Minimum detection efficiencies for loophole-free genuine nonlocality tests

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The certification of quantum nonlocality, which has immense significance in designing device-independent technologies, confronts severe experimental challenges. Detection loopholes, originating from the unavailability of perfect detectors, are one of the major issues among them. In the present study we focus on the minimum detection efficiency (MDE) required to detect various forms of genuine nonlocality, originating from the type of causal constraints imposed on the involved parties. In this context, we demonstrate that the MDE needed to manifest the recently suggested T_2-type nonlocality deviates significantly from perfection. Additionally, we have computed the MDE necessary to manifest Svetlichny's nonlocality, with the state-independent approach markedly reducing the previously established bound. Finally, considering the inevitable existence of noise we demonstrate the robustness of the imperfect detectors to certify T_2-type nonlocality.

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I. INTRODUCTION

Over the past 30 years, a distinctive category of information theory has emerged popularly known as device-independent (DI) technology, which surpasses several limitations of classical physics [1–13]. Admittedly, the foundational constituent behind such an ever expanding architecture was implanted in the seminal results of J. S. Bell [14,15], which derives a testable criterion for certifying the nonlocal aspect of any operational theory. In a nutshell, the nonlocality is a signature of multipartite correlations, incompatible with the classical outlook of the causal constraints imposed on the involved parties. For example, the pioneering nonlocality test as proposed by Clauser-Horne-Shimony-Holt (CHSH) considers a bipartite scenario, where spatially separated agents, supplemented with some nonsignalling correlation, are asked to produce a dichotomic random variable depending upon their individual classical inputs [16]. Finally, an inequality, in the form of their input-output statistics, is established. The violation of this inequality apparently leads to a contradiction with the causal constraint imposed on their individual input-output generation. However, relaxing the local-realistic description of their shared correlation, it is possible to explain the violation even under the same causal structures. This in turn certifies the nonlocal signature of that correlation. In the multipartite scenario, the event of local input-output generation for each of the parties can share various forms of causal constraints. (For the rest of the paper, whenever we say that the causal constraints imposed on some of the parties, it means that the same is imposed on the events of their local input-output generations.) Operationally, some of the parties may be allowed to communicate with each other but not with the rest. With an increasing number of parties, evidently, the possibility of such constraints and the associated complexities highly increases. In the most basic multipartite scenario, three spatially separated parties are asked to generate local random variables depending upon the input given to them locally. On the other hand, when there is no causal constraint imposed on two of the parties (personified as Bob and Charlie), then their local input-output statistics should be factorized between Alice vs Bob-Charlie when their measurements are spacelike separated from that of Alice. While any of their obtained statistics compatible with such a scenario is termed as bilocal (BL) correlations, an apparent contradiction of the same can be characterized by an inequality proposed by Svetlichny [17] and coined as genuinely nonlocal. In a more complicated scenario, one may further consider a temporal ordering in the input-output generation of the parties Bob and Charlie. Intuitively, this allows Charlie to make a redundant signaling to Bob when he is in the causal future of Bob and vice versa. It is instructive that such a causal constraint is stronger than that of the earlier one. Accordingly, the set of correlations compatible with these constraints, namely, the time-ordered bilocal (TOBL) correlations are strictly included in the set of BL correlations [18]. In isolation, the characterization of BL correlations has emerged over the past decade when an operational inconsistency of the Svetlichny's definition of bilocality is reported, and it has further fueled the development of various refined causal structures on genuine nonlocality [18,19,20].

Interestingly, quantum theory exhibits the signature of nonlocality in its simplest possible bipartite scenario. However, in the recent era of many-body physics, the importance of multipartite quantum correlations such as entanglement and nonlocality cannot be overemphasized. Notably, the genuine nonlocality has garnered significant attention, primarily owing to its implications in device-independent random number generation (DIRNG), device-independent quantum key distribution (DIQKD), and device-independent certification of

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genuine entanglement [21–27]. However, the experimental certification of the potential nonlocal signature of quantum correlations encounters serious challenges due to the possible loopholes in Bell test. One such major operational loophole is the detection loophole, which emerges due to an inefficient photon detector, used to record the local input-output statistics obtained by each of the spatially separated agents. In particular, there are possible instances where the imperfect detector may yield no click, i.e., an inclusive outcome. While one might consider overcoming this issue by simply discarding the no-click outcomes, such postselection requires an additional assumption, specifically the fair-sampling assumption. Without this assumption, even a local hidden variable model can violate Bell inequalities [28]. A more comprehensive and appropriate approach to addressing this challenge is to regard the no-click event as a potential valid outcome achievable during the Bell test. The second approach needs a cutoff on the detection efficiency to observe nonlocality. However, given an entangled state, accompanied by a set of observables may require further bound on the minimum detection efficiency. For instance, the Bell test involving two qubit maximally entangled state demands 83% detection efficiency, while a nonmaximally entangled state needs 67% [29–31]. Moreover, motivated by the experimental setup, a minimum detector efficiency for the asymmetric bipartite Bell test has also been reported [32,33]. The study of such bipartite nonlocality tests under inefficient detectors has gained further importance by developing a series of sophisticated experimental setups [34–38]. In spite of having several foundational and practical importance, in the multipartite scenario the nonlocality test with inefficient detectors is mostly restricted to its simplest form [39,40], with a recent addition for the Svetlichny-type genuine nonlocality test [41,42].

On the other hand, the resource theoretic construction of quantum nonlocality contradicts the existence of a unique nonlocal measure, even in its simplest scenario [43]. This further motivates us to investigate possible detection efficiencies for other possible tests of genuine nonlocality by imposing different causal constraints on the involved parties [18,19,20]. In this work, we consider various possibilities of nonlocality tests and derive the minimal possible sophistication one may require for the detectors to certify genuine nonlocality in the proposed scenario. With a brief discussion about the related genuine nonlocalities in Sec. II, we have formally introduced the relevant quantities in detection loophole-free nonlocality tests in Sec. III. Thereafter, we have established the necessary and sufficient conditions on detectors to demonstrate genuine nonlocality of the T_2 kind in Sec. IV. Notably, the results derived in this section encapsulate an experimentally motivated scenario for hybrid entanglement, where different quantum particles are allowed to be entangled via interparticle interactions [44–47]. Following this, a brief analysis of noise robustness has been presented for inefficient detectors. This analysis hints at a complementary relationship between detection efficiency and the potential range of one-parameter quantum settings (i.e., the state alongside triplets of measurement pairs) required to exhibit genuine nonlocality, particularly in the presence of unavoidable noise. Subsequently, through a state-independent generalized study, we have significantly reduced the previously established bound on detection efficiency for the Svetlichny-type nonlocality test in Sec. V. Finally, in Sec. VI we have concluded and discussed the further directions stemming from our work.

II. GENUINE NONLOCALITY UNDER VARIOUS CAUSAL CONSTRAINTS

Consider the simplest multipartite scenario, where three spatially separated agents perform local operations on their individual subsystems. Further, their space-time relation is restricted in such a way that no more than two of them can reside in the same light cone. As a consequence, any local-realistic hidden variable description simulating their local input-output statistics must be factorized in some convex combination of all three bipartitions. This in essence reads

$$P(abc|xyz) = \sum_{\lambda} q_{\lambda}\, P_{\lambda}(ab|xy)\, P_{\lambda}(c|z) + \sum_{\mu} q_{\mu}\, P_{\mu}(ac|xz)\, P_{\mu}(b|y) + \sum_{\nu} q_{\nu}\, P_{\nu}(bc|yz)\, P_{\nu}(a|x), \tag{1}$$

where $\{x, y, z\}$ and $\{a, b, c\}$ are the inputs and the outputs for the three parties: Alice, Bob, and Charlie, respectively. On a side note, throughout the paper, we have often used the symbols $\{A, B, C\}$ to denote the inputs. Also, $0 \leqslant q_{\lambda}, q_{\mu}, q_{\nu} \leqslant 1$ are the respective probabilities compatible with the causal constraint and $\sum_{\lambda} q_{\lambda} + \sum_{\mu} q_{\mu} + \sum_{\nu} q_{\nu} = 1$. In 1987, Svetlichny pointed out that all possible correlations that can be decomposed in the form of Eq. (1) respect the following inequality over their operational statistics [17]:

$$-\langle A_0B_0C_0\rangle - \langle A_1B_0C_0\rangle + \langle A_0B_1C_0\rangle - \langle A_1B_1C_0\rangle - \langle A_0B_0C_1\rangle + \langle A_1B_0C_1\rangle - \langle A_0B_1C_1\rangle - \langle A_1B_1C_1\rangle \leqslant 4, \tag{2}$$

where A_i, B_j, C_k with $\{i, j, k\} \in \{0, 1\}$ denote the measurements performed by Alice, Bob, and Charlie respectively. We also use the notation $\langle A_iB_jC_k\rangle$ to denote the expectation value of the two-outcome $\{\pm 1\}$ measurements performed by the respective parties on their local constituents. In contrast, any correlation violating the above inequality (2) can be characterized as Svetlichny-type genuine nonlocal.

It is instructive that for the above scenario the parties, sharing a common light cone (say, Bob and Charlie), are free to communicate with each other. However, such a relaxed causal constraint allows them to establish a bipartite nonlocality with Alice. With this operational inconsistency, one may further impose a definite causal order between input-output generation of Bob and Charlie, which, in effect, allows only a single direction of relevant communication between them [18,19]. Any correlation compatible with such a constraint is referred to be T_2 local and admits the following decomposition:

$$P(abc|xyz) = \sum_{\lambda} q_{\lambda} P_{\lambda}^{T_{AB}}(ab|xy) P_{\lambda}(c|z) + \sum_{\mu} q_{\mu} P_{\mu}^{T_{AC}}(ac|xz) P_{\mu}(b|y) + \sum_{\nu} q_{\nu} P_{\nu}^{T_{BC}}(bc|yz) P_{\nu}(a|x), \tag{3}$$

where $\{x, y, z\}$ and $\{a, b, c\}$ are the same as Eq. (1). The joint term $p_\lambda^{T_{AB}}(ab|xy) = p_\lambda^{A<B}(ab|xy)$ if the local statistics obtained by Alice is in the causal past of Bob and $p_\lambda^{T_{AB}}(ab|xy) = p_\lambda^{B<A}(ab|xy)$ when the order is reversed. All the other bipartite marginals can also be explained in a similar fashion. Notably, in both cases, there is (at most) one-way signaling. This refinement seems to offer a potential solution to sidestep the earlier mentioned issues that might arise within the Svetlichny scenario. Now if a probability distribution $p(abc|xyz)$ can be written in the form (3), then it is said to be T_2 local, otherwise it is called genuinely three-way nonlocal. In [19] the authors proposed an experimentally testable criteria to verify the compatibility of a tripartite correlation with such a time-ordered space-time structure:

$$\begin{aligned}\mathcal{B}_{T_2} := & -2\{P(00|A_1B_1) + P(00|B_1C_1) + P(00|A_1C_1)\}\\ & - P(000|A_0B_0C_1) - P(000|A_0B_1C_0)\\ & - P(000|A_1B_0C_0) + 2P(000|A_1B_1C_0)\\ & + 2P(000|A_1B_0C_1) + 2P(000|A_0B_1C_1)\\ & + 2P(000|A_1B_1C_1) \leqslant 0, \end{aligned} \tag{4}$$

where A_x, B_y, C_z are the dichotomic local observables with possible outcomes $\{0, 1\}$. A violation of the inequality (4) certifies the genuine nonlocality possessed by the correlation.

III. MINIMUM DETECTION EFFICIENCY

The requirement of a loophole-free experimental test for nonlocality can be classified into two different aspects: First, from the information theoretic perspective, it is required to obtain a nonlocal statistics out of a black box promising to generate a particular form of correlation or state. On the other hand, with a goal of distributing secret keys or, of generating random numbers, one may be eager to certify the nonlocal signature of an operational theory in a device-independent manner. Notably, while the first one conceives a state-dependent outlook, the latter one asks about the nature of the theory regardless of any possible preparations. These two different perspectives demand different forms of experimental sophistication, one of which is the efficiency of the detectors. However, in the following we will see that the latter one emerges as the optimized value of the earlier one, over all possible quantum states and measurements.

Let us consider one such Bell-inequality $\mathcal{B}$ for a nonlocality test, which is supposed to be verified on an N-partite entangled state ρ with ith party performing m_i numbers of incompatible measurements $\mathcal{M}_i := \{M_1, \dots, M_{m_i}\}$. Then the detection efficiency required for such a test is said to be the cutoff detection efficiency (CDE) and is denoted as

$$\eta^{\mathcal{B}} := \eta^{\mathcal{B}}\big(\rho, \{\mathcal{M}_k\}_{k=1}^N\big). \tag{5}$$

However, the minimum detection efficiency (MDE) for the same Bell test $\mathcal{B}$ can be identified as

$$\eta_{\min}^{\mathcal{B}} = \inf_{\rho, \{\mathcal{M}_k\}_{k=1}^N} \eta^{\mathcal{B}}\big(\rho, \{\mathcal{M}_k\}_{k=1}^N\big), \tag{6}$$

where the infimum is over the set of all possible N-partite entangled density matrices acting on the joint Hilbert space $\mathcal{H}$ and $\{\mathcal{M}_k\}_{k=1}^N$ are the set of all possible m_k incompatible measurements for the kth cite. Notably, CDE is a quantity of interest for the self-testing scenario, while MDE can be assigned with the device-independent certification of nonlocality in quantum theory.

Generally, the CDE is a very complicated nonlinear function of both states and measurements. Hence, the optimization involved in (6) is highly nontrivial, and the characterization of MDE for an arbitrary Bell-test becomes very challenging in general. However, motivated by the device-independent architectures and their implications, in the present work we derive the MDE for the detection of various tripartite genuine nonlocality. In particular, for the T_2-type nonlocality test we estimate $\eta_{\min}$ bypassing the optimization complexities related to CDE, while for the Svetlichny-type test, we adopt a rigorous numerical optimization.

Furthermore, in a real-life experimental setup, things become more intricate due to the inevitable existence of noise. If by $\{\mathbf{p}\}$ one denotes the set of parameters defining the noise, Eq. (6) is modified as

$$\eta_{\min}^{\mathcal{B}}(\{\mathbf{p}\}) = \inf_{\rho, \{\mathcal{M}_k\}_{k=1}^N} \eta^{\mathcal{B}}\big(\rho, \{\mathcal{M}_k\}_{k=1}^N, \{\mathbf{p}\}\big). \tag{7}$$

Clearly, in the laboratory the efficiencies of the detectors must follow the relation $\eta > \eta_{\min}^{\mathcal{B}}(\{\mathbf{p}\})$.

IV. MDE FOR GENUINE NONLOCALITY

Let us begin with a stronger causal constraint where the parties residing in the same light cone are bounded with a definite causal order to generate their local input-output statistics. As mentioned earlier, the incompatibility of a correlation in such a scenario can be certified by the violation of the inequality (4). To establish such a genuine three-way nonlocal signature of shared quantum correlation, we consider a situation where the associated parties will perform the local measurements with inefficient detectors with corresponding efficiencies η_A, η_B, and η_C. The following lemma derives the minimum requirement for detection efficiencies to violate (4).

Lemma 1. The spatially separated parties would be able to certify genuine three-way nonlocality in terms of the inequality $\mathcal{B}_{T_2}$, when the following condition holds:

$$(4\eta_A\eta_B\eta_C - \eta_A\eta_B - \eta_A\eta_C - \eta_B\eta_C) > 0. \tag{8}$$

Proof. From now onward, we will use $P'(\mathcal{O}|\mathcal{X})$ and $P(\mathcal{O}|\mathcal{X})$ for theoretical and observed probability, respectively, where input variable $\mathcal{X}$ produces output variable $\mathcal{O}$. Assuming the independence of each of the individual local detectors the observed probability for each individual can depend only upon the corresponding theoretical probability and the detection efficiency of the local detector. Now, if all the parties agree to assume $\mathcal{O} \equiv 1$ for the no-click instances, then $P(\mathcal{O}|\mathcal{X}) = \eta P'(\mathcal{O}|\mathcal{X})$, only when $\mathcal{O} \equiv 0$. This readily implies

$$\begin{aligned}P(000|A_1B_1C_1) &= \eta_A\eta_B\eta_C P'(000|A_1B_1C_1)\\ &\leqslant \eta_A\eta_B\eta_C \min_X P'(00|X),\end{aligned}$$

where $X \in \{A_1B_1, B_1C_1, A_1C_1\}$.

Again,

$$P(000|A_1B_1C_0) - P(00|A_1B_1)$$
$$= \eta_A\eta_B\eta_C P'(000|A_1B_1C_0) - \eta_A\eta_B P'(00|A_1B_1)$$
$$\leqslant \eta_A\eta_B\eta_C \min_X P'(00|X) - \eta_A\eta_B \min_X P'(00|X),$$

where the last inequality holds since $P(000|A_1B_1C_0) - P(00|A_1B_1) \leqslant 0$ and $\eta_C, \eta_A, \eta_B \leqslant 1$. Deriving the similar inequalities for the other pair of terms $P(000|A_1B_0C_1) - P(00|A_1C_1)$ and $P(000|A_0B_1C_1) - P(00|B_1C_1)$, we can rewrite the violation of $\mathcal{B}_{T_2}$ (4) as

$$(4\eta_A\eta_B\eta_C - \eta_A\eta_B - \eta_A\eta_C - \eta_B\eta_C) \min_X P'(00|X) > 0$$
$$\Rightarrow (4\eta_A\eta_B\eta_C - \eta_A\eta_B - \eta_A\eta_C - \eta_B\eta_C) > 0. \tag{9}$$

Hence, the above condition among the detector efficiencies becomes necessary to exhibit genuine three-way nonlocality. ■

Note that the derivation of the condition (9) is completely algebraic, without invoking any explicit structure of quantum theory. At this point, a pertinent question naturally arises regarding the sufficiency of the inequality (9) within the quantum theory—in other words, whether every triplet of $\{\eta_A, \eta_B, \eta_C\}$, satisfying $4\eta_A\eta_B\eta_C - \eta_A\eta_B - \eta_A\eta_C - \eta_B\eta_C > 0$, corresponds to CDE for $\mathcal{B}_{T_2}$. In the following, we will answer this question affirmatively.

Lemma 2. For every possible triplet of detection efficiencies $\{\eta_A, \eta_B, \eta_C\}$ with $(4\eta_A\eta_B\eta_C - \eta_A\eta_B - \eta_A\eta_C - \eta_B\eta_C) > 0$, there exists a single-parameter (θ) quantum setting which violates the $\mathcal{B}_{T_2}$ inequality.

By the *single-parameter (θ) quantum setting* we mean a class of genuinely entangled states along with two incompatible dichotomic measurements for each individual party, both parameterized by a single parameter θ.

Proof. Consider the three-qubit state

$$|\Psi_{ABC}(\theta)\rangle = k_\theta\left[(|011\rangle + |101\rangle + |110\rangle) + \frac{(1-3\cos\theta)}{\sin\theta}|111\rangle\right], \tag{10}$$

where k_θ is a function of θ, denoting the normalization of the state. Each party performs an identical set of incompatible measurements in the following bases:

$$X_0 \equiv \{|0\rangle, |1\rangle\},$$
$$X_1 \equiv \{\cos\theta|0\rangle + \sin\theta|1\rangle, \sin\theta|0\rangle - \cos\theta|1\rangle\},$$

where $X \in \{A, B, C\}$. Such a measurement setting readily implies

$$P(000|A_iB_jC_k) = 0,$$

where at least two of $\{i, j, k\}$ equals 0. On the other hand when two of $\{i, j, k\}$ equals 1, probabilities reduce to

$$P(000|A_iB_jC_k) = \eta_A\eta_B\eta_C k_\theta^2 \sin^4\theta,$$

and the following marginals become

$$P(00|X_1Y_1) = \eta_X\eta_Y k_\theta^2 \sin^4\theta\left(1 + \tan^2\frac{\theta}{2}\right),$$

where $X \neq Y \in \{A, B, C\}$.

Replacing the above expressions for the imperfect detectors, the l.h.s. of the inequality (4) reduces to the following:

$$k_\theta^2 \sin^4\theta\left[4\eta_A\eta_B\eta_C - \eta_A\eta_B - \eta_B\eta_C - \eta_A\eta_C - \tan^2\frac{\theta}{2}(\eta_A\eta_B + \eta_B\eta_C + \eta_A\eta_C)\right].$$

Therefore, to violate the inequality(4), we must have

$$\tan^2\frac{\theta}{2} < \frac{4\eta_A\eta_B\eta_C - \eta_A\eta_B - \eta_B\eta_C - \eta_A\eta_C}{\eta_A\eta_B + \eta_B\eta_C + \eta_A\eta_C}.$$

Therefore, for every possible value of $\{\eta_A, \eta_B, \eta_C\}$, satisfying the condition $4\eta_A\eta_B\eta_C - \eta_A\eta_B - \eta_B\eta_C - \eta_A\eta_C > 0$ it is possible to find a range of θ for which a three-way genuine nonlocality can be certified experimentally. Notably, the inequality on the detection efficiencies is strict, otherwise one may consider $\theta = n\pi$, $n \in \mathbb{Z}$ and the measurements X_0 and X_1 will become compatible. ■

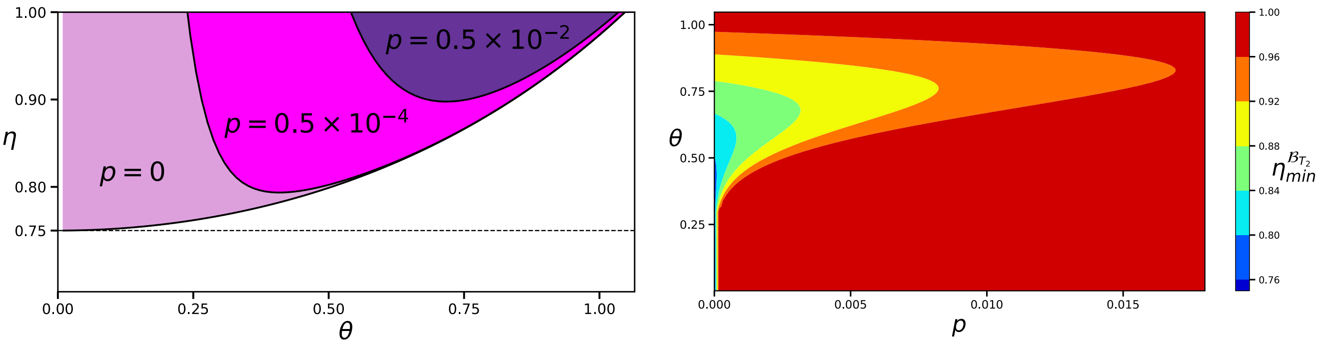
Now, we are at the position to estimate the MDE to experimentally certify the $\mathcal{B}_{T_2}$ nonlocality.

Theorem 1. The MDE's for $\mathcal{B}_{T_2}$ for each of the party satisfy the following relation:

$$4\left(\eta_{\min}^{\mathcal{B}_{T_2}}\right)_A\left(\eta_{\min}^{\mathcal{B}_{T_2}}\right)_B\left(\eta_{\min}^{\mathcal{B}_{T_2}}\right)_C - \left(\eta_{\min}^{\mathcal{B}_{T_2}}\right)_A\left(\eta_{\min}^{\mathcal{B}_{T_2}}\right)_B - \left(\eta_{\min}^{\mathcal{B}_{T_2}}\right)_A\left(\eta_{\min}^{\mathcal{B}_{T_2}}\right)_C - \left(\eta_{\min}^{\mathcal{B}_{T_2}}\right)_B\left(\eta_{\min}^{\mathcal{B}_{T_2}}\right)_C = 0.$$

Proof. The proof simply follows from Lemma 1 and 2. Note that Lemma 1 gives a necessary condition that must be satisfied by the three detectors whenever they are used to detect $\mathcal{B}_{T_2}$ nonlocality. On the other hand, it is shown in Lemma 2 that for any triplet of detectors with their efficiencies satisfying the condition (9), there exists a genuinely entangled quantum state along with a pair of measurements for each cite, whose $\mathcal{B}_{T_2}$ nonlocality violation can be certified experimentally. In other words, every possible triplet of detection efficiencies, satisfying condition (9), can be regarded as a CDE for $\mathcal{B}_{T_2}$ nonlocal inequality. Finally, following the definition of MDE in Eq. (6), we can conclude the proof. ■

Following the general MDE equation derived in Theorem 1, two different instances can be concluded. For the case of symmetric detection efficiencies, i.e., $\eta_A = \eta_B = \eta_C = \eta$, the inequality (4) can be violated if and only if $\eta > \eta_{\min}^{\mathcal{B}_{T_2}} = 75\%$. On the other hand, when a hybrid entanglement between different quantum particles is observed, then the MDE for each of the particles may demand different experimental sophistication. In such a scenario, if two of the detectors work perfectly, say, $\eta_A = \eta_B = 100\%$, then the inequality (4) can be violated if and only if $\eta_C > (\eta_{\min}^{\mathcal{B}_{T_2}})_C = 50\%$. Therefore, having perfect detectors for all three parties is not imperative. Indeed, with two of them being perfect, the genuine nonlocality can be established even if the imperfect one correctly detects the particle only half of the times. On the other hand, if none of the detectors is found to be perfect, then to detect such a nonlocal correlation each of the detectors must detect the particles with atleast 75% efficiency.



(a) (b)

FIG. 1. Noise robust MDE for $\mathcal{B}_{T_2}$ nonlocality. (a) η vs θ plot for different p. The solid black curve represents MDE, i.e., $\eta_{\min}^{\mathcal{B}_{T_2}}$. The detector should have detection efficiency within the shaded (color online) region to exhibit T_2 nonlocality. (b) Three-parameter plot of p vs θ vs $\eta_{\min}^{\mathcal{B}_{T_2}}$. It shows that $\mathcal{B}_{T_2}$ is fairly noise-tolerant (up to roughly 1.6%). Both figures have been drawn considering the symmetric case, i.e., for $\eta_A = \eta_B = \eta_C = \eta$.

Background noise

While the above analysis is restricted to pure entangled states, in the practical experimental setup the existence of background noise becomes unavoidable. The presence of noise in a preparation device is generally regarded as a junk state, prepared with a significantly low frequency. Without having any prior knowledge about the noise, a randomization over all possible junk states can be assumed as noise. In presence of such a *white noise* the tripartite state takes the form

$$\rho_{ABC} = (1-p)|\Psi(\theta)\rangle\langle\Psi(\theta)|_{ABC} + \frac{p}{8}\mathbb{I},$$

where the state $|\psi(\theta)\rangle$ is same as in Eq. (10). Naturally, the MDE $\eta_{\min}^{\mathcal{B}_{T2}}(\theta, p)$ [as introduced in Eq. (7)] becomes highly nonlinear function. However, we show that with a reasonable amount of detection efficiency $0.92 \leqslant \eta^{\mathcal{B}_{T2}} \leqslant 0.96$, one can expect tolerance even up to 1.6% background noise (see Fig. 1 for details).

V. MDE FOR SVETLICHNY's NONLOCALITY

We will now consider a genuine nonlocality experiment with relaxed causal constraints on involved parties, i.e., by invoking no-time orderings among their local input-output generation. The violation of Svetlichny's inequality (2) serves as a prominent benchmark for illustrating any incompatibility with such a causal structure and is identified as a Svetlichny-type genuinely nonlocal correlation. However, the experimental sophistication required to certify such a violation is pretty high: almost 97% of CDE for a three-particle GHZ state [41]. In the following, we will first derive an expression for CDE, depending on the statistics obtained by the individual parties. Thereafter, we numerically estimate the MDE considering any possible tripartite entangled preparation, which is significantly lower than the bound obtained in [41].

Let us first express the inequality (2) in terms of the probabilities of the outcomes. Relabeling the outcomes of each measurement $\{\pm 1\} \mapsto \{0, 1\}$, we can rewrite the inequality as

$$\begin{aligned} 4 &+ 8[m_{010} + m_{101} - (m_{000} + m_{001} + m_{011} + m_{100} + m_{110} \\ &+ m_{111})] + 8[q_{00} + q_{11} + r_{01} + r_{10} + s_{00} + s_{11}] \\ &- 4[a_0 + a_1 + b_0 + b_1 + c_0 + c_1] \leqslant 4, \end{aligned} \tag{11}$$

where

$$\begin{aligned} &m_{ijk} = P(000|A_iB_jC_k),\ q_{ij} = P(00|A_iB_j), \\ &r_{jk} = P(00|B_jC_k),\ s_{ik} = P(00|A_iC_k),\ a_i = P(0|A_i), \\ &b_j = P(0|B_j),\ c_k = P(0|C_k),\ \forall i, j, k \in \{0, 1\}. \end{aligned}$$

Theorem 2. For the Svetlichny-type nonlocality test, with symmetric detectors, the CDE becomes $\eta^{S} = \frac{\sqrt{\beta^2+4|\alpha|\gamma} - \mathrm{sgn}(\alpha)\times\beta}{2|\alpha|}$, where

$$\beta := 2\sum_{i,j,k}[\overline{(i\oplus j)}q_{ij} + (j\oplus k)r_{ij} + \overline{(i\oplus k)}s_{ik}],$$

$$\alpha := 2\sum_{i,j,k}(-1)^{(\bar{i}.\bar{j}+j.k+i.\bar{k})}m_{ijk},\ \gamma := \sum_{i,j,k}(a_i + b_j + c_k),$$

$$\mathrm{sgn}(\alpha) = 1,\ \text{for } \alpha \geqslant 0 \text{ and } \mathrm{sgn}(\alpha) = -1,\ \text{otherwise}.$$

Proof. The Svetlichny's inequality, in the form (11), can be violated with a properly chosen quantum setting, whenever

$$\begin{aligned} 8[&m_{010} + m_{101} - (m_{000} + m_{001} + m_{011} + m_{100} + m_{110} \\ &+ m_{111})] + 8[q_{00} + q_{11} + r_{01} + r_{10} + s_{00} + s_{11}] \\ &- 4[a_0 + a_1 + b_0 + b_1 + c_0 + c_1] > 0. \end{aligned}$$

Now, consider the situation where all three imperfect detectors are of the same detection efficiency η. Then replacing all the theoretical probabilities with the observed probabilities, we can rewrite the above inequality as

$$\begin{aligned} \sum_{i,j,k\in\{0,1\}} &\Big[(-1)^{(\bar{i}.\bar{j}+j.k+i.\bar{k})}2\eta^3 m_{ijk} + 2\eta^2\big\{\overline{(i\oplus j)}q_{ij} \\ &+ (j\oplus k)r_{jk} + \overline{(i\oplus k)}s_{ik}\big\} - \eta(a_i + b_j + c_k)\Big] > 0. \end{aligned}$$

The detection efficiency being always positive, the above inequality simplifies to

$$\alpha\eta^2 + \beta\eta - \gamma > 0,$$

where α, β, and γ are same as defined earlier.

Note that any quantum settings that exhibit Svetlichny-type nonlocality must satisfy the above inequality at least for $\eta = 1$, i.e., for the perfect detector. Hence $\alpha + \beta + \gamma > 0$. This, together with the fact that $\beta \geqslant 0$ and $\gamma \geqslant 0$, provides $\eta = \frac{\sqrt{\beta^2+4|\alpha|\gamma}-\mathbf{sgn}(\alpha)\times\beta}{2|\alpha|}$ for satisfying the equation $\alpha\eta^2 + \beta\eta - \gamma = 0$. Note that the $\text{sgn}(\alpha)$ function incorporates that α can be either positive or negative. Finally, noting that all the parameters α, β, γ are the parameters generated from the quantum settings, i.e., state and measurements, we can identify η as η^{S}—the CDE for Svetlichny-type nonlocality test. ■

Notably, unlike Theorem 1, the bound derived in Theorem 2 involves only the symmetric scenario, i.e., where all three detectors are of the same detection efficiency. Optimizing over all possible quantum settings numerically, it can be shown that the $\eta^S_{\text{min}} \simeq 88.1\%$ is sufficient for demonstrating Svetlichny-type nonlocality [48]. Additionally, when two of the detectors are perfect, then the numerical optimization reveals that the MDE corresponds to the third detector is $(\eta^S_{\text{min}})_C \simeq 51\%$ [48].

VI. DISCUSSION

In summary, we have estimated the minimum detection efficiency required for loophole-free tests of various forms of genuine nonlocality exhibited by multipartite quantum systems. In particular, taking the operational framework into account we have investigated both the time-ordered (T_2) genuine nonlocality, as well as the traditional Svetlichny-type genuine nonlocality tests. To this goal, we have primarily introduced the notion of cutoff detection efficiency which explicitly depends upon the quantum settings used to demonstrate a particular Bell nonlocality. Then systematically we have reached the minimum detection efficiency for the concerned nonlocality experiment, optimizing over all possible quantum states and measurement settings. Within this framework, our study reveals that, to demonstrate loophole-free T_2 nonlocality in symmetric cases, a detection efficiency of 75% at each site is both necessary and sufficient. This requirement can be relaxed to 50% when any two out of the three parties possess perfect detection efficiency. Interestingly, our results suggest that achieving the MDE for violating T_2 inequality is more feasible in the scenarios where the correlations nearly breach the inequality, instead of demonstrating the maximal violations. Further, we have shown that in the presence of a significantly low white noise, the range of quantum settings showing T_2 nonlocality sharply decreases. Nevertheless, for the given inefficient detectors and a permissible noise limit, our analysis characterizes the range of quantum settings viable to exhibit genuine nonlocality in experiments. On the other hand, we have also established that a detection efficiency of 88.1% is deemed sufficient for demonstrating Svetlichny nonlocality, which is well below the previously estimated one.

Though Svetlichny's inequality and T_2 inequality both are distinct facets of the set of the T_2 correlations, the set of all T_2 local correlations is a strict subset of that of the Svetlichny-local correlations. This in turn admits that the correlations violating Svetlichny's inequality are stronger than those violating the T_2 inequality and hence demand more sophistication in the involved experimental setup. Thus, as a consequence of our results, the investigation and comparison of MDE across distinct nonlocal classes can emerge as a compelling avenue for future research.

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[1] V. Scarani, The device-independent outlook on quantum physics (lecture notes on the power of Bell's theorem), Acta Physica Slovaca **62**, 347 (2012).

[2] A. K. Ekert, Quantum cryptography based on Bell's theorem, Phys. Rev. Lett. **67**, 661 (1991).

[3] J. Barrett, L. Hardy, and A. Kent, No signaling and quantum key distribution, Phys. Rev. Lett. **95**, 010503 (2005).

[4] A. Acín, N. Gisin, and L. Masanes, From Bell's theorem to secure quantum key distribution, Phys. Rev. Lett. **97**, 120405 (2006).

[5] S. Pironio, A. Acín, S. Massar, A. Boyer de la Giroday, D. N. Matsukevich, P. Maunz, S. Olmschenk, D. Hayes, L. Luo, T. A. Manning, and C. Monroe, Random numbers certified by Bell's theorem, Nature (London) **464**, 1021 (2010).

[6] R. Colbeck and R. Renner, Free randomness can be amplified, Nature Phys. **8**, 450 (2012).

[7] A. Chaturvedi and M. Banik, Measurement-device-independent randomness from local entangled states, Europhys. Lett. **112**, 30003 (2015).

[8] A. Mukherjee, A. Roy, S. S. Bhattacharya, S. Das, Md. R. Gazi, and M. Banik, Hardy's test as a device-independent dimension witness, Phys. Rev. A **92**, 022302 (2015).

[9] N. Brunner and N. Linden, Connection between Bell nonlocality and Bayesian game theory, Nat. Commun. **4**, 2057 (2013).

[10] A. Pappa, N. Kumar, T. Lawson, M. Santha, S. Zhang, E. Diamanti, and I. Kerenidis, Nonlocality and conflicting interest games, Phys. Rev. Lett. **114**, 020401 (2015).

[11] H. Buhrman, R. Cleve, S. Massar, and R. de Wolf, Nonlocality and communication complexity, Rev. Mod. Phys. **82**, 665 (2010).

[12] A. Roy, A. Mukherjee, T. Guha, S. Ghosh, S. S. Bhattacharya, and M. Banik, Nonlocal correlations: Fair and unfair

strategies in Bayesian games, Phys. Rev. A **94**, 032120 (2016).

[13] M. Banik, S. S. Bhattacharya, N. Ganguly, T. Guha, A. Mukherjee, A. Rai, and A. Roy, Two-qubit pure entanglement as optimal social welfare resource in Bayesian game, Quantum **3**, 185 (2019).

[14] J. S. Bell, On the Einstein Podolsky Rosen paradox, Phys. Phys. Fiz. **1**, 195 (1964).

[15] J. S. Bell, On the problem of hidden variables in quantum mechanics, Rev. Mod. Phys. **38**, 447 (1966).

[16] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Proposed experiment to test local hidden-variable theories, Phys. Rev. Lett. **23**, 880 (1969).

[17] G. Svetlichny, Distinguishing three-body from two-body nonseparability by a Bell-type inequality, Phys. Rev. D **35**, 3066 (1987).

[18] R. Gallego, L. E. Würflinger, A. Acín, and M. Navascués, Operational framework for nonlocality, Phys. Rev. Lett. **109**, 070401 (2012).

[19] J.-D. Bancal, J. Barrett, N. Gisin, and S. Pironio, Definitions of multipartite nonlocality, Phys. Rev. A **88**, 014102 (2013).

[20] S. Dutta, A. Mukherjee and M. Banik, Operational characterization of multipartite nonlocal correlations, Phys. Rev. A **102**, 052218 (2020).

[21] Q. Chen, S. Yu, C. Zhang, C. H. Lai, and C. H. Oh, Test of genuine multipartite nonlocality without inequalities, Phys. Rev. Lett. **112**, 140404 (2014).

[22] T. Holz, H. Kampermann, and D. Bruß, Genuine multipartite Bell inequality for device-independent conference key agreement, Phys. Rev. Res. **2**, 023251 (2020).

[23] J.-D. Bancal, N. Gisin, Y.-C. Liang, and S. Pironio, Device-independent witnesses of genuine multipartite entanglement, Phys. Rev. Lett. **106**, 250404 (2011).

[24] J.-D. Bancal, *On the Device-Independent Approach to Quantum Physics*, Springer Theses (Springer, Cham, 2013).

[25] Y.-C. Liang, D. Rosset, J.-D. Bancal, G. Pütz, T. J. Barnea, and N. Gisin, Family of Bell-like inequalities as device-independent witnesses for entanglement depth, Phys. Rev. Lett. **114**, 190401 (2015).

[26] E. Woodhead, B. Bourdoncle, A. Acín, Randomness versus nonlocality in the Mermin-Bell experiment with three parties, Quantum **2**, 82 (2018).

[27] L. Aolita, R. Gallego, A. Cabello, and A. Acín, Fully nonlocal, monogamous, and random genuinely multipartite quantum correlations, Phys. Rev. Lett. **108**, 100401 (2012).

[28] P. M. Pearle, Hidden-variable example based upon data rejection, Phys. Rev. D **2**, 1418 (1970).

[29] A. Garg and N. D. Mermin, Detector inefficiencies in the Einstein-Podolsky-Rosen experiment, Phys. Rev. D **35**, 3831 (1987).

[30] J.-Å. Larsson, Bell's inequality and detector inefficiency, Phys. Rev. A **57**, 3304 (1998).

[31] P. H. Eberhard, Background level and counter efficiencies required for a loophole-free Einstein-Podolsky-Rosen experiment, Phys. Rev. A **47**, R747(R) (1993).

[32] A. Cabello and J.-Å. Larsson, Minimum detection efficiency for a loophole-free atom-photon Bell experiment, Phys. Rev. Lett. **98**, 220402 (2007).

[33] N. Brunner, N. Gisin, V. Scarani, and C. Simon, Detection loophole in asymmetric Bell experiments, Phys. Rev. Lett. **98**, 220403 (2007).

[34] B. B. Blinov, D. L. Moehring, L.-M. Duan and C. Monroe, Observation of entanglement between a single trapped atom and a single photon, Nature (London) **428**, 153 (2004).

[35] B. Hanson, H. Bernien, A. E. Dréau, A. Reiserer, N. Kalb, M. S. Blok, J. Ruitenberg, R. F. L. Vermeulen, R. N. Schouten, C. Abellán *et al.*, Loophole-free Bell inequality violation using electron spins separated by 1.3 kilometres, Nature (London) **526**, 682 (2015).

[36] M. Giustina, M. A. M. Versteegh, S. Wengerowsky, J. Handsteiner, A. Hochrainer, K. Phelan, F. Steinlechner, J. Kofler, J.-Å. Larsson, C. Abellán *et al.*, Significant-loophole-free test of Bell's theorem with entangled photons, Phys. Rev. Lett. **115**, 250401 (2015).

[37] L. K. Shalm, E. Meyer-Scott, B. G. Christensen, P. Bierhorst, M. A. Wayne, M. J. Stevens, T. Gerrits, S. Glancy, D. R. Hamel, M. S. Allman *et al.*, Strong loophole-free test of local realism, Phys. Rev. Lett. **115**, 250402 (2015).

[38] S. Storz, J. Schär, A. Kulikov, P. Magnard, P. Kurpiers, J. Lütolf, T. Walter, A. Copetudo, K. Reuer, A. Akin *et al.*, Loophole-free Bell inequality violation with superconducting circuits, Nature (London) **617**, 265 (2023).

[39] J.-Å. Larsson and J. Semitecolos, Strict detector-efficiency bounds for n-site Clauser-Horne inequalities, Phys. Rev. A **63**, 022117 (2001).

[40] A. Cabello, D. Rodríguez, and I. Villanueva, Necessary and sufficient detection efficiency for the Mermin inequalities, Phys. Rev. Lett. **101**, 120402 (2008).

[41] Y. Xiang, H.-X. Wang, and F.-Y. Hong, Detection efficiency in the loophole-free violation of Svetlichny's inequality, Phys. Rev. A **86**, 034102 (2012).

[42] V. Gebhart and A. Smerzi, Coincidence postselection for genuine multipartite nonlocality: Causal diagrams and threshold efficiencies, Phys. Rev. A **106**, 062202 (2022).

[43] S. B. Ghosh, S. Roy Chowdhury, G. Kar, A. Roy, T. Guha, M. Banik, Quantum nonlocality: Multi-copy resource inter-convertibility & their asymptotic inequivalence, arXiv:2310.16386.

[44] H. Jeong, A. Zavatta, M. Kang, S.-W. Lee, L. S. Costanzo, S. Grandi, T. C. Ralph, M. Bellini, Generation of hybrid entanglement of light, Nat. Photonics **8**, 564 (2014).

[45] J. Wen, I. Novikova, C. Qian, C. Zhang, H. Du, Hybrid entanglement between optical discrete polarizations and continuous quadrature variables, Photonics **8**, 552 (2021).

[46] A. Feist, G. Huang, G. Arend, Y. Yang, J.-W. Henke, A. S. Raja, F. J. Kappert, R. N. Wang, H. Lourenço-Martins, Z. Qiu *et al.*, Cavity-mediated electron-photon pairs, Science **377**, 777 (2022).

[47] M. He and R. Malaney, Teleportation of hybrid entangled states with continuous-variable entanglement, Sci. Rep. **12**, 17169 (2022).

[48] The numerical optimization is mainly based on the NPA hierarchy [see Navascues *et al.*, New J. Phys. **10**, 073013 (2008)] and Seesaw algorithm. Interested readers can contact the corresponding author for the code.