Multiple scattering of coherent light in a random medium containing optical turbulence and randomly distributed particles

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We study multiple scattering of light in a random medium containing both optical turbulence and randomly distributed particles. Based on the form of dielectric correlation function attained, the Dyson equation and Bethe-Salpeter equation for scalar waves are derived. The Bethe-Salpeter equation is then expanded according to the number of scattering times and then simplified to a practical expression for calculations after manipulations. The numerical scheme allows to calculate the field-field correlation function of scalar wave multiple scattering by particles larger than wavelength, and that of propagation in a medium containing both optical turbulence and particles. Theoretical analysis and numerical calculations may have applications in light detection and ranging (LIDAR) and underwater optical systems.

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I. INTRODUCTION

Light in random media has become a widespread research area in the past decades [1–7]. A medium is called random if it varies with time and space, such as optical turbulence, a group of particles, and rough surfaces. When light is incident into these kinds of media, it goes along a totally random path, resulting in a random distribution of light field. Each kind of random media has been studied extensively, but it is also worthwhile to look into various combinations of them. Recent observations have shown that light scattering simultaneously by both rough surface and volumetric disorder leads to brand new effects, and the dependence of diffused light on statistical information of disorder is given [8]. Several papers reported the experimental results of light behaviors in a particles group, in underwater optical turbulence, and in the mixture of them [9–11].

Light fields in either optical turbulence or the particles group are governed by the Lippmann-Schwinger equation, in which the dielectric permittivity is a random variable [12]. In optical turbulence, the velocity field disturbs the stable temperature stratification (as well as the stable salinity stratification in ocean water), and leads to fluctuations of the spatial and temporal distribution of temperature (and salinity) and then fluctuations of the dielectric value [13,14]. These fluctuations are smooth and gently distort the incident light, leading to phenomena including scintillation, beam wander, and beam spread [3]. In a group of particles, however, there are abrupt changes of dielectric values at a boundary of each particle, which severely scatter the incident light [5,15,16]. A light beam will soon lose its incident direction, which is called diffusion [6,7], leading to intensity attenuation, speckle patterns, and weak localization [6,17,18].

Although these differences of light behavior in optical turbulence and particles bring different assumptions and treatments, the phenomena caused by them can both be interpreted with the statistical method, mainly involving statistical moments of field [3,6]. This makes it possible to theoretically describe light scattering in the presence of both optical turbulence and particles. For example, the first-order Born approximation and Rytov perturbation method were used in Ref. [19] to derive the second-order fluctuations of light in that case, and they gave the field correlation function in the case of spherical particles and von Kármán turbulence power spectrum. The intensity fluctuation of the reflected field from a layer of the particles group and turbulent atmosphere was investigated using the Rytov method [20]. Regarding the simulation approach, the authors of Ref. [21] studied the intensity distribution of the light beam and probability density distribution of intensity by Monte Carlo simulation, where the light beam passed alternately through the random phase screens and particles groups. In addition, temporal statistical properties of light scattered by particles suspended in a turbulent fluid were investigated, to be used as a tool for gaining information on the velocity field of turbulence [22]. The backscattering of acoustic waves by particles laden in a turbulent jet, as well as the situation where light scattering by an object submerged in inhomogeneous media have also been studied [23,24].

However, in the medium that optical turbulence and particles are totally mixed, a light field may be scattered by either the turbulent potential or particles potential, with random scattering times and at random positions. In addition, the probability of taking the potential of either type is supposed to be included in every scattering event. To analyze multiple scattering, a diagrammatic representation has been used to solve moments of fields in the particles group [6,7,25], while in optical turbulence, the Born approximation was used to discuss light propagation [3,14,26]. Accordingly, we adopt multiple scattering theory and the method of the Born series in

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this paper to investigate light behavior in the presence of both optical turbulence and particles. We also provide a numerical scheme and discuss on the average field and second-order field correlation function. The issue studied in this paper will benefit research on light scattering and transport in complex environments where optical turbulence and particles coexist.

This paper is organized as follows. In Sec. II we discuss the scattering process in the mixture of optical turbulence and particles and derive the expressions of dielectric correlation functions. In Sec. III we give the general expression of Bethe-Salpeter (BS) equation in the mixed medium. To enter numerical calculations, the BS equation is then expanded and transformed to Fourier space to obtain a practical expression. We also compare the theoretical framework proposed here with approaches in previous publications. Numerical calculations are performed in Sec. IV to plot second-order field correlation functions, and these functions are compared between the mixed medium and the pure medium. Finally, in Sec. V we conclude with the main results and some prospective applications.

II. AVERAGE FIELD OF LIGHT AND DIELECTRIC STATISTICS IN A RANDOM MEDIUM CONTAINING OPTICAL TURBULENCE AND PARTICLES

A. Dyson equation

A light field scattering in random media obeys the integral equation [12]

$$E(\mathbf{r}) = E_0(\mathbf{r}) + k_0^2 \int d\mathbf{r}' G_0(\mathbf{r}, \mathbf{r}')\delta\epsilon(\mathbf{r}')E(\mathbf{r}'), \tag{1}$$

which is the solution of scalar Helmholtz equation. Equation (1) is derived from the vector integral equation, and it holds when the depolarization effect is neglected [3]. $E(\mathbf{r})$ and $E_0(\mathbf{r})$ represent the total field and incident field at location $\mathbf{r}$, respectively. $k_0 = 2\pi/\lambda$ is the wave number in a host medium with λ the illuminating wavelength. $\delta\epsilon(\mathbf{r}') = \epsilon(\mathbf{r}') - \langle\epsilon(\mathbf{r}')\rangle$ means the fluctuation of the dielectric value at $\mathbf{r}'$ from the mean value, with $\langle\cdot\rangle$ the ensemble average. $G_0(\mathbf{r}, \mathbf{r}')$ is the free-space Green's function, and we have $d\mathbf{r}' = d^3r'$ in the three-dimensional (3D) problem. The iterative form of Eq. (1) implies multiple scattering, so we expand it and perform ensemble averaging. After some manipulation we obtain the average field [6,7,27]

$$\langle E(\mathbf{r})\rangle = \int d\mathbf{r}' \langle G(\mathbf{r}, \mathbf{r}_0)\rangle E_0(\mathbf{r}_0), \tag{2}$$

where $\langle\cdot\rangle$ means the ensemble average, and $\langle G\rangle$ is the average Green's function, obeying

$$\langle G(\mathbf{r}, \mathbf{r}_0)\rangle = G_0(\mathbf{r}, \mathbf{r}_0) + \int d\mathbf{r}' d\mathbf{r}'' \times G_0(\mathbf{r}, \mathbf{r}'')\Sigma(\mathbf{r}'', \mathbf{r}')\langle G(\mathbf{r}', \mathbf{r}_0)\rangle, \tag{3}$$

which is called the Dyson equation. In this equation, $\Sigma(\mathbf{r}'', \mathbf{r}')$ represents self-energy, consisting of all irreducible diagrams in diagrammatic representation [6]. In the weak-scattering regime, namely, $k_0\ell \gg 1$ where ℓ is the scattering mean free path, higher orders can be neglected and only the second term remains, yielding [2,25,28]

$$\Sigma(\mathbf{r}'', \mathbf{r}') \approx k_0^4 G_0(\mathbf{r}'', \mathbf{r}')B(\mathbf{r}'', \mathbf{r}'), \tag{4}$$

where $B(\mathbf{r}'', \mathbf{r}')$ is the second-order dielectric correlation function. In addition, the first term in the self-energy is zero because $\langle\delta\epsilon(\mathbf{r})\rangle = 0$. To calculate the average Green's function from the expression of self-energy, we require the Fourier transform of Eq. (3), which is given by [6,7]

$$\langle G(\mathbf{k})\rangle = \frac{1}{k^2 - k_0^2 - \Sigma(\mathbf{k})}, \tag{5}$$

where $\mathbf{k}$ is the coordinate in Fourier space. $\Sigma(\mathbf{k})$ satisfies $\Sigma(\mathbf{k}) \ll k_0^2$ in the weak-scattering regime, which means that the average Green's function in Eq. (5) is peaked around k_0. In a statistically translationally invariant and isotropic random medium, Σ is independent of the direction of $\mathbf{k}$ [6,29]. In this case we can keep $|\mathbf{k}| = k_0$ in the calculation of $\Sigma(\mathbf{k})$. Then the definition of the effective wave number k_{eff} is given by [6,7]

$$k_{\text{eff}}^2 = k_0^2 + \Sigma(k_0). \tag{6}$$

By noticing $|\Sigma(k_0)| \ll k_0^2$ we get [6,7]

$$k_{\text{eff}} \simeq k_0 + \frac{i}{2\ell}, \tag{7}$$

where $i = \sqrt{-1}$. The presence of ℓ results from the attenuation of a light field because of scattering, and this is always neglected in clear water or air optical turbulence (clear turbulence means that there are no other scattering media causing optical inhomogeneity except for turbulence).

B. Dielectric correlation functions

When introducing the average field, we are aware of the importance of dielectric correlation functions, which will be derived in this subsection. Optical turbulence and a group of particles both result in dielectric fluctuations. Since light travels much more quickly than the temporal variation of a scattering medium, the medium can be considered as a single realization for a single moment. Also, the dielectric value $\epsilon(\mathbf{r})$ in a specific position $\mathbf{r}$ in the presence of turbulence and particles can be regarded as a random variable. Considering the possibility of either turbulence and particle, $\epsilon(\mathbf{r})$ can be expressed by

$$\epsilon(\mathbf{r}) = \langle\epsilon(\mathbf{r})\rangle + \begin{cases} \delta\epsilon_t(\mathbf{r}), & \text{for turbulence,} \\ \delta\epsilon_p(\mathbf{r}), & \text{for particles,} \end{cases} \tag{8}$$

where we introduced $\delta\epsilon_t(\mathbf{r})$ and $\delta\epsilon_p(\mathbf{r})$ to represent the dielectric fluctuations originating from the turbulence potential and particle potential, respectively, for they cannot occupy one same location at the same time. $\langle\epsilon(\mathbf{r})\rangle$ means the ensemble average of $\epsilon(\mathbf{r})$ over the entire medium, with $\delta\epsilon(\mathbf{r}) = \epsilon(\mathbf{r}) - \langle\epsilon(\mathbf{r})\rangle$. Then the dielectric correlation function of the nth order is given by

$$B(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n) = \langle\delta\epsilon(\mathbf{r}_1)\delta\epsilon(\mathbf{r}_2)\ldots\delta\epsilon(\mathbf{r}_n)\rangle = \sum_{j_1, j_2, \ldots, j_n} \langle\delta\epsilon_{j_1}(\mathbf{r}_1)\delta\epsilon_{j_2}(\mathbf{r}_2)\ldots\delta\epsilon_{j_n}(\mathbf{r}_n)\rangle, \tag{9}$$

where $j_1, j_2, \ldots, j_n \in [t, p]$, with t and p corresponding to the turbulence and particles, respectively. Since we pay attention to how light behaves when the two media are combined, the dynamical interactions of turbulence with particles are neglected. Then Eq. (9) becomes

$$\langle[\delta\epsilon_t(\mathbf{r}_1)\ldots\delta\epsilon_t(\mathbf{r}_m)][\delta\epsilon_p(\mathbf{r}_1)\ldots\delta\epsilon_p(\mathbf{r}_n)]\rangle = \langle[\delta\epsilon_t(\mathbf{r}_1)\ldots\delta\epsilon_t(\mathbf{r}_m)]\rangle\langle[\delta\epsilon_p(\mathbf{r}_1)\ldots\delta\epsilon_p(\mathbf{r}_n)]\rangle. \quad (10)$$

Particularly, the second-order correlation function of Eq. (10) is given by

$$B(\mathbf{r}_1, \mathbf{r}_2) = B_t(\mathbf{r}_1, \mathbf{r}_2) + B_p(\mathbf{r}_1, \mathbf{r}_2) \quad (11)$$

in which we use B_t and B_p to refer to the correlation function of the turbulence and particles, respectively, and also used that $\langle\delta\epsilon_t(\mathbf{r}_1)\rangle\langle\delta\epsilon_p(\mathbf{r}_2)\rangle$ is approximately equal to zero. The calculations of B_t and B_p are different, which are given in Appendix A, in the case of spherical particles, and a given power spectrum of optical turbulence.

The dielectric correlation functions given in this subsection eliminates the need to distinguish whether each scattering event is due to the turbulence or particles. Now we only need to substitute Eq. (11) into Eq. (4) to get the expression of self-energy

$$\Sigma(\mathbf{r}'', \mathbf{r}') \approx k_0^4 G_0(\mathbf{r}'', \mathbf{r}')[B_p(\mathbf{r}'', \mathbf{r}') + B_t(\mathbf{r}'', \mathbf{r}')]. \quad (12)$$

III. FIELD-FIELD CORRELATIONS IN A RANDOM MEDIUM CONTAINING OPTICAL TURBULENCE AND PARTICLES

A. Bethe-Salpeter equation

A complete description of light transport requires not only average field but also field correlations. The spatial correlation function of electric field in random media obeys the Bethe-Salpeter equation [4,6,7]

$$\langle E(\mathbf{r})E^*(\mathbf{r}')\rangle = \langle E(\mathbf{r})\rangle\langle E^*(\mathbf{r}')\rangle + \int d\mathbf{r}_2 d\mathbf{r}_2' d\mathbf{r}_1 d\mathbf{r}_1' \langle G(\mathbf{r}, \mathbf{r}_2)\rangle\langle G^*(\mathbf{r}', \mathbf{r}_2')\rangle \times \Gamma(\mathbf{r}_2, \mathbf{r}_1; \mathbf{r}_2', \mathbf{r}_1')\langle E(\mathbf{r}_1)E^*(\mathbf{r}_1')\rangle, \quad (13)$$

where $\langle G\rangle$ represents the average Green's function, and $*$ denotes complex conjugation. Γ is the irreducible vertex, describing all possible scattering sequences between four points. The first term on the right-hand side of Eq. (13) represents the ballistic intensity, namely, the uncorrelated part. The second term is the correlated part, originating from multiple scattering process. Γ has infinite terms, but in the weak-scattering regime, namely, $k_0\ell \gg 1$, it reduces to

$$\Gamma(\mathbf{r}_2, \mathbf{r}_1; \mathbf{r}_2', \mathbf{r}_1') \approx k_0^4 B(\mathbf{r}_1, \mathbf{r}_1')\delta(\mathbf{r}_1 - \mathbf{r}_2)\delta(\mathbf{r}_1' - \mathbf{r}_2'), \quad (14)$$

which is called the ladder approximation. In the presence of both turbulence and particles, we substitute Eq. (11) and Eq. (14) into Eq. (13) to get the Bethe-Salpeter equation

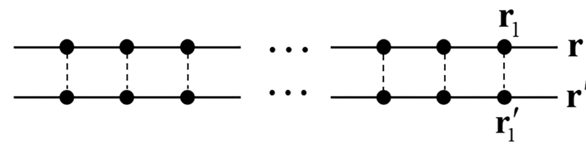

FIG. 1. Bethe-Salpeter equation in diagrammatic representation. A solid line represents a field. A circle represents a scattering potential. A dashed line means there are correlations between two potentials.

$$\langle E(\mathbf{r})E^*(\mathbf{r}')\rangle = \langle E(\mathbf{r})\rangle\langle E^*(\mathbf{r}')\rangle + k_0^4 \int d\mathbf{r}_1 d\mathbf{r}_1' \langle G(\mathbf{r}, \mathbf{r}_1)\rangle\langle G^*(\mathbf{r}', \mathbf{r}_1')\rangle \times [B_p(\mathbf{r}_1, \mathbf{r}_1') + B_t(\mathbf{r}_1, \mathbf{r}_1')]\langle E(\mathbf{r}_1)E^*(\mathbf{r}_1')\rangle, \quad (15)$$

whose diagrammatic representation is shown by Fig. 1. Each circle possibly takes either the turbulence potential or the particles potential in the medium containing both optical turbulence and a group of particles. A dashed line implies correlations between two potentials and corresponds to the dielectric correlation function in the BS equation.

In Eq. (15), the first term on the right-hand side is the ballistic intensity, decaying exponentially with scattering mean-free path along the propagation direction. The second part governs the correlations of two fields after multiple scattering, and it represents the phase correlations between two light fields propagating in close paths, if there are dielectric correlations in the scattering medium.

However, the BS equation cannot be solved exactly, unless $B(\mathbf{r}_1, \mathbf{r}_1')$ could be regarded as a delta function, corresponding to Gaussian white-noise model [6,7]. To calculate field-field correlations, we expand Eq. (15) into orders

$$\langle E(\mathbf{r})E^*(\mathbf{r}')\rangle = \langle E(\mathbf{r})\rangle\langle E^*(\mathbf{r}')\rangle + \sum_{m=1}^{\infty}\langle\delta E(\mathbf{r})\delta E^*(\mathbf{r}')\rangle_{(m)}, \quad (16)$$

where

$$\langle\delta E(\mathbf{r})\delta E^*(\mathbf{r}')\rangle_{(m)} = k_0^4 \int d\mathbf{r}_1 d\mathbf{r}_1' \langle\delta E(\mathbf{r}_1)\delta E^*(\mathbf{r}_1')\rangle_{(m-1)} \times \langle G(\mathbf{r}, \mathbf{r}_1)\rangle\langle G^*(\mathbf{r}', \mathbf{r}_1')\rangle[B_p(\mathbf{r}_1, \mathbf{r}_1') + B_t(\mathbf{r}_1, \mathbf{r}_1')] \quad (17)$$

represents the component that two fields experience m times correlated scattering events, also equivalent to the mth order of the Born series. Specifically, $\langle\delta E\delta E^*\rangle_{(m-1)}$ when $m = 1$ is actually the ballistic intensity $\langle E\rangle\langle E^*\rangle$. If a plane wave $E_0\exp(ik_0z)$ is incident, the average field at position z along its propagation distance is expressed by

$$\langle E(\mathbf{r})\rangle = E_0\exp(ik_0z)\exp\left(-\frac{z}{2\ell}\right), \quad (18)$$

using Eqs. (2) and (5). Then the ballistic intensity is given by

$$\langle E(\mathbf{r})\rangle\langle E^*(\mathbf{r}')\rangle = I_0 \exp[ik_0(z-z')]\exp\left[-\frac{(z+z')}{2\ell}\right], \quad (19)$$

with $I_0 = E_0^2$.

B. Numerical scheme

Equation (17) has illustrated that we can calculate $\langle\delta E\delta E^*\rangle_{(m)}$ by $\langle\delta E\delta E^*\rangle_{(m-1)}$, and thus calculate the total field-field correlation function approximately.

The form of Eq. (17) is not suitable for numerical calculations since it involves integrations over $\mathbf{r}_1$ and $\mathbf{r}_1'$, both of which are 3D coordinates. It is better to perform the calculation in Fourier space. When a plane wave is incident, the result could be expressed by (please refer to Appendix B for more details)

$$\begin{aligned}&\langle\delta E\delta E^*(\mathbf{k}_{(m)\perp},\eta_{(m)},0)\rangle_{(m)}\\&=\int\frac{d\mathbf{k}_{(m-1)\perp}}{(2\pi)^2}\frac{k_0^4B(\mathbf{k}_{(m)\perp}-\mathbf{k}_{(m-1)\perp},K_{(m)}-K_{(m-1)})}{4\left(K_{(m)}^2+K_{(m)}'^2\right)}\\&\quad\times\int_0^{\eta_{(m)}}\langle\delta E\delta E^*(\mathbf{k}_{(m-1)\perp},\eta_{(m-1)},0)\rangle_{(m-1)}\\&\quad\times e^{2K'_{(m)}[\eta_{(m-1)}-\eta_{(m)}]}d\eta_{(m-1)},\end{aligned} \quad (20)$$

where $\mathbf{k}_{(m)\perp}$ and $\eta_{(m)}$ represent the coordinates of $\langle\delta E\delta E^*\rangle_{(m)}$, with $\mathbf{k}_{(m)\perp}$ the transverse coordinates in Fourier space, and $\eta_{(m)}$ the propagation distance in real space. $B = B_t + B_p$. And $K_{(m)}$, $K'_{(m)}$ are given by ($K'_{(m)} > 0$)

$$\sqrt{k_0^2+\frac{ik_0}{\ell}-k_{(m)\perp}^2} = K_{(m)} + iK'_{(m)}, \quad (21)$$

$$\sqrt{k_0^2-\frac{ik_0}{\ell}-k_{(m)\perp}^2} = K_{(m)} - iK'_{(m)}. \quad (22)$$

The function

$$\begin{aligned}\langle\delta E\delta E^*(\mathbf{k}_{(m)\perp},\eta_{(m)},0)\rangle_{(m)} &= \int\langle\delta E\delta E^*(\Delta\mathbf{r}_{(m)\perp},\eta_{(m)},0)\rangle_{(m)}\\&\quad\times e^{-i\mathbf{k}_{(m)\perp}\cdot\Delta\mathbf{r}_{(m)\perp}}d\Delta\mathbf{r}_{(m)\perp}\end{aligned} \quad (23)$$

is a two-dimensional (2D) Fourier transform, with $\Delta\mathbf{r}_{(m)\perp} = \mathbf{r}_{(m)\perp} - \mathbf{r}'_{(m)\perp}$ the transverse separation distance of two observation points, where

$$\langle\delta E\delta E^*(\mathbf{r}_{(m)\perp},\eta_{(m)},\Delta z_{(m)})\rangle_{(m)}^{(p)} = \langle\delta E(\mathbf{r})\delta E^*(\mathbf{r}')\rangle_{(m)}^{(p)}. \quad (24)$$

Moreover, the first order of $\langle\delta E\delta E^*(\mathbf{k}_{(m)\perp},\eta_{(m)},0)\rangle_{(m)}$ is given by

$$\begin{aligned}&\langle\delta E\delta E^*(\mathbf{k}_{(1)\perp},\eta_{(1)},0)\rangle_{(1)}\\&=I_0\frac{e^{-2K'_{(1)}\eta_{(1)}}}{4\left(K_{(1)}^2+K_{(1)}'^2\right)}k_0^4B(\mathbf{k}_{(1)\perp},K_{(1)}-k_0)\int_0^{\eta_{(1)}}\\&\quad\times e^{-\frac{\eta_{(0)}}{\ell}}e^{2K'_{(1)}\eta_{(0)}}d\eta_{(0)}.\end{aligned} \quad (25)$$

Equation (20) has given a practical numerical scheme for $\langle\delta E\delta E^*\rangle_{(m)}$, which will be added to obtain $\langle\delta E\delta E^*\rangle$.

C. Comparison with existing approaches

The main result of this paper is the Bethe-Salpeter equation in the medium containing both optical turbulence and particles group, given by Eqs. (16) and (20). The method adopted stems directly from multiple scattering theory, so it immediately reduces to the expression of the BS equation if there is no turbulence. Moreover, we can also attain the field correlation function in clear water or air turbulence by setting ℓ in Eq. (16) to be infinity. After performing paraxial approximation onto the free-space Green's function, the result is actually the Born series used in optical turbulence.

Now we compare Eq. (16) with the results in previous publications. First, in Ref. [19] the authors investigated light-field transmission in the presence of both turbulence and particles, using the first-order Born approximation and Rytov method. When deriving the second-order correlation function of fields, a convolution of the correlation function of the field scattered by particles and the modulation of turbulence is performed. In this case, the BS equation developed here is basically identical to the main result in Ref. [19], by retaining $\langle\delta E(\mathbf{r})\delta E^*(\mathbf{r}')\rangle_{(m)}$ in Eq. (16) to second order, and keeping B_p in the first scattering event and B_t in the other scattering event. If optical turbulence and particles are totally mixed, we emphasize that their second-order dielectric correlation functions are approximately superimposed rather than multiplied. Also, Eq. (16) considers multiple scattering and light attenuation in Green's functions. Another difference is, the authors in Ref. [19] used the Rytov perturbation method to describe turbulent perturbations while we adopt the Born series in this paper.

In Ref. [21] the authors proposed a simulation method to investigate light propagation through optical turbulence and particles group. The random phase screen is used to model turbulence perturbation and the Monte Carlo method is used to simulate light scattering in particulate media with the Henyey-Greenstein phase function. The light beam passes through the phase screen and particles group alternately, but the situation considered in this paper is that the turbulence and particles are totally mixed. This means that the theoretical framework in this paper is more general, but the authors of Ref. [21] provided a practical simulation model instead of a theoretical approach.

IV. NUMERICAL ANALYSIS

The process of numerical calculations is demonstrated in Sec. III B. We will show the numerical results in this section. First, we denote the normalized field-field correlation functions $C_{(m)}(\Delta r_\perp)$ and $C(\Delta r_\perp)$ by

$$C_{(m)}(\Delta r_\perp) = \frac{\langle\delta E(\mathbf{r}_\perp)\delta E^*(\mathbf{r}'_\perp)\rangle_{(m)}}{\sum_{m=1}^M\langle|\delta E|^2\rangle_{(m)}}, \quad (26)$$

$$C(\Delta r_\perp) = \sum_{m=1}^M C_{(m)}(\Delta r_\perp), \quad (27)$$

where $\Delta r_\perp = |\mathbf{r}_\perp - \mathbf{r}'_\perp|$ is the distance between two observation points in the transverse plane and M is the maximum order in calculations.

The numerical calculations are first performed for the case without turbulence. We consider a situation that particles are spherical and identical in each group, with refractive

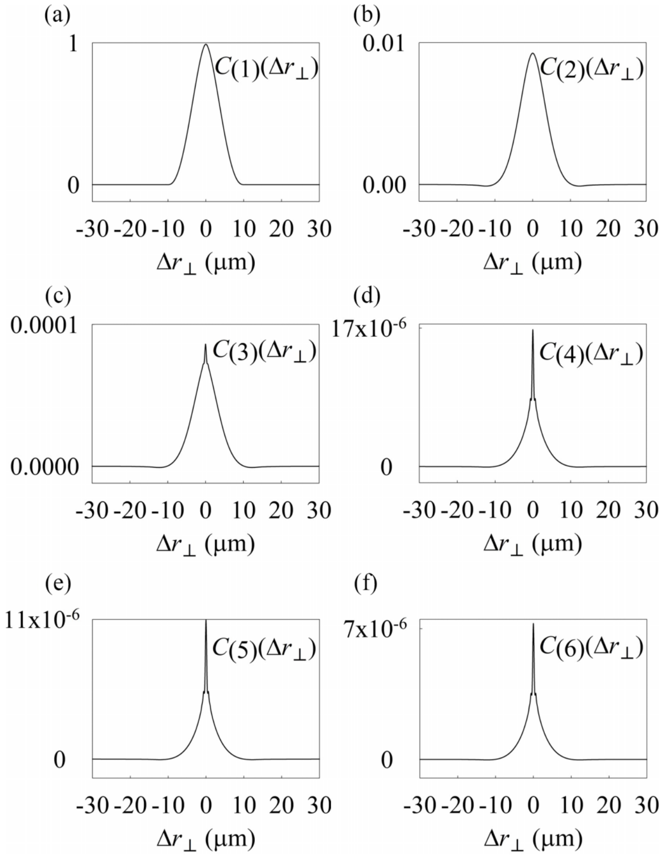

FIG. 2. First six orders of $C_{(m)}(\Delta r_\perp)$ when particle size is set to be 10 μm. Illuminating wavelength $\lambda = 532$ nm, refractive index of particle is 1.5, and refractive index of host medium is 1.34. Scattering mean free path $\ell = 0.205$ m, and propagation distance is 1 m.

index $n_p = 1.5$ and embedded in a host medium (oceanic water) with $n_0 = 1.34$. When calculating the ballistic intensity $\langle E\rangle\langle E^*\rangle$, scattering mean-free paths are obtained using Mie scattering coefficients.

Figure 2 shows the first six orders of $C_{(m)}(\Delta r_\perp)$ for 10 μm particles, from which we see that higher orders are much smaller than the lower, making their sum, namely, $C(\Delta r_\perp)$, soon converge. Then, Fig. 3 plots $C(\Delta r_\perp)$ for three different particle sizes, respectively. For the particles smaller than the illuminating wavelength, the results attained are consistent with those obtained through the exact theoretical analysis in previous publications [6,32]. For particles that are larger than wavelength but no more than 10 μm, the results illustrate that the spatial scale of the correlation function is approximately equal to the particle size.

The cases of particles larger than 10 μm are also calculated, but it turns out that $C(\Delta r_\perp)$ does not converge in this case. Instead, the intensity of higher orders is much more than the lower ones. On the other hand, those terms that are higher than third order have an identical curve form, which forces the total correlation function to have this form as well. It is possibly because a light field encounters more scattering times in a single particle if this particle has a larger size. This makes higher orders of $C_{(m)}(\Delta r_\perp)$ more important. We also found that the convergence is related to the difference of dielectric permittivity between scatterer and host medium.

Next, we perform numerical calculations for the cases that particles and optical turbulence coexist. We use three different power spectrum models of optical turbulence, as shown in Fig. 4. The peak width in the middle of each subfigure is consistent with Fig. 3(b), meaning that the contribution of particles still dominates diffuse intensity. On the other hand, the contribution of turbulence to $C(\Delta r_\perp)$ is smoothly distributed over a large spatial range, but its intensity is relatively much smaller. The reason is that turbulent eddies have much larger spatial scales than small particles while they have much weaker dielectric fluctuations. Thereby, the presence of particles brings significant influences on optical systems working in clear turbulence, including intensity attenuation and additional speckle patterns.

Figure 4 shows that both of the respective characteristics of turbulence and particles appear obviously in the correlation function, which results from the random distribution of scattering potentials of two types. In a medium that turbulence and particles are totally mixed, there is no sequential relationship for light scattering between two media. Instead, scattering events of each kind may occur at any position. When it comes to two correlated fields, they possibly encounter the same particles sequence to lead to a small-scale correlation, and also possibly encounter the same turbulent eddy to lead to

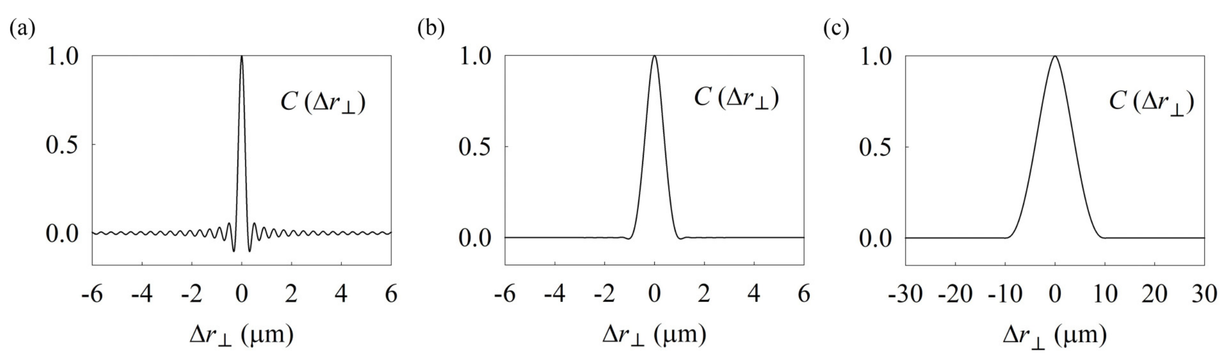

FIG. 3. Normalized correlation function $C(\Delta r_\perp)$ when the scattering medium consists of a group of particles. Particle sizes and mean free paths are set to be (a) 0.1 μm, $\ell = 0.200$ m, (b) 1 μm, $\ell = 0.265$ m, (c) 10 μm, $\ell = 0.205$ m, respectively. Illuminating wavelength $\lambda = 532$ nm, refractive index of particle is 1.5, and refractive index of host medium is 1.34. When calculating $C_{(m)}(\Delta r_\perp)$, the maximum order is up to $m = 6$. Propagation distance is 1 m.

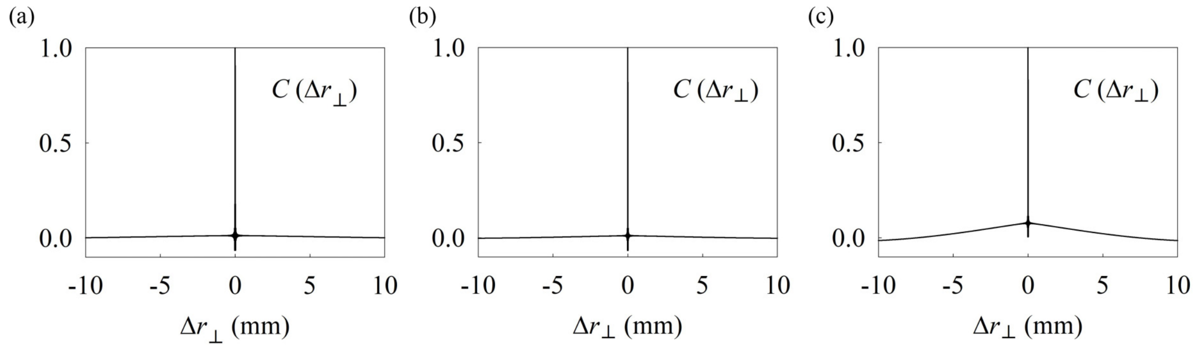

FIG. 4. Normalized correlation function $C(\Delta r_\perp)$ in the presence of both optical turbulence and 1 μm spherical particles. Refractive index power spectrum models used here are (a) von Kármán model [3], where the refractive index structural constant is set to be $C_n^2 = 1\times10^{-12}\ \mathrm{m}^{-2/3}$, and outer scale is 1 m. (b) Nikishovs' model [30], where the relative strength $\omega = -1$, Prandtl number $\mathrm{Pr}_T = 13.349$, Schmidt number $\mathrm{Pr}_S = 2393.2$, dissipation rate of the kinetic energy $\varepsilon = 1\times10^{-2}\ \mathrm{m}^2/\mathrm{s}^3$, kinematic viscosity $\nu = 18.534\times10^{-7}\ \mathrm{m}^2/\mathrm{s}$, temperature dissipation rates $\chi_T = 1\times10^{-5}\ \mathrm{K}^2/\mathrm{s}$, and Kolmogorov microscale $\eta = 15.885\times10^{-5}$ m. (c) Yao's model [31], where the parameters are set equally to that of Nikishovs'. Illuminating wavelength $\lambda = 532$ nm, refractive index of particle is 1.5, and refractive index of host medium is 1.34. Scattering mean free path $\ell = 0.265$ m, and propagation distance is 1 m. When calculating $C_{(m)}(\Delta r_\perp)$, the maximum order is up to $m = 6$.

a large-scale correlation. Both situations exist and may not necessarily affect each other. This is the main difference from the results in previous publications, such as Ref. [19] where a convolution of the correlation function of field scattered by particles and the modulation of turbulence was performed, and Ref. [21] that handled the two kinds of media sequentially. The method proposed in this article is capable of describing the statistical properties of the medium that optical turbulence and particles are totally mixed, as well as light scattering and transport in such a medium.

V. CONCLUSION

Light exhibits different behaviors when scattered by different scattering media, with particles and optical turbulence being two typical types. This paper proposes a theoretical method to describe light scattering in the situation that particles and optical turbulence are mixed and provides a numerical scheme. The theoretical method gives first-order and second-order statistical moments of the optical field by attaining the dielectric correlation function of the medium. Then, the BS equation representing the second-order moment of the field is given in a series expansion and enters numerical calculation after analytical derivation. The method proposed not only allows for the introduction of particle groups with particle sizes larger than the wavelength, but also approximately calculates the multiple scattering results when there are two different types of scatterers in the medium.

The actual calculation results indicate that the numerical scheme is feasible for particle sizes smaller than 10 micron (with a 0.45 difference in dielectric constant from the host medium). For a group of identical particles, the correlation scale of field correlation function is equal to the particle size. If there is also optical turbulence in the medium, the peak generated by particles still exists in the field correlation function, but the turbulence contribution will emerge with a larger spatial range and weaker intensity. These results reveal the property of light transport in a mixed medium and would be helpful in underwater optical communication and detection.

APPENDIX A: CALCULATION OF DIELECTRIC CORRELATION FUNCTIONS

In this Appendix, we demonstrate the calculation of the dielectric correlation function for the particles group, and also the process to get the dielectric correlation function from the refractive power spectrum in optical turbulence. We first consider a group of identical spherical particles, with dielectric permittivity ϵ_p, and the host medium has a dielectric permittivity ϵ_0. Then the dielectric correlation function is defined by

$$B_p(\mathbf{r},\mathbf{r}') = n_i \int \delta\epsilon_p(\mathbf{r}-\mathbf{r}_j)\delta\epsilon_p(\mathbf{r}'-\mathbf{r}_j)d\mathbf{r}_j, \qquad \text{(A1)}$$

with $\delta\epsilon_p = \epsilon_p - \epsilon_0$ and $\mathbf{r}_j$ is the position of the jth particle. $n_i = N/V$ means the density of particles, namely, the number of particles N in a volume V. A spherical particle has the potential that

$$\delta\epsilon_p(\mathbf{r}-\mathbf{r}_j) = \begin{cases} \delta\epsilon_p, & |\mathbf{r}-\mathbf{r}_j| \leqslant R_0, \\ 0, & |\mathbf{r}-\mathbf{r}_j| > R_0, \end{cases} \qquad \text{(A2)}$$

where R_0 is the radius of this particle. Taking the Fourier transform of Eq. (A2) leads to

$$\delta\epsilon_p(\mathbf{k}) = 4\pi\delta\epsilon_p R_0^3 \frac{j_1(kR_0)}{kR_0}, \qquad \text{(A3)}$$

with $j_1(\cdot)$ representing the spherical Bessel function of the first type and order 1. Also taking the Fourier transform of Eq. (A1) we get

$$B_p(\mathbf{k}) = n_i[\delta\epsilon_p(\mathbf{k})]^2 = \left(4\pi\delta\epsilon_p R_0^3\right)^2 n_i \left[\frac{j_1(kR_0)}{kR_0}\right]^2. \qquad \text{(A4)}$$

A particular case is, when R_0 is smaller than the illuminating wavelength, the Fourier transform of $B_p(\mathbf{k})$ will tend to be a delta function. The Fourier transform here is defined by

$$\tilde{f}(\mathbf{k}) = \int f(\mathbf{r})e^{-i\mathbf{k}\cdot\mathbf{r}}d\mathbf{r},$$
$$f(\mathbf{r}) = \int \tilde{f}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}}\frac{d\mathbf{k}}{(2\pi)^d}, \qquad \text{(A5)}$$

with d the dimension of the problem.

On the other hand, the statistics of optical turbulence is mainly about refractive index due to conventions. The refractive index in clear water/air turbulence is divided into two parts [3,14]

$$n(\mathbf{r}) = n_0 + \delta n(\mathbf{r}), \tag{A6}$$

where $n_0 = \langle n \rangle$ is the average refractive index and δn is the fluctuation part. (A more conventional notation is n_1, but we here use δn to ensure consistency of notations throughout the paper.) Then the dielectric permittivity is given by (the magnetic permeability is unity)

$$\epsilon(\mathbf{r}) = n^2(\mathbf{r}) \simeq n_0^2 + 2n_0\delta n(\mathbf{r}). \tag{A7}$$

Now the dielectric correlation function can be linked to refractive index correlation function by

$$\langle \delta\epsilon(\mathbf{r})\delta\epsilon(\mathbf{r}')\rangle = 4n_0^2\langle \delta n(\mathbf{r})\delta n(\mathbf{r}')\rangle. \tag{A8}$$

The refractive index power spectrum $\Phi_n(\mathbf{k})$ is defined by

$$\langle \delta n(\mathbf{r})\delta n(\mathbf{r}')\rangle = \int \Phi_n(\mathbf{k})\exp[i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')]d\mathbf{k}. \tag{A9}$$

Finally the dielectric correlation function can be expressed by

$$\begin{aligned} B_t(\mathbf{r},\mathbf{r}') &= \langle \delta\epsilon(\mathbf{r})\delta\epsilon(\mathbf{r}')\rangle \\ &= 4n_0^2\int \Phi_n(\mathbf{k})\exp[i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')]d^3k. \end{aligned} \tag{A10}$$

In addtion, in Fourier space

$$B_t(\mathbf{k}) = 4n_0^2(2\pi)^3\Phi_n(\mathbf{k}). \tag{A11}$$

APPENDIX B: ANALYTICAL CALCULATION OF $\langle \delta E\delta E^*\rangle_{(m)}$

The expression of the field-field correlation function needs to be simplified before getting into numerical calculation. In this Appendix, we demonstrate how to make it a suitable form for numerical calculations. The Fourier transform throughout this Appendix is given by Eq. (A5). First, the relation between the mth order and $(m-1)$th order is given by

$$\begin{aligned} \langle \delta E(\mathbf{r})\delta E^*(\mathbf{r}')\rangle_{(m)} = k_0^4\int d\mathbf{r}_1 d\mathbf{r}_1'\langle \delta E(\mathbf{r}_1)\delta E^*(\mathbf{r}_1')\rangle_{(m-1)} \\ \times\langle G(\mathbf{r},\mathbf{r}_1)\rangle\langle G^*(\mathbf{r}',\mathbf{r}_1')\rangle B(\mathbf{r}_1,\mathbf{r}_1'), \end{aligned} \tag{B1}$$

which is the same as Eq. (17). In translationally invariant random media, it is convenient to use cylindrical coordinates $(|\mathbf{r}_\perp|,\theta,z)$ if illuminated by a plane wave, and the z axis is aligned with the incident direction. Now we express every part in Eq. (B1) by Fourier integration, which is

$$\begin{aligned} \langle \delta E(\mathbf{r})\delta E^*(\mathbf{r}')\rangle_{(m)} = k_0^4\int d\mathbf{r}_1 d\mathbf{r}_1'\frac{d\mathbf{q}_\perp'}{(2\pi)^2}\frac{d\mathbf{k}}{(2\pi)^3}\frac{d\mathbf{k}'}{(2\pi)^3}\frac{d\mathbf{q}}{(2\pi)^3} \\ \times e^{i\mathbf{q}_\perp'\cdot(\mathbf{r}_{1\perp}-\mathbf{r}_{1\perp}')}\langle \delta E\delta E^*(\mathbf{q}_\perp',\eta_1,\Delta z_1)\rangle_{(m-1)} \\ \times\frac{e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}_1)}}{k^2-k_{\text{eff}}^2}\frac{e^{-i\mathbf{k}'\cdot(\mathbf{r}'-\mathbf{r}_1')}}{k'^2-k_{\text{eff}}^{*2}}e^{i\mathbf{q}\cdot(\mathbf{r}_1-\mathbf{r}_1')}B(\mathbf{q}), \end{aligned} \tag{B2}$$

where $\mathbf{k}$, $\mathbf{k}'$ and $\mathbf{q}$ are 3D coordinates in Fourier space, $\mathbf{q}_\perp'$ is the transverse coordinate in Fourier space, and

$$\begin{aligned} &\langle \delta E\delta E^*(\mathbf{q}_\perp',\eta_1,\Delta z_1)\rangle_{(m-1)} \\ &= \int d(\mathbf{r}_{1\perp}-\mathbf{r}_{1\perp}')e^{-i\mathbf{q}_\perp'\cdot(\mathbf{r}_{1\perp}-\mathbf{r}_{1\perp}')}\langle \delta E(\mathbf{r}_1)\delta E^*(\mathbf{r}_1')\rangle_{(m-1)}. \end{aligned} \tag{B3}$$

The field correlation function is circularly symmetric in transverse coordinates, but it possibly varies with the propagation distance $\eta = (z+z')/2$ and $\Delta z = z - z'$. Then we perform integration successively over $\mathbf{r}_{1\perp}$, $\mathbf{r}_{1\perp}'$, $\mathbf{k}_\perp'$, and $\mathbf{q}_\perp$, and obtain

$$\begin{aligned} \langle \delta E(\mathbf{r})\delta E^*(\mathbf{r}')\rangle_{(m)} = k_0^4\int dz_1 dz_1'\frac{d\mathbf{q}_\perp'}{(2\pi)^2}\frac{d\mathbf{k}}{(2\pi)^3}\frac{dk_z'}{(2\pi)}\frac{dq_z}{(2\pi)} \\ \times e^{i\mathbf{k}_\perp\cdot(\mathbf{r}_\perp-\mathbf{r}_\perp')}\langle \delta E\delta E^*(\mathbf{q}_\perp',\eta_1,\Delta z_1)\rangle_{(m-1)} \\ \times\frac{e^{ik_z(z-z_1)}}{k_z^2+k_\perp^2-k_{\text{eff}}^2}\frac{e^{-ik_z'(z'-z_1')}}{k_z'^2+k_\perp^2-k_{\text{eff}}^{*2}} \\ \times e^{iq_z(z_1-z_1')}B(\mathbf{k}_\perp-\mathbf{q}_\perp',q_z). \end{aligned} \tag{B4}$$

Next we integrate over k_z and k_z' to simplify the Green's function. These two integrations can be performed in the complex plane, by using contour integration and residue theorem [33]. The integration over k_z needs a contour in upper half plane while the integration over k_z' in lower half plane. The result is

$$\int\frac{dk_z}{(2\pi)}\frac{e^{ik_z(z-z_1)}}{k_z^2+k_\perp^2-k_{\text{eff}}^2} = \frac{ie^{i(K+iK')(z-z_1)}}{2(K+iK')}, \tag{B5}$$

$$\int\frac{dk_z'}{(2\pi)}\frac{e^{-ik_z'(z'-z_1')}}{k_z'^2+k_\perp^2-k_{\text{eff}}^{*2}} = -\frac{ie^{-i(K-iK')(z'-z_1')}}{2(K-iK')}. \tag{B6}$$

In these two integrations, the poles we used with the residue theorem are

$$\sqrt{k_{\text{eff}}^2-k_\perp^2} = K+iK', \tag{B7}$$

$$\sqrt{k_{\text{eff}}^{*2}-k_\perp^2} = K-iK', \tag{B8}$$

respectively, where $K' \geqslant 0$. Now we go back to $\langle \delta E\delta E^*\rangle_{(m)}$ with this simplification of the Green's functions and get

$$\begin{aligned} \langle \delta E(\mathbf{r})\delta E^*(\mathbf{r}')\rangle_{(m)} = k_0^4\int dz_1 dz_1'\frac{d\mathbf{q}_\perp'}{(2\pi)^2}\frac{d\mathbf{k}_\perp}{(2\pi)^2}\frac{dq_z}{(2\pi)} \\ \times e^{i\mathbf{k}_\perp\cdot(\mathbf{r}_\perp-\mathbf{r}_\perp')}\langle \delta E\delta E^*(\mathbf{q}_\perp',\eta_1,\Delta z_1)\rangle_{(m-1)} \\ \times\frac{e^{iK(z-z')}e^{-K'(z+z')}e^{-iK(z_1-z_1')}e^{K'(z_1+z_1')}}{4(K^2+K'^2)} \\ \times e^{iq_z(z_1-z_1')}B(\mathbf{k}_\perp-\mathbf{q}_\perp',q_z). \end{aligned} \tag{B9}$$

To continue, we make the change of variables $\eta_{(m)} = (z + z')/2$, $\eta_{(m-1)} = (z_1 + z_1')/2$ and $\Delta z_{(m-1)} = z_1 - z_1'$. Because the scale of dielectric correlation function is much less than the propagation distance (this assumption is proper in diffusive regime, while it also holds in turbulence by introducing Markov approximation), the integration over $\mathbf{r}_1$ and $\mathbf{r}_1'$ could be transformed to

$$\int dz_1 \int dz_1' = \int_0^{\eta_{(m)}} d\eta_{(m-1)} \int_{-\infty}^{\infty} d\Delta z_{(m-1)}. \quad \text{(B10)}$$

We also change the notations so that $\mathbf{k}_{(m-1)\perp} = \mathbf{q}_\perp'$, $\mathbf{k}_{(m)\perp} = \mathbf{k}_\perp$, $K_{(m)} = K$, and $K_{(m)}' = K'$. This change is to ensure that the subscript $_{(m)}$ is used to symbol coordinates belonging to $\langle \delta E \delta E^* \rangle_{(m)}$, and $_{(m-1)}$ for $\langle \delta E \delta E^* \rangle_{(m-1)}$. The upper limit of $\eta_{(m-1)}$ is set to be $\eta_{(m)}$, which is a result of neglecting backscatter effect. Thus we integrate over q_z, and then perform the Fourier transform of Eq. (B9), and obtain

$$\langle \delta E \delta E^*(\mathbf{k}_{(m)\perp}, \eta_{(m)}, \Delta z_{(m)}) \rangle_{(m)} = \frac{e^{iK_{(m)}\Delta z_{(m)}} e^{-2K_{(m)}'\eta_{(m)}}}{4\left(K_{(m)}^2 + K_{(m)}'^2\right)} k_0^4 \int \frac{d\mathbf{k}_{(m-1)\perp}}{(2\pi)^2} \int_{-\infty}^{\infty} d\Delta z_{(m-1)} \int_0^{\eta_{(m)}} d\eta_{(m-1)}$$
$$\times e^{-iK_{(m)}\Delta z_{(m-1)}} e^{2K_{(m)}'\eta_{(m-1)}} B(\mathbf{k}_{(m)\perp} - \mathbf{k}_{(m-1)\perp}, \Delta z_{(m-1)}) \langle \delta E \delta E^*(\mathbf{k}_{(m-1)\perp}, \eta_{(m-1)}, \Delta z_{(m-1)}) \rangle_{(m-1)}. \quad \text{(B11)}$$

This equation cannot be further simplified unless we know the incident light. Thereby, we consider an incident plane wave and substitute the ballistic intensity Eq. (19) into it to get the first order and mth order

$$\langle \delta E \delta E^*(\mathbf{k}_{(1)\perp}, \eta_{(1)}, \Delta z_{(1)}) \rangle_{(1)} = I_0 \frac{e^{iK_{(1)}\Delta z_{(1)}} e^{-2K_{(1)}'\eta_{(1)}}}{4\left(K_{(1)}^2 + K_{(1)}'^2\right)} k_0^4 B(\mathbf{k}_{(1)\perp}, K_{(1)} - k_0) \int_0^{\eta_{(1)}} e^{-\frac{\eta_{(0)}}{\ell}} e^{2K_{(1)}'\eta_{(0)}} d\eta_{(0)}, \quad \text{(B12)}$$

$$\cdots, \quad \text{(B13)}$$

$$\langle \delta E \delta E^*(\mathbf{k}_{(m)\perp}, \eta_{(m)}, \Delta z_{(m)}) \rangle_{(m)} = I_0 e^{iK_{(m)}\Delta z_{(m)}} e^{-2K_{(m)}'\eta_{(m)}} \int \frac{d\mathbf{k}_{(m-1)\perp}}{(2\pi)^2} \frac{k_0^4 B(\mathbf{k}_{(m)\perp} - \mathbf{k}_{(m-1)\perp}, K_{(m)} - K_{(m-1)})}{4\left(K_{(m)}^2 + K_{(m)}'^2\right)} \cdots$$
$$\int \frac{d\mathbf{k}_{(1)\perp}}{(2\pi)^2} \frac{k_0^4 B(\mathbf{k}_{(2)\perp} - \mathbf{k}_{(1)\perp}, K_{(2)} - K_{(1)})}{4\left(K_{(2)}^2 + K_{(2)}'^2\right)} \int \frac{d\mathbf{k}_{(0)\perp}}{(2\pi)^2} (2\pi)^2 \delta(\mathbf{k}_{(0)\perp})$$
$$\times \frac{k_0^4 B(\mathbf{k}_{(1)\perp} - \mathbf{k}_{(0)\perp}, K_{(1)} - K_{(0)})}{4\left(K_{(1)}^2 + K_{(1)}'^2\right)}$$
$$\times \int_0^{\eta_{(m)}} \cdots \left\{ \int_0^{\eta_{(2)}} \left\{ \int_0^{\eta_{(1)}} e^{-\frac{\eta_{(0)}}{\ell}} e^{2K_{(1)}'\eta_{(0)}} d\eta_{(0)} \right\} e^{-2K_{(1)}'\eta_{(1)}} e^{2K_{(2)}'\eta_{(1)}} d\eta_{(1)} \right\} \cdots$$
$$e^{-2K_{(m-1)}'\eta_{(m-1)}} e^{2K_{(m)}'\eta_{(m-1)}} d\eta_{(m-1)}, \quad \text{(B14)}$$

where $K_{(0)} = k_0$, and $\sqrt{k_{\text{eff}}^2 - k_{(m)\perp}^2} = K_{(m)} + iK_{(m)}'$, $K_{(m)}' > 0$. In attaining this equation, we perform the integrations over $\Delta z_{(0)}$, $\Delta z_{(1)}$,..., $\Delta z_{(m-1)}$. By writing down the form of each order, we are able to find the general form of $\langle \delta E \delta E^* \rangle_{(m)}$. Then setting $\Delta z_{(m)} = 0$, and comparing the expression of the $(m-1)$th order to the mth order, we get the following relation:

$$\langle \delta E \delta E^*(\mathbf{k}_{(m)\perp}, \eta_{(m)}, 0) \rangle_{(m)} = \int \frac{d\mathbf{k}_{(m-1)\perp}}{(2\pi)^2} \frac{k_0^4 B(\mathbf{k}_{(m)\perp} - \mathbf{k}_{(m-1)\perp}, K_{(m)} - K_{(m-1)})}{4\left(K_{(m)}^2 + K_{(m)}'^2\right)}$$
$$\times \int_0^{\eta_{(m)}} \langle \delta E \delta E^*(\mathbf{k}_{(m-1)\perp}, \eta_{(m-1)}, 0) \rangle_{(m-1)} e^{2K_{(m)}'[\eta_{(m-1)} - \eta_{(m)}]} d\eta_{(m-1)}. \quad \text{(B15)}$$

Thus, Eq. (B15) is the final expression we will use in the numerical scheme to calculate $\langle \delta E \delta E^* \rangle_{(m)}$. Moreover, if necessary, we can also calculate

$$\langle \delta E \delta E^*(\mathbf{k}_{(m)\perp}, \eta_{(m)}, \Delta z_{(m)}) \rangle_{(m)} = e^{iK_{(m)}\Delta z_{(m)}} \int \frac{d\mathbf{k}_{(m-1)\perp}}{(2\pi)^2} \frac{k_0^4 B(\mathbf{k}_{(m)\perp} - \mathbf{k}_{(m-1)\perp}, K_{(m)} - K_{(m-1)})}{4\left(K_{(m)}^2 + K_{(m)}'^2\right)}$$
$$\times \int_0^{\eta_{(m)}} \langle \delta E \delta E^*(\mathbf{k}_{(m-1)\perp}, \eta_{(m-1)}, 0) \rangle_{(m-1)} e^{2K_{(m)}'[\eta_{(m-1)} - \eta_{(m)}]} d\eta_{(m-1)}. \quad \text{(B16)}$$

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