Psi(x) = a^{1111}\partial_j\partial^j\partial_k\partial^k\Psi(x), \quad \text{(A26)}$$

which is the same expression we get by also inserting Eq. (A25) into Eq. (A23), where the latter equation is the term under investigation before the transformation.

Moreover, the operator $(\partial_1^2 + \partial_2^2 + \partial_3^2)^2$ is actually the square of the Laplacian, i.e., $(\nabla^2)^2 \equiv \nabla^2(\nabla^2)$. Since the Laplacian operator is invariant under any rotation, we have that Eq. (A26) is covariant under any rotation as well.

Using Eqs. (A21) and (A26), and renaming the constant coefficients, we can write the most general linear fourth-order partial differential equation that is covariant under arbitrary spatial rotations as follows:

$$\widetilde{A}\partial_0^4\Psi(x) + \widetilde{B}\partial_j\partial^j\partial_k\partial^k\Psi(x) - \widetilde{C}\partial_0^2\partial_j\partial^j\Psi(x) + \widetilde{a}\partial_0^3\Psi(x) - \widetilde{b}\partial_0\partial_j\partial^j\Psi(x) + A\partial_0^2\Psi(x) - B\partial_j\partial^j\Psi(x) + C\partial_0\Psi(x) + f(x)\Psi(x) = 0. \quad \text{(A27)}$$

We need now to analyze the covariance under a Galilean boost. Using Eqs. (A9) and (A10), we see that one of the terms coming from $\widetilde{A}\partial_0^4\Psi(x)$ after the boost is

$$-4\beta g(x)\widetilde{A}\partial_0^3\partial_x\Psi(x). \quad \text{(A28)}$$

A mixed derivative term proportional to $\partial_0^3\partial_x\Psi(x)$ does not come from any other transformed term of Eq. (A27). This is true because the other terms in Eq. (A27) are either at most of third order in the derivatives or second order in the time derivative when the order of the derivative is higher than three. Hence, since this term is absent before the boost, we can only guarantee covariance under Galilean transformations if

$$\widetilde{A} = 0. \quad \text{(A29)}$$

If we now use Eqs. (A9) and (A10), one of the terms stemming from $-\widetilde{C}\partial_0^2\partial_j\partial^j\Psi(x)$ after the boost is

$$2\beta g(x)\widetilde{C}\partial_0\partial_x\partial_j\partial^j\Psi(x). \quad \text{(A30)}$$

Since we must have $\widetilde{A} = 0$ and the other terms in Eq. (A27) are either at most of third order in the derivatives or do not have a time derivative, a similar term proportional to $\partial_0\partial_x\partial_j\partial^j\Psi(x)$ will not appear after the boost. Thus, since a term like this is not present before the transformation, we have

$$\widetilde{C} = 0 \quad \text{(A31)}$$

to preserve the covariance of Eq. (A27) under a Galilean boost.

A similar argument used to prove Eq. (A12) applies here as well. This leads to

$$\widetilde{a} = 0 \tag{A32}$$

to preserve covariance under boosts.

Combing the results given by Eqs. (A29), (A31), and (A32), the differential equation (A27) becomes

$$\widetilde{B}\partial_j\partial^j\partial_k\partial^k\Psi(x) - \widetilde{b}\partial_0\partial_j\partial^j\Psi(x) + A\partial_0^2\Psi(x) - B\partial_j\partial^j\Psi(x) + C\partial_0\Psi(x) + f(x)\Psi(x) = 0. \tag{A33}$$

It is worth mentioning that after the boost, the fourth-order term $\widetilde{B}\partial_j\partial^j\partial_k\partial^k\Psi(x)$ will provide similar terms to those coming from $-\widetilde{b}\partial_0\partial_j\partial^j\Psi(x)$. This is why the argument used in the previous subsection to rule out the latter term is no longer valid. Similarly, since the latter term is now present, after the boost it will give similar terms to those coming from $A\partial_0^2\Psi(x)$. This is why now the presence of a pure second-order time derivative will not contradict Galilean covariance and why the proof of the main text ruling it out no longer applies.

Applying Eqs. (A9) and (A10) to Eq. (A33), after a long but straightforward calculation, similar to the ones we explicitly showed in Secs. III and IV, we get that we can only guarantee covariance if

$$\widetilde{B} = \frac{\overline{A}B^2}{\overline{C}^2}, \tag{A34}$$

$$\overline{b} = \frac{2\overline{A}B}{\overline{C}}, \tag{A35}$$

$$g(\mathbf{r}', t') = \exp\left[\frac{\overline{C}}{2mB}\left(\frac{mv^2t'}{2} + m\mathbf{v}\cdot\mathbf{r}'\right)\right], \tag{A36}$$

where $A = c^2\overline{A}$, $C = c\overline{C}$, and $\widetilde{b} = c\overline{b}$. We also used that $x^{0'} = ct'$ to arrive at the last equation.

To fix the value of $\overline{C}/B$ we require that the plane wave solution to Eq. (A33) satisfies the free-particle dispersion relation (31), where the dispersion relation is a consequence of de Broglie's wave-particle duality hypothesis. Inserting the ansatz (33) into Eq. (A33), we obtain the dispersion relation (31) if

$$\frac{\overline{C}}{B} = \frac{i2m}{\hbar}, \tag{A37}$$

$$f(\mathbf{r}, t) = 0. \tag{A38}$$

Note that with this value for $\overline{C}/B$, $g(\mathbf{r}', t')$ given by Eq. (A36) is exactly the one we obtained for the standard Schrödinger equation [cf. Eq. (38)]. In other words, the wave functions of the fourth-order equation and the Schrödinger equation have the same transformation law under a Galilean boost.

Using Eqs. (A34), (A35), and (A37), the wave equation can be written as

$$\frac{\overline{A}\hbar^2}{4m^2}\nabla^4\Psi + \frac{i\hbar\overline{A}}{m}\nabla^2\partial_t\Psi - \overline{A}\partial_t^2\Psi - B\nabla^2\Psi - \frac{i2mB}{\hbar}\partial_t\Psi - f\Psi = 0, \tag{A39}$$

where $\nabla^4 \equiv \nabla^2(\nabla^2)$, i.e., the square of the Laplacian.

It is interesting to note that if $\overline{A} = 0$, or equivalently $A = 0$, we recover the standard Schrödinger equation by properly adjusting the free parameter B, i.e., by choosing $B = -\hbar^2/(2m)$ and by setting $f = V(\mathbf{r}, t)$. Since the constant A multiplies the second-order time derivative, its suppression leads immediately to the Schrödinger equation, even if we go up to fourth order.

Moreover, writing the Schrödinger equation as

$$\hat{\mathcal{S}}\Psi = 0, \tag{A40}$$

where

$$\hat{\mathcal{S}} = \frac{\hbar^2}{2m}\nabla^2 + i\hbar\partial_t - V \tag{A41}$$

is what we call the Schrödinger operator, we obtain for a constant V that

$$\hat{\mathcal{S}}^2 = \frac{\hbar^4}{4m^2}\nabla^4 + \frac{i\hbar^3}{m}\nabla^2\partial_t - \hbar^2\partial_t^2 - V\frac{\hbar^2}{m}\nabla^2 - i2V\hbar\partial_t + V^2. \tag{A42}$$

Comparing Eq. (A42) with the operator acting on Ψ that generates the wave equation (A39), we realize that they are equal if

$$\overline{A} = \hbar^2, \tag{A43}$$

$$B = \frac{V\hbar^2}{m}, \tag{A44}$$

$$f = -V^2. \tag{A45}$$

In other words, for a free particle ($V = 0$), or a particle in a constant potential, the fourth-order wave equation (A39) is obtained from "squaring" the Schrödinger equation. Specifically, it is obtained by squaring the Schrödinger operator $\hat{\mathcal{S}}$. The Schrödinger equation is given by $\hat{\mathcal{S}}\Psi = 0$ and the fourth-order equation by $\hat{\mathcal{S}}^2\Psi = 0$ if $\overline{A}$, B, and f are given by Eqs. (A43) to (A45). The latter relation also implies that any solution to the Schrödinger equation is also a solution to the fourth-order equation. Indeed, if Ψ is a solution to the Schrödinger equation we have $\hat{\mathcal{S}}\Psi = 0$. But $\hat{\mathcal{S}}^2\Psi = \hat{\mathcal{S}}(\hat{\mathcal{S}}\Psi)$. Therefore, we must have $\hat{\mathcal{S}}^2\Psi = 0$, proving that Ψ is also a solution to the fourth-order equation.

Finally, if V is not constant but depends on the position, namely, if $V = V(\mathbf{r})$, one of the terms of the square of the Schrödinger operator is

$$-\frac{\hbar^2}{m}\nabla V\cdot\nabla\Psi. \tag{A46}$$

Looking at Eq. (A39), we realize that this type of mixed derivative is absent from it. Therefore, it is not possible, in general, to make the square of the Schrödinger equation ($\hat{\mathcal{S}}^2\Psi = 0$) equivalent to the fourth-order wave equation (A39) if V is not a constant.

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