Universal *p*-wave tetramers in low-dimensional fermionic systems with three-body interaction

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Inspired by the narrow Feshbach resonance in systems with the two-body interaction, we propose a two-channel model of three-component fermions with a three-body interaction that takes into account finite-range effects in low dimensions. Within this model, the *p*-wave Efimov-like effect in the four-body sector is predicted in fractional dimensions between $d = 1$ and $d = 2$. The impact of the finite-range interaction on the formation of the four-body bound states in $d = 1$ is also discussed in detail.

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I. INTRODUCTION

Few-body quantum physics is known to be drastically different from its classical counterpart. The most famous example is the Efimov effect [1–3] realized, as an infinite tower of three-body bound states, in the three-boson system at resonant two-body interaction. Similar behavior is found [4,5] in the three-body sector of two-component fermions. In the fermionic case, the Efimov effect occurs in the odd-wave (typically *p*-wave) channels on a certain interval of particle mass ratios. Being well-established in three dimensions, this effect generally has a limitation on spatial dimensionality [6]. Even for nonequal particle masses the universal three-body bound states emerge in the noninteger dimensions between $2 < d < 4$. Exactly in two dimensions, the super Efimov effect was found [7] in a system of spin-polarized fermions with resonant *p*-wave interaction. Attempts to find universal Efimov-like behavior in lower dimensions necessarily require the mixed-dimension geometries [8] or higher-order few-body interactions, namely, four-body [9] in one-dimensional (1D) systems and three-body [10] in two-dimensional (2D) systems. In the latter cases, the system parameters should be highly fine-tuned to provide the resonant highest-order interaction and vanishing of all lower-order couplings.

In recent years, research in this field has shifted to fractional dimensions. Particularly, interesting analytical results for a model with a zero-range two-body potential were obtained in Refs. [11–13]. The existence of Efimov-like bound states with a characteristic scaling law in the four-body sector was reported in Ref. [14] for a bosonic system with a resonant three-body interaction in noninteger ($1 < d < 2$) dimensions. Independently of the spatial dimension, for particles interacting through a realistic two-body potential, three-body forces typically appear [15–17] as an effective induced interaction. The latter becomes dominant [18] when the two-body potential is tuned to a zero crossing. For effective field theories above an upper critical dimension, the introduction of three-body terms in the Lagrangians is mandatory to provide the ultraviolet (UV) completeness of these theories. In context of the Efimov physics, such a completion was proposed in seminal papers by Bedaque *et al.* [19,20] and more recently in Ref. [21] for a model of 1D spin-polarized fermions with the contact two-body *p*-wave interaction. An active investigation of 1D few- and many-body systems with three-body interactions started with the publication of several articles [22–25], describing general properties of SU(3) fermions and (mostly) three-body states of bosons. Similarly to 2D systems of particles interacting with each other by a two-body δ (pseudo)potential [26], the three-body contact interaction exhibits the quantum scale anomaly in 1D systems, which predetermines universal properties and thermodynamics of dilute bosons [27,28] and fermions [29,30]. In the thermodynamic limit, the three-body interaction is responsible for the formation of the quantum droplet state [31,32] in the 1D system of bosons and for the crossover transition [33] from tightly bound vacuum trimers to the so-called Cooper triples.

The present work deals with the two-channel model of three-component fermionic particles with unequal masses that interact through the three-body forces in the fractional dimension. The simplified version of the proposed setup in harmonic trapping potential was studied in Ref. [34] using the high-temperature series at finite densities of constituents. In particular, we are interested in finding regions in the parameter space for the emergence of the four-body bound states and revealing their peculiarities. Previously, important aspects of the few-body physics of SU(3) fermions with a contact three-body interaction in 1D systems were discussed by McKenney and Drut in Ref. [35]. In this context, we complement these previous studies by revealing the bound states of a system of different fermionic species with unequal masses and generalize these results to the case of finite interaction ranges and higher dimensions.

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II. MODEL AND RENORMALIZATION

We consider a model of three-component Galilean-invariant fermions with different masses m_σ ($\sigma = 1$, 2, and 3) in d-dimensional space. All two-body interactions between particles are assumed to be suppressed, and only the three-body potential [15] is switched on between fermions of different sorts. In order to take into account the finite-range effects of the three-body interaction, we consider a two-channel model that is very similar to that of the narrow Feshbach resonance but in the three-body sector:

$$H = \sum_\sigma \int_{\mathbf{p}} \varepsilon_\sigma(\mathbf{p}) f^\dagger_{\sigma,\mathbf{p}} f_{\sigma,\mathbf{p}} + \int_{\mathbf{p}} \left[\frac{\mathbf{p}^2}{2M} + \delta\nu_\Lambda\right] c^\dagger_{\mathbf{p}} c_{\mathbf{p}}$$
$$+ g \int_{\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3} c^\dagger_{\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3} f_{3,\mathbf{p}_3} f_{2,\mathbf{p}_2} f_{1,\mathbf{p}_1} + \text{H.c.}, \quad (2.1)$$

where $\varepsilon_\sigma(\mathbf{p}) = \frac{\mathbf{p}^2}{2m_\sigma}$, $\int_{\mathbf{p}} = \int \frac{d\mathbf{p}}{(2\pi)^d}$, $M = m_1 + m_2 + m_3$ is the mass of the composite fermion, and g and $\delta\nu_\Lambda$ are the coupling and the detuning. The latter depends on the UV cutoff Λ that restricts the upper integration limit in the above momentum integrals. The anticommutators of the field operators are fixed as follows, $\{c_{\mathbf{p}}, c^\dagger_{\mathbf{p}'}\} = (2\pi)^d \delta(\mathbf{p} - \mathbf{p}')$ and $\{f_{\sigma,\mathbf{p}}, f^\dagger_{\sigma',\mathbf{p}'}\} = (2\pi)^d \delta_{\sigma,\sigma'} \delta(\mathbf{p} - \mathbf{p}')$, and are equal to zero for all other pairs. Before we proceed with the four-body bound and scattering states, let us first consider the simplest interacting system of three fermions. The latter solution sets up the correct renormalization of the cutoff-dependent detuning $\delta\nu_\Lambda$ and the relation of our model to the system with three-body δ-like interaction [14]. The three-body state that describes particles with zero total momentum can be written down as follows:

$$|c\rangle = A c^\dagger_{\mathbf{0}} |0\rangle$$
$$+ \int_{\mathbf{p}_1,\mathbf{p}_2} A_{\mathbf{p}_1,\mathbf{p}_2,-\mathbf{p}_1-\mathbf{p}_2} f^\dagger_{1,\mathbf{p}_1} f^\dagger_{2,\mathbf{p}_2} f^\dagger_{3,-\mathbf{p}_1-\mathbf{p}_2} |0\rangle, \quad (2.2)$$

where $|0\rangle$ is the vacuum state, and amplitudes A and $A_{\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3}$ are subject to the Schrödinger equation. Denoting energy of the system by $\mathcal{E}$, we have

$$[\delta\nu_\Lambda - \mathcal{E}]A + g \int_{\mathbf{p}_1,\mathbf{p}_2} A_{\mathbf{p}_1,\mathbf{p}_2,-\mathbf{p}_1-\mathbf{p}_2} = 0, \quad (2.3)$$

$$\left[\sum_\sigma \varepsilon_\sigma(\mathbf{p}_\sigma) - \mathcal{E}\right] A_{\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3} + gA = 0. \quad (2.4)$$

The nontrivial solution to these coupled equations for the bound states ϵ_g (negative $\mathcal{E}$'s) reads

$$\mathcal{D}(\epsilon_g) = 0, \quad (2.5)$$

where for convenience we have introduced an auxiliary function,

$$\mathcal{D}(\mathcal{E}) = \delta\nu_\Lambda - \mathcal{E} - g^2 \int_{\mathbf{p}} \Pi_{23}(\mathbf{p}|\mathcal{E}), \quad (2.6)$$

here and below $\Pi_{23}(\mathbf{p}|\mathcal{E}) = \Pi_{23}(\mathcal{E} - \frac{\mathbf{p}^2}{2M_{23}} - \varepsilon_1(\mathbf{p}))$, $\Pi_{23}(\mathbf{p}, \mathbf{p}'|\mathcal{E}) = \Pi_{23}(\mathcal{E} - \frac{(\mathbf{p}+\mathbf{p}')^2}{2M_{23}} - \varepsilon_1(\mathbf{p}) - \varepsilon_1(\mathbf{p}')), \ldots,$ with $M_{23} = m_2 + m_3$, and $\Pi_{23}(\mathcal{E}) = \int_{\mathbf{p}} \frac{1}{\varepsilon_2(\mathbf{p}) + \varepsilon_3(\mathbf{p}) - \mathcal{E}}$. First, let us discuss the limit of broad resonance. Keeping energy fixed while setting $g \to \infty$, we can identify the three-body bare coupling constant $g^{-1}_{3,\Lambda} = -\delta\nu_\Lambda/g^2$ which absorbs the UV divergence of the integral in $\mathcal{D}(\mathcal{E})$ (see the Appendix for more details). This procedure relates the observable three-body coupling g_3 to the three-body bound-state energy ϵ_∞ at the broad resonance:

$$g_3^{-1} = -\frac{\Gamma(1-d)}{(2\pi)^d} \left(\frac{m_1 m_2 m_3}{M}\right)^{d/2} |\epsilon_\infty|^{d-1}, \quad (2.7)$$

where $\Gamma(z)$ is the gamma function [36]. Restoring the finite magnitude of g and repeating the above calculation procedure, one obtains the transcendental equation for the composite fermion's bound-state energy at narrow resonance:

$$\epsilon_g + \frac{g^2}{g_3}\left[1 - \left(\frac{\epsilon_g}{\epsilon_\infty}\right)^{d-1}\right] = 0. \quad (2.8)$$

There is always a single solution to this equation for positive g_3's. At large coupling g it recovers the broad-resonant result ϵ_∞, while in a case of small g's it asymptotically behaves as $\epsilon_g \approx -g^2/g_3$. The unitarity limit, $\epsilon_g = 0$, is reached at infinite g_3 for all dimensions above $d = 1$. The one-dimensional limit of Eq. (2.8) is also well-defined:

$$\epsilon_g = \frac{g^2}{2\pi}\sqrt{\frac{m_1 m_2 m_3}{M}} \ln\left(\frac{\epsilon_g}{\epsilon_\infty}\right), \quad (2.9)$$

although it is not for the three-body coupling (2.7). From Eq. (2.9) it can be readily concluded that $|\epsilon_g| \leqslant |\epsilon_\infty|$. Finally, the Galilean invariance allows us to generalize the above results to a nonzero composite fermion momentum $\mathbf{p}$: the only modification is the energy shift $\mathcal{E} \to \mathcal{E} - \frac{\mathbf{p}^2}{2M}$ in Eq. (2.6).

III. FOUR-BODY PROBLEM

The wave function of an arbitrary four-body (composite fermion $+f_1$ atom) state with zero center-of-mass momentum reads

$$|f_1 c\rangle = \int_{\mathbf{p}} B_{\mathbf{p}} f^\dagger_{1,\mathbf{p}} c^\dagger_{-\mathbf{p}} |0\rangle + \int_{\mathbf{p}_1,\mathbf{p}'_1,\mathbf{p}_2} B_{\mathbf{p}_1\mathbf{p}'_1,\mathbf{p}_2,-\mathbf{p}_1-\mathbf{p}'_1-\mathbf{p}_2}$$
$$\times f^\dagger_{1,\mathbf{p}_1} f^\dagger_{1,\mathbf{p}'_1} f^\dagger_{2,\mathbf{p}_2} f^\dagger_{3,-\mathbf{p}_1-\mathbf{p}'_1-\mathbf{p}_2} |0\rangle, \quad (3.10)$$

where $B_{\mathbf{p}_1\mathbf{p}'_1,\mathbf{p}_2,-\mathbf{p}_1-\mathbf{p}'_1-\mathbf{p}_2} = -B_{\mathbf{p}'_1\mathbf{p}_1,\mathbf{p}_2,-\mathbf{p}_1-\mathbf{p}'_1-\mathbf{p}_2}$ is the antisymmetric function of the first two arguments. By acting with the Hamiltonian H on the ansatz (3.10) one obtains the following system of two coupled equations,

$$\left[\varepsilon_1(\mathbf{p}) + \frac{\mathbf{p}^2}{2M} + \delta\nu_\Lambda - \mathcal{E}\right] B_{\mathbf{p}}$$
$$+ 2g \int_{\mathbf{p}_1,\mathbf{p}_2} B_{\mathbf{p}\mathbf{p}_1,\mathbf{p}_2,-\mathbf{p}-\mathbf{p}_1-\mathbf{p}_2} = 0, \quad (3.11)$$

$$\left[\sum_\sigma \varepsilon_\sigma(\mathbf{p}_\sigma) + \varepsilon_1(\mathbf{p}'_1) - \mathcal{E}\right] B_{\mathbf{p}_1\mathbf{p}'_1,\mathbf{p}_2,\mathbf{p}_3}$$
$$+ \frac{g}{2}[B_{\mathbf{p}_1} - B_{\mathbf{p}'_1}] = 0, \quad (3.12)$$

for the wave function with eigenvalue $\mathcal{E}$. For negative energies $\mathcal{E} = \epsilon_4$ (four-body bound states), Eqs. (3.11) and (3.12) can be readily transformed into a single integral equation for the function $B_{\mathbf{p}}$,

$$\mathcal{D}_1(\mathbf{p}|\epsilon_4)B_{\mathbf{p}} + g^2 \int_{\mathbf{q}} \Pi_{23}(\mathbf{p}, \mathbf{q}|\epsilon_4)B_{\mathbf{q}} = 0, \tag{3.13}$$

with the shorthand notation for function $\mathcal{D}_1(\mathbf{p}|\mathcal{E}) = \mathcal{D}(\mathcal{E} - \varepsilon_1(\mathbf{p}) - \frac{\mathbf{p}^2}{2M})$. Because of the Fermi statistics, the four-body bound states can only occur in the odd-wave channels. The one requiring the shallowest potential well is the p wave with the wave function of the following form $B_{\mathbf{p}} = (\mathbf{np}/p)B_p$ (here B_p depends on modulus $\mathbf{p}$ with $\mathbf{n}$ being a unit vector). As it is expected for states with the orbital quantum number $l = 1$, this is not a single wave function, but d mutually orthogonal states parameterized by different $\mathbf{n}_i$ satisfying the condition $\mathbf{n}_i\mathbf{n}_j = \delta_{ij}$. Plugging the p-wave harmonics back into Eq. (3.13) and performing the d-dimensional hyperangle integration, we obtained the one-dimensional integral equation (see the Appendix) for amplitude B_p.

A. $d > 1$

Before discussing numerical results, let us consider the limit of broad resonance $g \to \infty$ and the disappearing three-body bound state $g_3 \to \infty$. In the case of bosons, this limit is characterized by the universal log-periodic wave function that signals an emergence [14] of the Efimov-like effect in the four-body sector. A very similar situation should be observed in our system. However, its fermionic nature (and, consequently, nonzero total angular momentum) demands the total mass of the second and third particles M_{23} should be small enough in comparison to m_1 in order to provide the effective attracting potential for the trapping of four particles. The scaling limit ($g, g_3 \to \infty$) supposes the power-law behavior $B_p = 1/p^{1-d/2\pm\eta}$ of the wave function. Then, the integral equation transforms into the algebraic one on parameter η:

$$1 = \frac{\Gamma\left(1 - \frac{d/2+\eta}{2}\right)\Gamma\left(1 - \frac{d/2-\eta}{2}\right)m_1 M^{2d-3}}{-\Gamma(1-d)\Gamma(1+d/2)M_{23}^{d-1}(M+m_1)^{d-1}} \times {}_2F_1\left(1 - \frac{d/2+\eta}{2}, 1 - \frac{d/2-\eta}{2}; \frac{2+d}{2}; \frac{m_1^2}{M^2}\right), \tag{3.14}$$

with ${}_2F_1(a, b; c; z)$ being the hypergeometric function [36]. Truly imaginary solutions $\eta = i\eta_0$ determine a region (see Fig. 1) of the four-body p-wave Efimov-like effect in the considered system, with the universal ratio of energy levels

$$\epsilon_4^{(n)}/\epsilon_4^{(n+1)} = e^{2\pi/\eta_0} \quad (n \gg 1). \tag{3.15}$$

Note that η_0 depends only on the spatial dimension and the ratio of total mass of the second and third fermions M_{23} to mass of the first particle m_1. Equation (3.14) possesses solutions for any dimensions $1 < d < 2$ at arbitrarily large m_1/M_{23} (typical magnitudes of mass ratio are $m_1/M_{23} \approx 840$ and $m_1/M_{23} \approx 500$ for $d = 1.1$ and $d = 1.999$, respectively). Extremely close to $d = 1$, the lower threshold mass ratio for an emergence of Efimov-like effect can be asymptotically obtained as $m_1/M_{23} \approx 0.757 \times 2^{1/(1-d)}$, combining analytical and numerical approaches. Typical dependence of the scaling factor e^{π/η_0} on the mass ratio for specific dimensions and the d-behavior of exponent η_0 are presented in Fig. 2. The

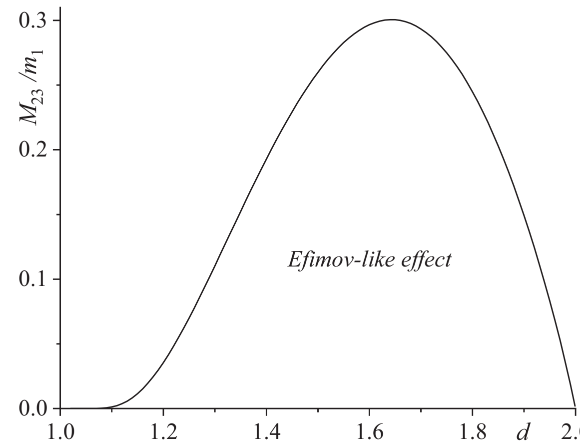

FIG. 1. Region where the p-wave Efimov-like effect in the four-body sector of the three-component Fermi system with the three-body interaction emerges.

asymptotic form of the appropriate wave functions is a linear combination of two partial solutions:

$$B_p = \frac{\sin[\eta_0 \ln(p/\Lambda_0)]}{p^{1-d/2}}, \tag{3.16}$$

where Λ_0 is an arbitrary momentum scale related to the lowest Efimov state. An emergent discrete scale invariance of the asymptotic expression (3.16) for the wave function in the scaling region is the characteristic feature of the Efimov-like effects.

To verify the above predictions about the behavior of our model in the four-particle sector, we have numerically solved the integral equation (3.13) in the p-wave channel by discretizing its kernel. The parameters of the system were specially chosen to reveal the Efimov behavior as simply as possible. In particular, we set $d = 1.5$ and mass ratios $M_{23}/m_1 = 1/50$ small enough to provide a strong induced attractive potential between two f_1 atoms. The three-body binding energy is sent to zero, $\epsilon_g \to 0$, and coupling

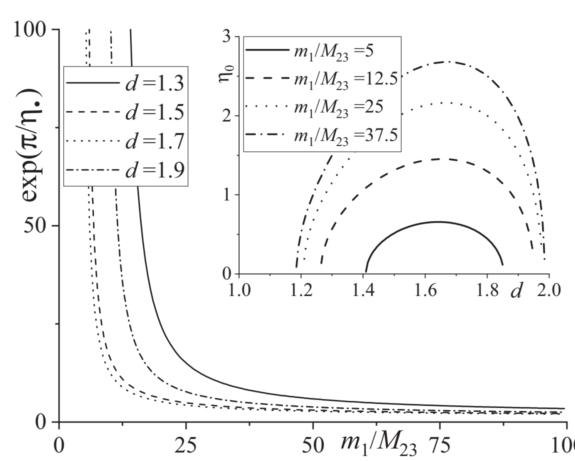

FIG. 2. The scaling factor e^{π/η_0} as a function of mass ratio m_1/M_{23}. The insert shows typical dependence of an exponent η_0 on the spatial dimension.

TABLE I. The first five eigenvalues $\epsilon_4^{(n)}$ (in units of $\frac{m_1+M}{2m_1Mr_0^2}$) of the four-body problem in the p-wave channel. Numerical calculations were performed in $d = 1.5$ and for mass ratios $M_{23} = m_1/50$.

n	$\lvert\epsilon_4^{(n)}\rvert$	$\epsilon_4^{(n-1)}/\epsilon_4^{(n)}$
0	4.2250	
1	3.4802×10^{-1}	12.141
2	3.5671×10^{-2}	9.7564
3	3.8086×10^{-3}	9.3668
4	4.0949×10^{-4}	9.3008
⋮	⋮	⋮
∞	0	9.2909

$g = \sqrt{\frac{m_1m_2m_3}{M}}\frac{1}{r_0^{2-d}}$ is parametrized by scale r_0, which is related to the effective range. Actually, this is the only dimensionful parameter at unitary limit $g_3^{-1} = 0$. The four-body p-wave energies $\epsilon_4^{(n)}$ are measured in units of $\frac{m_1+M}{2m_1Mr_0^2}$, and the numerical prefactors for the first five levels are gathered in Table I.For the identification of the level, the wave functions were also calculated (see Fig. 3). In the third column, we show the ratio of neighboring eigenenergies (3.15) which should tend to $e^{2\pi/\eta_0} = 9.290\,926\ldots$ at large n. It is seen, however, that already for the fourth excited state this quantity $\epsilon_4^{(3)}/\epsilon_4^{(4)}$ deviates from the universal value by a tenth of a percent.

It was shown in Ref. [37] that the three-body system containing two identical fermions interacting with an exterior particle via a zero-range two-body potential possesses the non-Efimov p-wave bound states at some mass ratios. These trimers exhibit a continuous scale invariance and were also found [38,39] in the quasi-2D geometries. The later studies [40] added a third type of trimer to the universal phase diagram of the three-body system. At this point, we can speculate, since the problem requires a more detailed investigation which is out of the scope of the present study, that a similar variety of tetramers should be expected in our system above the Efimov-like region in Fig. 1.

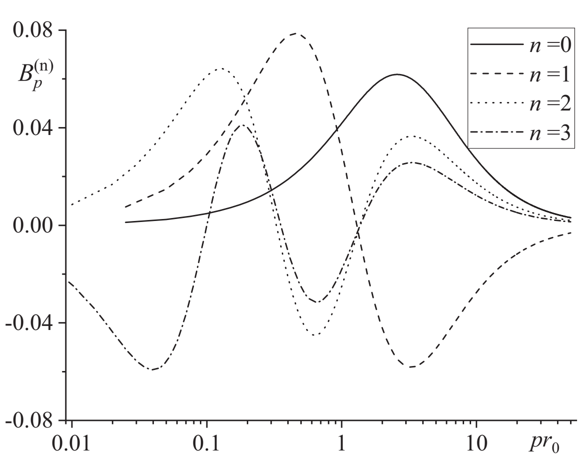

FIG. 3. The ground-state and the first few excited states' wave functions $B_p^{(n)}$ (un-normalized) of the four-body problem in the p-wave channel.

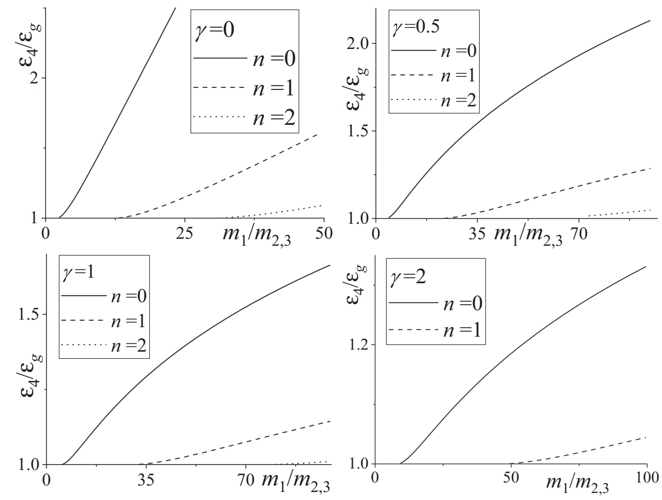

FIG. 4. Four-body energy levels (in units of ϵ_g) as functions of mass ratio $m_1/m_{2,3}$ at the different effective ranges [parametrized by $\gamma = \ln(\frac{\epsilon_\infty}{\epsilon_g})$] of the three-body interaction.

B. $d = 1$

The one-dimensional case is of particular interest due to potential realization in experiments and/or numerical simulations. According to the previous section, there is no four-body Efimov-like effect in the considered system in 1D, but a strong imbalance between masses of atoms may lead to the formation of the bound states. Projection onto p-wave states in the four-body Schrödinger equation (3.13) in $d = 1$ reduces to choosing the odd $B_{\mathbf{p}} = -B_{-\mathbf{p}}$ solutions. An appropriate eigenvalue ϵ_4 at finite ranges ($g < \infty$) of the interaction potential should be deeper than the three-body bound-states ϵ_g. Therefore, it is natural to pick dimensionless units in a way that the four-body energies are measured in units of ϵ_g, and the effective range is determined by closeness of the three-body bound-state energy to its broad-resonance value $\gamma = \ln(\frac{\epsilon_\infty}{\epsilon_g})$. Then $\gamma = 0$ corresponds to the zero-range $r_0 = 0$ case, while $\gamma > 0$ refers to the model with narrow resonance. Introducing the momentum scale p_0 such that $\frac{m_1+M}{2m_1M}p_0^2 = |\epsilon_g|$, one can readily rewrite Eq. (3.13) in dimensionless units in 1D as

$$\int_{-\infty}^{\infty} dq \frac{b_q}{\sqrt{\left(q - \frac{m_1}{M}p\right)^2 + \left(1 - \frac{m_1^2}{M^2}\right)(p^2 + \lambda)}} = [\gamma(p^2 + \lambda - 1) + \ln(p^2 + \lambda)]b_p, \tag{3.17}$$

with $\lambda = \epsilon_4/\epsilon_g$ being the dimensionless eigenvalue and $B_{p_0p} = b_p$. Recall that due to the fermionic nature of the considered system, solutions for b_p should be searched on a class of odd functions. The results of numerical diagonalization are presented in Fig. 4. Since, both m_2 and m_3 enter Eq. (3.17) only in combination M_{23}, we can freely take them equal to each other, $m_2 = m_3$. Our calculations demonstrate that even at the broad resonance $\gamma = 0$, which is the most favorable limit for the existence of the four-body bound states, the first level emerges at $m_1/m_2 \approx 2.5$ (the second and the third ones at ≈13.0 and ≈31.5, respectively). These tetramers are universal; i.e., their energies ϵ_4 are proportional to ϵ_∞ at any coupling strength, and the proportionality factor depends

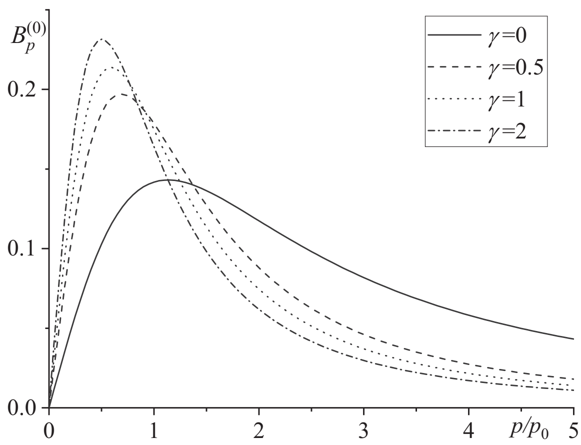

FIG. 5. The four-body ground-state wave functions $B_p^{(0)}$ (unnormalized) at different values of the effective range and fixed mass ratio $m_1/M_{23} = 7.75$ in 1D.

only on the mass ratio m_1/M_{23}. A qualitatively similar situation is realized [41] in the three-body system composed of two identical fermions interacting through the two-body contact potential with a particle of another sort. With an increase of the effective range, these values shift towards larger m_1/m_2 ratios. Particularly, when $\epsilon_g = \epsilon_\infty/e$, the first three tetramers are found at $m_1/m_2 \approx 5.5$, $m_1/m_2 \approx 32.5$, and $m_1/m_2 \approx 78.5$. We have also obtained the four-body ground-state wave functions in 1D (see Fig. 5) with $m_1/m_2 = 15.5$ at various effective ranges. For comparison, the first two four-body eigenstates are depicted in Fig. 6 for zero $\gamma = 0$ and nonzero $\gamma = 0.5$ effective ranges of the three-body interaction.

Figure 4 (especially zero-range case) reveals an intrinsic dependence of the eigenvalues at the large mass imbalance. Indeed, the integral in Eq. (3.17) possesses a nonintegrable singularity at $q = \frac{m_1}{M} p$ when $m_1/M = 1$. This fact together with parity properties of the wave function b_p allows for the asymptotic (with logarithmic precision) determination of the four-body bound states:

$$\gamma(\lambda - 1) + \ln\lambda - \ln\frac{C_\gamma}{1 - m_1^2/M^2} + \cdots = 0. \tag{3.18}$$

The constant C_γ should be determined for every four-body energy level separately by imposing the solution to Eq. (3.18) to be consistent with the large-$\frac{m_1}{M_{23}}$ tail of the eigenvalue behavior found numerically. From Eq. (3.18) one immediately recognizes the linear dependence of ϵ_4 on $m_1/m_{2,3}$ (in the limit $m_1/m_{2,3} \gg 1$) at broad resonance. Although the asymptotic solution (3.18) qualitatively explains the eigenvalue behavior at large mass imbalance, the logarithmic accuracy does not allow the quantitative description of the numerically calculated curves in Fig. 4 at intermediate mass ratios.

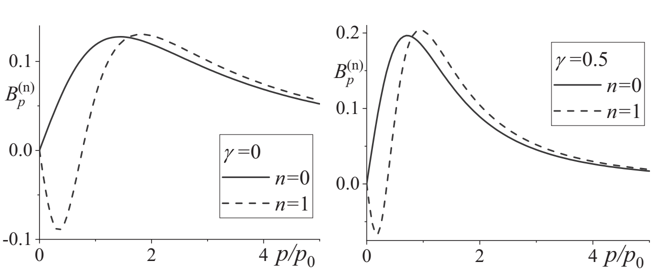

FIG. 6. Four-body wave functions (unnormalized) of the first two energy levels for mass ratio $m_1/M_{23} = 15.25$ at two effective ranges $\gamma = 0$ (left) and $\gamma = 0.5$ (right).

IV. SUMMARY

In conclusion, we have proposed an effective model of contact three-body interaction that includes the effects of finiteness of the potential range. Applying this effective description to the three-component Fermi system with the suppressed two-body interactions in the fractional dimension above $d = 1$, we have predicted the emergence of the Efimov-like physics in the p-wave channel of the four-body sector. Being substantially suppressed below $d \approx 1.1$ this effect can be, in principle, observed in any dimension between $d = 1$ and $d = 2$ at a huge mass imbalance of constituent particles. Analytic estimations in the scaling limit are supported by numerically exact calculations for finite effective ranges at unitarity. The detailed analysis of the one-dimensional problem revealed the necessary conditions for the occurrence of negative eigenvalues in the four-body spectrum, in the case of both broad and narrow resonances. Particularly, it is shown that depending on the mass ratios of fermions with three-body interactions, one can, in principle, observe an arbitrarily large number of the tetramer levels. The effect is suppressed for the nonzero effective ranges towards larger mass imbalance.

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APPENDIX

For self-sufficiency of the material presented, all important explicit expressions for functions used in the main text are collected in this section. In arbitrary d, the $f_2 - f_3$ vacuum bubble reads

$$\Pi_{23}(\mathcal{E}) = \frac{\Gamma(1 - d/2)}{(2\pi)^{d/2}} \left(\frac{m_2 m_3}{M_{23}}\right)^{d/2} (-\mathcal{E})^{d/2-1}. \tag{A1}$$

By substituting $\Pi_{23}(\mathcal{E})$ into Eq. (2.6), we see that the integral over $\mathbf{p}$ diverges at UV. To cure this issue we can define the detuning parameter $\delta\nu_\Lambda$ as follows:

$$\delta\nu_\Lambda - g^2 \int_{\mathbf{p}} \Pi_{23}(\mathbf{p}|0) = -\frac{g^2}{g_3}, \tag{A2}$$

with the right-hand side being finite. It is easy to relate, by utilizing equation $\mathcal{D}(\epsilon_g) = 0$, the newly introduced parameter g_3 to the three-body bound-state energy at broad resonance

($g \to \infty$, while keeping ϵ_∞ finite):

$$-\frac{g^2}{g_3} - g^2 \int_{\mathbf{p}} [\Pi_{23}(\mathbf{p}|\epsilon_\infty) - \Pi_{23}(\mathbf{p}|0)] = 0. \tag{A3}$$

Of course, the coupling g falls out from the final result, and calculating the remaining convergent integral one obtains Eq. (2.7) in the main text. With g_3 (and $\delta\nu_\Lambda$) being related to ϵ_∞, we can repeat the calculations of the three-body bound-state energies ϵ_g at finite g's [see Eq. (2.8)].

The p-wave partial harmonics of function $\Pi_{23}(\mathbf{p}, \mathbf{q}|\epsilon_4)$ in Eq. (3.13) is found after the solid-angle averaging:

$$\frac{1}{\Omega_d} \int d\Omega_d \frac{\mathbf{pq}}{pq} \Pi_{23}(\mathbf{p}, \mathbf{q}|\epsilon_4)$$
$$= \pi_{23}(p, q|\epsilon_4)$$
$$= -\frac{\Gamma(2-d/2)}{(2\pi)^{d/2} d} \left(\frac{m_2 m_3}{M_{23}}\right)^{d/2}$$
$$\times \frac{z\, {}_2F_1\left(1-\frac{d}{4}, \frac{3}{2}-\frac{d}{4}; 1+\frac{d}{2}; z^2\right)}{\left(\frac{p^2+q^2}{2m_1 M_{23}/M} + |\epsilon_4|\right)^{1-d/2}}, \tag{A4}$$

where $z = \frac{pq/M_{23}}{\frac{p^2+q^2}{2m_1 M_{23}/M} + |\epsilon_4|}$. Then the final one-dimensional integral equation utilized for numerical calculations in arbitrary dimension d reads

$$\mathcal{D}_1(\mathbf{p}|\epsilon_4) B_p = \frac{\Omega_d}{(2\pi)^d} \int_0^\infty dq q^{d-1} \pi_{23}(p, q|\epsilon_4) B_q, \tag{A5}$$

with the function in the right-hand side explicitly given by $\mathcal{D}_1(\mathbf{p}|\epsilon_4) = \frac{p^2}{2m_1} + \frac{p^2}{2M} + |\epsilon_4| + \frac{g^2}{g_3}[\frac{(\frac{p^2}{2m_1} + \frac{p^2}{2M} + |\epsilon_4|)^{d-1}}{|\epsilon_\infty|^{d-1}} - 1]$. In the $d = 1$ case the above equation reduces, after passing to dimensionless variables, to Eq. (3.17) in the main text.

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