Adjoint master equation for multitime correlators

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(Received 16 October 2023; accepted 18 April 2024; published 8 May 2024)

The quantum regression theorem is a powerful tool for calculating the multitime correlators of operators of open quantum systems whose dynamics can be described in Markovian approximation. However, the scope of the quantum regression theorem is limited by a particular time order of the operators in multitime correlators and does not include out-of-time-ordered correlators. In this work, we obtain an adjoint master equation for multitime correlators that applies to out-of-time-ordered correlators. We show that this equation can be derived for various approaches to the description of the dynamics of open quantum systems, such as the global or local approach. We show that the adjoint master equation for multitime correlators is self-consistent. Namely, the final equation does not depend on how the operators are grouped inside the correlator and it coincides with the quantum regression theorem for the particular time ordering of the operators.

DOI: 10.1103/PhysRevA.109.052207

I. INTRODUCTION

Out-of-time-ordered correlators (OTOCs) hold a special place in many-body quantum physics. Originally, OTOC was introduced in [1] to analyze the relation between the semi-classical and quantum description of the electrons in superconductors. In recent years, OTOCs received growing attention because of their deep connection with the entanglement in quantum systems [2–14]. OTOCs can be used to quantitatively characterize the quantum chaos [15–25] and to study some other problems in quantum systems [26–32]. OTOCs with more than two operators dependent on time are important in the quantum chaos theory because they are used to define the quantum generalized Lyapunov exponent [33]. In addition, it is possible to measure OTOC experimentally [34–44].

An experimentally feasible quantum system is never completely isolated from its environment. Therefore, the calculation of correlators should take into account the open nature of the quantum system. When the system dynamics can be described in the Born-Markov approximation, the quantum regression theorem is applicable for calculations of multitime correlation functions of operators. However, the quantum regression theorem cannot be applied to OTOCs [45,46]. In [47], the general quantum regression theorem (GQRT) for OTOC was developed. The derivation of GQRT relies on Heisenberg-Langevin equations and requires explicit knowledge of the corresponding noise operators, which generally demands the knowledge of the system's eigenstates and eigenenergies. Thus, it may be difficult to apply GQRT for complex quantum systems.

An alternative method to describe the dynamics of an open quantum system is the Lindblad master equation [45]. In the Born-Markov approximation, the Heisenberg-Langevin equation and Lindblad master equation are equivalent [48]. Each of these two methods has its advantages and disadvantages. Usually, one of these methods is preferable for a particular problem. When the relaxation superoperators strongly depend on the interaction constant between subsystems of a complex open quantum system, it is necessary to investigate the ratios between the relaxation rates and this interaction constant. When the interaction constant is less than relaxation rates, one uses the so-called local approach [49–53], in the opposite case one uses the global approach [45,49–51,54]. In the intermediate case, one can use a partially secular approach [55,56] or the approach based on the perturbation theory [57]. For this type of problem, the Lindblad master equation is preferable. In this regard, it is important to expand the scope of applicability of the quantum regression theorem to OTOCs in the formalism of the Lindblad master equation.

In this work, we derive the adjoint master equation for multitime correlators. We write explicitly the equation for OTOCs with an arbitrary number of operators. We show that the adjoint master equation for multitime correlators is self-consistent, such that the resultant equation does not depend on how we group the operators inside the trace function. The derived adjoint master equation for multitime correlators gives the same result as the standard quantum regression theorem when the latter is applicable. Recently, a similar framework was used in [46] to analyze the quantum regression theorem with a main focus on non-Markovian dynamics. In this paper, we mainly focus on explicitly going beyond four-point correlators, prooving the self-consistency of the adjoint master equation for multitime correlators, and verifying the compatibility of the equation with the approximate approaches to the relaxation operators.

II. DECOMPOSITION OF THE SYSTEM-RESERVOIR INTERACTION

We consider the system interacting with the reservoir. The Hamiltonian of the system is $\hat{H}_S$, the Hamiltonian of the

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reservoir is $\hat{H}_R$, and the Hamiltonian of the system-reservoir interaction is $\hat{H}_{SR}$. We assume that the evolution of the whole system is Hermitian and governed by the Hamiltonian

$$\hat{H} = \hat{H}_S + \hat{H}_R + \hat{H}_{SR}. \tag{1}$$

We consider the system Hamiltonian to be time-independent. Additionally, we consider the interaction of the system with only one reservoir such that the interaction Hamiltonian is

$$\hat{H}_{SR} = \hbar\lambda\hat{S}\hat{R}, \tag{2}$$

where $\hat{S}$ is the operator of the system, $\hat{R}$ is the operator of the reservoir, and λ is the interaction constant. One reservoir with the interaction Hamiltonian (2) is the simplest possible case, which, however, can be straightforwardly generalized to the multiple reservoirs.

The Hamiltonian (1) governs the time evolution of any operator $\hat{D}$ in Heisenberg representation

$$\hat{D}(t_2) = e^{i\hat{H}(t_2-t_1)/\hbar}\hat{D}(t_1)e^{-i\hat{H}(t_2-t_1)/\hbar}, \tag{3}$$

where t_1 and t_2 are arbitrary moments in time. The equation for time evolution (3) remains the same if the operator $\hat{D}$ is the system's operator, the reservoir's operator, or the product of both types.

In the general case, it is rarely possible to establish an explicit form of the operator (3), even if, at the initial moment, the operator $\hat{D}$ was only an operator of the system.

Usually, two conditions accompany the study of the open quantum system: (1) it is the evolution of the system that is important, whereas the evolution of the reservoir is not of interest; (2) the system has a small number of degrees of freedom, while the reservoir, on the contrary, has a large number of degrees of freedom. When both conditions are met, one can exploit the Born-Markov approximation and write effective equations for the averages and correlators for the system operators tracing out the reservoir's degrees of freedom.

To derive the equation for the correlators, we have to find the decomposition of the Hermitian time evolution of the operator $\hat{S}$ form the right-hand side of the Eq. (2) [45]. Typically, there are two possible options for such a decomposition. The first one is the *exact* decomposition of the operator $\hat{S}$,

$$e^{i\hat{H}_S t/\hbar}\hat{S}e^{-i\hat{H}_S t/\hbar} = \hat{C}_0 + \left(\sum_{j=1}^{M}\hat{C}_j e^{-i\omega_j t} + \text{H.c.}\right). \tag{4}$$

The second option is the *approximate* decomposition of the operator $\hat{S}$

$$e^{i\hat{H}_S t/\hbar}\hat{S}e^{-i\hat{H}_S t/\hbar} \approx \hat{C}_0 + \left(\sum_{j=1}^{M}\hat{C}_j e^{-i\omega_j t} + \text{H.c.}\right). \tag{5}$$

In Eqs. (4) and (5) $\omega_j \neq 0$, $\omega_j \neq \omega_k$ at $j \neq k$, and $\hat{C}_0^\dagger = \hat{C}_0$. The exact decomposition (4) is used in the global approach to the Lindblad master equation [45,49–51,54], whereas the approximate decomposition (5) is used in the local approach [49–53], partially secular approach [56], and the approach based on perturbation theory [57].

III. INTERACTION REPRESENTATION

There are two coexisting parts that determine the time evolution of the correlators. The first part originates from the Hamiltonian $\hat{H}_S$ and preserves the energy in the system. Thus, this part is associated with the Hermitian evolution. The second part originates from the Hamiltonians $\hat{H}_{SR}$ and $\hat{H}_R$, causing the energy flow between the system and reservoir and/or destruction of the phase in the system. Thus, this part is associated with non-Hermitian evolution.

We use the interaction representation to divide the Hermitian and non-Hermitian parts of the evolution. For arbitrary operator $\hat{D}$ its interaction representation $\hat{D}'(t)$ is defined by

$$\hat{D}'(t) = e^{i(\hat{H}_S+\hat{H}_R)t/\hbar}\hat{D}e^{-i(\hat{H}_S+\hat{H}_R)t/\hbar}. \tag{6}$$

The relation between the full quantum dynamics of the operator with its interaction representation follows from Eq. (3)

$$\hat{D}(t) = \hat{V}^\dagger(t)\hat{D}'(t)\hat{V}(t). \tag{7}$$

Here $\hat{V}(t) = e^{i(\hat{H}_S+\hat{H}_R)t/\hbar}e^{-i\hat{H}t/\hbar}$ and it can be obtained from the equation

$$\frac{d\hat{V}(t)}{dt} = -\frac{i}{\hbar}\hat{H}'_{SR}(t)\hat{V}(t), \tag{8}$$

with the initial condition $\hat{V}(0) = \hat{1}$. According to Eqs. (2) and (6), $\hat{H}'_{SR}(t) = \hbar\lambda\hat{S}'(t)\hat{R}'(t)$. Thus, we can explicitly find the approximate evolution of the operator $\hat{V}(t+\Delta t)$ up to the second order in the interaction constant λ,

$$\hat{V}(t+\Delta t) \approx \hat{V}(t) + \hat{W}(t,\Delta t), \tag{9}$$

where $W(t,\Delta t) = -i\lambda\hat{V}_1(t,\Delta t) - \lambda^2\hat{V}_2(t,\Delta t)$ and

$$\hat{V}_1(t,\Delta t) = \int_t^{t+\Delta t} dt_1\hat{S}'(t_1)\hat{R}'(t_1)\hat{V}(t), \tag{10}$$

$$\hat{V}_2(t,\Delta t) = \int_t^{t+\Delta t} dt_1 \int_t^{t_1} dt_2\hat{S}'(t_1)\hat{S}'(t_2)\hat{R}'(t_1)\hat{R}'(t_2)\hat{V}(t). \tag{11}$$

The approximate time evolution (9) is the key to determining the effective non-Hermitian dynamics of the system. Indeed, with this equation for any operator of the system $\hat{B}$, we can determine the relation between its values at time t and at time $t+\Delta t$

$$\hat{B}(t+\Delta t) \approx \Delta t\frac{i}{\hbar}[\hat{H}_S(t),\hat{B}(t)] + \{\hat{V}^\dagger(t) + \hat{W}^\dagger(t,\Delta t)\}\hat{B}'(t)\{\hat{V}(t) + \hat{W}(t,\Delta t)\}, \tag{12}$$

where we use $[\hat{B}(t),\hat{H}_R(t)] = 0$. Note that we define the Hamiltonians $\hat{H}_S(t)$ and $\hat{H}_R(t)$ according to Eq. (3).

We use Eq. (12) to derive the quantum regression theorem and the more general adjoint master equation for multitime correlators.

IV. QUANTUM REGRESSION THEOREM

In this section, we derive the standard quantum regression theorem for the correlation function

$$\langle \hat{A}_1 \hat{B}_1(t+\tau)\hat{A}_2\rangle, \tag{13}$$

where $\hat{A}_1$, $\hat{B}_1(t+\tau)$, and $\hat{A}_2$ are the operators of the system and $\hat{A}_1$ and $\hat{A}_2$ are taken at times prior to t and do not depend on τ. We assume that we know the correlation function (13) at $\tau = 0$, and we are to determine its subsequent evolution at $\tau > 0$.

To derive the quantum regression theorem, we consider the connection between the correlation functions $\langle \hat{A}_1 \hat{B}_1(t+\tau)\hat{A}_2\rangle$ and $\langle \hat{A}_1 \hat{B}_1(t+\tau+\Delta\tau)\hat{A}_2\rangle$. Equation (12) leads to

$$\begin{aligned}
&\langle \hat{A}_1 \hat{B}_1(t+\tau+\Delta\tau)\hat{A}_2\rangle \\
&\approx \Delta\tau \frac{i}{\hbar}\langle \hat{A}_1 [\hat{H}_S(t+\tau), \hat{B}_1(t+\tau)]\hat{A}_2\rangle + \langle \hat{A}_1 \{\hat{V}^\dagger(t+\tau) \\
&\quad + \hat{W}^\dagger(t+\tau, \Delta\tau)\}\hat{B}_1'(t+\tau)\{\hat{V}(t+\tau) \\
&\quad + \hat{W}(t+\tau, \Delta\tau)\}\hat{A}_2\rangle.
\end{aligned} \tag{14}$$

For Eq. (14), we can apply assumptions and methods similar to those used for the derivation of the Lindblad master equation [45,54]. In particular, we set certain limitations for the time $\Delta\tau$, use the Born and Markov approximations, and assume that at any time t the density matrix is factorized $\hat{\rho} = \hat{\rho}_S(t)\hat{\rho}_R^{\text{th}}$, the operator of the reservoir has the zero mean $\text{tr}_R[\hat{R}(t)\hat{\rho}_R^{\text{th}}] = 0$. We also suppose that we know either exact (4) or approximate (5) decomposition of the system operator in interaction representation such that the operators $\hat{C}_j$ are known. In the right-hand side of Eq. (14), we preserve only terms that either do not depend on $\Delta\tau$ or linear in $\Delta\tau$ and obtain the standard quantum regression theorem (see Appendix A)

$$\frac{d\langle \hat{A}_1 \hat{B}_1(t+\tau)\hat{A}_2\rangle}{d\tau} = \langle \hat{A}_1 \mathcal{L}_{t+\tau}[\hat{B}_1(t+\tau)]\hat{A}_2\rangle, \tag{15}$$

where we introduce the adjoint Lindblad superoperator

$$\begin{aligned}
\mathcal{L}_t[\hat{B}] &= \frac{i}{\hbar}[\hat{H}_S(t), \hat{B}] + L_{\sqrt{\gamma_0}\hat{C}_0(t)}[\hat{B}] \\
&\quad + \sum_{j=1}^{M} L_{\sqrt{\gamma_j^\downarrow}\hat{C}_j(t)}[\hat{B}] + \sum_{j=1}^{M} L_{\sqrt{\gamma_j^\uparrow}\hat{C}_j^\dagger(t)}[\hat{B}],
\end{aligned} \tag{16}$$

$$L_{\hat{C}}[\hat{B}] = \hat{C}^\dagger \hat{B}\hat{C} - \tfrac{1}{2}\hat{C}^\dagger\hat{C}\hat{B} - \tfrac{1}{2}\hat{B}\hat{C}^\dagger\hat{C}, \tag{17}$$

where the dissipation rates γ_0, $\gamma_j^\downarrow$, and $\gamma_j^\uparrow$ are considered in more details in Appendix A.

For $\hat{A}_1 = \hat{1}$ and $\hat{A}_2 = \hat{1}$, Eq. (15) reduces to the equation for the mean of the operator $\hat{B}_1(t+\tau)$. For $\hat{A}_1 = \hat{B}^\dagger(t)$, Eq. (15) leads to the standard quantum regression theorem for correlation functions $\langle \hat{B}_1^\dagger(t)\hat{B}_1(t+\tau)\rangle$ that can be found in many textbooks [45,48].

V. ADJOINT MASTER EQUATION FOR MULTITIME CORRELATORS

The quantum regression theorem cannot be applied to calculate OTOCs. In this section, we address this issue and derive an adjoint master equation for multitime correlators that is applicable to OTOCs. The derivation itself is a generalization of the derivation of the quantum regression theorem presented in the previous section. To illustrate the main ideas behind this derivation we first consider a correlator with two time-dependent operators. Then we consider a correlator with an arbitrary number of operators depending on time.

A. Time evolution of two operators

The simplest correlator to which the quantum regression theorem is not applicable is

$$\langle \hat{A}_1 \hat{B}_1(t+\tau)\hat{A}_2\hat{B}_2(t+\tau)\hat{A}_3\rangle, \tag{18}$$

where $\hat{A}_1$, $\hat{B}_1(t+\tau)$, $\hat{A}_2$, $\hat{B}_2(t+\tau)$, and $\hat{A}_3$ are the operators of the system, and $\hat{A}_1$, $\hat{A}_2$, and $\hat{A}_3$ contain only times prior to t and do not depend on τ. We assume we know the correlator (18) at $\tau = 0$.

The connection between the correlator

$$\langle \hat{A}_1 \hat{B}_1(t+\tau)\hat{A}_2\hat{B}_2(t+\tau)\hat{A}_3\rangle,$$

and the correlator

$$\langle \hat{A}_1 \hat{B}_1(t+\tau+\Delta\tau)\hat{A}_2\hat{B}_2(t+\tau+\Delta\tau)\hat{A}_3\rangle,$$

follows from Eq. (12)

$$\begin{aligned}
\langle \hat{A}_1 \hat{B}_1(t+\tau+\Delta\tau)\hat{A}_2\hat{B}_2(t+\tau+\Delta\tau)\hat{A}_3\rangle &\approx \Delta\tau\frac{i}{\hbar}\langle \hat{A}_1[\hat{H}_S(t+\tau), \hat{B}_1(t+\tau)]\hat{A}_2\hat{B}_2(t+\tau)\hat{A}_3\rangle \\
&\quad + \Delta\tau\frac{i}{\hbar}\langle \hat{A}_1\hat{B}_1(t+\tau)\hat{A}_2[\hat{H}_S(t+\tau), \hat{B}_2(t+\tau)]\hat{A}_3\rangle + \langle \hat{A}_1\{\hat{V}^\dagger(t+\tau) \\
&\quad + \hat{W}^\dagger(t+\tau, \Delta\tau)\}\hat{B}_1'(t+\tau)\{\hat{V}(t+\tau) + \hat{W}(t+\tau, \Delta\tau)\}\hat{A}_2\{\hat{V}^\dagger(t+\tau) \\
&\quad + \hat{W}^\dagger(t+\tau, \Delta\tau)\}\hat{B}_2'(t+\tau)\{\hat{V}(t+\tau) + \hat{W}(t+\tau, \Delta\tau)\}\hat{A}_3\rangle.
\end{aligned} \tag{19}$$

This equation allows us to obtain the adjoint master equation in almost the same way we derive the quantum regression theorem (15). The details of this derivation are presented in Appendix B. We note, that in this derivation we again suppose that either exact (4) or approximate (5) decomposition such that the operators $\hat{C}_j$ are known. There is one notable difference between this derivation and the derivation of the quantum regression theorem given in the previous section. Namely, in this derivation, we have to take into account the combinations of operators $\hat{W}(t+\tau, \Delta\tau)$ and $\hat{W}^\dagger(t+\tau, \Delta\tau)$ that belong to different operators

$\hat{B}'_j(t+\tau)$ in the correlator (19). The resultant adjoint master equation for the correlator (18) is

$$\frac{d\langle\hat{A}_1\hat{B}_1(t+\tau)\hat{A}_2\hat{B}_2(t+\tau)\hat{A}_3\rangle}{d\tau} = \langle\hat{A}_1\mathcal{L}_{t+\tau}[\hat{B}_1(t+\tau)]\hat{A}_2\hat{B}_2(t+\tau)\hat{A}_3\rangle + \langle\hat{A}_1\hat{B}_1(t+\tau)\hat{A}_2\mathcal{L}_{t+\tau}[\hat{B}_2(t+\tau)]\hat{A}_3\rangle$$
$$+ \gamma_0\langle\hat{A}_1[\hat{C}_0,\hat{B}_1](t+\tau)\hat{A}_2[\hat{B}_2,\hat{C}_0](t+\tau)\hat{A}_3\rangle + \sum_{j=1}^{M}\gamma_j^{\downarrow}\langle\hat{A}_1[\hat{C}_j^{\dagger},\hat{B}_1](t+\tau)\hat{A}_2[\hat{B}_2,\hat{C}_j](t+\tau)\hat{A}_3\rangle$$
$$+ \sum_{j=1}^{M}\gamma_j^{\uparrow}\langle\hat{A}_1[\hat{C}_j,\hat{B}_1](t+\tau)\hat{A}_2[\hat{B}_2,\hat{C}_j^{\dagger}](t+\tau)\hat{A}_3\rangle, \tag{20}$$

where we denote $[\hat{C},\hat{B}](t+\tau) = [\hat{C}(t+\tau),\hat{B}(t+\tau)]$. The last three terms in the right-hand side of the Eq. (20) containing commutators between the operators $\hat{C}_j$ and $\hat{B}_m$ are the main difference between quantum regression and the adjoint master equation for multitime correlators.

We stress that in Eq. (20) the operators $\hat{C}_j$ may correspond to the exact decomposition (4) or to the approximate decomposition (5). In both cases, the Eq. (20) remains the same.

For $\hat{A}_1 = \hat{1}$, Eq. (20) reproduces the results obtained in [46,47].

B. Time evolution of *n* operators

In this subsection, we present the adjoint master equation for the correlation function containing an arbitrary number of time-dependent operators

$$\langle\hat{A}_1\hat{B}_1(t+\tau)\dots\hat{A}_n\hat{B}_n(t+\tau)\hat{A}_{n+1}\rangle, \tag{21}$$

where $\hat{A}_j$, $\hat{B}_j(t+\tau)$ are the operators of the system and $\hat{A}_j$ may contain only times prior to t and does not depend on τ. We assume that the correlation function (21) at $\tau = 0$ is known. Below, we also use the notation $\langle\hat{A}_1\hat{B}_1(t+\tau)\dots\hat{A}_{n+1}\rangle$ for the correlation function (21).

In this case, the derivation is the same as in the previous subsection. As a result, we obtain the adjoint master equation for the correlation function (21)

$$\frac{d\langle\hat{A}_1\hat{B}_1(t+\tau)\dots\hat{A}_n\hat{B}_n(t+\tau)\hat{A}_{n+1}\rangle}{d\tau} = \sum_{m=1}^{n}\mathcal{L}_{t+\tau}^{(m)}[\langle\hat{A}_1\hat{B}_1(t+\tau)\dots\hat{A}_n\hat{B}_n(t+\tau)\hat{A}_{n+1}\rangle]$$
$$+ \sum_{m_1=1}^{n-1}\sum_{m_2=m_1+1}^{n}\mathcal{M}_{t+\tau}^{(m_1,m_2)}[\langle\hat{A}_1\hat{B}_1(t+\tau)\dots\hat{A}_{n+1}\rangle], \tag{22}$$

where

$$\mathcal{L}_{t+\tau}^{(m)}[\langle\hat{A}_1\hat{B}_1(t+\tau)\dots\hat{A}_n\hat{B}_n(t+\tau)\hat{A}_{n+1}\rangle] = \langle\hat{A}_1\dots\hat{A}_m\mathcal{L}_{t+\tau}[\hat{B}_m(t+\tau)]\hat{A}_{m+1}\dots\hat{A}_{n+1}\rangle, \tag{23}$$

$$\mathcal{M}_{t+\tau}^{(m_1,m_2)}[\langle\hat{A}_1\hat{B}_1(t+\tau)\dots\hat{A}_{n+1}\rangle] = \gamma_0\langle\hat{A}_1\dots[\hat{C}_0,\hat{B}_{m_1}](t+\tau)\dots[\hat{B}_{m_2},\hat{C}_0](t+\tau)\dots\hat{A}_{n+1}\rangle$$
$$+ \sum_{j=1}^{M}\gamma_j^{\downarrow}\langle\hat{A}_1\dots[\hat{C}_j^{\dagger},\hat{B}_{m_1}](t+\tau)\dots[\hat{B}_{m_2},\hat{C}_j](t+\tau)\dots\hat{A}_{n+1}\rangle$$
$$+ \sum_{j=1}^{M}\gamma_j^{\uparrow}\langle\hat{A}_1\dots[\hat{C}_j,\hat{B}_{m_1}](t+\tau)\dots[\hat{B}_{m_2},\hat{C}_j^{\dagger}](t+\tau)\dots\hat{A}_{n+1}\rangle. \tag{24}$$

The first term in the right-hand side of Eq. (22) corresponds to the dynamics of the solitary operators $\hat{B}_j(t)$. The second term in the right-hand side of Eq. (22) describes the nontrivial time evolution of the correlator due to interplay between the operators $\hat{B}_j(t)$. These terms are vital for OTOCs and provide the self-consistency of the adjoint master equation for multitime correlators.

VI. SELF-CONSISTENCY OF THE ADJOINT MASTER EQUATION FOR MULTITIME CORRELATORS

We show the self-consistency of the adjoint master equation for multitime correlators for particular correlator

$$\langle\hat{B}_1(t+\tau)\hat{B}_2(t+\tau)\rangle, \tag{25}$$

where $\hat{B}_1$ and $\hat{B}_2$ are the operators of the system.

We can obtain the equation for this correlation function (25) in two different ways. First, we can consider the correlator (25) as the correlator (18) with $\hat{A}_1 = \hat{A}_2 = \hat{A}_3 = \hat{1}$, $\hat{B}_2 = \hat{1}$, and $\hat{B}_1 = (\hat{B}_1\hat{B}_2)$. In this case, Eq. (20) gives

$$\frac{d\langle \hat{B}_1(t+\tau)\hat{B}_2(t+\tau)\rangle}{d\tau} = \langle \mathcal{L}_{t+\tau}[\hat{B}_1(t+\tau)\hat{B}_2(t+\tau)]\rangle. \tag{26}$$

Second, we can consider the correlator (25) as the correlator (18) with $\hat{A}_1 = \hat{A}_2 = \hat{A}_3 = \hat{1}$. In this case, Eq. (20) gives

$$\frac{d\langle \hat{B}_1(t+\tau)\hat{B}_2(t+\tau)\rangle}{d\tau} = \langle \mathcal{L}_{t+\tau}[\hat{B}_1(t+\tau)]\hat{B}_2(t+\tau)\rangle + \langle \hat{B}_1(t+\tau)\mathcal{L}_{t+\tau}[\hat{B}_2(t+\tau)]\rangle + \gamma_0\langle[\hat{C}_0, \hat{B}_1](t+\tau)[\hat{B}_2, \hat{C}_0](t+\tau)\rangle$$
$$+ \sum_{j=1}^{M}\gamma_j^{\downarrow}\langle[\hat{C}_j^{\dagger}, \hat{B}_1](t+\tau)[\hat{B}_2, \hat{C}_j](t+\tau)\rangle + \sum_{j=1}^{M}\gamma_j^{\uparrow}\langle[\hat{C}_j, \hat{B}_1](t+\tau)[\hat{B}_2, \hat{C}_j^{\dagger}](t+\tau)\rangle. \tag{27}$$

The self-consistency of the adjoint master equation for multi-time correlators means that (i) Eq. (26) is equivalent to Eq. (27) and (ii) the adjoint master equation for multitime correlators gives the same result as the standard quantum regression theorem (15) for the correlator (13) with $\hat{A}_1 = \hat{A}_2 = \hat{1}$ and $\hat{B}_1 = (\hat{B}_1\hat{B}_2)$. Thus, the adjoint master equation for the correlation function (25) is self-consistent.

In Appendix C, we prove the self-consistency of the adjoint master equation in the general case.

VII. CONCLUSION

In this work, we derived an adjoint master equation for multitime correlators. This equation is preferable for the usage of different approaches to relaxation operators such as the global approach [45,49–51,54], the local approach [49–53], the partially secular approach [56], and the approach based on the perturbation theory [57]. Indeed, the presented derivation does not depend on whether we use the exact or approximate expansion of the operator of interaction between the system and reservoir. Such flexibility reflects that the adjoint master equation for the multitime correlators is independent of the particular relaxation operators.

We showed that the derived adjoint master equation for multitime correlators is self-consistent: the final equation for a correlator does not depend on how we group the operators inside the correlator and it coincides with the quantum regression theorem when the latter is applicable. This equation does not apply to the out-of-time-ordered correlators in open quantum systems. Also, the adjoint master equation for multitime correlators allows calculating of the correlator $\langle \hat{A}_1(t_1)\hat{A}_2(t_2)\ldots\ \hat{A}_n(t_n)\rangle$ with arbitrary relations between t_1, t_2,..., t_n. The exact algorithm for $n = 4$ is presented in [47].

In this work, we explicitly derived the adjoint master equation for one reservoir interacting with the system. The derivation in the case of multiple reservoirs is straightforward.

ACKNOWLEDGMENTS

The research was financially supported by a grant from the Russian Science Foundation (Project No. 20-72-10057). V.Y.S. thanks the Foundation for the Advancement of Theoretical Physics and Mathematics “Basis.”

APPENDIX A: TRANSFORMATION OF THE CORRELATORS IN THE RIGHT-HAND SIDE OF EQ. (14)

Here, we consider the second term in the right-hand side of Eq. (14)

$$\langle \hat{A}_1\{\hat{V}^{\dagger}(t+\tau) + \hat{W}^{\dagger}(t+\tau, \Delta\tau)\}\hat{B}_1'(t+\tau)\{\hat{V}(t+\tau) + \hat{W}(t+\tau, \Delta\tau)\}\hat{A}_2\rangle, \tag{A1}$$

and trace out the reservoirs' degrees of freedom preserving the terms up to λ^2. In this Appendix, we use all the standard assumptions about the reservoir given in the textbook [45]. We also explicitly listed these assumptions in Sec. IV. We consider the terms proportional to λ^0, λ^1 and λ^2 separately.

To transform the term of Eq. (A1) proportional to λ^0 we use Eq. (7) and obtain

$$\langle \hat{A}_1\hat{V}^{\dagger}(t+\tau)\hat{B}_1'(t+\tau)\hat{V}(t+\tau)\hat{A}_2\rangle = \langle \hat{A}_1\hat{B}_1(t+\tau)\hat{A}_2\rangle. \tag{A2}$$

All the terms of Eq. (A1) proportional to λ^1 are zero because $\mathrm{tr}_R(\hat{R}(t)\hat{\rho}_R^{\mathrm{th}}) = 0$. Namely,

$$\lambda^2\langle \hat{A}_1\hat{V}_1^{\dagger}(t+\tau, \Delta\tau)\hat{B}_1'(t+\tau)\hat{V}(t+\tau)\hat{A}_2\rangle = 0, \tag{A3}$$

$$\lambda^2\langle \hat{A}_1\hat{V}^{\dagger}(t+\tau)\hat{B}_1'(t+\tau)\hat{V}_1(t+\tau, \Delta\tau)\hat{A}_2\rangle = 0. \tag{A4}$$

The terms of Eq. (A1) proportional to λ^2 are

$$\lambda^2\langle \hat{A}_1\hat{V}_1^{\dagger}(t+\tau, \Delta\tau)\hat{B}_1'(t+\tau)\hat{V}_1(t+\tau, \Delta\tau)\hat{A}_2\rangle, \tag{A5}$$

$$\lambda^2\langle \hat{A}_1\hat{V}_2^{\dagger}(t+\tau, \Delta\tau)\hat{B}_1'(t+\tau)\hat{V}(t+\tau)\hat{A}_2\rangle, \tag{A6}$$

$$\lambda^2\langle\hat{A}_1\hat{V}^\dagger(t+\tau)\hat{B}_1'(t+\tau)\hat{V}_2(t+\tau,\Delta\tau)\hat{A}_2\rangle. \tag{A7}$$

We consider the correlator Eq. (A5) in details. This correlator can be transformed in the following way:

$$\begin{aligned}
&\lambda^2\langle\hat{A}_1\hat{V}_1^\dagger(t+\tau,\Delta\tau)\hat{B}_1'(t+\tau)\hat{V}_1(t+\tau,\Delta\tau)\hat{A}_2\rangle\\
&=\lambda^2\int_{t+\tau}^{t+\tau+\Delta\tau}dt_1\int_{t+\tau}^{t+\tau+\Delta\tau}dt_2\langle\hat{A}_1\hat{V}^\dagger(t+\tau)\hat{S}'(t_1)\hat{R}'(t_1)\hat{B}_1'(t+\tau)\hat{S}'(t_2)\hat{R}'(t_2)\hat{V}(t+\tau)\hat{A}_2\rangle\\
&=\lambda^2\int_{t+\tau}^{t+\tau+\Delta\tau}dt_1\int_{t+\tau}^{t+\tau+\Delta\tau}dt_2\langle\hat{A}_1\hat{R}'(t_1)[\hat{C}_0(t+\tau)+\sum_{j=1}^{M}\hat{C}_j(t+\tau)e^{-i\omega_j(t_1-t-\tau)}\\
&\quad+\sum_{j=1}^{M}\hat{C}_j^\dagger(t+\tau)e^{i\omega_j(t_1-t-\tau)}]\hat{B}(t+\tau)[\hat{C}_0(t+\tau)+\sum_{k=1}^{M}\hat{C}_k(t+\tau)e^{-i\omega_k(t_2-t-\tau)}+\sum_{k=1}^{M}\hat{C}_k^\dagger(t+\tau)e^{i\omega_k(t_2-t-\tau)}]\hat{R}'(t_2)\hat{A}_2\rangle,
\end{aligned}\tag{A8}$$

where we use the definition Eq. (10), decomposition Eq. (4) [or Eq. (5)], and relation between interaction representation and full evolution given by Eq. (7). We also replace $\hat{V}^\dagger(t+\tau)\hat{R}'(t_1)$ and $\hat{R}'(t_2)\hat{V}(t+\tau)$ with $\hat{R}'(t_1)\hat{V}^\dagger(t+\tau)$ and $\hat{V}(t+\tau)\hat{R}'(t_2)$ for $t_1, t_2 \geqslant t$ because the corresponding commutators are proportional to λ and can be omitted. Following [45], we assume that density matrix of the system and the reservoir are factorized at any moment in time. Thus,

$$\begin{aligned}
\lambda^2\langle\hat{A}_1\hat{V}_1^\dagger(t+\tau,\Delta\tau)\hat{B}_1'(t+\tau)\hat{V}_1(t+\tau,\Delta\tau)\hat{A}_2\rangle &= \lambda^2\int_0^{\Delta\tau}d\tau_1\int_0^{\Delta\tau}d\tau_2\langle\hat{R}'(\tau_1)\hat{R}'(\tau_2)\rangle\langle\hat{A}_1[\hat{C}_0(t+\tau)+\sum_{j=1}^{M}\hat{C}_j(t+\tau)e^{-i\omega_j\tau_1}\\
&\quad+\sum_{j=1}^{M}\hat{C}_j^\dagger(t+\tau)e^{i\omega_j\tau_1}]\hat{B}(t+\tau)[\hat{C}_0(t+\tau)+\sum_{k=1}^{M}\hat{C}_k(t+\tau)e^{-i\omega_k\tau_2}\\
&\quad+\sum_{k=1}^{M}\hat{C}_k^\dagger(t+\tau)e^{i\omega_k\tau_2}]\hat{A}_2\rangle,
\end{aligned}\tag{A9}$$

where we use $\langle\hat{R}'(\tau_1)\hat{R}'(\tau_2)\rangle = \langle\hat{R}'(t_1)\hat{R}'(t_2)\rangle$. We integrate the correlator of the reservoir operators over time and obtain [45]

$$\lambda^2\int_0^{\Delta\tau}d\tau_1\int_0^{\Delta\tau}d\tau_2\langle\hat{R}'(\tau_1)\hat{R}'(\tau_2)\rangle e^{i\omega_A\tau_1}e^{-i\omega_B\tau_2} = \Delta\tau\delta_{\omega_A,\omega_B}\begin{cases}\gamma_A^\downarrow, & \text{if } \omega_A > 0,\\ \gamma_A^\uparrow, & \text{if } \omega_A < 0,\\ \gamma_0, & \text{if } \omega_A = 0.\end{cases}\tag{A10}$$

Finally, we have

$$\begin{aligned}
\lambda^2\langle\hat{A}_1\hat{V}_1^\dagger(t+\tau,\Delta\tau)\hat{B}_1'(t+\tau)\hat{V}_1(t+\tau,\Delta\tau)\hat{A}_2\rangle &= \Delta\tau\gamma_0\langle\hat{A}_1\hat{C}_0(t+\tau)\hat{B}_1(t+\tau)\hat{C}_0(t+\tau)\hat{A}_2\rangle\\
&\quad+\Delta\tau\sum_{j=1}^{M}\gamma_j^\downarrow\langle\hat{A}_1\hat{C}_j^\dagger(t+\tau)\hat{B}_1(t+\tau)\hat{C}_j(t+\tau)\hat{A}_2\rangle\\
&\quad+\Delta\tau\sum_{j=1}^{M}\gamma_j^\uparrow\langle\hat{A}_1\hat{C}_j(t+\tau)\hat{B}_1(t+\tau)\hat{C}_j^\dagger(t+\tau)\hat{A}_2\rangle.
\end{aligned}\tag{A11}$$

The consideration of the correlators (A6) and (A7) is similar to Eq. (A5). Here, we present the result of tracing out the reservoirs' degrees of freedom in these correlators

$$\begin{aligned}
\lambda^2\langle\hat{A}_1\hat{V}_2^\dagger(t+\tau,\Delta\tau)\hat{B}_1'(t+\tau)\hat{V}(t+\tau)\hat{A}_2\rangle &= \Delta\tau\frac{1}{2}\gamma_0\langle\hat{A}_1\hat{C}_0(t+\tau)\hat{C}_0(t+\tau)\hat{B}_1(t+\tau)\hat{A}_2\rangle\\
&\quad+\Delta\tau\frac{1}{2}\sum_{j=1}^{M}\gamma_j^\downarrow\langle\hat{A}_1\hat{C}_j^\dagger(t+\tau)\hat{C}_j(t+\tau)\hat{B}_1(t+\tau)\hat{A}_2\rangle\\
&\quad+\Delta\tau\frac{1}{2}\sum_{j=1}^{M}\gamma_j^\uparrow\langle\hat{A}_1\hat{C}_j(t+\tau)\hat{C}_j^\dagger(t+\tau)\hat{B}_1(t+\tau)\hat{A}_2\rangle,
\end{aligned}\tag{A12}$$

$$\lambda^2\langle\hat{A}_1\hat{V}^\dagger(t+\tau)\hat{B}_1'(t+\tau)\hat{V}_2(t+\tau,\Delta\tau)\hat{A}_2\rangle = \Delta\tau\frac{1}{2}\gamma_0\langle\hat{A}_1\hat{B}_1(t+\tau)\hat{C}_0(t+\tau)\hat{C}_0(t+\tau)\hat{A}_2\rangle + \Delta\tau\frac{1}{2}\sum_{j=1}^{M}\gamma_j^\downarrow\langle\hat{A}_1\hat{B}_1(t+\tau)\hat{C}_j^\dagger(t+\tau)\hat{C}_j(t+\tau)\hat{A}_2\rangle + \Delta\tau\frac{1}{2}\sum_{j=0}^{M}\gamma_j^\uparrow\langle\hat{A}_1\hat{B}_1(t+\tau)\hat{C}_j(t+\tau)\hat{C}_j^\dagger(t+\tau)\hat{A}_2\rangle. \tag{A13}$$

Using Eq. (A2), Eqs. (A3) and (A4), and Eqs. (A11) and (A13), we derive the quantum regression theorem (15) from Eq. (14).

APPENDIX B: TRANSFORMATION OF THE CORRELATORS IN THE RIGHT-HAND SIDE OF EQ. (19)

Here, we present the transformation of the correlator in the right-hand side of Eq. (19) that traces out the reservoirs' degrees of freedom and preserves the terms up to λ^2. We consider the correlator on the right-hand side of Eq. (19) that has the form

$$\langle\hat{A}_1\{\hat{V}^\dagger(t+\tau)+\hat{W}^\dagger(t+\tau,\Delta\tau)\}\hat{B}_1'(t+\tau)\{\hat{V}(t+\tau)+\hat{W}(t+\tau,\Delta\tau)\}\hat{A}_2\{\hat{V}^\dagger(t+\tau) + \hat{W}^\dagger(t+\tau,\Delta\tau)\}\hat{B}_2'(t+\tau)\{\hat{V}(t+\tau)+\hat{W}(t+\tau,\Delta\tau)\}\hat{A}_3\rangle, \tag{B1}$$

and decompose it in correlators proportional to λ^0, λ^1, λ^2.

The term of the correlator (B1) proportional to λ^0 is

$$\langle\hat{A}_1\hat{V}^\dagger(t+\tau)\hat{B}_1'(t+\tau)\hat{V}(t+\tau)\hat{A}_2\hat{V}^\dagger(t+\tau)\hat{B}_2'(t+\tau)\hat{V}(t+\tau)\hat{A}_3\rangle = \langle\hat{A}_1\hat{B}_1(t+\tau)\hat{A}_2\hat{B}_2(t+\tau)\hat{A}_3\rangle. \tag{B2}$$

All the terms of the correlator (B1) proportional to λ^1 are zero because $\mathrm{tr}_R(\hat{R}(t)\hat{\rho}_R^{\mathrm{th}}) = 0$. Namely,

$$\lambda\langle\hat{A}_1\hat{V}_1^\dagger(t+\tau,\Delta\tau)\hat{B}_1'(t+\tau)\hat{V}(t+\tau)\hat{A}_2\hat{V}^\dagger(t+\tau)\hat{B}_2'(t+\tau)\hat{V}(t+\tau)\hat{A}_3\rangle = 0, \tag{B3}$$

$$\lambda\langle\hat{A}_1\hat{V}^\dagger(t+\tau)\hat{B}_1'(t+\tau)\hat{V}_1(t+\tau,\Delta\tau)\hat{A}_2\hat{V}^\dagger(t+\tau)\hat{B}_2'(t+\tau)\hat{V}(t+\tau)\hat{A}_3\rangle = 0, \tag{B4}$$

$$\lambda\langle\hat{A}_1\hat{V}^\dagger(t+\tau)\hat{B}_1'(t+\tau)\hat{V}(t+\tau)\hat{A}_2\hat{V}_1^\dagger(t+\tau,\Delta\tau)\hat{B}_2'(t+\tau)\hat{V}(t+\tau)\hat{A}_3\rangle = 0, \tag{B5}$$

$$\lambda\langle\hat{A}_1\hat{V}^\dagger(t+\tau)\hat{B}_1'(t+\tau)\hat{V}(t+\tau)\hat{A}_2\hat{V}^\dagger(t+\tau)\hat{B}_2'(t+\tau)\hat{V}_1(t+\tau,\Delta\tau)\hat{A}_3\rangle = 0. \tag{B6}$$

The terms of the correlator (B1) proportional to λ^2 are

$$\lambda^2\langle\hat{A}_1\hat{V}_1^\dagger(t+\tau,\Delta\tau)\hat{B}_1'(t+\tau)\hat{V}_1(t+\tau,\Delta\tau)\hat{A}_2\hat{V}^\dagger(t+\tau)\hat{B}_2'(t+\tau)\hat{V}(t+\tau)\hat{A}_3\rangle, \tag{B7}$$

$$\lambda^2\langle\hat{A}_1\hat{V}_2^\dagger(t+\tau,\Delta\tau)\hat{B}_1'(t+\tau)\hat{V}(t+\tau)\hat{A}_2\hat{V}^\dagger(t+\tau)\hat{B}_2'(t+\tau)\hat{V}(t+\tau)\hat{A}_3\rangle, \tag{B8}$$

$$\lambda^2\langle\hat{A}_1\hat{V}^\dagger(t+\tau)\hat{B}_1'(t+\tau)\hat{V}_2(t+\tau,\Delta\tau)\hat{A}_2\hat{V}^\dagger(t+\tau)\hat{B}_2'(t+\tau)\hat{V}(t+\tau)\hat{A}_3\rangle, \tag{B9}$$

$$\lambda^2\langle\hat{A}_1\hat{V}^\dagger(t+\tau)\hat{B}_1'(t+\tau)\hat{V}(t+\tau)\hat{A}_2\hat{V}_1^\dagger(t+\tau,\Delta\tau)\hat{B}_2'(t+\tau)\hat{V}_1(t+\tau,\Delta\tau)\hat{A}_3\rangle \tag{B10}$$

$$\lambda^2\langle\hat{A}_1\hat{V}^\dagger(t+\tau)\hat{B}_1'(t+\tau)\hat{V}(t+\tau)\hat{A}_2\hat{V}_2^\dagger(t+\tau,\Delta\tau)\hat{B}_2'(t+\tau)\hat{V}(t+\tau)\hat{A}_3\rangle, \tag{B11}$$

$$\lambda^2\langle\hat{A}_1\hat{V}^\dagger(t+\tau)\hat{B}_1'(t+\tau)\hat{V}(t+\tau)\hat{A}_2\hat{V}^\dagger(t+\tau)\hat{B}_2'(t+\tau)\hat{V}_2(t+\tau,\Delta\tau)\hat{A}_3\rangle, \tag{B12}$$

$$\lambda^2\langle\hat{A}_1\hat{V}_1^\dagger(t+\tau,\Delta\tau)\hat{B}_1'(t+\tau)\hat{V}(t+\tau)\hat{A}_2\hat{V}_1^\dagger(t+\tau,\Delta\tau)\hat{B}_2'(t+\tau)\hat{V}(t+\tau)\hat{A}_3\rangle, \tag{B13}$$

$$\lambda^2\langle\hat{A}_1\hat{V}_1^\dagger(t+\tau,\Delta\tau)\hat{B}_1'(t+\tau)\hat{V}(t+\tau)\hat{A}_2\hat{V}^\dagger(t+\tau)\hat{B}_2'(t+\tau)\hat{V}_1(t+\tau,\Delta\tau)\hat{A}_3\rangle, \tag{B14}$$

$$\lambda^2\langle\hat{A}_1\hat{V}^\dagger(t+\tau)\hat{B}_1'(t+\tau)\hat{V}_1(t+\tau,\Delta\tau)\hat{A}_2\hat{V}_1^\dagger(t+\tau,\Delta\tau)\hat{B}_2'(t+\tau)\hat{A}_3\hat{V}(t+\tau)\rangle, \tag{B15}$$

$$\lambda^2\langle\hat{A}_1\hat{V}^\dagger(t+\tau)\hat{B}_1'(t+\tau)\hat{V}_1(t+\tau,\Delta\tau)\hat{A}_2\hat{V}^\dagger(t+\tau)\hat{B}_2'(t+\tau)\hat{V}_1(t+\tau,\Delta\tau)\hat{A}_3\rangle. \tag{B16}$$

The correlators (B7) to (B9) are similar to the correlators (A5) to (A7). These correlators correspond to the term

$$\langle\hat{A}_1\mathcal{L}_{t+\tau}[\hat{B}_1(t+\tau)]\hat{A}_2\hat{B}_2(t+\tau)\hat{A}_3\rangle \tag{B17}$$

in Eq. (20). The same is true for the correlators (B10) to (B12) that corresponds to the term

$$\langle\hat{A}_1\hat{B}_1(t+\tau)\hat{A}_2\mathcal{L}_{t+\tau}[\hat{B}_2(t+\tau)]\hat{A}_3\rangle \tag{B18}$$

in Eq. (20). The correlators (B13) to (B16) do not directly correspond to the correlators (A5) to (A7). The consideration similar to one conducted for (A5) in Appendix A can be done for the correlators (B13) to (B16) leading to the following result:

$$\begin{aligned}
&\lambda^2\langle\hat{A}_1\hat{V}_1^\dagger(t+\tau,\Delta\tau)\hat{B}_1'(t+\tau)\hat{V}(t+\tau)\hat{A}_2\hat{V}_1^\dagger(t+\tau,\Delta\tau)\hat{B}_2'(t+\tau)\hat{V}(t+\tau)\hat{A}_3\rangle\\
&\quad=\Delta\tau\gamma_0\langle\hat{A}_1\hat{C}_0(t+\tau)\hat{B}_1(t+\tau)\hat{A}_2\hat{C}_0(t+\tau)\hat{B}_2(t+\tau)\hat{A}_3\rangle\\
&\qquad+\Delta\tau\sum_{j=1}^{M}\gamma_j^\downarrow\langle\hat{A}_1\hat{C}_j^\dagger(t+\tau)\hat{B}_1(t+\tau)\hat{A}_2\hat{C}_j(t+\tau)\hat{B}_2(t+\tau)\hat{A}_3\rangle\\
&\qquad+\Delta\tau\sum_{j=1}^{M}\gamma_j^\uparrow\langle\hat{A}_1\hat{C}_j(t+\tau)\hat{B}_1(t+\tau)\hat{A}_2\hat{C}_j^\dagger(t+\tau)\hat{B}_2(t+\tau)\hat{A}_3\rangle,
\end{aligned}\tag{B19}$$

$$\begin{aligned}
&\lambda^2\langle\hat{A}_1\hat{V}_1^\dagger(t+\tau,\Delta\tau)\hat{B}_1'(t+\tau)\hat{V}(t+\tau)\hat{A}_2\hat{V}^\dagger(t+\tau)\hat{B}_2'(t+\tau)\hat{V}_1(t+\tau,\Delta\tau)\hat{A}_3\rangle\\
&\quad=\Delta\tau\gamma_0\langle\hat{A}_1\hat{C}_0(t+\tau)\hat{B}_1(t+\tau)\hat{A}_2\hat{B}_2(t+\tau)\hat{C}_0(t+\tau)\hat{A}_3\rangle\\
&\qquad+\Delta\tau\sum_{j=1}^{M}\gamma_j^\downarrow\langle\hat{A}_1\hat{C}_j^\dagger(t+\tau)\hat{B}_1(t+\tau)\hat{A}_2\hat{B}_2(t+\tau)\hat{C}_j(t+\tau)\hat{A}_3\rangle\\
&\qquad+\Delta\tau\sum_{j=1}^{M}\gamma_j^\uparrow\langle\hat{A}_1\hat{C}_j(t+\tau)\hat{B}_1(t+\tau)\hat{A}_2\hat{B}_2(t+\tau)\hat{C}_j^\dagger(t+\tau)\hat{A}_3\rangle,
\end{aligned}\tag{B20}$$

$$\begin{aligned}
&\lambda^2\langle\hat{A}_1\hat{V}^\dagger(t+\tau)\hat{B}_1'(t+\tau)\hat{V}_1(t+\tau,\Delta\tau)\hat{A}_2\hat{V}_1^\dagger(t+\tau,\Delta\tau)\hat{B}_2'(t+\tau)\hat{A}_3\hat{V}(t+\tau)\rangle\\
&\quad=\Delta\tau\gamma_0\langle\hat{A}_1\hat{B}_1(t+\tau)\hat{C}_0(t+\tau)\hat{A}_2\hat{C}_0(t+\tau)\hat{B}_2(t+\tau)\hat{A}_3\rangle\\
&\qquad+\Delta\tau\sum_{j=1}^{M}\gamma_j^\downarrow\langle\hat{A}_1\hat{B}_1(t+\tau)\hat{C}_j^\dagger(t+\tau)\hat{A}_2\hat{C}_j(t+\tau)\hat{B}_2(t+\tau)\hat{A}_3\rangle\\
&\qquad+\Delta\tau\sum_{j=1}^{M}\gamma_j^\uparrow\langle\hat{A}_1\hat{B}_1(t+\tau)\hat{C}_j(t+\tau)\hat{A}_2\hat{C}_j^\dagger(t+\tau)\hat{B}_2(t+\tau)\hat{A}_3\rangle,
\end{aligned}\tag{B21}$$

$$\begin{aligned}
&\lambda^2\langle\hat{A}_1\hat{V}^\dagger(t+\tau)\hat{B}_1'(t+\tau)\hat{V}_1(t+\tau,\Delta\tau)\hat{A}_2\hat{V}^\dagger(t+\tau)\hat{B}_2'(t+\tau)\hat{V}_1(t+\tau,\Delta\tau)\hat{A}_3\rangle\\
&\quad=\Delta\tau\gamma_0\langle\hat{A}_1\hat{B}_1(t+\tau)\hat{C}_0(t+\tau)\hat{A}_2\hat{B}_2(t+\tau)\hat{C}_0(t+\tau)\hat{A}_3\rangle\\
&\qquad+\Delta\tau\sum_{j=1}^{M}\gamma_j^\downarrow\langle\hat{A}_1\hat{B}_1(t+\tau)\hat{C}_j^\dagger(t+\tau)\hat{A}_2\hat{B}_2(t+\tau)\hat{C}_j(t+\tau)\hat{A}_3\rangle\\
&\qquad+\Delta\tau\sum_{j=1}^{M}\gamma_j^\uparrow\langle\hat{A}_1\hat{B}_1(t+\tau)\hat{C}_j(t+\tau)\hat{A}_2\hat{B}_2(t+\tau)\hat{C}_j^\dagger(t+\tau)\hat{A}_3\rangle.
\end{aligned}\tag{B22}$$

These correlators correspond to the terms

$$\begin{aligned}
&\gamma_0\langle\hat{A}_1[\hat{C}_0,\hat{B}_1](t+\tau)\hat{A}_2[\hat{B}_2,\hat{C}_0](t+\tau)\hat{A}_3\rangle+\sum_{j=1}^{M}\gamma_j^\downarrow\langle\hat{A}_1[\hat{C}_j^\dagger,\hat{B}_1](t+\tau)\hat{A}_2[\hat{B}_2,\hat{C}_j](t+\tau)\hat{A}_3\rangle\\
&+\sum_{j=1}^{M}\gamma_j^\uparrow\langle\hat{A}_1[\hat{C}_j,\hat{B}_1](t+\tau)\hat{A}_2[\hat{B}_2,\hat{C}_j^\dagger](t+\tau)\hat{A}_3\rangle
\end{aligned}\tag{B23}$$

in Eq. (20).

APPENDIX C: SELF-CONSISTENCY OF THE ADJOINT MASTER EQUATION FOR MULTITIME CORRELATORS IN GENERAL CASE

Here, we prove the self-consistency of the adjoint master equation Eq. (22). To do this we consider correlator (21) with

$$\hat{B}_m=\hat{T}_1^{(m)}\dots\hat{T}_{p_m}^{(m)}.\tag{C1}$$

Using the property of the commutator

$$[\hat{C}, \hat{T}_1 \dots \hat{T}_p] = \sum_{k=1}^{p} \hat{T}_1 \dots \hat{T}_{k-1}[\hat{C}, \hat{T}_k]\hat{T}_{k+1} \dots \hat{T}_p, \tag{C2}$$

and mathematical induction for p_m one can proof that

$$\begin{aligned}\mathcal{L}^{(m)}_{t+\tau}[\langle \hat{A}_1\hat{B}_1(t+\tau)\dots \hat{A}_n\hat{B}_n(t+\tau)\hat{A}_{n+1}\rangle] &= \sum_{k=1}^{p_m} \mathcal{L}^{(m,k)}_{t+\tau}[\langle \hat{A}_1\hat{B}_1(t+\tau)\dots \hat{A}_n\hat{B}_n(t+\tau)\hat{A}_{n+1}\rangle] \\ &+ \sum_{k_1=1}^{p_m-1}\sum_{k_2=k_1+1}^{p_m} \mathcal{M}^{(m,k_1,m,k_2)}_{t+\tau}[\langle \hat{A}_1\hat{B}_1(t+\tau)\dots \hat{A}_n\hat{B}_n(t+\tau)\hat{A}_{n+1}\rangle,]\end{aligned} \tag{C3}$$

and

$$\mathcal{M}^{(m_1,m_2)}_{t+\tau}[\langle \hat{A}_1\hat{B}_1(t+\tau)\dots \hat{A}_n\hat{B}_n(t+\tau)\hat{A}_{n+1}\rangle] = \sum_{k_1=1}^{p_{m_1}}\sum_{k_2=1}^{p_{m_2}} \mathcal{M}^{(m_1,k_1,m_2,k_2)}_{t+\tau}[\langle \hat{A}_1\hat{B}_1(t+\tau)\dots \hat{A}_n\hat{B}_n(t+\tau)\hat{A}_{n+1}\rangle], \tag{C4}$$

where we denote

$$\mathcal{L}^{(m,k)}_{t+\tau}[\langle \hat{A}_1\hat{B}_1(t+\tau)\dots \hat{A}_n\hat{B}_n(t+\tau)\hat{A}_{n+1}\rangle] = \langle \hat{A}_1\hat{B}_1(t+\tau)\dots \hat{A}_m \dots \mathcal{L}_{t+\tau}[\hat{T}^{(m)}_k(t+\tau)]\dots \hat{A}_{m+1}\dots \hat{A}_n\hat{B}_n(t+\tau)\hat{A}_{n+1}\rangle, \tag{C5}$$

and

$$\begin{aligned}&\mathcal{M}^{(m_1,k_1,m_2,k_2)}_{t+\tau}[\langle \hat{A}_1\hat{B}_1(t+\tau)\dots \hat{A}_n\hat{B}_n(t+\tau)\hat{A}_{n+1}\rangle] \\ &= \gamma_0\langle \hat{A}_1\hat{B}_1(t+\tau)\dots\hat{A}_{m_1}\dots[\hat{C}_0, \hat{T}^{(m_1)}_{k_1}](t+\tau)\dots\hat{A}_{m_1+1}\dots\hat{A}_{m_2}\dots[\hat{T}^{(m_2)}_{k_2}, \hat{C}_0](t+\tau)\dots\hat{A}_{m_2+1}\dots\hat{A}_n\hat{B}_n(t+\tau)\hat{A}_{n+1}\rangle \\ &+ \sum_{j=1}^{M}\gamma^{\downarrow}_j\langle \hat{A}_1\hat{B}_1(t+\tau)\dots\hat{A}_{m_1}\dots[\hat{C}^{\dagger}_j, \hat{T}^{(m_1)}_{k_1}](t+\tau)\dots\hat{A}_{m_1+1}\dots\hat{A}_{m_2}\dots[\hat{T}^{(m_2)}_{k_2}, \hat{C}_j](t+\tau)\dots\hat{A}_{m_2+1}\dots\hat{A}_n\hat{B}_n(t+\tau)\hat{A}_{n+1}\rangle \\ &+ \sum_{j=1}^{M}\gamma^{\uparrow}_j\langle \hat{A}_1\hat{B}_1(t+\tau)\dots\hat{A}_{m_1}\dots[\hat{C}_j, \hat{T}^{(m_1)}_{k_1}](t+\tau)\dots\hat{A}_{m_1+1}\dots\hat{A}_{m_2}\dots[\hat{T}^{(m_2)}_{k_2}, \hat{C}^{\dagger}_j](t+\tau)\dots\hat{A}_{m_2+1}\dots\hat{A}_n\hat{B}_n(t+\tau)\hat{A}_{n+1}\rangle.\end{aligned} \tag{C6}$$

Using equations (C3) and (C4) one can rewrite Eq. (22) as

$$\begin{aligned}&\frac{d\langle \hat{A}_1\hat{B}_1(t+\tau)\dots\hat{A}_n\hat{B}_n(t+\tau)\hat{A}_{n+1}\rangle}{d\tau} \\ &= \sum_{m=1}^{n}\sum_{k=1}^{p_m}\mathcal{L}^{(m,k)}_{t+\tau}[\langle \hat{A}_1\hat{T}^{(1)}_1(t+\tau)\dots\hat{T}^{(p_1)}_1(t+\tau)\dots\hat{A}_n\hat{T}^{(1)}_n(t+\tau)\dots\hat{T}^{(p_n)}_n(t+\tau)\hat{A}_{n+1}\rangle] \\ &+ \sum_{m_1=1}^{n-1}\sum_{m_2=m_1+1}^{n}\sum_{k_1=1}^{p_{m_1}}\sum_{k_2=1}^{p_{m_2}}\mathcal{M}^{(m_1,k_1,m_2,k_2)}_{t+\tau}[\langle \hat{A}_1\hat{T}^{(1)}_1(t+\tau)\dots\hat{T}^{(p_1)}_1(t+\tau)\dots\hat{A}_n\hat{T}^{(1)}_n(t+\tau)\dots\hat{T}^{(p_n)}_n(t+\tau)\hat{A}_{n+1}\rangle] \\ &+ \sum_{m=1}^{n}\sum_{k_1=1}^{p_m-1}\sum_{k_2=k_1+1}^{p_m}\mathcal{M}^{(m,k_1,m,k_2)}_{t+\tau}[\langle \hat{A}_1\hat{T}^{(1)}_1(t+\tau)\dots\hat{T}^{(p_1)}_1(t+\tau)\dots\hat{A}_n\hat{T}^{(1)}_n(t+\tau)\dots\hat{T}^{(p_n)}_n(t+\tau)\hat{A}_{n+1}\rangle].\end{aligned} \tag{C7}$$

Representing $\hat{B}_m$ as $\hat{T}^{(m)}_1 \dots \hat{T}^{(m)}_{p_m}$ and using only Eq. (22), we come to the same equation. It proves the self-consistency of the adjoint master equation for multitime correlators.

[1] A. I. Larkin and Y. N. Ovchinnikov, Quasiclassical method in the theory of superconductivity, Sov. Phys. JETP **28**, 1200 (1969).

[2] S. H. Shenker and D. Stanford, Multiple shocks, J. High Energy Phys. 12 (2014) 046.

[3] S. H. Shenker and D. Stanford, Stringy effects in scrambling, J. High Energy Phys. 05 (2015) 132.

[4] D. J. Luitz and Y. Bar Lev, Information propagation in isolated quantum systems, Phys. Rev. B **96**, 020406(R) (2017).

[5] A. A. Patel, D. Chowdhury, S. Sachdev, and B. Swingle, Quantum butterfly effect in weakly interacting diffusive metals, Phys. Rev. X **7**, 031047 (2017).
[6] N. Yunger Halpern, Jarzynski-like equality for the out-of-time-ordered correlator, Phys. Rev. A **95**, 012120 (2017).
[7] S. V. Syzranov, A. V. Gorshkov, and V. Galitski, Out-of-time-order correlators in finite open systems, Phys. Rev. B **97**, 161114(R) (2018).
[8] M. Gärttner, P. Hauke, and A. M. Rey, Relating out-of-time-order correlations to entanglement via multiple-quantum coherences, Phys. Rev. Lett. **120**, 040402 (2018).
[9] A. Nahum, J. Ruhman, and D. A. Huse, Dynamics of entanglement and transport in one-dimensional systems with quenched randomness, Phys. Rev. B **98**, 035118 (2018).
[10] C.-J. Lin and O. I. Motrunich, Out-of-time-ordered correlators in a quantum Ising chain, Phys. Rev. B **97**, 144304 (2018).
[11] V. Khemani, A. Vishwanath, and D. A. Huse, Operator spreading and the emergence of dissipative hydrodynamics under unitary evolution with conservation laws, Phys. Rev. X **8**, 031057 (2018).
[12] T. Rakovszky, F. Pollmann, and C. W. von Keyserlingk, Diffusive hydrodynamics of out-of-time-ordered correlators with charge conservation, Phys. Rev. X **8**, 031058 (2018).
[13] N. Yunger Halpern, B. Swingle, and J. Dressel, Quasiprobability behind the out-of-time-ordered correlator, Phys. Rev. A **97**, 042105 (2018).
[14] N. Roy and A. Sharma, Entanglement entropy and out-of-time-order correlator in the long-range Aubry–André–Harper model, J. Phys.: Condens. Matter **33**, 334001 (2021).
[15] D. A. Roberts and D. Stanford, Diagnosing chaos using four-point functions in two-dimensional conformal field theory, Phys. Rev. Lett. **115**, 131603 (2015).
[16] J. Maldacena, S. H. Shenker, and D. Stanford, A bound on chaos, J. High Energy Phys. 08 (2016) 106.
[17] J. Maldacena and D. Stanford, Remarks on the Sachdev-Ye-Kitaev model, Phys. Rev. D **94**, 106002 (2016).
[18] D. Stanford, Many-body chaos at weak coupling, J. High Energy Phys. 10 (2016) 009.
[19] K. Hashimoto, K. Murata, and R. Yoshii, Out-of-time-order correlators in quantum mechanics, J. High Energy Phys. 10 (2017) 138.
[20] D. Chowdhury and B. Swingle, Onset of many-body chaos in the O(*N*) model, Phys. Rev. D **96**, 065005 (2017).
[21] N. Tsuji, T. Shitara, and M. Ueda, Out-of-time-order fluctuation-dissipation theorem, Phys. Rev. E **97**, 012101 (2018).
[22] J. Chávez-Carlos, B. López-del-Carpio, M. A. Bastarrachea-Magnani, P. Stránský, S. Lerma-Hernández, L. F. Santos, and J. G. Hirsch, Quantum and classical Lyapunov exponents in atom-field interaction systems, Phys. Rev. Lett. **122**, 024101 (2019).
[23] S. Pilatowsky-Cameo, J. Chávez-Carlos, M. A. Bastarrachea-Magnani, P. Stránský, S. Lerma-Hernández, L. F. Santos, and J. G. Hirsch, Positive quantum Lyapunov exponents in experimental systems with a regular classical limit, Phys. Rev. E **101**, 010202(R) (2020).
[24] E. M. Fortes, I. García-Mata, R. A. Jalabert, and D. A. Wisniacki, Signatures of quantum chaos transition in short spin chains, Europhys. Lett. **130**, 60001 (2020).
[25] I. García-Mata, R. A. Jalabert, and D. A. Wisniacki, Out-of-time-order correlations and quantum chaos, Scholarpedia **18**, 55237 (2023).
[26] M. Heyl, F. Pollmann, and B. Dóra, Detecting equilibrium and dynamical quantum phase transitions in ising chains via out-of-time-ordered correlators, Phys. Rev. Lett. **121**, 016801 (2018).
[27] R. de Mello Koch, J.-H. Huang, C.-T. Ma, and H. J. Van Zyl, Spectral form factor as an OTOC averaged over the heisenberg group, Phys. Lett. B **795**, 183 (2019).
[28] Z.-H. Sun, J.-Q. Cai, Q.-C. Tang, Y. Hu, and H. Fan, Out-of-time-order correlators and quantum phase transitions in the rabi and dicke models, Ann. Phys. (Leipzig) **532**, 1900270 (2020).
[29] J. Hu and S. Wan, Out-of-time-ordered correlation in the anisotropic Dicke model, Commun. Theor. Phys. **73**, 125703 (2021).
[30] A. V. Kirkova, D. Porras, and P. A. Ivanov, Out-of-time-order correlator in the quantum rabi model, Phys. Rev. A **105**, 032444 (2022).
[31] I. V. Panyukov, V. Y. Shishkov, and E. S. Andrianov, Second-order autocorrelation function of spectrally filtered light from an incoherently pumped two-level system, Ann. Phys. (Leipzig) **534**, 2100286 (2022).
[32] I. V. Panyukov, V. Y. Shishkov, and E. S. Andrianov, Controlling purity, indistinguishability, and quantum yield of an incoherently pumped two-level system by spectral filters, Phys. Rev. A **108**, 023711 (2023).
[33] S. Pappalardi and J. Kurchan, Quantum bounds on the generalized lyapunov exponents, Entropy **25**, 246 (2023).
[34] B. Swingle, G. Bentsen, M. Schleier-Smith, and P. Hayden, Measuring the scrambling of quantum information, Phys. Rev. A **94**, 040302(R) (2016).
[35] G. Zhu, M. Hafezi, and T. Grover, Measurement of many-body chaos using a quantum clock, Phys. Rev. A **94**, 062329 (2016).
[36] J. Li, R. Fan, H. Wang, B. Ye, B. Zeng, H. Zhai, X. Peng, and J. Du, Measuring out-of-time-order correlators on a nuclear magnetic resonance quantum simulator, Phys. Rev. X **7**, 031011 (2017).
[37] M. Gärttner, J. G. Bohnet, A. Safavi-Naini, M. L. Wall, J. J. Bollinger, and A. M. Rey, Measuring out-of-time-order correlations and multiple quantum spectra in a trapped-ion quantum magnet, Nat. Phys. **13**, 781 (2017).
[38] K. X. Wei, C. Ramanathan, and P. Cappellaro, Exploring localization in nuclear spin chains, Phys. Rev. Lett. **120**, 070501 (2018).
[39] K. A. Landsman, C. Figgatt, T. Schuster, N. M. Linke, B. Yoshida, N. Y. Yao, and C. Monroe, Verified quantum information scrambling, Nature (London) **567**, 61 (2019).
[40] M. Niknam, L. F. Santos, and D. G. Cory, Sensitivity of quantum information to environment perturbations measured with a nonlocal out-of-time-order correlation function, Phys. Rev. Res. **2**, 013200 (2020).
[41] C. M. Sánchez, A. K. Chattah, K. X. Wei, L. Buljubasich, P. Cappellaro, and H. M. Pastawski, Perturbation independent decay of the loschmidt echo in a many-body system, Phys. Rev. Lett. **124**, 030601 (2020).
[42] X. Nie, B.-B. Wei, X. Chen, Z. Zhang, X. Zhao, C. Qiu, Y. Tian, Y. Ji, T. Xin, D. Lu *et al.*, Experimental observation of equilibrium and dynamical quantum phase transitions via out-of-time-ordered correlators, Phys. Rev. Lett. **124**, 250601 (2020).

[43] X. Mi, P. Roushan, C. Quintana, S. Mandra, J. Marshall, C. Neill, F. Arute, K. Arya, J. Atalaya, R. Babbush *et al.*, Information scrambling in quantum circuits, Science **374**, 1479 (2021).

[44] J. Braumüller, A. H. Karamlou, Y. Yanay, B. Kannan, D. Kim, M. Kjaergaard, A. Melville, B. M. Niedzielski, Y. Sung, A. Vepsäläinen *et al.*, Probing quantum information propagation with out-of-time-ordered correlators, Nat. Phys. **18**, 172 (2022).

[45] H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press on Demand, 2002).

[46] S. Khan, B. K. Agarwalla, and S. Jain, Quantum regression theorem for multi-time correlators: A detailed analysis in the heisenberg picture, Phys. Rev. A **106**, 022214 (2022).

[47] P. D. Blocher and K. Mølmer, Quantum regression theorem for out-of-time-ordered correlation functions, Phys. Rev. A **99**, 033816 (2019).

[48] M. O. Scully and S. Zubairy, *Quantum Optics* (Cambridge University Press, Cambridge, England, 1997).

[49] A. Levy and R. Kosloff, The local approach to quantum transport may violate the second law of thermodynamics, Europhys. Lett. **107**, 20004 (2014).

[50] P. P. Hofer, M. Perarnau-Llobet, L. D. M. Miranda, G. Haack, R. Silva, J. B. Brask, and N. Brunner, Markovian master equations for quantum thermal machines: Local versus global approach, New J. Phys. **19**, 123037 (2017).

[51] J. O. González, L. A. Correa, G. Nocerino, J. P. Palao, D. Alonso, and G. Adesso, Testing the validity of the 'local' and 'global' GKLS master equations on an exactly solvable model, Open Syst. Inf. Dyn. **24**, 1740010 (2017).

[52] N. Shammah, S. Ahmed, N. Lambert, S. De Liberato, and F. Nori, Open quantum systems with local and collective incoherent processes: Efficient numerical simulations using permutational invariance, Phys. Rev. A **98**, 063815 (2018).

[53] G. De Chiara, G. Landi, A. Hewgill, B. Reid, A. Ferraro, A. J. Roncaglia, and M. Antezza, Reconciliation of quantum local master equations with thermodynamics, New J. Phys. **20**, 113024 (2018).

[54] V. Y. Shishkov, E. S. Andrianov, A. A. Pukhov, A. P. Vinogradov, and A. A. Lisyansky, Relaxation of interacting open quantum systems, Phys.-Usp. **62**, 510 (2019).

[55] M. Cattaneo, G. L. Giorgi, S. Maniscalco, and R. Zambrini, Local versus global master equation with common and separate baths: Superiority of the global approach in partial secular approximation, New J. Phys. **21**, 113045 (2019).

[56] I. V. Vovchenko, V. Y. Shishkov, A. A. Zyablovsky, and E. S. Andrianov, Model for the description of the relaxation of quantum-mechanical systems with closely spaced energy levels, JETP Lett. **114**, 51 (2021).

[57] V. Y. Shishkov, E. S. Andrianov, A. A. Pukhov, A. P. Vinogradov, and A. A. Lisyansky, Perturbation theory for lindblad superoperators for interacting open quantum systems, Phys. Rev. A **102**, 032207 (2020).