Non-Abelian phase and geometric force in a quantum-classical hybrid system

Yang Liu,[1] Y. N. Zhang,[2] X. X. Yi,[1] and H. D. Liu[1,*]
[1]Center for Quantum Sciences and School of Physics, Northeast Normal University, Changchun 130024, China
[2]School of Science, Shenyang University of Technology, Shenyang 110870, China

(Received 24 February 2024; revised 8 April 2024; accepted 17 April 2024; published 2 May 2024)

It is well known that the dynamics of a hybrid system, which consists of a quantum subsystem and a classical subsystem, can be studied by transforming the quantum subsystem into a classical one. Based on our previous work [Phys. Rev. A **102**, 032213 (2020)], we present the formalism of the quantum-classical hybrid system containing a quantum subsystem with energy degeneracy. By simulating the non-Abelian dynamics and geometric phases for the quantum subsystem with classical non-Abelian dynamics and geometric angles, the effect of the non-Abelian gauge field and the geometric phase of the quantum (sub)system on the dynamics of the classical subsystem is revealed by a simple canonical transformation. Remarkably, the non-Abelian angle geometric vector potential is found not only to induce the Wilczek-Zee phase in the dynamics of the quantum subsystem but also to give rise to a Lorentz-like non-Abelian geometric force. To illustrate the dynamics of the hybrid system, we examine the interaction between a spin-half particle and a magnetic field coupled with a magnetic particle. The explicit expression of the non-Abelian magneticlike gauge field is presented, introducing a different form of non-Abelian geometric force in the realm of quantum-classical dynamics.

DOI: 10.1103/PhysRevA.109.052203

I. INTRODUCTION

In the study of the adiabatic evolution of general quantum systems, the Berry phase has stood for several decades as a profound concept in quantum mechanics which depends on the geometry of the parameter space of the Hamiltonian since its notion was introduced by Berry [1–3]. With the generalization of the definition of the geometric phase [4–9], the investigation of this phenomenon has gained increasing prominence across diverse fields, including the quantum Hall effect [10], molecular dynamics [11], linear-response theory [12], adiabatic passage [13,14], and adiabatic shortcut techniques [15], but not limited to branches of physics [16–24]. Furthermore, there is a mechanics analog of the Berry phase in classical integrable systems called Hannay's angle which is the additional angle shift acquired over the evolution of a system undergoing slow changes of the parameters [6,25]. It is already known that the Berry phase possesses a geometric connection with Hannay's angle under a semiclassical approximation [26]. Consequently, considerable efforts have been directed toward unraveling the implications for quantum-classical correspondence.

The Hilbert space of quantum systems has been demonstrated to share the same mathematical structure with the phase space of canonical classical Hamiltonians [27,28]. This revelation allows for some fundamental properties of quantum mechanics such as wave functions and observable symmetries to be seamlessly mapped into a classical system without losing any physics [29–36]. As a result, a set of applications simulating quantum dynamics with classical oscillators has arisen [37–40], since the analogy between the time-dependent Schrödinger equation and the classical Hamilton equation was pointed out by Dirac. Moreover, the quantum-classical mapping (QCM) does not rest upon nondegenerate quantum (sub)systems [41]. Since there is an absence of energy gaps in quantum degenerate systems, there will be a nonadiabatic evolution in the degenerate subspace [42] giving rise to an effective non-Abelian gauge field [43] and a non-Abelian phase called the Wilczek-Zee (WZ) phase [4]. After the definition of the WZ phase was clarified, it has garnered increasing attention in the domains of quantum computation and quantum control [44–50], and the simulation of quantum systems with energy degeneracy using classical resonant oscillators has also been achieved [41,51].

Previous studies primarily focused on nondegenerate quantum-classical hybrid systems [52–65] or the simulation of the dynamics of the quantum degenerate systems [42,45–50]. Other insights have pointed out that both subsystems of a hybrid system can be treated classically [3,66,67], and they influence each other not only through subsystem-subsystem coupling interactions but also via a vector potential [68,69]. For quantum degenerate systems, the WZ gauge potential can be put into the framework of classical mechanics by summing its mean value of each quantum subspace which is just the geometric angle induced by classical non-Abelian evolution [41,51]. In this context, we are motivated to address some questions: How can we describe the interaction between subsystems of a hybrid system with degeneracy in the quantum subsystem? What is the role of the non-Abelian gauge potential in affecting the dynamics of the classical subsystem? These questions serve as the focal point of our investigation.

This paper is organized as follows. In Sec. II B, we present the dynamical evolution and WZ phases of the quantum

*liuhd100@nenu.edu.cn

degenerate subsystem of a hybrid system in the framework of classical resonant oscillators based on QCM. In Sec. II C, we briefly outline the theoretical framework of the generation of the non-Abelian geometric force in the classical dynamics of a hybrid system containing a quantum degenerate subsystem. Then, the general form of the non-Abelian geometric force is provided. To illustrate our result above, we study a cantilever-spin system in a single spin detection experiment [68,70,71], where the classical subsystem is a magnetic particle and the quantum subsystem is a spin-half particle in a magnetic field replacing the Pauli matrices by the Dirac matrices as an example in Sec. III. Finally, we conclude our results in Sec. IV.

II. NON-ABELIAN ADIABATIC DYNAMICS IN A HYBRID SYSTEM

According to the Born-Oppenheimer approximation (BOA), the Hamiltonian of a hybrid coupled system, which consists of a fast quantum subsystem with energy degeneracy and a slow classical subsystem, can be written as [34]

$$H_{\text{hybrid}} = \langle\psi|\hat{H}_1(\boldsymbol{Q}|\psi\rangle + H_2(\boldsymbol{P}, \boldsymbol{Q}), \tag{1}$$

where $|\psi\rangle$ is the state of the quantum subsystem, and $\boldsymbol{P}$ is the momentum of the slow classical subsystem. The coupling of the two subsystems is indicated by the dependency of the Hamiltonian of the fast quantum subsystem $\hat{H}_1(\boldsymbol{Q})$ on $\boldsymbol{Q}$, which is the coordinate of the slow classical subsystem.

A. Dynamics and Non-Abelian phase in a quantum subsystem

First, we focus on the adiabatic dynamics of a fast non-Abelian quantum subsystem governed by $\hat{H}_1(\boldsymbol{Q})$, with the adiabatic parameters $\boldsymbol{Q}$. The instantaneous spectrum is determined by the instantaneous eigenequation

$$\hat{H}_1(\boldsymbol{Q})|E_{ka}(\boldsymbol{Q})\rangle = E_k(\boldsymbol{Q})|E_{ka}(\boldsymbol{Q})\rangle, \tag{2}$$

where $E_k(\boldsymbol{Q})$ and $|E_{ka}(\boldsymbol{Q})\rangle$ are the degenerated eigenenergy and energy eigenstate, respectively. $a = 1, 2, \ldots, N$ with the degree of degeneracy N. For an initial state

$$|\varphi_k(\boldsymbol{Q}_0)\rangle = \sum_a \varphi_{ka}(\boldsymbol{Q}_0)|E_{ka}(\boldsymbol{Q}_0)\rangle, \tag{3}$$

the states driven by the Schrödinger equation would be

$$|\varphi_k(\boldsymbol{Q})\rangle = \sum_a \varphi_{ka}(\boldsymbol{Q})e^{-\frac{i}{\hbar}\int_0^t E_k(\boldsymbol{Q})d\tau} \sum_{b=1}^{n} U_{ab}^k|E_{kb}(\boldsymbol{Q})\rangle, \tag{4}$$

where $U_{ab}^k = \mathcal{P}e^{i\int_c A_{ab}^k(\boldsymbol{Q})\cdot dX}$ is the geometric WZ phase factor given by the path-ordered integral, and the geometric WZ potential which reads [4,27]

$$\boldsymbol{A}_{ab}^k(\boldsymbol{Q}) = i\langle E_{ka}(\boldsymbol{Q})|\partial_{\mathbf{X}} E_{kb}(\boldsymbol{Q})\rangle \tag{5}$$

is the matrix element of the non-Abelian gauge potential $\boldsymbol{A}^k$. As the slowly varying parameters $\boldsymbol{Q} = (Q_1, Q_2, Q_3)$ are the coordinates of the slow classical system, the gauge potential $\boldsymbol{A}^k(\boldsymbol{Q})$ is related to a curvature $F_{\alpha\beta}$ or an effective "magnetic" field $\boldsymbol{B}$ as [4,27,41,43]

$$B_\gamma^k = \tfrac{1}{2}\varepsilon_{\alpha\beta\gamma}F_{\alpha\beta}^k,$$
$$F_{\alpha\beta}^k = \partial_{Q_\alpha}A_\beta^k - \partial_{Q_\beta}A_\alpha^k - \frac{i}{\hbar}\left[A_\alpha^k, A_\beta^k\right] \quad (\alpha, \beta, \gamma = 1, 2, 3), \tag{6}$$

where the non-Abelian term $-\frac{i}{\hbar}[A_\alpha^k, A_\beta^k]$ emerges since the matrices A_α^k do not commute with each other.

B. QCM of Non-Abelian adiabatic dynamics and angle in a hybrid system

To study the dynamics of the whole hybrid system, we next map this quantum non-Abelian dynamics of the quantum subsystem into a classical one [41]. The dynamical evolution of the fast quantum subsystem governed by $\hat{H}_1(\boldsymbol{Q})$ on a bare basis $\{|n\rangle\}$ can be described by the Schrödinger equation

$$i\hbar\frac{dc_n}{dt} = \frac{\partial H_1}{\partial c_n}, \tag{7}$$

with $|\psi\rangle = \sum_n c_n|n\rangle$ and $H_1(\boldsymbol{p}, \boldsymbol{q}; \boldsymbol{Q}) = \langle\psi|\hat{H}_1|\psi\rangle$. By decomposing the probability amplitudes c_n into real and imaginary parts $c_n = (q_n + ip_n)/\sqrt{2\hbar}$, the Schrödinger equation can be rewritten as

$$\dot{q}_n = \frac{\partial H_1}{\partial p_n}, \quad \dot{p}_n = -\frac{\partial H_1}{\partial q_n}, \tag{8}$$

where q_n and p_n can be regarded as the "position variable" and "momentum variable," respectively.

On the other hand, the quantum states $|\psi\rangle$ can also be expanded in terms of instantaneous eigenstates

$$|\psi\rangle = \sum_{ka} \varphi_{ka}|E_{ka}(\boldsymbol{Q})\rangle, \tag{9}$$

where $\{|E_{ka}(\boldsymbol{Q})\rangle\}$ is determined by Eq. (2). Then, one can introduce the angle-action variables $(\boldsymbol{\theta}, \boldsymbol{I})$ by $|\varphi_{ka}\rangle = \sqrt{I_{ka}/\hbar}e^{-i\theta_{ka}}$ [32,68,72], and they satisfy the similar structure of canonical equations

$$\dot{\theta}_{ka} = \frac{\partial\mathcal{H}_1}{\partial I_{ka}} + \frac{\partial}{\partial I_{ka}}\left[\frac{\partial S(\boldsymbol{q}, \boldsymbol{I}; \boldsymbol{Q})}{\partial\boldsymbol{Q}}\cdot\dot{\boldsymbol{Q}}\right],$$
$$\dot{I}_{ka} = -\frac{\partial}{\partial\theta_{ka}}\left[\frac{\partial S(\boldsymbol{q}, \boldsymbol{I}; \boldsymbol{Q})}{\partial\boldsymbol{Q}}\cdot\dot{\boldsymbol{Q}}\right], \tag{10}$$

where the mean value of energy $\mathcal{H}_1(\boldsymbol{I}; \boldsymbol{Q}) = \sum_{ka} E_{ka}(\boldsymbol{Q})I_{ka}/\hbar$ and S is a generating function induced by the canonical transformation $(\boldsymbol{p}, \boldsymbol{q}) \rightarrow (\boldsymbol{\theta}, \boldsymbol{I})$.

The unitary transformation connecting the two bases $\{|n\rangle\}$ and $\{|E_{ka}(\boldsymbol{Q}, \boldsymbol{X}_1)\rangle\}$,

$$\boldsymbol{c} = \boldsymbol{U}\boldsymbol{\varphi}, \tag{11}$$

corresponds to the classical canonical transformation $(\boldsymbol{p}, \boldsymbol{q}) \rightarrow (\boldsymbol{\theta}, \boldsymbol{I})$,

$$p_n(t) = \sum_{ka}\sqrt{2I_{ka}}\{\cos\theta_{ka}\text{Im}[U_{n,ka}(t)] - \sin\theta_{ka}\text{Re}[U_{n,ka}(t)]\},$$
$$q_n(t) = \sum_{ka}\sqrt{2I_{ka}}\{\cos\theta_{ka}\text{Re}[U_{n,ka}(t)] + \sin\theta_{ka}\text{Im}[U_{n,ka}(t)]\}. \tag{12}$$

Therefore, the Hamiltonian in the adiabatic basis $\{|E_{ka}(\boldsymbol{Q})\rangle\}$ can be obtained as $\hat{H}_{1a} = \hat{U}^\dagger \hat{H}_1 \hat{U} - i\hbar \hat{U}^\dagger \dot{\hat{U}}$ with $|E_{ka}(\boldsymbol{Q})\rangle = \hat{\boldsymbol{U}}|n\rangle$. Correspondingly, under the canonical transformation Eq. (12), the Hamiltonian $H_1(\boldsymbol{p}, \boldsymbol{q}; \boldsymbol{Q})$ becomes $\mathcal{H}_1(\boldsymbol{I}; \boldsymbol{Q})$ but differs from the old one by

$$H_1(\boldsymbol{p}, \boldsymbol{q}; \boldsymbol{Q}) = \bar{H}_1(\boldsymbol{\theta}, \boldsymbol{I}; \boldsymbol{Q}) = \mathcal{H}_1(\boldsymbol{I}; \boldsymbol{Q}) + \frac{\partial S_1(\boldsymbol{q}, \boldsymbol{I}; \boldsymbol{Q})}{\partial \boldsymbol{t}}, \tag{13}$$

where the partial differential of generating function S_1 can be written as

$$\frac{\partial S_1(\boldsymbol{q}, \boldsymbol{I}; \boldsymbol{Q})}{\partial t} = \left[\frac{\partial \mathscr{S}_1}{\partial \boldsymbol{Q}} - \boldsymbol{p}(\boldsymbol{\theta}, \boldsymbol{I}; \boldsymbol{Q}) \frac{\partial \boldsymbol{q}(\boldsymbol{\theta}, \boldsymbol{I}; \boldsymbol{Q})}{\partial \boldsymbol{Q}}\right] \cdot \dot{\boldsymbol{Q}}. \tag{14}$$

By a cyclic evolution of $\boldsymbol{\theta}$, the single-valued function $\mathscr{S}_1$ will be trivial gradient vanished for zero contribution to the equations of motion.

By taking the averaging principle for the non-Abelian classical dynamics by averaging

$$\langle \cdots \rangle_\theta = \begin{cases} \cdots & (k = k'), \\ \oint d\theta & (k \neq k'), \end{cases} \tag{15}$$

the average over $\boldsymbol{\theta}$ of the new Hamiltonian of the total system after the canonical transformation $(\boldsymbol{p}, \boldsymbol{q}) \to (\boldsymbol{\theta}, \boldsymbol{I})$ can be written as

$$\mathscr{H} = \mathcal{H}_1(\boldsymbol{I}; \boldsymbol{Q}) + H_2(\boldsymbol{P}, \boldsymbol{Q}) - \left\langle \boldsymbol{p}(\boldsymbol{\theta}, \boldsymbol{I}; \boldsymbol{Q}) \frac{\partial \boldsymbol{q}(\boldsymbol{\theta}, \boldsymbol{I}; \boldsymbol{Q})}{\partial \boldsymbol{Q}} \right\rangle_{\boldsymbol{\theta}} \cdot \dot{\boldsymbol{Q}}, \tag{16}$$

where the first two terms are the effective Hamiltonian, and the latter is the classical non-Abelian angle gauge potential

$$\boldsymbol{\mathcal{A}}(\boldsymbol{\theta}, \boldsymbol{I}; \boldsymbol{Q}) = \left\langle \boldsymbol{p}(\boldsymbol{\theta}, \boldsymbol{I}; \boldsymbol{Q}) \frac{\partial \boldsymbol{q}(\boldsymbol{\theta}, \boldsymbol{I}; \boldsymbol{Q}}{\partial \boldsymbol{Q}} \right\rangle_{\boldsymbol{\theta}} \tag{17}$$

which is just the one-form for the sum of the mean value of the WZ potential $A^k_{ab} = i\langle E_{ka}|\partial E_{kb}\rangle$ in each subspace

$$\boldsymbol{\mathcal{A}}(\boldsymbol{\theta}, \boldsymbol{I}; \boldsymbol{Q}) = \sum_{kab} \varphi^*_{ka}(\boldsymbol{\theta}, \boldsymbol{I}) A^k_{ab}(\boldsymbol{Q}) \varphi_{kb}(\boldsymbol{\theta}, \boldsymbol{I}) = \sum_k \boldsymbol{\varphi}^\dagger_{\boldsymbol{k}}(\boldsymbol{\theta}, \boldsymbol{I}) \boldsymbol{A}^k(\boldsymbol{X}) \boldsymbol{\varphi}_{\boldsymbol{k}}(\boldsymbol{\theta}, \boldsymbol{I}). \tag{18}$$

The WZ phase in the quantum subsystem can then be represented by a classical non-Abelian geometric angle [41]

$$\Delta\theta_{ka} = -\frac{\partial}{\partial I_{ka}} \oint \boldsymbol{\mathcal{A}}(\boldsymbol{\theta}, \boldsymbol{I}; \boldsymbol{Q}) \cdot d\boldsymbol{Q}. \tag{19}$$

It is worth noticing that the angle gauge potential $\mathcal{A}$ is no longer a linear function of action variables $\boldsymbol{I}$ and contains angle variables $\boldsymbol{\theta}$. Therefore, the Poisson brackets between the component matrices $\mathcal{A}$ in the parameter space are

$$\{\mathcal{A}_\alpha, \mathcal{A}_\beta\} \equiv \sum_{ka} \frac{\partial \mathcal{A}_\alpha}{\partial \theta_{ka}} \frac{\partial \mathcal{A}_\beta}{\partial I_{ka}} - \frac{\partial \mathcal{A}_\beta}{\partial \theta_{ka}} \frac{\partial \mathcal{A}_\alpha}{\partial I_{ka}} \neq 0. \tag{20}$$

The classical non-Abelian curvature can be defined as

$$W_{\alpha\beta} = \partial_{Q_\alpha} \mathcal{A}_\beta - \partial_{Q_\beta} \mathcal{A}_\alpha + \{\mathcal{A}_\alpha, \mathcal{A}_\beta\} = \sum_k \boldsymbol{\varphi}^\dagger_{\boldsymbol{k}}(\boldsymbol{\theta}, \boldsymbol{I}) \boldsymbol{F}^k(\boldsymbol{X}) \boldsymbol{\varphi}_{\boldsymbol{k}}(\boldsymbol{\theta}, \boldsymbol{I}), \tag{21}$$

which corresponds to the mean value of $F^k_{\alpha\beta}$ for the quantum degenerate system.

C. Geometric forces in a non-Abelian hybrid system

For hybrid systems in which the Hamiltonian of the classical subsystem is $H_2 = \boldsymbol{P}^2/2M + V_2(\boldsymbol{Q})$, the quantum subsystem will provide an additional vector potential in momenta $\boldsymbol{P}$ [68] by introducing a canonical transformation from $\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{P}, \boldsymbol{Q}$ to $\boldsymbol{I}, \boldsymbol{\theta}, \boldsymbol{\mathcal{P}}, \boldsymbol{Q}$,

$$\boldsymbol{p} = \frac{\partial S}{\partial \boldsymbol{q}}, \quad \boldsymbol{\theta} = \frac{\partial S}{\partial \boldsymbol{I}}, \quad \boldsymbol{P} = \frac{\partial S}{\partial \boldsymbol{Q}} = \frac{\partial \boldsymbol{S}}{\partial \boldsymbol{Q}} + \boldsymbol{\mathcal{P}}, \quad \boldsymbol{Q} = \frac{\partial F}{\partial \boldsymbol{\mathcal{P}}} = \boldsymbol{Q}, \tag{22}$$

with a generating function

$$S = S_1(\boldsymbol{q}, \boldsymbol{I}, \boldsymbol{Q}) + \boldsymbol{q} \cdot \boldsymbol{P}. \tag{23}$$

After the averaging principle, the equations of motion for canonical variables $\boldsymbol{I}, \boldsymbol{\theta}$ can be written as [41,62,68]

$$\dot{\boldsymbol{I}} = \dot{\boldsymbol{Q}} \cdot \frac{\partial \bar{A}}{\partial \boldsymbol{\theta}} = 0, \quad \dot{\boldsymbol{\theta}} = \frac{\partial \mathcal{H}_1}{\partial \boldsymbol{I}} - \dot{\boldsymbol{Q}} \cdot \frac{\partial \bar{A}}{\partial \boldsymbol{I}}, \tag{24}$$

and the classical canonical variables $\boldsymbol{P}, \boldsymbol{Q}$ can be given as

$$\dot{\boldsymbol{\mathcal{P}}} = -\frac{\partial \mathcal{H}_1}{\partial \boldsymbol{Q}} - \frac{\partial V_2}{\partial \boldsymbol{Q}} + \dot{\boldsymbol{Q}} \cdot \frac{\partial \bar{\boldsymbol{\mathcal{A}}}}{\partial \boldsymbol{Q}}, \quad \dot{\boldsymbol{Q}} = \dot{\boldsymbol{Q}} = \frac{\boldsymbol{\mathcal{P}} - \bar{\boldsymbol{\mathcal{A}}}}{M}. \tag{25}$$

One then can write a more explicit form for the equation of coordinate $\boldsymbol{Q}$ as

$$m\ddot{\boldsymbol{Q}} = -\frac{\partial \mathcal{H}_1}{\partial \boldsymbol{Q}} - \frac{\partial V_2}{\partial \boldsymbol{Q}} + \dot{\boldsymbol{Q}} \times \boldsymbol{\mathcal{B}}, \tag{26}$$

where the extra geometric term is brought by the effective non-Abelian magnetic field $\boldsymbol{\mathcal{B}}$ with

$$\mathcal{B}_\gamma = \tfrac{1}{2} \varepsilon_{\alpha\beta\gamma} W_{\alpha\beta} \quad (\alpha, \beta, \gamma = 1, 2, 3). \tag{27}$$

III. EXAMPLE

To illustrate the formalism explicitly, we consider a simple hybrid system where the classical subsystem is a magnetic particle with magnetic moment m_F pointing in the negative direction of the z axis moves freely in the xy plane, and the quantum subsystem is a four-level system with magnetic moment μ placed below the plane at distance d as shown in Fig. 1 [68,70]. The total Hamiltonian reads

$$H = \langle \psi | \hat{H}_1 | \psi \rangle + H_2, \tag{28}$$

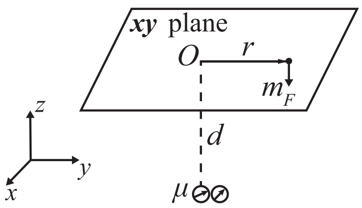

FIG. 1. Sketch of the hybrid system. The origin of the coordinate system is placed in the xy plane above the quantum subsystem. As the classical magnetic particle moves freely in the xy plane, the two interacting spin-half particles placed beneath the plane with a distance of d feel the magnetic field from the classical magnetic particle.

and the Hamiltonian operator for the quantum subsystem with twofold degeneracy is [42,73,74]

$$\hat{H}_1 = -\mu \boldsymbol{\gamma} \cdot \boldsymbol{B}, \tag{29}$$

where the Dirac matrices

$$\gamma_k = \sigma_1 \otimes \sigma_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}, \tag{30}$$

with the Pauli matrices σ_k. This Hamiltonian may describe two interacting spin-half particles since $\gamma_k = \sigma_1 \otimes \sigma_k$ or a single four-level particle in the magnetic field $\{B_x, B_y, B_z\} = \frac{\mu_0 m_F \{3xd, 3yd, 2d^2 - r^2\}}{4\pi (d^2+r^2)^{5/2}}$ with $r^2 = x^2 + y^2$ which is provided by the coordinates of the classical subsystem because of the magnetic dipolar interaction [42,68], whose Hamiltonian is

$$H_2 = \frac{P^2}{2m} + V_2(\boldsymbol{r}), \tag{31}$$

with $\boldsymbol{r} = \{x, y\}$ and the mass of the particle m. Therefore, the coordinates x and y act as the adiabatic parameters for the quantum subsystem. For the quantum subsystem, the Hamiltonian $\hat{H}_1$ has two eigenvalues $E_\pm = \pm \mu B$, with two degenerate eigenstates each,

$$|E_{-1}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{B_x - iB_y}{B} \\ -\frac{B_z}{B} \\ 0 \\ 1 \end{pmatrix}, \quad |E_{-2}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{B_z}{B} \\ \frac{B_x + iB_y}{B} \\ 1 \\ 0 \end{pmatrix},$$
$$|E_{+1}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} -\frac{B_x - iB_y}{B} \\ \frac{B_z}{B} \\ 0 \\ 1 \end{pmatrix}, \quad |E_{+2}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} -\frac{B_z}{B} \\ -\frac{B_x + iB_y}{B} \\ 1 \\ 0 \end{pmatrix}, \tag{32}$$

where $B = \sqrt{B_x^2 + B_y^2 + B_z^2} = \frac{\mu_0 m_F (4d^2 + r^2)}{4\pi (d^2 + r^2)^{3/2}}$.

The geometric WZ gauge potential (18) can be written as [4]

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}^- & 0 \\ 0 & \mathbf{A}^+ \end{pmatrix}, \tag{33}$$

where

$$\mathbf{A}^- = \mathbf{A}^+ = \begin{pmatrix} \frac{-B_y \dot{B}_x + B_x \dot{B}_y}{2B^2} & \frac{-iB_z \dot{B}_x + B_z \dot{B}_y + (iB_x - B_y)\dot{B}_z}{2B^2} \\ \frac{iB_z \dot{B}_x + B_z \dot{B}_y - (iB_x + B_y)\dot{B}_z}{2B^2} & \frac{B_y \dot{B}_x - B_x \dot{B}_y}{2B^2} \end{pmatrix} \tag{34}$$

are non-Abelian vector potentials in the two degenerate subspaces.

Next, we transform this quantum adiabatic dynamics into a classical one. The equivalent classical Hamiltonian function H_1 can be given as the average of Eq. (29) in the bare basis $\{|n\rangle\}$, i.e.,

$$\begin{aligned} H_1 = &-\frac{\mu}{\hbar}[(q_1 q_4 + p_1 p_4 + q_2 q_3 + p_2 p_3)B_x. \\ &+ (q_1 p_4 - p_1 q_4 + q_3 p_2 - p_3 q_2)B_y \\ &+ (q_1 q_3 + p_1 p_3 - q_2 q_4 - p_2 p_4)B_z]. \end{aligned} \tag{35}$$

The canonical transformation $(\boldsymbol{p}, \boldsymbol{q}) \to (\boldsymbol{\theta}, \boldsymbol{I})$ corresponding to the unitary transformation $\{|n\rangle\} \to \{|E_k\rangle\}$ can be given by Eq. (12),

$$\begin{aligned} p_1 =& -\sqrt{I_{-1}}\left(\frac{B_y}{B}\cos\theta_{-1} + \frac{B_x}{B}\sin\theta_{-1}\right) - \sqrt{I_{-2}}\left(\frac{B_z}{B}\sin\theta_{-2}\right) \\ &+ \sqrt{I_{+1}}\left(\frac{B_y}{B}\cos\theta_{+1} + \frac{B_x}{B}\sin\theta_{+1}\right) + \sqrt{I_{+2}}\left(\frac{B_z}{B}\sin\theta_{+2}\right), \\ q_1 =& \sqrt{I_{-1}}\left(\frac{B_x}{B}\cos\theta_{-1} - \frac{B_y}{B}\sin\theta_{-1}\right) + \sqrt{I_{-2}}\left(\frac{B_z}{B}\cos\theta_{-2}\right) \\ &- \sqrt{I_{+1}}\left(\frac{B_x}{B}\cos\theta_{+1} - \frac{B_y}{B}\sin\theta_{+1}\right) - \sqrt{I_{+2}}\left(\frac{B_z}{B}\cos\theta_{+2}\right), \\ p_2 =& \sqrt{I_{-1}}\left(\frac{B_z}{B}\sin\theta_{-1}\right) + \sqrt{I_{-2}}\left(\frac{B_y}{B}\cos\theta_{-2} - \frac{B_x}{B}\sin\theta_{-2}\right) \\ &- \sqrt{I_{+1}}\left(\frac{B_z}{B}\sin\theta_{+1}\right) - \sqrt{I_{+2}}\left(\frac{B_y}{B}\cos\theta_{+2} - \frac{B_x}{B}\sin\theta_{+2}\right), \\ q_2 =& -\sqrt{I_{-1}}\left(\frac{B_z}{B}\cos\theta_{-1}\right) + \sqrt{I_{-2}}\left(\frac{B_x}{B}\cos\theta_{-2} + \frac{B_y}{B}\sin\theta_{-2}\right) \\ &+ \sqrt{I_{+1}}\left(\frac{B_z}{B}\cos\theta_{+1}\right) - \sqrt{I_{+2}}\left(\frac{B_x}{B}\cos\theta_{+2} + \frac{B_y}{B}\sin\theta_{+2}\right), \\ p_3 =& -\sqrt{I_{-2}}\sin\theta_{-2} - \sqrt{I_{+2}}\sin\theta_{+2}, \\ q_3 =& \sqrt{I_{-2}}\cos\theta_{-2} + \sqrt{I_{+2}}\cos\theta_{+2}, \\ p_4 =& -\sqrt{I_{-1}}\sin\theta_{-1} - \sqrt{I_{+1}}\sin\theta_{+1}, \\ q_4 =& \sqrt{I_{-1}}\cos\theta_{-1} + \sqrt{I_{+1}}\cos\theta_{+1}. \end{aligned} \tag{36}$$

The mean value of the Hamiltonian after this canonical transformation can be obtained as

$$H_1 = \mathcal{H}_1 + \boldsymbol{\mathcal{A}} \cdot \dot{\boldsymbol{Q}}, \tag{37}$$

with

$$\mathcal{H}_1 = \frac{\mu B}{\hbar}(-I_{-1} - I_{-2} + I_{+1} + I_{+2}). \tag{38}$$

By the averaging principle (15), the non-Abelian angle gauge potential (18) can be written as

$$\mathcal{A} = \frac{3d}{2(d^2+r^2)(4d^2+r^2)}\{-3dy(I_{-1}-I_{-2}+I_{+1}-I_{+2})+2(2d^2+x^2-y^2)[\sqrt{I_{-1}I_{-2}}\sin\Delta\theta_- + \sqrt{I_{+1}I_{+2}}\sin\Delta\theta_+] + 4xy[\sqrt{I_{-1}I_{-2}}\cos\Delta\theta_- + \sqrt{I_{+1}I_{+2}}\cos\Delta\theta_+]\}\cdot\hat{e}_x + \frac{3d}{2(d^2+r^2)(4d^2+r^2)}\{3dx(I_{-1}-I_{-2}+I_{+1}-I_{+2}) + 2(2d^2-x^2+y^2)[\sqrt{I_{-1}I_{-2}}\cos\Delta\theta_- + \sqrt{I_{+1}I_{+2}}\cos\Delta\theta_+] + 4xy[\sqrt{I_{-1}I_{-2}}\sin\Delta\theta_- + \sqrt{I_{+1}I_{+2}}\sin\Delta\theta_+]\}\cdot\hat{e}_y, \tag{39}$$

where $\Delta\theta_- = \theta_{-1}-\theta_{-2}$ and $\Delta\theta_+ = \theta_{+1}-\theta_{+2}$. The dynamics of the action and angle variables of the degenerate quantum subsystem can be calculated by Eq. (24) straightforwardly,

$$\dot{I}_{\pm1} = -\dot{I}_{\pm2} = \frac{3d\sqrt{I_{\pm1}I_{\pm2}}}{(d^2+r^2)(4d^2+r^2)}(-G_{\pm1}\dot{x}+G_{\pm2}\dot{y}), \tag{40}$$

$$\begin{aligned}\dot{\theta}_{\pm1} &= -\frac{\partial\mathcal{A}}{\partial I_{\pm1}} + \frac{\partial\mathcal{H}_1}{\partial I_{\pm1}}\\ &= -\frac{3d}{2(d^2+r^2)(4d^2+r^2)}\left[\left(-3dy+K_{\pm1}\sqrt{\frac{I_{\pm2}}{I_{\pm1}}}\right)\dot{x} + \left(3dx+K_{\pm2}\sqrt{\frac{I_{\pm2}}{I_{\pm1}}}\right)\dot{y}\right] \pm \frac{\mu\mu_0 m_F\sqrt{4d^2+r^2}}{4\pi\hbar(d^2+r^2)^2},\\ \dot{\theta}_{\pm2} &= -\frac{\partial\mathcal{A}}{\partial I_{\pm2}} + \frac{\partial\mathcal{H}_1}{\partial I_{\pm2}}\\ &= -\frac{3d}{2(d^2+r^2)(4d^2+r^2)}\left[\left(3dy+K_{\pm1}\sqrt{\frac{I_{\pm1}}{I_{\pm2}}}\right)\dot{x} + \left(-3dx+K_{\pm2}\sqrt{\frac{I_{\pm1}}{I_{\pm2}}}\right)\dot{y}\right] \pm \frac{\mu\mu_0 m_F\sqrt{4d^2+r^2}}{4\pi\hbar(d^2+r^2)^2},\end{aligned} \tag{41}$$

with $G_{\pm1} \equiv 2xy\sin(\theta_{\pm1}-\theta_{\pm2}) - (2d^2+x^2-y^2)\cos(\theta_{\pm1}-\theta_{\pm2})$, $G_{\pm2} \equiv 2xy\cos(\theta_{\pm1}-\theta_{\pm2}) - (2d^2-x^2+y^2)\sin(\theta_{\pm1}-\theta_{\pm2})$, $K_{\pm1} \equiv 2xy\cos(\theta_{\pm1}-\theta_{\pm2}) + (2d^2+x^2-y^2)\sin(\theta_{\pm1}-\theta_{\pm2})$, and $K_{\pm2} \equiv 2xy\sin(\theta_{\pm1}-\theta_{\pm2}) + (2d^2-x^2+y^2)\cos(\theta_{\pm1}-\theta_{\pm2})$. Notably, the sums of action variables $I_{-1}+I_{-2}$ and $I_{+1}+I_{+2}$ remain conserved, while the individual action variables do not. The non-Abelian geometric angles can then be derived by solving

$$\begin{aligned}\Delta\theta_{\pm1} &= \int -\frac{3d}{2(d^2+r^2)(4d^2+r^2)}\left[\left(-3dy+K_{\pm1}\sqrt{\frac{I_{\pm2}}{I_{\pm1}}}\right)dx + \left(3dx+K_{\pm2}\sqrt{\frac{I_{\pm2}}{I_{\pm1}}}\right)dy\right],\\ \Delta\theta_{\pm2} &= \int -\frac{3d}{2(d^2+r^2)(4d^2+r^2)}\left[\left(3dy+K_{\pm1}\sqrt{\frac{I_{\mp1}}{I_{\pm2}}}\right)dx + \left(-3dx+K_{\pm2}\sqrt{\frac{I_{\pm1}}{I_{\pm2}}}\right)dy\right].\end{aligned} \tag{42}$$

To illustrate the non-Abelian adiabatic evolution of the quantum subsystem, we choose the parameters as

$$x = r\cos\left(\pi - \pi\sin\frac{\pi t}{\tau}\right), \quad y = r\sin\left(\pi - \pi\sin\frac{\pi t}{\tau}\right). \tag{43}$$

As shown in Figs. 2(a) and 2(b), the action variables $I_{\pm i}$ ($i = 1, 2$) of each quantum subsystem no longer remain invariant during adiabatic evolution. However, the respective sums of the action variables, $I_- = I_{-1}+I_{-2}$ and $I_+ = I_{+1}+I_{+2}$, are conserved under adiabatic evolution. We also analytically solving the Schrödinger equation driven by the Hamiltonian in the adiabatic basis

$$\hat{H}_{1a} = \hat{U}^\dagger\hat{H}_1\hat{U} - i\hbar\hat{U}^\dagger\dot{\hat{U}} = \mu\begin{pmatrix}-B & 0 & 0 & 0\\ 0 & -B & 0 & 0\\ 0 & 0 & B & 0\\ 0 & 0 & 0 & B\end{pmatrix} + \hbar\begin{pmatrix}\mathbf{M} & -\mathbf{M}\\ -\mathbf{M} & \mathbf{M}\end{pmatrix}, \tag{44}$$

where

$$\mathbf{M} = \begin{pmatrix}\frac{B_y\dot{B}_x - B_x\dot{B}_y}{2B^2} & \frac{iB_z\dot{B}_x - B_z\dot{B}_y - (iB_x-B_y)\dot{B}_z}{2B^2}\\ \frac{-iB_z\dot{B}_x - B_z\dot{B}_y + (iB_x+B_y)\dot{B}_z}{2B^2} & \frac{-B_y\dot{B}_x + B_x\dot{B}_y}{2B^2}\end{pmatrix}. \tag{45}$$

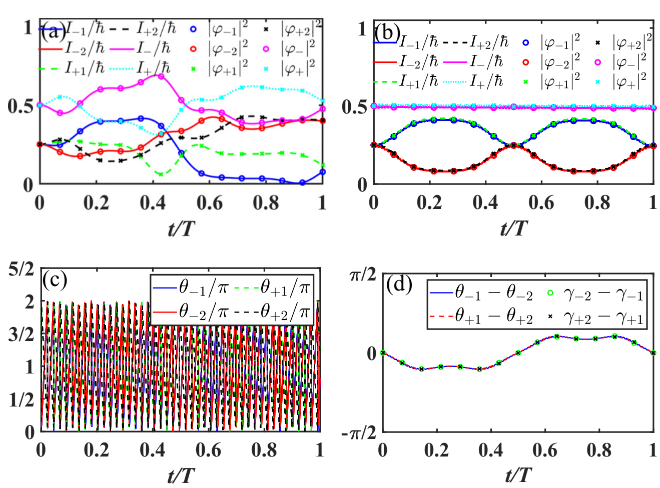

FIG. 2. The evolution of the action variables and probabilities in one period T with (a) $\tau = 100/\Omega$ and (b) $\tau = 5000/\Omega$. The evolution of (c) the angle variables (wrapped to the interval $[0, 2\pi]$), (d) the classical angle differences $\Delta\theta_\pm$, and the phase differences $\Delta\gamma_\pm$ in one period T with $\tau = 5000/\Omega$. The initial condition and other parameters are chosen as $I_{\pm1} = I_{\pm2} = \hbar/4$, $\theta_{\pm1} = \theta_{\pm2} = 0$, $\mu B/\hbar = \Omega$, and $d = 4r$.

It is obvious that the time-dependent probabilities $\varphi_{\pm i} = \langle E_{\pm i}|\psi\rangle$ of a generic quantum state on the eigenstates $|E_{\pm i}\rangle$ are in accord with the action variables $I_{\pm i}$, and the total probabilities $|\varphi_{+1}|^2 + |\varphi_{+2}|^2$ and $|\varphi_{-1}|^2 + |\varphi_{-2}|^2$ are unchanged as I_- and I_+. Moreover, the adiabatic dynamics of I_{-1} and I_{+1} are identical, as are those of I_{-2} and I_{+2}. The angle variables will accumulate dynamic angles and non-Abelian geometric angles during the adiabatic evolution, as shown in Fig. 2(c). To illustrate the non-Abelian geometric effect, we draw the classical angle differences after QCM $\Delta\theta_+ = \theta_{+1} - \theta_{+2} = \Delta\theta_{+1} - \Delta\theta_{+2}$, and $\Delta\theta_- = \theta_{-1} - \theta_{-2} = \Delta\theta_{-1} - \Delta\theta_{-2}$ in Fig. 2(d). These non-Abelian effects of phase differences corresponding to the same eigenfrequencies $\pm\mu B$ are due to the degeneracy of the eigenenergy spectrum of the quantum subsystem $\hat{H}_1$. Additionally, the quantum phase differences $\Delta\gamma_+ = \gamma_{+2} - \gamma_{+1} = \Delta\gamma_{+2} - \Delta\gamma_{+1}$, and $\Delta\gamma_- = \gamma_{-2} - \gamma_{-1} = \Delta\gamma_{-2} - \Delta\gamma_{-1}$, which are also shown in Fig. 2(d), are consistent with the classical angle differences. Thus the feasibility of the non-Abelian QCM method is demonstrated.

Next, we study these non-Abelian effects on the dynamics of the classical subsystem. We obtain the total Hamiltonian (16) for the hybrid system (28) under the averaging principle as

$$\begin{aligned}\mathscr{H} &= \frac{\mu B}{\hbar}(-I_{-1} - I_{-2} + I_{+1} + I_{+2}) + \frac{(\boldsymbol{P} - \boldsymbol{\mathcal{A}})^2}{2m} \\ &= \mu B(-|\varphi_{-1}|^2 - |\varphi_{-2}|^2 + |\varphi_{+1}|^2 + |\varphi_{+2}|^2) \\ &\quad + \frac{(\boldsymbol{P} - \boldsymbol{\mathcal{A}})^2}{2m},\end{aligned} \tag{46}$$

with $V_2(\boldsymbol{r}) = 0$. By Eqs. (26), (27), and (39), the equations of motion for the classical subsystem are

$$\begin{aligned} m\ddot{x} &= -\frac{3\mu\mu_0 m_F x(5d^2 + r^2)(-|\varphi_{-1}|^2 - |\varphi_{-2}|^2 + |\varphi_{+1}|^2 + |\varphi_{+2}|^2)}{4\pi(d^2 + r^2)^3\sqrt{4d^2 + r^2}} + \dot{y}\mathcal{B}_z, \\ m\ddot{y} &= -\frac{3\mu\mu_0 m_F y(5d^2 + r^2)(-|\varphi_{-1}|^2 - |\varphi_{-2}|^2 + |\varphi_{+1}|^2 + |\varphi_{+2}|^2)}{4\pi(d^2 + r^2)^3\sqrt{4d^2 + r^2}} - \dot{x}\mathcal{B}_z, \end{aligned} \tag{47}$$

where

$$\begin{aligned}\mathcal{B}_z =& \frac{\hbar 9d^2(2d^2 + r^2)}{2(d^2 + r^2)^2(4d^2 + r^2)^2}\{(2d^2 - r^2)(|\varphi_{-1}|^2 - |\varphi_{-2}|^2 + |\varphi_{+1}|^2 - |\varphi_{+2}|^2) \\ &- 3dx(\varphi_{-1}^*\varphi_{-2} + \varphi_{-1}\varphi_{-2}^* + \varphi_{+1}^*\varphi_{+2} + \varphi_{+1}\varphi_{+2}^*) + 3idy[-\varphi_{-1}^*\varphi_{-2} + \varphi_{-1}\varphi_{-2}^* - \varphi_{+1}^*\varphi_{+2} + \varphi_{+1}\varphi_{+2}^*]\}\end{aligned} \tag{48}$$

is a non-Abelian magneticlike field. In contrast to the geometric force brought by the Abelian effective magnetic field [68], the non-Abelian effective magnetic field is not exclusively dictated by the classical subsystem dynamics. Rather, it is also influenced by the quantum subsystem dynamics. This influence stems from the nonconservation of probabilities $P_{\pm i} = |\varphi_{\pm i}|^2$ on the eigenstates and the emerging interference terms in the expression of $\mathcal{B}_z$.

To illustrate the effect of the non-Abelian potential of the quantum subsystem, we choose the initial motion of the particle in the classical subsystem as

$$\begin{aligned} x(0) &= 0, \quad y(0) = 0, \\ \dot{x}(0) &= 0.03d/t_f, \quad \dot{y}(0) = 0.04d/t_f, \end{aligned} \tag{49}$$

where t_f is the total evolution time. As shown in Fig. 3(a), the dynamics of the quantum subsystem is adiabatic, since the populations $P_- = |\varphi_{-1}|^2 + |\varphi_{-2}|^2$ and $P_+ = |\varphi_{+1}|^2 + |\varphi_{+2}|^2$ on the two degenerate energy levels and the population differences $\Delta P_- = |\varphi_{-1}|^2 - |\varphi_{-2}|^2$ and $\Delta P_+ = |\varphi_{+1}|^2 - |\varphi_{+2}|^2$ between the degenerate states of the same eigenenergies are consistent with those under the BOA and remain nearly unchanged. In the meantime, the effective magnetic field $\mathcal{B}_z$ evolves as shown in Fig. 3(b). The non-Abelian nature of this field (48) lies in its dependence not only on the evolution of probability amplitudes $\varphi_{\pm i}$ in the quantum subsystem induced by the degeneracy but also on the position of the magnetic particle in the classical subsystem.

For the initial condition (49), the dynamics of the classical subsystem driven by H_2 will not change the symmetry of the trajectory of the particle in the real space as shown in Fig. 3(d). While influenced by the non-Abelian effect field $\mathcal{B}_z$, the motion of the slow particle is presented in Fig. 3(c) which is not a straight line but a curve and is in agreement with the trajectory without BOA. Thus the validity of this method is demonstrated. It is worth noting that this geometric force is brought by $\mathcal{B}_z$ breaking the real-space symmetry by the non-real-space dynamics of the spin states' non-Abelian phases, akin to the Abelian case [68].

IV. CONCLUSION

We have demonstrated that the versatility of the approach to present the quantum-classical hybrid system in the general theoretical framework of classical theory can be generalized to a process with energy degeneracy in the quantum subsystem. To interpret the impact of the non-Abelian gauge potential induced by the nonadiabatic evolution in the degenerate subspace, we employ a special canonical transformation to transform the quantum subsystem into a classical counterpart. Following this canonical transformation, a distinct vector potential emerges, inducing a non-Abelian magneticlike field attributed to the non-Abelian geometric force, along with a non-Abelian geometric angle, which is just the sum of the mean values of the WZ phase in the individual quantum degenerate subspace. To illustrate our theoretical framework,

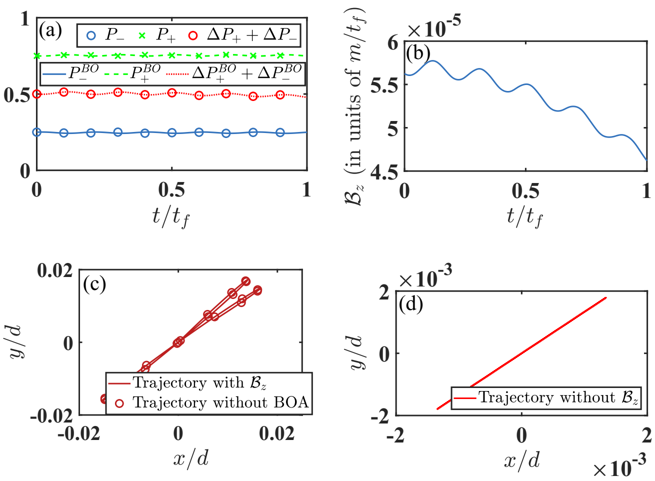

FIG. 3. The time evolution of (a) the probabilities on the eigenstates of the quantum subsystem under the BOA, $P_-^{\rm BO} = |\varphi_{-1}|^2 + |\varphi_{-2}|^2$, $P_+^{\rm BO} = |\varphi_{+1}|^2 + |\varphi_{+2}|^2$, and $\Delta P_-^{\rm BO} + \Delta P_+^{\rm BO} = |\varphi_{-1}|^2 - |\varphi_{-2}|^2 + |\varphi_{+1}|^2 - |\varphi_{+2}|^2$, and them without the BOA, $P_- = |\varphi_{-1}|^2 + |\varphi_{-2}|^2$, $P_+ = |\varphi_{+1}|^2 + |\varphi_{+2}|^2$, and $\Delta P_- + \Delta P_+ = |\varphi_{-1}|^2 - |\varphi_{-2}|^2 + |\varphi_{+1}|^2 - |\varphi_{+2}|^2$. The time evolution of (b) the effective magnetic field $\mathcal{B}_z$. The trajectories of motion in the classical subsystem (c) with $\mathcal{B}_z$, without the BOA, and (d) without $\mathcal{B}_z$. The motion of the particle with initial condition $x(0) = 0$, $\dot{x}(0) = 0.03d/t_f$, $y(0) = 0$, $\dot{y}(0) = 0.04d/t_f$. The parameters are chosen as $3\mu\mu_0 m_F t_f^2/(\pi m d^5) = -1.28 \times 10^4$, $\hbar t_f/(md^2) = 800$.

we apply it to a general hybrid system in which the classical subsystem is a magnetic particle and the quantum subsystem is a spin-half particle in a magnetic field replacing the Pauli matrices with the Dirac matrices as an example. The explicit formula for the non-Abelian magneticlike field depending on the non-Abelian geometric force is given. The result shows that even though the dynamics of the quantum subsystem is adiabatic, the population differences between the degenerate states of the same eigenenergies and interference will bring a non-Abelian effective magnetic field to the classical subsystem. This non-Abelian field depends not only on the position of the magnetic particle in the classical subsystem but also on the evolution of probability amplitudes in the quantum subsystem induced by the degeneracy. The trajectory of the particle in real space can be significantly influenced by the non-Abelian effect field. The non-real-space dynamics of the spin states' non-Abelian phases generate a non-Abelian geometric force breaking the real-space symmetry. Our theory could be expected to find applications in the non-Abelian quantum-classical hybrid dynamics with a fast subsystem and a slow subsystem, such as non-Abelian spin-electron dynamics [69] and non-Abelian spin torque [75].

ACKNOWLEDGMENTS

This work is supported by the National Natural Science Foundation of China (NSFC) (Grants No. 11875103 and No. 12147206).

[1] S. Pancharatnam, Generalized theory of interference, and its applications, Proc. Indian Acad. Sci. Sect. A **44**, 247 (1956).
[2] H. C. Longuet-Higgins, U. Öpik, M. H. L. Pryce, and R. A. Sack, Studies of the Jahn-Teller effect. II. The dynamical problem, Proc. R. Soc. London Ser. A **244**, 1 (1958).
[3] M. V. Berry, Quantal phase factors accompanying adiabatic changes, Proc. R. Soc. London Ser. A **392**, 45 (1984).
[4] F. Wilczek and A. Zee, Appearance of gauge structure in simple dynamical systems, Phys. Rev. Lett. **52**, 2111 (1984).
[5] J. H. Hannay, Angle variable holonomy in adiabatic excursion of an integrable Hamiltonian, J. Phys. A: Math. Gen. **18**, 221 (1985).
[6] Y. Aharonov and J. Anandan, Phase change during a cyclic quantum evolution, Phys. Rev. Lett. **58**, 1593 (1987).
[7] E. Sjöqvist, A. K. Pati, A. Ekert, J. S. Anandan, M. Ericsson, D. K. L. Oi, and V. Vedral, Geometric phases for mixed states in interferometry, Phys. Rev. Lett. **85**, 2845 (2000).
[8] X. X. Yi, L. C. Wang, and T. Y. Zheng, Berry phase in a composite system, Phys. Rev. Lett. **92**, 150406 (2004).
[9] D. M. Tong, E. Sjöqvist, L. C. Kwek, and C. H. Oh, Kinematic approach to the mixed state geometric phase in nonunitary evolution, Phys. Rev. Lett. **93**, 080405 (2004).
[10] R. B. Laughlin, The relationship between high-temperature superconductivity and the fractional quantum Hall effect, Science **242**, 525 (1988).
[11] E. Sjöqvist and M. Hedström, Noncyclic geometric phase, coherent states, and the time-dependent variational principle: Application to coupled electron-nuclear dynamics, Phys. Rev. A **56**, 3417 (1997).
[12] S. R. Jain and A. K. Pati, Adiabatic geometric phases and response functions, Phys. Rev. Lett. **80**, 650 (1998).
[13] H. Wu, X.-R. Huang, C.-S. Hu, Z.-B. Yang, and S.-B. Zheng, Rydberg-interaction gates via adiabatic passage and phase control of driving fields, Phys. Rev. A **96**, 022321 (2017).
[14] M.-T. Zhou, J.-L. Liu, P.-F. Sun, Z.-Y. An, J. Li, X.-H. Bao, and J.-W. Pan, Experimental creation of single Rydberg excitations via adiabatic passage, Phys. Rev. A **102**, 013706 (2020).
[15] M. V. Berry, Transitionless quantum driving, J. Phys. A: Math. Theor. **42**, 365303 (2009).
[16] A. Tomita and R. Y. Chiao, Observation of Berry's topological phase by use of an optical fiber, Phys. Rev. Lett. **57**, 937 (1986).
[17] G. Garcia de Polavieja, Noncyclic geometric phase shift for quantal revivals, Phys. Rev. Lett. **81**, 1 (1998).
[18] P. Bruno, V. K. Dugaev, and M. Taillefumier, Topological Hall effect and Berry phase in magnetic nanostructures, Phys. Rev. Lett. **93**, 096806 (2004).
[19] S.-Q. Shen, Spin Hall effect and Berry phase in two-dimensional electron gas, Phys. Rev. B **70**, 081311(R) (2004).
[20] Y. Zhang, Y.-W. Tan, H. L. Stormer, and P. Kim, Experimental observation of the quantum Hall effect and Berry's phase in graphene, Nature (London) **438**, 201 (2005).
[21] M. N. Leuenberger and E. R. Mucciolo, Berry-phase oscillations of the Kondo effect in single-molecule magnets, Phys. Rev. Lett. **97**, 126601 (2006).

[22] G. González and M. N. Leuenberger, Berry-phase blockade in single-molecule magnets, Phys. Rev. Lett. **98**, 256804 (2007).

[23] N. Read, Non-Abelian adiabatic statistics and Hall viscosity in quantum Hall states and $p_x + ip_y$ paired superfluids, Phys. Rev. B **79**, 045308 (2009).

[24] A. Raczyński, J. Zaremba, and S. Zielińska-Raczyńska, Berry phase in controlled light propagation and storage, Phys. Rev. A **97**, 043861 (2018).

[25] A. Messiah, *Quantum Mechanics* (Elsevier, Amsterdam, 1961).

[26] M. V. Berry, Classical adiabatic angles and quantal adiabatic phase, J. Phys. A: Math. Gen. **18**, 15 (1985).

[27] D. Chruscinski and A. Jamiolkowski, *Geometric Phases in Classical and Quantum Mechanics*, Vol. 36 (Springer, Berlin, 2012).

[28] V. I. Arnold, *Mathematical Methods of Classical Mechanics*, Graduate Texts in Mathematics Vol. 60 (Springer, New York, 1978).

[29] P. A. M. Dirac, The quantum theory of the emission and absorption of radiation, Proc. R. Soc. London Ser. A **114**, 243 (1927).

[30] F. Strocchi, Complex coordinates and quantum mechanics, Rev. Mod. Phys. **38**, 36 (1966).

[31] A. Heslot, Quantum mechanics as a classical theory, Phys. Rev. D **31**, 1341 (1985).

[32] S. Weinberg, Testing quantum mechanics, Ann. Phys. **194**, 336 (1989).

[33] J. Liu, B. Wu, and Q. Niu, Nonlinear evolution of quantum states in the adiabatic regime, Phys. Rev. Lett. **90**, 170404 (2003).

[34] B. Wu, J. Liu, and Q. Niu, Geometric phase for adiabatic evolutions of general quantum states, Phys. Rev. Lett. **94**, 140402 (2005).

[35] V. Husain, I. Javed, S. S. Seahra, and N. X, Motivating semiclassical gravity: A classical-quantum approximation for bipartite quantum systems, Phys. Rev. D **108**, 086033 (2023).

[36] M. Amin and M. A. Walton, Quantum-classical dynamical brackets, Phys. Rev. A **104**, 032216 (2021).

[37] J. S. Briggs and A. Eisfeld, Equivalence of quantum and classical coherence in electronic energy transfer, Phys. Rev. E **83**, 051911 (2011); Coherent quantum states from classical oscillator amplitudes, Phys. Rev. A **85**, 052111 (2012); Quantum dynamics simulation with classical oscillators, **88**, 062104 (2013); J. S. Briggs, Equivalent emergence of time dependence in classical and quantum mechanics, *ibid.* **91**, 052119 (2015).

[38] M. Radonjić, S. Prvanović, and N. Burić, System of classical nonlinear oscillators as a coarse-grained quantum system, Phys. Rev. A **84**, 022103 (2011); Emergence of classical behavior from the quantum spin, **85**, 022117 (2012); Hybrid quantum-classical models as constrained quantum systems, **85**, 064101 (2012).

[39] N. Burić, D. B. Popović, M. Radonjić, and S. Prvanović, Orbits of hybrid systems as qualitative indicators of quantum dynamics, Phys. Lett. A **378**, 1081 (2014).

[40] D. Arsenović, N. Burić, D. B. Popović, M. Radonjić, and S. Prvanović, Positive-operator-valued measures in the Hamiltonian formulation of quantum mechanics, Phys. Rev. A **91**, 062114 (2015).

[41] Y. N. Zhang, J. Shen, H. D. Liu, and X. X. Yi, Non-Abelian quantum adiabatic dynamics and phase simulation with classical resonant oscillators, Phys. Rev. A **102**, 032213 (2020).

[42] G. Rigolin and G. Ortiz, Adiabatic perturbation theory and geometric phases for degenerate systems, Phys. Rev. Lett. **104**, 170406 (2010); Adiabatic theorem for quantum systems with spectral degeneracy, Phys. Rev. A **85**, 062111 (2012); Degenerate adiabatic perturbation theory: Foundations and applications, *ibid.* **90**, 022104 (2014).

[43] J. Ruseckas, G. Juzeliūnas, P. Öhberg, and M. Fleischhauer, Non-Abelian gauge potentials for ultracold atoms with degenerate dark states, Phys. Rev. Lett. **95**, 010404 (2005).

[44] R. Unanyan, M. Fleischhauer, B. W. Shore, and K. Bergmann, Robust creation and phase-sensitive probing of superposition states via stimulated Raman adiabatic passage (STIRAP) with degenerate dark states, Opt. Commun. **155**, 144 (1998).

[45] J. Pachos, P. Zanardi, and M. Rasetti, Non-Abelian Berry connections for quantum computation, Phys. Rev. A **61**, 010305(R) (1999).

[46] K. Singh, D. M. Tong, K. Basu, J. L. Chen, and J. F. Du, Geometric phases for nondegenerate and degenerate mixed states, Phys. Rev. A **67**, 032106 (2003).

[47] J. Gong and S. A. Rice, General method for complete population transfer in degenerate systems, Phys. Rev. A **69**, 063410 (2004).

[48] K. Takahashi, Transitionless quantum driving for spin systems, Phys. Rev. E **87**, 062117 (2013).

[49] L. S. Simeonov and N. V. Vitanov, Generation of non-Abelian geometric phases in degenerate atomic transitions, Phys. Rev. A **96**, 032102 (2017).

[50] K. Snizhko, R. Egger, and Y. Gefen, Non-Abelian Berry phase for open quantum systems: Topological protection versus geometric dephasing, Phys. Rev. B **100**, 085303 (2019).

[51] Q. Zhang, Quantum adiabatic evolution with energy degeneracy levels, Phys. Rev. A **93**, 012116 (2016).

[52] C. A. Mead and D. G. Truhlar, On the determination of Born-Oppenheimer nuclear motion wave functions including complications due to conical intersections and identical nuclei, J. Chem. Phys. **70**, 2284 (1979).

[53] H. Kuratsuji and S. Iida, Effective action for adiabatic process: Dynamical meaning of Berry and Simon's phase, Prog. Theor. Phys. **74**, 439 (1985).

[54] M. Stone, Born-Oppenheimer approximation and the origin of Wess-Zumino terms: Some quantum-mechanical examples, Phys. Rev. D **33**, 1191 (1986).

[55] E. Gozzi and W. D. Thacker, Classical adiabatic holonomy and its canonical structure, Phys. Rev. D **35**, 2398 (1987).

[56] Y. Aharonov, E. Ben-Reuven, S. Popescu, and D. Rohrlich, Perturbative induction of vector potentials, Phys. Rev. Lett. **65**, 3065 (1990).

[57] M. V. Berry and J. M. Robbins, Chaotic classical and half-classical adiabatic reactions: Geometric magnetism and deterministic friction, Proc. R. Soc. London Ser. A **442**, 659 (1993).

[58] R. G. Littlejohn and S. Weigert, Adiabatic motion of a neutral spinning particle in an inhomogeneous magnetic field, Phys. Rev. A **48**, 924 (1993).

[59] A. Peres and D. R. Terno, Hybrid classical-quantum dynamics, Phys. Rev. A **63**, 022101 (2001).

[60] P. C. Hammel, Seeing single spins, Nature (London) **430**, 300 (2004).

[61] M. J. W. Hall and M. Reginatto, Interacting classical and quantum ensembles, Phys. Rev. A **72**, 062109 (2005).

[62] H. D. Liu, S. L. Wu, and X. X. Yi, Berry phase and Hannay's angle in a quantum-classical hybrid system, Phys. Rev. A **83**, 062101 (2011).

[63] H.-T. Elze, Linear dynamics of quantum-classical hybrids, Phys. Rev. A **85**, 052109 (2012).

[64] V. Gil and L. L. Salcedo, Canonical bracket in quantum-classical hybrid systems, Phys. Rev. A **95**, 012137 (2017).

[65] M. Kantner, M. Mittnenzweig, and T. Koprucki, Hybrid quantum-classical modeling of quantum dot devices, Phys. Rev. B **96**, 205301 (2017).

[66] C. P. Slichter, *Principles of Magnetic Resonance*, Springer Series in Solid-State Sciences Vol. 1 (Springer, Berlin, 1978).

[67] G. P. Berman, D. I. Kamenev, and V. I. Tsifrinovich, Stationary cantilever vibrations in oscillating-cantilever-driven adiabatic reversals: Magnetic-resonance-force-microscopy technique, Phys. Rev. A **66**, 023405 (2002).

[68] Q. Zhang and B. Wu, General approach to quantum-classical hybrid systems and geometric forces, Phys. Rev. Lett. **97**, 190401 (2006).

[69] N. Lenzing, A. I. Lichtenstein, and M. Potthoff, Emergent non-Abelian gauge theory in coupled spin-electron dynamics, Phys. Rev. B **106**, 094433 (2022).

[70] H. J. Mamin, R. Budakian, B. W. Chui, and D. Rugar, Detection and manipulation of statistical polarization in small spin ensembles, Phys. Rev. Lett. **91**, 207604 (2003).

[71] D. Rugar, R. Budakian, H. J. Mamin, and B. W. Chui, Single spin detection by magnetic resonance force microscopy, Nature (London) **430**, 329 (2004).

[72] S. Weinberg, Precision tests of quantum mechanics, Phys. Rev. Lett. **62**, 485 (1989).

[73] A. P. Schnyder, S. Ryu, A. Furusaki, and A. W. W. Ludwig, Classification of topological insulators and superconductors in three spatial dimensions, Phys. Rev. B **78**, 195125 (2008).

[74] S. N. Biswas, A realization of Berry's non-Abelian gauge field, Phys. Lett. B **228**, 440 (1989).

[75] M. Elbracht, S. Michel, and M. Potthoff, Topological spin torque emerging in classical spin systems with different timescales, Phys. Rev. Lett. **124**, 197202 (2020).