# Error-corrected quantum repeaters with Gottesman-Kitaev-Preskill qudits

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The Gottesman-Kitaev-Preskill (GKP) code offers the possibility to encode higher-dimensional qudits into individual bosonic modes with, for instance, photonic excitations. Since photons enable the reliable transmission of quantum information over long distances and since GKP states subject to photon loss can be recovered to some extent, the GKP code has found recent applications in theoretical investigations of quantum communication protocols. While previous studies have primarily focused on GKP qubits, the possible practical benefits of higher-dimensional GKP qudits are hitherto widely unexplored. In this paper, we carry out performance analyses for three quantum repeater protocols based on GKP qudits including concatenations with a multiqudit quantum polynomial code. We find that the potential data transmission gains for qudits are often hampered by their decreased GKP error-correcting capabilities. However, we also identify parameter regimes in which having access to an increased number of quantum levels per mode can enhance the theoretically achievable secret-key rate of the quantum repeater. Some of our protocols share the attractive feature that local processing and complete error syndrome identification are realizable without online squeezing. Provided a supply of suitable multimode GKP states is available, this can be realized with a minimal set of passive linear optical operations, even when the logical qudits are composed of many physical qudits.

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# I. INTRODUCTION

Quantum technologies rely on the availability of precisely controllable quantum systems, e.g., qubits, which can be realized with various physical implementations. In 2000, Gottesman, Kitaev, and Preskill (GKP) proposed a method to encode finite-dimensional quantum systems (qudits) into quantum-mechanical harmonic oscillators [1]. More recent theoretical developments include further proposals and assessments of GKP state preparation with superconducting devices [2,3]. After years of experimental progress, GKP qubits finally have been demonstrated in superconducting microwave cavities [4–6] and in the harmonic motion of ions [7,8].

In the optical domain, on the other hand, preparing GKP states is notoriously difficult, although there is a very recent experimental demonstration of quantum optical states with some distinct GKP-type features [9]. The main problem is that reliable and strong nonlinearities are required but not readily available. In one approach, Gaussian boson sampling [10,11], one exploits that measurements can induce nonlinear effects. Here, Gaussian resource states are combined via passive linear optics and partially read out via photon-number resolving measurements. In this way, high-quality optical GKP states can be obtained, albeit only probabilistically. Gaussian boson sampling requires detectors with a sufficiently high level of photon-number resolution as well as increasingly complex

While the best method for creating optical GKP states has not yet been identified, it is safe to assume that their physical realization will require extremely sophisticated experimental procedures. Once such technology is available, however, it will be comparatively straightforward to extend it to higherdimensional GKP qudits and to concatenated multiqubit or -qudit GKP codes. For example, multiple GKP qubits can be entangled via Gaussian operations [1]. Furthermore, ordinary beam splitters enable the generation of certain collective GKP ancilla states such as Bell states with GKP qubits [20] or qudits [21], as well as the collective detection of their error syndromes [21]. To guide such future experiments, we find it meaningful to investigate the performance of advanced multiqudit GKP protocols in the realm of quantum communication.

The GKP encoding enables the correction of small displacement errors of the oscillator's quadratures, in particular, those that originate from typical Gaussian error channels such

linear circuits [10,11]. To shift the experimental burden associated with this, alternative approaches have been proposed [12,13]. If non-Gaussian resource states or non-Gaussian optical elements are available, a recursive application of short linear circuits and homodyne measurements is sufficient for the preparation of GKP states [14–16]. There also exist alternatives which do not rely on measurements at all [16,17]. A final option is to combine photon-subtraction- and homodynebased elements to convert many-mode Gaussian cluster states into non-Gaussian few-mode states which can be further processed into GKP states [18]. Such an approach is compatible with measurement-based, continuous-variable quantum computation [16,19].

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as amplitude damping or photon loss. However, large displacement errors cannot be avoided completely, especially for realistic, finitely squeezed GKP states. This can cause misidentification of error syndromes, which leads to discrete logical errors on the affected GKP qudits.

In order to correct such errors, a higher-level quantum error-correcting code (QECC) can be employed to encode a few logical qudits into a larger number of physical GKP qudits [3,22-27]. Hereby, the error-correction capability of the higher-level QECC can benefit from analog information in the single-qudit GKP syndrome measurements [22–25]. In order to satisfy the quantum Singleton bound  $n - k \ge 2(d - d)$ 1), every QECC with code parameters [n, k, d] must trade off the number of correctable (arbitrary) single-qudit errors against the number of physical qudits per logical qudit, which are given by |(d-1)/2| and n/k, respectively [28–30]. An optimal tradeoff is obtained by those QECCs that meet the quantum Singleton bound with equality and are called maximum distance separable (MDS) codes. While, for qubits, the only [31] nontrivial (i.e.,  $d \ge 3$  and  $k \ge 1$ ) MDS code encodes one logical qubit into five physical qubits [32], there is a plethora of MDS codes for higher-dimensional qudits. Such QECCs are explicitly available in the form of quantum polynomial codes, which exist for every qudit dimension being a prime power [33–36].

Currently, experimental realizations of long-distance quantum communication protocols are limited by the rapid decay of photonic signals that are sent through optical fibers. This process is formally described by a pure-loss bosonic channel whose secret-key capacity scales linearly with its transmittance  $\eta$  [37]. More precisely, it is given by  $-\log_2(1-\eta) \approx$  $1.44 \eta$  [38]. In consequence, the secret-key rate (SKR) of point-to-point quantum key distribution (QKD) is exponentially suppressed in the length *L* of an optical fiber, which typically has a transmittance of  $\eta = \exp(-L/22 \text{ km})$ .

To overcome this problem, quantum repeaters have been proposed [39]. By introducing repeater stations, a long channel is split into multiple shorter ones. To cope with the loss, different strategies have been conceptualized and, subsequently, been classified into three so-called generations of quantum repeaters [40]. These generations fundamentally differ in their speed of operation and in the level of technological maturity required for their realization.

First-generation quantum repeaters are based on heralded, probabilistic entanglement distribution [39]. Once a Bell pair is successfully distributed between two neighboring repeater stations, it is stored in local quantum memories where it resides until a second Bell pair, which connects the two repeater stations to a third one, is created. Whenever two parts of different Bell pairs are present in a single repeater station, entanglement swapping can be executed, which results in a single Bell pair ranging over a larger distance. This process is repeated until a long-distance Bell pair is shared between Alice and Bob. In addition to channel loss, unavoidable operational gate and storage errors pose a challenge for quantum repeaters. To cope with such errors, first-generation quantum repeaters employ nested entanglement purification [41], a probabilistic protocol for the distillation of multiple lowfidelity Bell pairs into a smaller number of states with higher fidelities, involving two-way classical communication. In the

fiber (for both classical and quantum signaling). To avoid this slowdown, second-generation quantum re*peaters* [42] replace entanglement purification by QECCs for the local memories. With this modification in place, the rate bottleneck is now posed by classical communication between neighboring repeater stations, which are separated by a distance of  $L_0$ . Only after a failed entanglement distribution attempt has been heralded, the quantum memories can be freed up for the next attempt. Therefore, the improved upper bound on the repetition rate is now given by  $c/L_0$ , which is typically on the order of 1 kHz for  $L_0 \approx 100$  km to 1 MHz for  $L_0 \approx 100$  m. The only possibility to speed up the classical two-way communication is to reduce  $L_0$ , i.e., to invest in a larger number of realistically imperfect repeater stations whose quantum information must be consistently protected by the QECC.

Finally, *third-generation quantum repeaters* enable ultrafast quantum communication as they dispense with the temporary storage of quantum information and classical two-way communication altogether [43–45]. Instead, these repeaters employ QECCs to correct both channel losses and operational errors. The repetition rates in this case are only limited by the speed of state preparations, local gate operations, and measurements in the individual repeater stations. Whereas the preparation of QECC-encoded multiphoton states typically relies on some form of light-matter interaction, all other components of a third-generation quantum repeater can, in principle, be realized in an all-optical fashion [46–50].

In this paper, we theoretically analyze the performance of third-generation quantum repeaters based on optical GKP qudits. Our investigation also includes cases where the GKP code is concatenated with a higher-level QECC. Here, we focus on quantum polynomial codes that previously have been considered in combination with multimode (MM) and Fock-encoded qudits [51–54]. For GKP-encoded states, similar performance studies have only been carried out in the special case of qubits [55–57]. Our paper thus closes the gap between these two approaches to a certain extent as it offers a treatment of the remaining case of GKP qudits. The consideration of qudits, which can transmit more quantum information per channel use than qubits, is particularly attractive in the context of GKP and third-generation quantum repeaters, due to the existence of hardware-efficient GKP-qudit operations and syndrome extraction routines based on linear-optical elements alone. In this way, the only fundamental experimental challenge that remains is to provide a supply of suitable multimode GKP ancilla states, a problem that can be tackled independently.

This paper is structured as follows. In Sec. II, we describe the details of our paper: we begin with introducing the repeater protocols under investigation in Secs. II A and II B and proceed with our noise model in Sec. II C. In Sec. III, we present the secret-key rates obtainable with the different GKP qudit repeater protocols and discuss the influence of various experimental parameters. Finally, in Sec. IV, we summarize our results and conclude with a recommendation of the most promising quantum repeater protocol based on GKP qudits as identified in this paper.

## **II. SETTING**

GKP codes encode a D-dimensional qudit within the Fock space  $\mathcal{F}$  of a quantum-mechanical harmonic oscillator [1]. We denote the annihilation operator of the oscillator by  $\hat{a}$  and its quadrature operators by  $\hat{p} = \frac{i}{\sqrt{2}}(\hat{a}^{\dagger} - \hat{a})$  and  $\hat{q} = \frac{1}{\sqrt{2}}(\hat{a}^{\dagger} + \hat{a})$ . For simplicity, we focus in this paper on the square GKP code, which is defined as the *D*-dimensional subspace of  $\mathcal{F}$ that is invariant under the action of  $S_X = \exp(-i\sqrt{2\pi D}\hat{p})$ and  $S_Z = \exp(i\sqrt{2\pi D}\hat{q})$ . By repeated nondestructive measurements of the stabilizer operators  $S_X$  and  $S_Z$ , followed by appropriate displacement operations (or, at least, through tracking of the corresponding generalized Pauli frame), one can enforce the state of the oscillator to (effectively) remain in the GKP code space. Since the logical Pauli operators of the square GKP code are given by  $X = \exp(-i\sqrt{\frac{2\pi}{D}}\hat{p})$  and  $Z = \exp(i\sqrt{\frac{2\pi}{D}}\hat{q})$ , it is thereby possible (in the idealizing limit of perfect GKP states) to correct arbitrary displacement errors that are smaller than  $\sqrt{\pi/2D}$  in magnitude. To implement two-qudit gates between GKP qudits, one can utilize common two-mode Gaussian gates. For example, on the level of GKP qudits, the bosonic CSUM gate,  $\exp(-i\hat{q}_1\hat{p}_2)$ , acts as a two-qudit controlled-X gate,  $CX = \sum_{k=0}^{D-1} |k\rangle \langle k|_1 \otimes X_2^k$ . A similarly defined CZ gate is implemented by means of a CPHASE gate,  $\exp(i\hat{q}_1\hat{q}_2)$ .

# A. Repeater protocols

Since GKP qudits can be encoded into photons, which are the ideal carriers of "flying" quantum information propagating at maximal speed, they have been envisioned in the context of quantum communication [56–58]. In this paper, we analyze and compare the performance of three third-generation quantum repeater protocols introduced in the following subsections. For each protocol, the term "qudit" may either refer to a bare (physical) GKP qudit or to an ensemble of multiple GKP qudits encoding a single (logical) qudit using a higher-level QECC, in particular, in combination with Knill's error-correction-by-teleportation procedure [21,59]. Even in the absence of a higher-level QECC, our protocols should be regarded as instances of error-corrected (third-generation) quantum repeaters, as the availability of GKP syndrome information facilitates the correction of displacement errors.

#### 1. Two-way teleportation protocol with classical postamplification

The first of the three quantum repeater chains under investigation is portrayed in Fig. 1(a). For this protocol, every repeater station prepares a pair of qudits in a (logical) Bell state. One of the qudits is sent in the direction of the next repeater station, while the other one is sent backward. In the middle between two neighboring repeater stations, the forward- and backward-propagating qudits are joined in a (logical) Bell measurement, which is implementable on the physical level with (transversal) beam splitters and two homodyne detectors per physical Bell measurement [21]. During the transmission from the repeater stations to the central Bell measurement apparatus, the states of the qudits are altered due to the finite transmittance of the optical fiber channel. For the general case of many physical qudits representing one logical qudit, the optical loss channels act individually and independently upon the different modes of the physical multimode state that propagates through each fiber segment. To facilitate a direct comparison with the other protocols, we denote the channel transmittance by  $\sqrt{\eta} = \exp(-L_0/2L_{\text{att}})$ , as the relevant length of the fiber is given by  $L_0/2$  here. Throughout this paper,  $L_0$  denotes the distance between two adjacent repeater stations, and  $L_{\text{att}} = 22 \text{ km}$  is the attenuation length of a typical fiber at the telecommunication wavelength of 1550 nm. In order to compensate for the loss-induced state change (with damped quadrature amplitudes), the classical measurement signal of the Bell measurements needs to be correspondingly amplified by a factor of  $\sqrt{\eta}^{-1}$  before decoding the GKP syndrome. Overall, this protocol produces an imperfect Bell pair ranging from one end of the repeater chain to the other. Note that classical communication is only needed for postprocessing and, therefore, it does not slow down the repetition rates of this protocol. Further note that for the case of a logical qudit composed of many physical qudits, classical postamplification is performed individually for each physical Bell measurement to obtain the syndrome of the higher-level QECC [21].

#### 2. One-way teleportation protocol with optical preamplification

As a modification of the protocol from Sec. II A 1, we also consider a quantum repeater chain where the Bell measurements are executed within the repeater stations [see Fig. 1(b)]. Here, only one qudit per Bell pair is transmitted through the fiber channel. This time, the transmittance is given by  $\eta = \exp(-L_0/L_{att})$  because the traveling distance of the photons now covers a full repeater segment, i.e., twice the distance as in the previous scenario. To cope with the fiber losses, an optical preamplification channel  $\mathcal{A}(\eta^{-1})$  is individually and independently applied to each (physical) GKP mode before it is sent through the fiber; this step replaces the classical postamplification of the measurement signal from Sec. II A 1.

# 3. One-way half-teleportation protocol with optical preamplification

The utilization of a Bell measurement (protocols described in Secs. II A 1 and II A 2) provides GKP syndrome information for both quadratures. This facilitates the correction of displacement errors on the level of the (physical and logical) GKP qudits. For the final repeater chain under consideration, on the other hand, every repeater station is responsible for preparing and measuring only a single logical GKP qudit [see Fig. 1(c)]. This protocol has two core components. First, a lower-level GKP error correction converts naturally occurring Gaussian displacement errors into Pauli errors on the physical qudits as we will explain in Sec. II C 3. Second, a higher-level QECC is utilized to cope with the resulting Pauli errors. At the start of the repeater chain, Alice prepares two higher-level logical qudits in the state  $|\overline{+}\rangle = \sum_{k} |k\rangle / \sqrt{D}$  and entangles them with a logical  $\overline{CZ}$  gate. Since we restrict ourselves to quantum polynomial codes, the  $\overline{CZ}$  gate admits



FIG. 1. Unit cells of the quantum repeater protocols considered in this paper. The transmittance  $\eta = \exp(-L_0/L_{att})$  of the bosonic pure-loss channel  $\mathcal{L}(\eta)$  is exponentially suppressed in the distance  $L_0$  between adjacent repeater stations (dashed blue boxes). Here, the qudits can be either individual GKP qudits or logical qudits that are composed of multiple GKP qudits by means of a higher-level  $[n, 1, d]_D$  QECC. (a) In the two-way teleportation protocol, every repeater station prepares two qudits in the maximally entangled state  $|\Phi\rangle = \frac{1}{\sqrt{D}} \sum_{k=0}^{D-1} |k, k\rangle$ . One of the two qudits is sent forward and the other one backward. After propagating a distance of  $L_0/2$ , at which each physical mode has been subject to a loss channel  $\mathcal{L}(\sqrt{\eta})$ , a Bell measurement (BM) is performed. (b) Also in the one-way teleportation protocol, two qudits are prepared in state  $|\Phi\rangle$ . In contrast to (a), only one of the qudits is sent to an adjacent repeater station. To compensate for loss, a quantum-limited amplification channel  $\mathcal{A}(\eta^{-1})$  with gain  $\eta^{-1}$  is applied to each of the physical GKP modes. After propagating a distance of  $L_0$ , a BM combines the forward-moving qudit with the stationary qudit of the subsequent repeater station. (c) The one-way half-teleportation protocol is a GKP adaptation of a previously studied discrete-variable protocol [54]. Here, we add measurements to convert displacement errors into Pauli errors. Overlined ancilla states represent codewords of the higher-level  $[n, 1, d]_D$  QECC, while ancilla states without overscore stand for GKP codewords. The  $\overline{CX}$  gates correspond to transversal CSUM gates and the  $\overline{CZ}$  gates correspond to semitransversal CPHASE gates. Measurements of q and p denote the measurement of the position and momentum quadrature, respectively. Loss and amplifier channels are again to be understood to act individually and independently on the physical GKP modes.

a semitransversal implementation with favorable errorspreading properties (see Appendix A for details). Alice stores one of the logical qudits and to the second one she applies a quantum-limited amplifier with gain  $\eta^{-1}$  to each of the physical GKP modes before she sends them through a lossy fiber of transmittance  $\eta$  to the first repeater station, where the incoming logical qudit is entangled with a new logical qudit in state  $|\overline{+}\rangle$ . A subsequent destructive (physical) quditwise p measurement effectively transfers the encoded quantum information onto the next qudit and simultaneously delivers syndrome information involving Xstabilizers. These steps are then repeated at every repeater station. Besides yielding higher-level X syndromes, the pmeasurements are also responsible for providing lower-level GKP syndrome information p mod  $\sqrt{2\pi/D}$ . The physical CZ gates propagate Gaussian p errors on one mode into qerrors on the next one. To prevent these q errors from merging with q errors that occur at the subsequent transmission, we introduce an additional ancilla-based GKP syndrome measurement in every repeater station (see Appendix C2 for details). To complete the protocol, all measurement results are communicated to Bob who applies a suitable correction operator depending on the measurement outcomes [53]. Assuming N is even and in the absence of errors, this protocol is equivalent to N/2 teleportation subroutines spread over N + 1 different laboratories. For this reason, we refer to this protocol as *half teleportation*.

## B. Some comments on potential realizations of qudit repeaters

To compensate for fiber loss, it is crucial to amplify the signal. For the two protocols in Secs. II A 1 and II A 2, one may opt between optical preamplification and classical postamplification. For the half-teleportation protocol in Sec. II A 3, on the other hand, optical preamplification is the only option. This is because the GKP qudits need to be correctly scaled, i.e., they need to be in the GKP code space up to a displacement, before the *CZ* gate is applied. Since classical postamplification can be carried out conveniently in software, lacking this option may be considered as a disadvantage of the half-teleportation protocol.

While we analyze their performance for GKP qudits, these protocols can be straightforwardly adapted to other qudit encodings, such as MM qudits, which have been experimentally demonstrated in the context of (repeaterless) higher-dimensional quantum key distribution in the form of orbital angular momentum [60] and time-bin qudits [61]. Two of the three repeater protocols under consideration rely on Bell measurements. For GKP qudits, a deterministic Bell measurement can easily be implemented with static linear optics by employing a balanced beam splitter and continuous-variable homodyne measurements [21]. Experimental implementations of Bell measurements for MM-encoded qudits, on the other hand, are disproportionately more involved. Moreover, deterministic CX gates for MM qudits require strong nonlinearities that are typically mediated through auxiliary matter qudits, which reduces the achievable repetition rates to the order of MHz. This is in stark contrast to all-optical implementations that can reach GHz repetition rates. An attempt to circumvent this shortcoming of MM qudits is based on probabilistic linear optical Bell measurements, enabling an all-optical error-correction step at every repeater station [46–50]. Such probabilistic Bell measurements cannot exceed 50% for MM qubits in the simplest setting without additional resources such as photonic ancilla states [62-64]. For a deterministic Bell measurement, nonlinear optics is required. Furthermore, probabilistic unambiguous state discrimination measurement of the corresponding two-qudit Bell states, making only use of linear optics and photon counting without ancilla photons, is impossible for MM qudits with D > 2[65,66]. Therefore, overall, the GKP concept and the GKPbased QR protocols presented in this paper represent a way to combine an increased communication capacity based on photonic qudit encoding with an enhanced loss (and error) robustness based on photonic qudit quantum error correction.

#### C. Noise model

GKP codes are designed to correct displacement errors. As we review next, this allows us to model photon loss and imperfect GKP state preparation with incoherent Gaussian displacement channels. For our error analyses, it will suffice to keep track of their variances.

## 1. Transmission loss, coupling, and measurement inefficiencies

The bosonic pure-loss channel  $\mathcal{L}(\eta)$  is commonly used to model fiber loss and coupling inefficiencies in quantum communication protocols [67,68]. This loss channel arises when an optical mode is coupled via a beam splitter of transmittance  $\eta$  to an environmental mode initialized in the vacuum state which is traced out after the interaction, i.e.,

$$\mathcal{L}(\eta): \rho \longmapsto \mathrm{Tr}_{\mathrm{env}} \left( U_{\mathrm{BS}}^{(\eta)}(\rho \otimes |\mathrm{vac}\rangle \langle \mathrm{vac}|) U_{\mathrm{BS}}^{(\eta) \uparrow} \right), \quad (1)$$

where  $U_{\rm BS}^{(\eta)} = \exp[\arccos(\sqrt{\eta})(\hat{a}_1\hat{a}_2^{\dagger} + \hat{a}_1^{\dagger}\hat{a}_2)]$ . When  $\mathcal{L}(\eta)$  is applied to a GKP state, its quadratures are damped, which shrinks the GKP lattice. To rescale the lattice, one has to amplify the signal. Depending on whether this amplification

is carried out optically before  $\mathcal{L}(\eta)$ , optically after  $\mathcal{L}(\eta)$ , or classically after the measurement of a quadrature operator, the effective error channel on the GKP subspace is altered.

For the one-way protocols in Secs. II A 2 and II A 3, we consider the usage of an optical amplification channel  $\mathcal{A}(\eta^{-1})$ . Such an amplification channel with gain  $\eta^{-1}$  can be obtained by coupling an optical mode using a two-mode squeezing (TMS) operation to an ancillary mode in the vacuum state which is traced out after the interaction, i.e.,

$$\mathcal{A}(\eta^{-1}): \rho \longmapsto \operatorname{Tr}_{\operatorname{anc}}\left(U_{\operatorname{TMS}}^{(\eta^{-1})}(\rho \otimes |\operatorname{vac}\rangle\langle \operatorname{vac}|)U_{\operatorname{TMS}}^{(\eta^{-1})\dagger}\right), \quad (2)$$

where  $U_{\text{TMS}}^{(\eta^{-1})} = \exp[\operatorname{arcosh}(\sqrt{\eta}^{-1})(\hat{a}_1\hat{a}_2 - \hat{a}_1^{\dagger}\hat{a}_2^{\dagger})]$ . If  $\mathcal{A}(\eta^{-1})$ is applied *after*  $\mathcal{L}(\eta)$ , the result is a Gaussian displacement channel with variance  $\sigma^2 = (1 - \eta)/\eta$  [55]. If  $\mathcal{A}(\eta^{-1})$ is applied *before*  $\mathcal{L}(\eta)$ , however, the variance is improved to  $\sigma^2 = 1 - \eta$  as this avoids amplifying noise that occurs during transmission [58]. In our analyses of the one-way protocols, we will therefore consider the latter strategy. Furthermore, we will assume a total transmittance of  $\eta_{\text{tot}} = \eta_c \eta_{\text{det}} \exp(-L_0/L_{\text{att}})$ , where  $L_{\text{att}} = 22 \text{ km}$  is the attenuation length, and  $\eta_c$  and  $\eta_{\text{det}}$  denote the efficiencies for coupling into the fiber and homodyne measurement, respectively. Homodyne measurements work reliably and efficiencies of 99% have already been demonstrated in an experiment for broadband light [69], which is relevant for obtaining a high repetition rate. For this reason, we will assume  $\eta_c \eta_{\text{det}} = 0.99$ unless stated otherwise.

For the two-way teleportation-based protocol in Sec. II A 1, it is possible and beneficial to replace  $\mathcal{A}(\sqrt{\eta}^{-1})$  with a classical amplification of the measured signal. Effectively, this turns the loss into a Gaussian error channel with variance  $\sigma^2 = 1/\sqrt{\eta_{\text{tot}}} - 1$  [57], where  $\eta_{\text{tot}} = \eta_c^2 \eta_{\text{det}}^2 \exp(-L_0/L_{\text{att}})$  takes into account that, in a two-way protocol, two signals are coupled into the fiber.

Finally note that, in principle, one could alternatively make use of noiseless linear amplification (NLA) [70]. However, this would add further complications as NLA—by nature of the no-cloning theorem [71]—is an inherently probabilistic process.

#### 2. Approximate GKP states

The second, important noise contribution arises during the preparation of GKP states. In position basis, the state vector of an ideal square GKP qudit takes the form

$$|j\rangle = \sum_{k \in \mathbb{Z}} \left| \hat{q} = \sqrt{\frac{2\pi}{D}} (j + Dk) \right\rangle, \tag{3}$$

where  $j \in \{0, ..., D-1\}$  labels a computational basis state. These ideal states are unphysical as they are neither normalizable nor superpositions of finite-width peaks. To describe normalizable, physical instances of GKP states, we instead consider approximate GKP states for which multiple realizations that are essentially equivalent have been proposed  $[1,72,73]^1$ . Normalizability can be restored using an

<sup>&</sup>lt;sup>1</sup>The state given in Eq. (4) is not symmetric under exchange of position and momentum. However, this state can be squeezed by a factor of  $\sqrt{1 + \kappa^2 \Delta^2}$  to obtain the parametrization given in Eq. (5).

overall slowly decaying Gaussian envelope, and the delta peaks can be approximated with (a still infinite number of) highly squeezed Gaussian peaks. This results in approximate GKP states of the form

$$|\tilde{j}\rangle \propto \sum_{k \in \mathbb{Z}} \exp\left(-\frac{\pi \kappa^2}{D}(j+Dk)^2\right)$$
$$\times \int_{-\infty}^{\infty} dq \exp\left(-\frac{\left(q - \sqrt{\frac{2\pi}{D}}(j+Dk)\right)^2}{2\Delta^2}\right) |\hat{q} = q\rangle,$$
(4)

where  $\Delta$  and  $\kappa$  are squeezing parameters corresponding to the peaks' width in position and momentum representation, respectively. Alternatively,  $|\tilde{j}\rangle$  can be interpreted as an ideal GKP state  $|j\rangle$  to which coherent Gaussian displacements have been applied, i.e.,

$$\begin{split} |\tilde{j}\rangle \propto \int_{\mathbb{R}^2} du \, dv \exp\left(-\frac{1}{2}\left(\frac{u^2}{\gamma^2} + \frac{v^2}{\delta^2}\right) \\ &+ i\left(\frac{-u\hat{p} + v\hat{q}}{\sqrt{2}}\right) \right) |j\rangle, \end{split}$$
(5)

where the squeezing parameters  $\gamma$  and  $\delta$  are in one-to-one correspondence to  $\Delta$  and  $\kappa$  (see Theorem 1 in Ref. [73]). In this paper, we only consider the symmetric case of  $\gamma = \delta$ . As a further simplification, we assume incoherent Gaussian displacements with variance  $\sigma_{sq}^2 = \frac{\delta^2}{2}$ , which can be understood as a twirling-approximation (see Appendix A in Ref. [23]). Numerical simulations confirm that such an approximation does not overestimate the approximate GKP state's fidelity [74]. Following Refs. [23,72,75], we define the *squeezing parameter* (given in dB),

$$s_{\rm GKP} = -10 \log_{10} \left( \frac{\sigma_{\rm sq}^2}{\sigma_{\rm vac}^2} \right), \tag{6}$$

where  $\sigma_{\rm vac}^2 = 1/2$  denotes the quadrature variance of the vacuum state.

By means of a higher-level QECC, it is possible to concatenate multiple approximate GKP qudits, each of which is modeled by an ideal GKP state followed by Gaussian squeezing errors, into a single logical qudit. The corresponding unitary encoding circuit may redistribute the error probabilities between the modes, which in principle leads to correlated errors [53]. The resulting error probabilities have a complicated dependence on the selected encoding circuit, thus, they cannot be easily captured in full generality in our analytical model. Therefore, we leave such details for future work. For the purpose of the present investigation, we are satisfied with a noise model where unphysical, ideal GKP states are first encoded using a higher-level QECC and, afterward, physicality is restored by applying Gaussian squeezing channels individually and independently distributed to each qudit, as motivated above.

#### 3. Converting Gaussian noise into Pauli errors

The purpose of the GKP error-correction step shown in Fig. 1 is to discretize the continuous displacement errors that

build up on the GKP qudits. In general, a single-qudit Pauli error channel is completely described by its joint error probability distribution of X and Z errors [53]. We denote such a distribution by

$$\mathcal{P}(X,Z) = \begin{pmatrix} P(X^0,Z^0) & \dots & P(X^0,Z^{D-1}) \\ \vdots & \ddots & \vdots \\ P(X^{D-1},Z^0) & \dots & P(X^{D-1},Z^{D-1}) \end{pmatrix}.$$
 (7)

In the special case of a square-lattice GKP qudit, we denote by  $\mathcal{P}_{sq}(X, Z)$  the Pauli error channel that results from a Gaussian noise channel with zero mean and a covariance matrix  $\Sigma_{sq} = \sigma^2 \mathbb{I}$  (with respect to q and p). For  $\mathcal{P}_{sq}(X, Z)$ , we find that X and Z errors are independent because the same is true for the two Gaussian random variables describing q and p shifts. In other words, the matrix  $\mathcal{P}_{sq}(X, Z) = \mathcal{P}_{sq}(X) \otimes \mathcal{P}_{sq}(Z)$  factors into the outer product of the error probability vectors that store the marginal distributions of X and Z errors. By symmetry of the square lattice, we have  $\mathcal{P}_{sq}(X) = \mathcal{P}_{sq}(Z)$ . The probability to suffer  $k \in \{0, \ldots, D-1\}$  shifts can be expressed as

$$P_{\text{sq}}(X^{k}, \sigma^{2}) = \sum_{j \in \mathbb{Z}} \int_{\sqrt{\frac{2\pi}{D}}(jD+k+\frac{1}{2})}^{\sqrt{\frac{2\pi}{D}}(jD+k+\frac{1}{2})} \frac{1}{\sqrt{2\pi\sigma^{2}}} \exp\left(-\frac{q^{2}}{2\sigma^{2}}\right) dq$$
$$= \sum_{j \in \mathbb{Z}} \frac{1}{2} \left( \operatorname{erf}\left(\sqrt{\frac{2\pi}{D}} \frac{jD+k+\frac{1}{2}}{\sigma}\right) - \operatorname{erf}\left(\sqrt{\frac{2\pi}{D}} \frac{jD+k-\frac{1}{2}}{\sigma}\right) \right), \tag{8}$$

where  $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-q^2) dq$  is the error function. For our purposes, it is sufficient to keep only the three terms with  $|j| \leq 1$ .

#### 4. Error sources that can be neglected

To simplify the presentation of our results, we neglect errors in two-qudit gates. For the two protocols from Secs. II A 1 and II A 2, all two-qudit gates can be implemented via simple and reliable 50 : 50 beam splitters whose errors are dominated by other noise processes. To execute the half-teleportation protocol from Sec. II A 3, on the other hand, one would need to implement *CZ* gates by applying Gaussian operations which have higher errors. Nevertheless, we can neglect these errors because—as it will turn out in Sec. III B—even under the optimistic assumption of perfect *CZ* gates (which benefits the half-teleportation protocol), this protocol performs worse than the protocols from Secs. II A 1 and II A 2. Our final conclusion about which protocol is most promising, thus, remains unaffected.

#### **III. SECRET-KEY RATES OF QUANTUM REPEATERS**

The central figure of merit that we employ to compare the performance of different repeater protocols is the SKR per channel use. More precisely, we use  $\log_2(D) - H(\mathcal{P}_{\text{final}})$ , D 1

which is a lower bound on the two-way capacity [38], where

$$H(\mathcal{P}_{\text{final}}) = -\sum_{j,k=0}^{D-1} P_{\text{final}}(X^j, Z^k) \log_2(P_{\text{final}}(X^j, Z^k))$$
(9)

denotes the Shannon entropy of the error probability distribution  $\mathcal{P}_{\text{final}}$  on the final Bell pair distributed among Alice and Bob. Note that this bound can be achieved by a qudit generalization (using D + 1 bases, assuming D to be prime) of the six-state protocol [76] in the asymptotic limit, where almost every round the same basis is used [77]. Moreover, if X and Z errors are independent, the same rate is obtainable with a generalization of the BB84 protocol [78] (two bases, arbitrary D).<sup>2</sup>

## A. Repeater performance with GKP error correction only

For near-term applications, it is certainly more convenient to operate a quantum repeater with bare GKP qudits and not with multiple GKP qudits in a QECC. To guide such initial experiments, we begin our discussion with this important special case. For the two protocols considered with bare GKP qudits, which are described in Secs. II A 1 and II A 2, lower-level error correction is performed via a teleportation step on the logical level of the GKP code, which leads to independent X and Z errors. As mentioned above, the SKR per channel use is thus given by  $\log_2(D) - H(\mathcal{P}_{\text{final}})$  not only for the generalized six-state protocol (D prime) but also for the generalized BB84 protocol (D arbitrary). The precise value of  $H(\mathcal{P}_{\text{final}})$ has a complicated dependence on the repeater spacing  $L_0$ , on the total repeater length L, on the squeezing parameter  $s_{GKP}$ that characterizes approximate GKP states, and on the qudit dimension D. However, we can numerically assess  $H(\mathcal{P}_{\text{final}})$ ; see Appendix **B** for details.

In Fig. 2, we show the optimal choice (color coded) of qudit dimension D for different values of L and  $s_{\text{GKP}}$ , where  $L_0 = 500 \,\mathrm{m}$  is fixed. Using inset lines, we also display the corresponding (maximal) value of the SKR per channel use. As expected, the key rate vanishes if the GKP approximation is too bad (small  $s_{GKP}$ ) or too much loss accumulates (large L). Since increasing the squeezing poses a core experimental challenge, the smallest value of  $s_{GKP}$  at which a nonzero SKR can be achieved is of particular interest. Below  $s_{GKP} = 10 \text{ dB}$ , neither protocol is suitable for generating secret keys. For both protocols and for every fixed value of L, we observe that GKP qubits (D = 2) represent the leading contender for near-term quantum repeaters based on the GKP code. To some degree, this result is surprising because in the ideal case, the SKR per channel use is given by  $\log_2(D)$ , and increasing the qudit dimension would be beneficial. In the presence of noise, however, higher-dimensional GKP qudits have a severe disadvantage of decreased error-correction capabilities: a D-dimensional GKP qudit can only correct displacement errors that are smaller than  $\sqrt{\pi/2D}$  in magnitude. Only in





FIG. 2. Optimal dimension *D* of bare GKP qudits utilized in a quantum repeater line with coupling and measurement efficiencies  $\eta_c \eta_{det} = 99\%$  and an intermediate repeater spacing of  $L_0 = 500$  m, where the (a) one-way or (b) two-way teleportation protocol is used. For each choice of total repeater length *L* and squeezing parameter  $s_{\text{GKP}}$ , the qudit dimension is adjusted such that the SKR per channel use,  $\log_2(D) - H(\mathcal{P}_{\text{final}})$ , is optimized (inset lines). In the parameter regions of D = 1, it is not possible to generate secret keys.

the regime of very small errors, i.e., where the qubit GKP protocol has almost reached its maximum performance of  $\log_2(D) - H(\mathcal{P}_{\text{final}}) = \log_2(2) - 0 = 1.0$ , it is beneficial to employ qutrits (D = 3) instead of qubits. To see such benefits at all, we need at least  $s_{\rm GKP} \gtrsim 18 \, \rm dB$ . For repeater lines of modest lengths of a few dozen kilometers, however, larger squeezing levels of 20-25 dB are required to compensate for additional loss. At some value of L, loss errors become so severe that only an unrealistically disproportional improvement of  $s_{\text{GKP}}$  could compensate them. For the one-way protocol in Fig. 2(a), qutrits cease to be the optimal option for repeaters longer than a few hundred kilometers, whereas the two-way protocol in Fig. 2(b) can still benefit from qutrits even for repeaters exceeding  $L = 10\,000$  km. For the latter, however, a squeezing level above 30 dB is required, which will only be available in the long term (if at all). The reason for the better performance of the two-way protocol is the lower

<sup>&</sup>lt;sup>2</sup>The secret-key fraction is given by  $I(A, B) - I(A, E) = \log_2(D) - H(\overline{q}_{01}) - I(A, E)$ , where expressions of  $H(\overline{q}_{01})$  and the mutual information I(A, E) between Alice and Eve are provided in Eqs. (5) and (7) of Ref. [77].

required amplification factor  $\sqrt{\eta}^{-1}$  in the usage of the classical postamplification, as discussed in Sec. II C 1.

Finally note that, in our error analysis, we distinguish the cases of even and odd qudit dimensions. Only if D is even, we can leverage a beneficial linear-optics protocol for the generation of GKP Bell pairs (see Appendix B). For very short repeater chains, we indeed observe that GKP qudits with D even outperform those with D odd. For larger values of L, however, loss errors begin to dominate and parameter regions emerge where the optimal SKR is obtained by odd-dimensional GKP qudits.

# B. Repeater performance with both GKP and higher-level error correction

In comparison to the experimental challenge of creating high-quality GKP qudits in the first place, concatenating multiples of them into a single logical qudit by means of a higher-level QECC is relatively straightforward. In the following, we study the performance of third-generation quantum repeaters that make use of  $[D, 1, \frac{D+1}{2}]_D$  quantum polynomial codes ( $D \ge 3$  prime), as reviewed in Appendix A. Since the distance of a quantum polynomial code is given by  $d = \frac{D+1}{2}$ , which is optimal in terms of the Singleton bound, any collection of errors that affect no more than  $\lfloor \frac{d-1}{2} \rfloor = \lfloor \frac{D-1}{4} \rfloor$ qudits can be corrected. Thus, it makes sense for us to limit the discussion to prime qudit dimensions where D-1 is a multiple of 4. For error patterns that affect more than  $\frac{D-1}{4}$ qudits, we assume (as a worst-case approximation) the occurrence of a uniformly random logical error. This maximizes the Shannon entropy  $H(\mathcal{P}_{\text{final}})$  and lower bounds the achievable SKR,  $\log_2(D) - H(\mathcal{P}_{\text{final}})$ ; see Appendix C for details.

In Fig. 3, we plot the (lower bound on the) SKR per logical channel use as a function of *L*, where  $L_0 = 100$  m is fixed. For each of the three repeater protocols introduced in Sec. II A, we show the SKR for D = 5 (black), D = 13 (red), D = 17 (green), and D = 29 (cyan). For any fixed value of *D*, we again (as in Fig. 2) observe that the two-way teleportation protocol (dash-dotted curve) from Sec. II A 1 performs best. It is followed by the one-way teleportation protocol is the one-way half-teleportation protocol (solid curve) from Sec. II A 3. We attribute the poor performance of the latter protocol to the fact that it employs only half as many (compared to the other protocols) logical measurements, which facilitate the correction of errors.

Recall from Sec. III A that for bare GKP repeaters, the decreased error-correcting capabilities render higherdimensional qudits unfeasible for near-term applications. Since the code distance  $d = \frac{D+1}{2}$  grows with *D*, one could expect that concatenating bare GKP qudits with quantum polynomial codes would turn the tide. We see that this is not the case: for an optimistic but conceivable value of  $s_{\text{GKP}} =$ 20 dB, we see in Fig. 3(a) that only the smallest code with D =5 achieves a nonzero SKR for repeater lengths L > 70 km. To assess the performance of larger codes, we need to assume exorbitant squeezing levels, e.g.,  $s_{\text{GKP}} = 30$  dB as in Fig. 3(b). In this scenario, the [[5, 1, 3]]<sub>5</sub> code operates near its maximum performance of  $\log_2(5) \approx 2.3$  for all considered values of *L*. Depending on the distance *L*, the largest value of  $\log_2(D) =$ 





FIG. 3. Lower bound on the SKR per logical channel use, log<sub>2</sub>(*D*) – *H*( $\mathcal{P}_{\text{final}}$ ), as a function of the total length *L* for a quantum repeater line with coupling and measurement efficiencies  $\eta_c \eta_{\text{det}} =$ 99%, an intermediate repeater spacing of  $L_0 = 100$  m, and squeezing levels of (a)  $s_{\text{GKP}} = 20$  dB or (b)  $s_{\text{GKP}} = 30$  dB. Here, the GKP code is concatenated with a  $[D, 1, \frac{D+1}{2}]_D$  quantum polynomial code. The highlighted area shows the achievable SKR per physical channel use of a bare GKP repeater as in Fig. 2(b).

 $H(\mathcal{P}_{\text{final}})$  is obtained by a different code: until  $L \approx 100$  km, the  $[29, 1, 15]_{29}$  code achieves a value beyond the optimal performance of  $\log_2(17) \approx 4.1$  of the  $[\![17, 1, 9]\!]_{17}$  code. The latter starts to lose performance after a few thousand kilometers, where it falls behind the  $[13, 1, 7]_{13}$  code. For comparison, we also show in Fig. 3 the performance of the two-way repeater protocol with bare GKP qudits (shaded region), where we select the value of D that optimizes the SKR, as in Fig. 2(b). For  $s_{GKP} = 20 \text{ dB}$  in Fig. 3(a), bare GKP ququarts (D = 4) are optimal until  $L \approx 200$  km. For longer repeaters, too much loss accumulates, and lower-dimensional GKP codes with higher error-correcting capabilities become beneficial: in a small range of L, bare qutrits are the optimal choice, but already for  $L \gtrsim 300$  km, qubits perform best. As before, this advantage of even dimensions over odd ones is due to improved Bell state availability [21]. For  $s_{GKP} = 30 \text{ dB}$  in Fig. 3(b), losses are less of an issue and eight-dimensional GKP qudits are optimal

until  $L \approx 20$  km. For  $30 \leq L \leq 200$  km, D = 6 is optimal. For  $1000 \leq L \leq 50\,000$  km, a bare GKP repeater should operate with D = 4.

It is important to stress that, from a practical perspective and for the considered parameters, it is not useful to employ higher-level QECCs if the application is QKD. For example, if  $s_{\text{GKP}} = 30 \text{ dB}$  and L = 1000 km, the two-way teleportation protocol with a logical  $[17, 1, 9]_{17}$  code achieves the largest rate of about four secret bits per logical channel use. To accomplish this, however, 17 GKP qudits (entangled in a QECC), i.e., 17 GKP-encoded and entangled optical modes, need to be transmitted. With an even lower experimental effort, one could simply transmit in parallel 17 bare GKP ququarts, i.e., 17 GKP-encoded but unentangled optical modes, each of which establishes almost two secret bits. In other words, here the best bare protocol is more efficient than the best higher-level encoded one by a factor of about 8.5.

#### 1. Optimal choice of the repeater spacing

In our discussion of Fig. 3, we have pointed out that no practical benefit is to be expected when switching from bare GKP qudits to a higher-level QECC if the repeater spacing is fixed to  $L_0 = 100 \text{ m}$ . This raises the question of how the choice of  $L_0$  influences this conclusion. Since implementation cost scales with the total number  $N = L/L_0$  of repeater stations, here we focus on SKR/N as a figure of merit. In a commercial setting, SKR/N is roughly proportional to the return on investment. In Fig. 4, we plot SKR/N as a function of  $L_0$  for a quantum repeater line of fixed length L =2000 km. The colors and line styles have the same meaning as in Fig. 3. This time, we assume a more optimistic value of  $\eta_c \eta_{det} = 99.9\%$ , which benefits higher-level QECCs. Despite this optimistic assumption, we still find that (for QKD) bare GKP qudits outperform those encoded into quantum polynomial codes. For example, for  $s_{GKP} = 20 \text{ dB}$  in Fig. 4(a), the  $[5, 1, 3]_5$  code performs best among the quantum polynomial codes and reaches the optimal value of SKR/N at a repeater spacing of  $L_0 \approx 0.55$  km. For this optimal repeater configuration, the  $[5, 1, 3]_5$  code can generate approximately 1.8 secret bits by transmitting five GKP ququints (D = 5). In the same setting, one can generate almost 5.0 secret bits by transmitting five bare GKP qubits (not shown). The same behavior is observed for  $s_{\text{GKP}} = 30 \text{ dB}$  in Fig. 4(b), where the  $[5, 1, 3]_5$ code now achieves approximately 2.0 secret bits per logical channel use at the optimal operating point of  $L_0 \approx 1.1$  km. In the same setting, transmitting five bare GKP qutrits would generate more than 5.6 secret bits.

From Fig. 4, we can also infer the maximal repeater spacing at which the secret-key rate drops to zero. For the considered parameters, the best higher-level encoded protocol, i.e., the two-way protocol from Sec. II A 1 with the  $[5, 1, 3]_5$  code and  $s_{\text{GKP}} = 30 \text{ dB}$ , is operational for all values of  $L_0 < 1.5 \text{ km}$ ; however,  $L_0 \approx 1.1 \text{ km}$  is most effective. For the one-way protocols from Secs. II A 2 and II A 3, the  $[5, 1, 3]_5$  code already fails for  $L_0 \approx 0.7 \text{ km}$ . As expected, we find that better repeaters (larger  $s_{\text{GKP}}$ , smaller *D*) allow for a larger repeater spacing.



FIG. 4. Lower bound on the SKR per logical channel use, log<sub>2</sub>(*D*) – *H*( $\mathcal{P}_{\text{final}}$ ), normalized by the number *N* of repeater segments (*N* – 1 repeater stations) as a function of the repeater spacing *L*<sub>0</sub> for a repeater line of total length *L* = *NL*<sub>0</sub> = 2000 km, coupling and measurement efficiencies  $\eta_c \eta_{\text{det}} = 99.9\%$ , and squeezing levels of (a)  $s_{\text{GKP}} = 20 \text{ dB}$  or (b)  $s_{\text{GKP}} = 30 \text{ dB}$ . Here, the GKP code is concatenated with a  $[D, 1, \frac{D+1}{2}]_D$  quantum polynomial code.

#### 2. Identifying and overcoming noise bottlenecks

Before one takes a great effort of building a quantum repeater based on GKP qudits, it is important to ensure that the experimental building blocks work sufficiently well. There are multiple components for which improvements might be beneficial or even necessary. Thus, it is important to identify and remove the noise bottleneck, which would otherwise diminish the performance. Since we have already discussed the impact of fiber loss, here we focus on input noise that arises from approximate GKP state preparation, coupling losses, and measurement errors. As explained in Sec. II C, these error processes can be modeled by Gaussian noise. Errors propagate through the circuit and eventually accumulate on individual measurement results in the repeater stations, which for the two-way postamplification protocol from Sec. II A 1 can be described by a Gaussian channel with variance

$$\sigma_{\rm eff}^2 = 3\sigma_{\rm sq}^2 + \frac{1 - \eta_c \eta_{\rm det}}{\eta_c \eta_{\rm det}} \exp\left(\frac{L_0}{2L_{\rm att}}\right). \tag{10}$$



FIG. 5. (a) Variance  $\sigma_{\text{eff}}^2$  of Gaussian noise effectively affecting each physical GKP qudit as a function of the squeezing level  $s_{\text{GKP}}$ , the coupling efficiency  $\eta_c$ , and the measurement efficiency  $\eta_{\text{det}}$ . (b) Lower bound on the SKR per logical channel use,  $\log_2(D) - H(\mathcal{P}_{\text{final}})$ , as a function of  $\sigma_{\text{eff}}^2$  for a quantum repeater line with a total transmission distance of L = 5000 km and a repeater spacing of  $L_0 = 500$  m, where the two-way protocol in combination with a  $[D, 1, \frac{D+1}{2}]_D$  quantum polynomial code is utilized.

Indeed, there are three sources from which GKP state preparation errors can propagate to the measurements, which leads to the first term in Eq. (10). The second term in Eq. (10) accounts for coupling losses: since the variance (incorporating both coupling and fiber channel losses) of a length- $L_0$  link in the two-way protocol is given by  $[\eta_c \eta_{det} \exp(-L_0/2L_{att})]^{-1} - 1$ , the noise difference between a link with coupling errors and without is given by

$$\left(\left(\eta_{c}\eta_{det}\exp\left(-\frac{L_{0}}{2L_{att}}\right)\right)^{-1}-1\right)-\left(\left(\exp\left(-\frac{L_{0}}{2L_{att}}\right)\right)^{-1}-1\right)$$
$$=\frac{1-\eta_{c}\eta_{det}}{\eta_{c}\eta_{det}}\exp\left(\frac{L_{0}}{2L_{att}}\right).$$
(11)

In Fig. 5(a), we plot  $\sigma_{\text{eff}}^2$  as a function of  $s_{\text{GKP}}$  and  $\eta_c \eta_{\text{det}}$  for  $L_0 = 0.5$  km. Note that this plot would change only slightly for other values of  $L_0 \leq 1$  km. Recall that  $\sigma_{\text{sq}}^2$  and  $s_{\text{GKP}}$  can be converted into each other via Eq. (6). Here, we assume

a repeater spacing of  $L_0 = 500$  m; however, the situation remains almost unchanged if  $L_0$  takes any other value between 1 m and 1 km. The contour lines in Fig. 5 can be used to infer whether one should work on improving  $s_{\rm GKP}$  or  $\eta_c \eta_{\rm det}$ : since moving along a contour line does not improve performance, a series of improvements should instead correspond to a path orthogonal to the contour lines. For example, for  $s_{\rm GKP} = 6 \,\mathrm{dB}$  and  $\eta_c \eta_{\rm det} = 0.99$ , we have  $\sigma_{\rm eff}^2 \approx 0.4$ , which can be reduced to  $\sigma_{\rm eff}^2 \approx 0.2$  if the GKP approximation is improved to  $s_{\rm GKP} = 9 \,\mathrm{dB}$ ; increasing  $\eta_c \eta_{\rm det}$ , on the other hand, would not help at all. Conversely, if coupling losses dominate, e.g.,  $s_{\rm GKP} = 30 \,\mathrm{dB}$  and  $\eta_c \eta_{\rm det} = 0.92$ , the variance  $\sigma_{\rm eff}^2 \approx 0.1$  can be reduced by a factor of 2 if coupling efficiencies are improved to  $\eta_c \eta_{\rm det} = 0.97$ ; increasing  $s_{\rm GKP}$ , however, would show no significant effect here.

To study how the combination of all error mechanisms in Eq. (10) affects the performance of error-corrected quantum repeaters with GKP qudits, we depict in Fig. 5(b) the influence of  $\sigma_{\rm eff}^2$  on the SKR obtained with the two-way protocol from Sec. II A 1 for an error-corrected quantum repeater line with  $L = 5000 \text{ km}, L_0 = 500 \text{ m}, \text{ and a } [D, 1, \frac{D+1}{2}]_D \text{ code. Here,}$ each physical GKP qudit in every repeater station is affected by a Gaussian channel with variance  $\sigma_{\rm eff}^2$ . As before, we find that a larger value of D allows both for a larger SKR per logical channel use in the low-noise regime and for a smaller noise level to be tolerated before the SKR drops to zero. We also observe that the parameter range of  $\sigma_{eff}^2$  where the SKR drops from its optimal value to zero is alarmingly small. This effect is most pronounced for the  $[5, 1, 3]_5$  code, which has almost optimal performance until  $\sigma_{\rm eff}^2 \approx 0.01$  but already for  $\sigma_{\rm eff}^2 \approx 0.02$  its SKR is equal to zero. This showcases that moderate improvements can have a huge impact if they address a noise bottleneck.

# 3. Leveraging lower-level syndrome information to improve higher-level error correction

So far, we have independently treated the error-correction procedures of lower-level GKP qudits and the higher-level QECC. More precisely, we assumed that, in the first step, displacement errors on the physical GKP qudits are removed. This may or may not result in a logical GKP qudit error. Then, in a second step, the higher-level  $[n, 1, d]_D$  code deals with potential errors on the GKP qudits: the correction succeeds if the number of errors with unknown locations is not larger than  $\frac{d-1}{2}$ . In this final subsection, we investigate the more general case, where the locations of some of the errors are known. The modified error-correction routine succeeds whenever  $t_k + 2t_u < d$ , where  $t_k$  and  $t_u$  denote the number of errors with known and unknown locations, respectively. To obtain some information about error location, one can exploit the continuous, "analog" results of the homodyne measurements in the repeater stations [22,24]. If a displacement error of the form  $\exp(i\epsilon \hat{p})$  occurs, the homodyne measurement of  $\hat{q}$ reveals the value of  $\epsilon$  modulo  $\sqrt{2\pi/D}$ , which we call analog GKP syndrome. In particular, every displacement error with  $|\epsilon| < \sqrt{\pi/2D}$  can be corrected. The probability of successful error correction is large if  $\epsilon$  is small. When an error of magnitude  $\epsilon \approx \sqrt{\pi/2D}$  occurs, however, the situation is less clear. Borrowing ideas from Ref. [79], we introduce a discarding parameter  $\gamma \in [0, 1]$ , and treat any instances of  $\epsilon$  which are closer than  $\sqrt{\pi/2D}(1-\gamma)$  from the boundary of two bins as an erasure error with a known location in the *n*-qudit register. In the case  $\gamma = 1$ , we do not discard any qudits, which corresponds to the strategy considered so far. The other extreme,  $\gamma = 0$ , corresponds to the absurd approach where all qudits are always discarded.

The advantage of this modification is that, for every qudit that is not discarded, the probabilities for errors (with unknown locations) are improved from Eq. (8) to

$$P_{\mathrm{sq}}^{(\gamma)}(X^k,\sigma^2) \propto \sum_{j \in \mathbb{Z}} \int_{\sqrt{\frac{2\pi}{D}}(jD+k-\frac{\gamma}{2})}^{\sqrt{\frac{2\pi}{D}}(jD+k+\frac{\gamma}{2})} \frac{\exp\left(-\frac{q^2}{2\sigma^2}\right)}{\sqrt{2\pi\sigma^2}} dq, \quad (12)$$

where the proportionality constant follows from  $\sum_{k=0}^{D-1} P_{\text{sq}}^{(\gamma)}(X^k, \sigma^2) = 1$ . This improvement comes at the expense that we have to introduce an erasure error with probability

$$p_{\text{discard}} = 1 - \sum_{k=0}^{D-1} \sum_{j \in \mathbb{Z}} \int_{\sqrt{\frac{2\pi}{D}} (jD+k-\frac{\gamma}{2})}^{\sqrt{\frac{2\pi}{D}} (jD+k+\frac{\gamma}{2})} \frac{\exp\left(-\frac{q^2}{2\sigma^2}\right)}{\sqrt{2\pi\sigma^2}} dq; \quad (13)$$

however, we can still exploit our knowledge about the location of this error.

Denote the probability that a single GKP qudit is free of errors by  $p_0 = P_{sq}^{(\gamma)}(X^0, \sigma^2)$ . Then, the condition  $t_k + 2t_u < d$  and basic combinatorics lead to the probability of a failed error-correction attempt

$$p_{\text{fail}}(\gamma) = 1 - \sum_{t_k=0}^{d-1} \binom{n}{t_k} p_{\text{discard}}^{t_k} (1 - p_{\text{discard}})^{n-t_k} \\ \times \sum_{t_u=0}^{t_{u,\text{max}}} \binom{n-t_k}{t_u} p_0^{n-t_k-t_u} (1 - p_0)^{t_u}, \quad (14)$$

where  $t_{u,max} = \lfloor (d - t_k - 1)/2 \rfloor$  is the maximal number of correctable errors with *unknown* locations, assuming that  $t_k$  erasures with *known* locations occurred, and *n* is the number of physical GKP qudits.

In Fig. 6, we show how the logical failure rate (red) depends on the discarding parameter  $\gamma$  for a [13, 1, 7] code. For each physical GKP qudit, we assume that all noise combined (stemming, e.g., from GKP approximation, coupling, or transmission) corresponds to a fairly small but finite variance  $\sigma^2 = 0.01$  of the overall Gaussian noise channel. For  $\gamma = 1$ , i.e., without discarding (black), the failure rate has a remarkably low value of  $p_{\text{fail}} \approx 5 \times 10^{-11}$ , which is due to the low level of noise and the high error-correcting distance of d = 7. We observe a local minimum at  $\gamma_{opt} \approx 0.82$ , where the failure rate is improved by more than an order of magnitude to  $p_{\text{fail}}(\gamma_{\text{opt}}) \approx 3 \times 10^{-12}$ . If  $\gamma$  is decreased below  $\gamma_{\text{opt}}$ , we begin to introduce more erasures than the QECC can deal with, and the failure rate increases. On the other hand, if  $\gamma$ is increased above  $\gamma_{opt}$ , then the error rates  $P_{sq}^{(\gamma)}(X^k, \sigma^2)$  start to deteriorate. This causes an increasing amount of errors with unknown locations and leads to the rise in  $p_{fail}$ .

A curious effect in Fig. 6 is that the performance in the seldom-discarding regime ( $\gamma > 0.96$ ) is worse than in the never-discarding case ( $\gamma = 1$ ). We attribute this to the



FIG. 6. Failure probability  $p_{\text{fail}}$  for decoding the result of a logical measurement for a  $[\![13, 1, 7]\!]_{13}$  code as a function of the discarding parameter  $\gamma$ . Each physical GKP qudit is subject to Gaussian noise with variance  $\sigma^2 = 0.01$ .

fact that, for  $0.96 < \gamma < 1.0$ , those cases dominate where only a single erasure error is introduced, i.e.,  $t_k = 1$  and the number of correctable errors with unknown locations is decreased to  $t_{u,max} = 2$ . At the same time, the error probabilities  $P_{sq}^{(\gamma)}(X^k, \sigma^2)$  are only slightly improved because they continuously depend on  $\gamma$ . Thus, the performance is worse than for the naive approach with  $\gamma = 1$ , i.e.,  $t_k = 0$  and  $t_{u,max} = 3$ .

## IV. CONCLUSION AND OUTLOOK

In this paper, we have analyzed the performance of third-generation quantum repeaters that operate with higherdimensional GKP qudits. We have focused on the GKP square lattice and also considered concatenations with quantum polynomial codes.

A priori, GKP qudits and quantum polynomial codes seem to be a perfect match in the context of quantum repeaters. For one, GKP codes can be encoded into photons and are known for their good performance against photon loss, which is crucial for repeaters. Second, quantum polynomial codes are optimal in the sense that they achieve the Singleton bound; however, they require higher-dimensional qudits-a feature that the GKP encoding has to offer. A more technical fact, which is a deal breaker for standard approaches of quantum computation measuring stabilizers by measuring attached ancilla qubits, is that the stabilizer group of quantum polynomial codes is generated by operators whose support extends over almost all physical qudits. This makes it practically impossible to nondestructively measure the stabilizer operators—a crucial subroutine of fault-tolerant quantum computation. For quantum repeaters, destructively measuring all qudits individually suffices to extract the necessary stabilizer information, i.e., high-weight stabilizer operators are not an issue for quantum repeaters. For these reasons, one could have expected that quantum repeaters with GKP qudits concatenated with a quantum polynomial code would perform well in tandem.

A posteriori, however, our paper revealed that the decreased lower-level error-correcting capabilities of higherdimensional GKP qudits severely limit their potential benefits. While this finding might disappoint to a certain extent, it can be regarded as good news for experimentalists. Indeed, the most promising GKP repeater protocol identified in this paper is also the one which is the easiest to implement. The missing component that is currently holding back an experimental realization of such repeaters is efficient sources of high-quality GKP states [9]. Once such source is available; however, there will be no need for quantum memories or classical two-way communication. Therefore, the achievable repetition rates will only be limited by fast optical elements for the local processing of GKP qudits.

Our recommendation for a first experimental target is a repeater protocol (Sec. II A 1) that makes use of bare (i.e., without higher-level code) two-dimensional GKP qubits. Admittedly, these qubits will require challenging squeezing levels beyond 10 dB. However, the identified protocol has the advantage of readily available syndrome measurements based on balanced beam splitters and homodyne measurements alone. Furthermore, this protocol is compatible with rescaling the GKP lattice in classical software, whereas other protocols would require optical amplifiers to compensate for the loss. In the medium-to-long term, when squeezing levels above 20 dB will be available, we found that bare GKP qutrits can potentially outperform GKP qubits for meaningful repeater lengths. Only in the very long term, if squeezing levels around 30 dB can possibly be reached, we expect some benefit from concatenating multiple GKP qudits using quantum polynomial codes, however only for tasks like entanglement distribution where utmost fidelities are important. For the application of quantum key distribution, on the other hand, our analysis showed that it is typically more cost effective to operate bare GKP qudit repeaters instead.

With regard to potential experimental realizations, a useful feature of bare qudits is that GKP Bell pairs, which enable teleportation-based syndrome detection and error correction, can be directly created by applying a balanced beam splitter upon two suitable, individual single-mode GKP or grid states [20,21]. In the case of GKP qudits that are concatenated with a higher-level code, the complete syndrome information can still be obtained in a single linear-optics step; neither online squeezing operations nor additional GKP ancilla states for higher-level syndrome extraction are required. This only works, however, if suitable higher-level Bell pairs are available [21].

An interesting open research direction is to study the performance of multimode GKP codes that do *not* arise as a concatenation of physical GKP states and a higher-level stabilizer code [21,80–83]. For such an analysis, theoretical insights about multimode Gaussian channels might become important [84]. Moreover, one could analyze how bosonic encodings other than GKP perform in a quantum repeater setting, e.g., cat codes [85,86], spherical codes [87], etc. [88,89]. Finally, it would be interesting to explore how third-generation quantum repeaters with GKP qudits compare to first- or second-generation repeaters.

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D.M. initiated the project as a whole and suggested exploring the idea of concatenating GKP qudits with polynomial codes. F.S. designed the repeater protocols, derived the analytical model, performed the numerics, and created the figures. F.S. and D.M. designed the study, interpreted the results, and wrote the paper. P.v.L. supported research and development and helped preparing the paper.

## APPENDIX A: QUANTUM POLYNOMIAL CODES

In this Appendix, we review two variations of the quantum polynomial code. We start with a version by Cleve, Gottesman, and Lo (CGL) in Appendix A1, which is the one used throughout this paper. For the interested reader, we also review an alternative construction by Aharonov and Ben-Or (ABO) in Appendix A2 to which, however, our error analysis does not immediately apply. We conclude in Appendix A3 with a discussion of why, in the case of GKP qudit repeaters, the Cleve-Gottesman-Lo construction is more promising than the Aharonov–Ben-Or construction.

#### 1. Leading-term construction

In Ref. [33], CGL make use of a family of  $[[n, 1, d]]_D$ QECCs, where *D* has to be an odd prime and  $n = 2d - 1 \le D$ , which they call quantum polynomial codes. In this paper, we concentrate on the case n = D as this maximizes the distance,  $d = \frac{D+1}{2}$ . For every element  $a \in \mathbb{F}_D = \{0, \dots, D-1\}$ , a logical basis state is defined as

$$|\bar{a}\rangle_{\text{CGL}} = \frac{1}{\sqrt{D^{d-1}}} \sum_{\substack{\lambda_0, \dots, \lambda_{d-2} \in \mathbb{F}_D \\ \lambda_{d-1} = a}} |f_{\lambda}(0), \dots, f_{\lambda}(n-1)\rangle, \quad (A1)$$

where for every vector  $\boldsymbol{\lambda} = (\lambda_0, \dots, \lambda_{d-1}) \in (\mathbb{F}_D)^d$  the polynomial

$$f_{\lambda}(T) = \lambda_0 + \lambda_1 T + \lambda_2 T^2 + \ldots + \lambda_{d-1} T^{d-1}$$
(A2)

has been defined. Since the logical information is encoded into the leading coefficient  $\lambda_{d-1} = a$  of the polynomials in the superposition of Eq. (A1), we refer to this variation of the quantum polynomial code as the *leading-term construction*.

While we have already reviewed stabilizer group and logical operators of this code in Appendix A of Ref. [53], here we elaborate on the implementation of the logical *CZ* gate. Since this code is a CSS code with the property that for every *X* stabilizer there is a *Z* stabilizer with the same support (including multiplicities, which matter in the qudit case), it follows from the conjugation rules  $CZ(X \otimes I)CZ = X \otimes Z$  and  $CZ(Z \otimes I)CZ = X \otimes Z$ 

 $I)CZ = Z \otimes I$  that  $CZ^{\otimes n}$  preserves the stabilizer group under conjugation. Hence,  $CZ^{\otimes n}$  is a logical gate. However,  $CZ^{\otimes n}$ acts as  $\overline{CZ}^{-1}$  on the logical level because the exponent vector  $\mathbf{i} = (i_1, \ldots, i_D) = (0, 1, 2^{d-1}, \ldots, (D-1)^{d-1})$  of  $\overline{X} = X^{i_1} \otimes \ldots \otimes X^{i_D} =: X^{\mathbf{i}}$  differs from the one of  $\overline{Z} = Z^{-\mathbf{i}}$  by its sign. In other words, in order to apply  $\overline{CZ}$  on the logical level, one does not have to apply a *CZ* gate on every pair of physical qubits (which would be called a transversal implementation of the *CZ* gate) but a  $CZ^{-1}$  gate (which is called a semitransversal implementation in Ref. [34]). In this case, semitransversality does not lead to any disadvantage; however, the situation is different for the construction reviewed next.

## 2. Constant-term construction

In contrast to Eq. (A1), where the logical digit  $a \in \mathbb{F}_D$  is encoded into the leading term of a collection of polynomials, ABO use an alternative construction in Ref. [34]. They encode the logical information into the coefficient  $\lambda_0 = f(0)$  of the constant term of the polynomials, i.e., they define the logical basis states

$$|\bar{a}\rangle_{\text{ABO}} = \frac{1}{\sqrt{D^{d-1}}} \sum_{\substack{\lambda_1, \dots, \lambda_{d-1} \in \mathbb{F}_D \\ \lambda_0 = a}} |f_{\lambda}(1), \dots, f_{\lambda}(n)\rangle, \quad (A3)$$

for every  $a \in \mathbb{F}_D$ . This results in an  $[\![2d - 1, 1, d]\!]_D$  code for every n = 2d - 1 < D. Note that the same construction has been used in Eq. (8.8) of Ref. [36]. In Eq. (4.5) of Ref. [34] it is shown that, for this code, the logical  $\overline{CZ}$  gate can be semitransversally applied via  $CZ^{c_1} \otimes \ldots \otimes CZ^{c_n}$ , where  $c_1, \ldots, c_n \in \mathbb{F}_D$  are defined as the interpolation coefficients with the property that

$$g(0) = \sum_{k=1}^{n} c_k g(k)$$
 (A4)

holds true for all polynomials g with deg(g)  $\leq n - 1$ . While these interpolation coefficients have been defined only abstractly in Ref. [34], it is of course important to know them if one wants to implement the logical  $\overline{CZ}$  gate for this code. Hence, we compute them in the following proposition.

*Proposition.* An explicit expression of the interpolation coefficients is given by  $c_k = (-1)^{k+1} \binom{n}{k} \mod D$  for every  $k \in \{1, ..., n\}$ .

*Proof.* We also define  $c_0 = -1 = (-1)^{1} \binom{n}{0}$  and denote the coefficients of the arbitrary polynomial by  $\mu_i$ , i.e.,  $g(x) = \sum_{i=0}^{n-1} \mu_i x^i$ . Then, Eq. (A4), which defines the interpolation coefficients, is equivalent to

$$\sum_{i=0}^{n-1} \mu_i \sum_{k=0}^n c_k k^i = 0.$$
 (A5)

With our choice of  $c_k$ , Eq. (A5) is fulfilled because of

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} k^i = 0,$$
 (A6)

which is a special case (with  $a_j = \delta_{i,j}$  and i < n) of Eq. (5.42) from Ref. [90].

## 3. Some semitransversal gates are better than others

The reason why we have carried out our repeater analysis for the quantum polynomial code from Appendix A1 is that its semitransversal *CZ* gate propagates errors more favorably than the code from Appendix A2. Indeed, to implement the gate *CZ<sup>c</sup>* for any  $c \in \mathbb{F}_D$  on a GKP qudit, one has to apply the gate CPHASE<sup>c</sup> on the physical level. If a Gaussian *p* error with variance  $\sigma^2$  was present on one of the optical modes, it will induce *q* error of variance  $c^2 \times \sigma^2$  on the other optical mode. For the leading-term construction this is fine because c = -1in this case. For the constant-term construction, however, errors would unfavorably amplify because the interpolation coefficients  $c_k = (-1)^{k+1} {n \choose k} \mod D$  can take values that grow linearly in *D*.

# APPENDIX B: ERROR ANALYSIS OF BARE GKP REPEATERS

We begin our error analysis by reviewing how Gaussian displacement errors of the form  $\exp(\epsilon \hat{q}_i)$  and  $\exp(\epsilon \hat{p}_i)$ , where  $\epsilon \in \mathbb{R}$  is the error magnitude, propagate across CSUM and CPHASE gates. The CSUM gate  $\exp(-i\hat{q}_1\hat{p}_2)$  acts as  $CX_{1,2} = \sum_{k=0}^{D-1} |k\rangle\langle k|_1 \otimes X_2^k$  on GKP qudits, while the CPHASE gate,  $\exp(i\hat{q}_1\hat{q}_2)$ , implements  $CZ_{1,2} = \sum_{k=0}^{D-1} |k\rangle\langle k|_1 \otimes Z_2^k$  [1]. Hereby,  $X = \sum_{k=0}^{D-1} |k + 1 \mod D\rangle\langle k|$  and  $Z = \sum_{k=0}^{D-1} (e^{2\pi i/D})^k |k\rangle\langle k|$  denote the unitary generalizations of the qubit Pauli *X* and *Z* gates to the case of *D*-dimensional qudits. It is well known that single-qudit Pauli errors are propagated across *CX* and *CZ* gates via

$$CZ_{1,2}X_1 = X_1Z_2 CZ_{1,2},$$
  

$$CX_{1,2}X_1 = X_1X_2 CX_{1,2},$$
  
and 
$$CX_{1,2}Z_2 = Z_1^{\dagger}Z_2 CX_{1,2}$$
(B1)

(see, e.g., Refs. [1,53]). The error propagation rules of Eq. (B1) have their bosonic analogs: applying the Baker-Campbell-Hausdorff formula yields

$$\exp(i\hat{q}_1\hat{q}_2)\exp(i\hat{p}_1) = \exp(i(\hat{p}_1 - \hat{q}_2))\exp(i\hat{q}_1\hat{q}_2),$$
  

$$\exp(-i\hat{q}_1\hat{p}_2)\exp(i\hat{p}_1) = \exp(i(\hat{p}_1 + \hat{p}_2))\exp(-i\hat{q}_1\hat{p}_2),$$
  
and 
$$\exp(-i\hat{q}_1\hat{p}_2)\exp(i\hat{q}_2) = \exp(i(\hat{q}_2 - \hat{q}_1))\exp(-i\hat{q}_1\hat{p}_2).$$
  
(B2)

In the two repeater protocols from Secs. II A 1 and II A 2, every repeater station is responsible for performing a Bell measurement. This is achieved by a beam splitter, followed by two homodyne measurements. For both of these homodyne measurements, the results are postprocessed (binned) into a measurement outcome of the GKP qudit. Errors on the GKP qudit lead to errors on the measurement outcomes. The latter can be described by a Pauli error channel  $\mathcal{P}_{sq}(X, \sigma^2)$ , as in Eq. (8), where the variance  $\sigma^2$  comprises all Gaussian noise contributions that have propagated to the measurement device. As discussed in Sec. II C, we take the following error sources into account. (1) Loss that arises when GKP qudits are coupled into an optical fiber. The resulting coupling efficiency is denoted by  $\eta_c$ .

(2) Loss that arises during transmission. If the traveling distance is  $L_0$ , the associated transmittance is given by  $\eta = \exp(-L_0/L_{\text{att}})$ , where  $L_{\text{att}}$  is the attenuation distance.

(3) Effective loss that models the inefficiency of the homodyne measurement. The resulting detector efficiency is denoted by  $\eta_{det}$ .

(4) Unavoidable approximation errors of square GKP qudits. These are modeled by a Gaussian channel of variance  $\sigma_{sq}^2$ .

Since beam splitters only require passive linear optical elements, we assume they work perfectly. In addition, we ignore errors stemming from Gaussian elements, i.e., from CSUM and CPHASE gates.

For the two-way teleportation protocol from Sec. II A 1, transmission and coupling losses lead to a Gaussian error channel with variance  $\frac{1}{\eta_c \eta_{det} \sqrt{\eta}} - 1$  (see Sec. II C 1). Furthermore, there are three GKP state preparations in the causal light cone of any given measurement. All in all, this amounts to a final variance of  $\sigma_{2\text{-way}}^2 = 3\sigma_{sq}^2 + \frac{1}{\eta_c \eta_{det} \sqrt{\eta}} - 1$ . For the one-way teleportation protocol from Sec. II A 2,

For the one-way teleportation protocol from Sec. II A 2, the only difference is that the Gaussian error channel arising from losses now has a variance of  $1 - \eta_c \eta_{det} \eta$  (see Sec. II C 1). Therefore, the final variance is given by  $\sigma_{1-way}^2 = 3\sigma_{sq}^2 + 1 - \eta_c \eta_{det} \eta$ .

If *D* is even, it is possible to directly generate a two-qudit GKP Bell pair by applying a balanced beam splitter to two grid states [20,21]. Unlike general Gaussian transformations, this linear optical transformation does not amplify the noise. In consequence, the above variances are improved to  $\sigma_{2\text{-way}}^2 = 2\sigma_{\text{sq}}^2 + \frac{1}{\eta_c\eta_{\text{det}}\sqrt{\eta}} - 1$  and  $\sigma_{1\text{-way}}^2 = 2\sigma_{\text{sq}}^2 + 1 - \eta_c\eta_{\text{det}}\eta$ .

On the physical level, every Bell measurement is composed of two homodyne measurements. Errors on the measurement of one quadrature effectively propagate into X errors on Bob's qudits, while those of the other quadrature lead to Z errors. By symmetry, the final probability distributions for X and Z errors coincide, and it suffices to compute it in one case. Ignoring finite-size effects<sup>3</sup> and potential correlations between the error probabilities of different repeater stations, we estimate the final X error distribution  $\mathcal{P}_{\text{final}}(X) =$  $\mathcal{P}_{sq}^{*N}(X, \sigma^2)$  on Bob's qudit as the N-fold discrete convolution of  $\mathcal{P}_{sq}(X, \sigma^2)$ , where N denotes the number of repeater stations. We expect that this estimate captures the general behavior of the performance of GKP qudit repeaters. In principle, computing this convolution can be sped up by diagonalizing the corresponding error-probability matrix [53]. For our purposes, however, a direct implementation is sufficient. Then, we compute the outer product  $\mathcal{P}_{\text{final}}(X, Z) =$  $\mathcal{P}_{\text{final}}(X) \otimes \mathcal{P}_{\text{final}}(Z)$ . The secret-key rate of the repeater line, finally, is given by  $\log_2(D) - H[\mathcal{P}_{\text{final}}(X, Z)] = \log_2(D) - 2$  $H(\mathcal{P}_{\text{final}}(X)).$ 

# APPENDIX C: ERROR ANALYSIS OF GKP REPEATERS WITH HIGHER-LEVEL CODES

In this Appendix, we lift our error analysis from Appendix B to the logical level. First, we discuss in Appendix C1 the two repeater protocols from Secs. II A 1 and II A 2. In Appendix C2, we discuss the optimal placement of the lower-level GKP measurements for the third protocol from Sec. II A 3 and analyze its performance.

# 1. Logical performance of GKP qudits concatenated with quantum polynomial codes

In Appendix **B**, we showed that the error probability distribution for measurements in repeater stations is given by  $\mathcal{P}_{sq}(X, \sigma^2)$ , where  $\sigma_{2\text{-way}}^2 = 3\sigma_{sq}^2 + \frac{1}{\eta_c \eta_{det} \sqrt{\eta}} - 1$  and  $\sigma_{1\text{-way}}^2 = 3\sigma_{sq}^2 + 1 - \eta_c \eta_{det} \eta$  for the two-way and one-way teleportation protocol, respectively. When the protocol is lifted to its logical version, we still find the same error distribution for each of the measurements of the physical GKP qudits (of which there are *D*). This is because of the semitransversality of the *CZ* gate for this code (see Appendix A1 for details).

Here, we consider a simple decoder that only corrects errors occurring on a number of qudits not more than half the distance  $d = \frac{D+1}{2}$ . Thus, the probability that a correctable error pattern occurs at a repeater station is given by

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$$p_{\rm cor} = \sum_{k=0}^{\frac{d-1}{2}} {D \choose k} p_0^{D-k} (1-p_0)^k,$$
(C1)

where  $p_0 = P_{sq}(X^0, \sigma^2)$ , as in Eq. (8). If the decoding attempt fails, we replace the measured state with a state such that the entropy of the overall density matrix is maximized (as a worst-case approximation). In other words, with probability  $1 - p_{cor}$ , we insert a logical error, uniformly at random from the set  $\{1, \ldots, D-1\}$ . Therefore, the error probability distribution on measurement outcomes in any repeater station is given by

$$P_{\rm rep}(X^k) = \begin{cases} p_{\rm cor} & \text{if } k = 0\\ \frac{1}{D-1}(1-p_{\rm cor}) & \text{otherwise.} \end{cases}$$
(C2)

If the probability of errors is so large that  $p_{cor} < \frac{1}{D-1}(1-p_{cor})$ , we replace Eq. (C2) by the uniform distribution. As in Appendix B, we ignore correlations between error distributions on different repeater stations, and estimate the final error distribution of the encoded repeater line as  $\mathcal{P}_{final}(X, Z) = \mathcal{P}_{rep}^{*N}(X) \otimes \mathcal{P}_{rep}^{*N}(Z)$ .

# 2. Error analysis of the half-teleportation protocol for various placements of GKP syndrome measurements

In this Appendix, we discuss how introducing additional ancilla-based measurements of lower-level GKP stabilizers can improve the performance of the one-way half-teleportation protocol with optical preamplification from Sec. II A 3. Such measurements are pictured in Fig. 1(c) of the main text. As discussed in Sec. II C 1, every transmission from one repeater station to the next is associated with

<sup>&</sup>lt;sup>3</sup>In principle, the measurements near the ends of the repeater line have smaller error probabilities. Ignoring this slightly underestimates performance, however, the difference is vanishingly small for a large number of repeater stations.



FIG. 7. Propagation of Gaussian errors for the half-teleportation protocol *without* additional GKP stabilizer measurements. Because of periodic boundary conditions, we have  $\sigma_{in,p}^2 = \sigma_{out,p}^2 = \sigma_{sq}^2$  and  $\sigma_{in,q}^2 = \sigma_{out,q}^2 = 2\sigma_{sq}^2 + 1 - \eta$ . Therefore, the variance of *q* errors reaching the *p* measurements is given by  $\sigma_{sq}^2 + \sigma_{in,q}^2 + 1 - \eta = 3\sigma_{sq}^2 + 2(1 - \eta)$ .

a Gaussian error channel with variance  $\sigma_{loss}^2 = 1 - \eta_c \eta_{det} \eta$ , where  $\eta = \exp(-L_0/L_{att})$ . In every repeater station, all incoming GKP qudits are measured in the *p* quadrature. Before this, however, each GKP qudit is coupled via a physical CPHASE<sup>†</sup> gate to a qudit in the next logical block. Since the CPHASE<sup>†</sup> gate spreads *p* errors into *q* errors, but *q* errors are not propagated to the next mode, every error source only has a limited range. A *p* error that arises during one transmission does not directly affect *p* measurements on the qudit it occurred to; however, it propagates into a *q* error on the subsequent GKP qudit. Furthermore, a *p* error during GKP state preparation backpropagates through the CPHASE<sup>†</sup> gate and causes a *q* error on the readout of the preceding GKP qudit.

In the plain version (without lower-level GKP stabilizer measurements), errors on physical readouts (in the repeater stations) follow an error distribution  $\mathcal{P}_{sq}(Z, 2\sigma_{loss}^2 +$  $3\sigma_{sq}^2$ ), where the variance takes noise from two transmissions and three GKP state preparations into account. By introducing a lower-level GKP stabilizer measurement in every repeater station, we can correct displacement errors after a single transmission. In this way, we effectively avoid combining the two transmission loss channels. Instead, all Gaussian errors in one quadrature are replaced by the discrete Pauli error channel from Eq. (8). Such discrete qudit Pauli errors will propagate to the measurements in the usual way [53]. Depending on where in the repeater station we place the ancilla-based GKP stabilizer measurement, the final error distribution will vary. We discuss four options.

(i) No additional GKP stabilizer measurements are performed (see Fig. 7 for the error analysis).

(ii) After every CZ gate, the (physical) target qudit is subjected to a GKP stabilizer measurement of  $S_X = \exp(-i\sqrt{2\pi D\hat{p}})$ . This is achieved by preparing an ancillary GKP qudit in state  $|0\rangle$ , applying a CSUM gate from the ancilla to the repeater qudit, and a *p* measurement of the ancilla GKP qudit (see Fig. 8 for the error analysis).

(iii) *Before* every *CZ* gate, the control qudit is subjected to a GKP stabilizer measurement of  $S_Z = \exp(i\sqrt{2\pi D}\hat{q})$ . This is achieved by preparing a GKP ancilla in state  $|+\rangle$ , applying a CSUM gate from the repeater qudit to the ancilla, followed by a *q* measurement of the ancilla (see Fig. 9 for the error analysis).

(iv) We *alternate* between options (ii) and (iii); see Fig. 10 for the error analysis.

In option (i), the error analysis from Appendix C1 with  $\sigma^2 = 2\sigma_{loss}^2 + 3\sigma_{sq}^2$  applies (see Fig. 7). Both in option (ii) and (iii), which we refer to as *symmetric* placements of the GKP stabilizer measurements, it turns out that every *p* measurement is subject to two discrete Pauli error channels as in Eq. (8), one having variance  $2\sigma_{sq}^2 + \sigma_{loss}^2$  and the other one  $4\sigma_{sq}^2 + \sigma_{loss}^2$ . Thus, the error analysis from Appendix C1 applies after we insert

$$p_{0}^{\text{sym}} = \sum_{k=0}^{D-1} P_{\text{sq}} (X^{k}, 2\sigma_{\text{sq}}^{2} + \sigma_{\text{loss}}^{2}) \\ \times P_{\text{sq}} (X^{D-k}, 4\sigma_{\text{sq}}^{2} + \sigma_{\text{loss}}^{2})$$
(C3)

into Eq. (C1). Finally, in option (iv) both GKP stabilizer and logical measurements are subject to Gaussian errors with

$$\begin{array}{c|c} \sigma_{\mathrm{in},p}^{2} & \mathcal{A}\left(\eta^{-1}\right) & \mathcal{L}\left(\eta\right) & \sigma_{\mathrm{in},p}^{2} + 1 - \eta & \sigma_{\mathrm{in},p}^{2} + 1 - \eta & \sigma_{\mathrm{sn},q}^{2} & \sigma$$

FIG. 8. Propagation of Gaussian errors for the half-teleportation protocol with additional GKP stabilizer measurements *after* every *CZ* gate. Because of periodic boundary conditions, we have  $\sigma_{in,p}^2 = \sigma_{out,p}^2 = 2\sigma_{sq}^2$  and  $\sigma_{in,q}^2 = \sigma_{out,q}^2 = \sigma_{sq}^2$ . Therefore, the variance of *q* errors reaching the *p* measurements is given by  $\sigma_{sq}^2 + \sigma_{in,q}^2 + 1 - \eta = 2\sigma_{sq}^2 + 1 - \eta$ . In addition to these continuous displacement errors, a discrete Pauli error channel  $\mathcal{P}_{sq}(Z, \sigma_{GKP}^2)$  leads to lower-level logical errors on every *X* measurement, where  $\sigma_{GKP}^2 = 2\sigma_{sq}^2 + \sigma_{in,p}^2 + 1 - \eta = 4\sigma_{sq}^2 + 1 - \eta$  is the variance of *q* errors reaching the lower-level GKP stabilizer measurement.



FIG. 9. Propagation of Gaussian errors for the half-teleportation protocol with additional GKP stabilizer measurements *before* every *CZ* gate. Because of periodic boundary conditions, we have  $\sigma_{in,p}^2 = \sigma_{out,p}^2 = \sigma_{sq}^2$  and  $\sigma_{in,q}^2 = \sigma_{out,q}^2 = 2\sigma_{sq}^2$ . Therefore, the variance of *q* errors reaching the *p* measurements is given by  $2\sigma_{sq}^2 + \sigma_{in,q}^2 + 1 - \eta = 4\sigma_{sq}^2 + 1 - \eta$ . In addition to these continuous displacement errors, a discrete Pauli error channel  $\mathcal{P}_{sq}(Z, \sigma_{GKP}^2)$  leads to lower-level logical errors on every *X* measurement, where  $\sigma_{GKP}^2 = \sigma_{sq}^2 + \sigma_{in,p}^2 + 1 - \eta = 2\sigma_{sq}^2 + 1 - \eta$  is the variance of *p* errors reaching the lower-level GKP stabilizer measurement. Originally, the lower-level GKP stabilizer measurement results in a discrete Pauli error channel  $\mathcal{P}_{sq}(X, \sigma_{GKP}^2)$ , which is then propagated to a Pauli error channel  $\mathcal{P}_{sq}(Z, \sigma_{GKP}^2)$  in the next segment due to the *CZ* gate.

variance  $3\sigma_{sq}^2 + \sigma_{loss}^2$ . This time, we thus have to insert

$$p_0^{\text{alt}} = \sum_{k=0}^{D-1} P_{\text{sq}} (X^k, 3\sigma_{\text{sq}}^2 + \sigma_{\text{loss}}^2) P_{\text{sq}} (X^{D-k}, 3\sigma_{\text{sq}}^2 + \sigma_{\text{loss}}^2)$$
(C4)

into Eq. (C1).

In Fig. 11, we show how the placement of GKP stabilizer measurements influences the performance of the half-teleportation protocol, using the exact same setting as in Fig. 4 of the main text. Overall, the situation is very similar to that in Fig. 4: for  $s_{\text{GKP}} = 20 \text{ dB}$  in Fig. 11(a), only the  $[5, 1, 3]_5$  code (black) offers a nonzero SKR, whereas for  $s_{\text{GKP}} = 30 \text{ dB in Fig. 11(b)}$  also the  $[[13, 1, 7]]_{13}$  code (red) and the  $[\![17, 1, 9]\!]_{17}$  code (green) have the potential to distribute secret keys. We see in Fig. 11 that an alternating placement of GKP stabilizer measurements (solid curves) leads to the highest values of SKR/N. For both option (ii) and (iii), the symmetric placements (dotted curves) are governed by Eq. (C3), and therefore lead to the same performance. We see that not performing any additional GKP stabilizer measurements (dash-dotted curve) leads to the lowest performance, which is easily explained by the large variance  $2\sigma_{loss}^2 + 3\sigma_{sq}^2$ . The other options break the term  $2\sigma_{loss}^2$  and, therefore, perform better. For the symmetric placement, the bottleneck is posed by the term  $4\sigma_{sq}^2$  in Eq. (C3), which is worse than  $3\sigma_{sq}^2$  in

Eq. (C4) for the alternating placement. This explains why the latter performs best. For a large squeezing value of  $s_{\text{GKP}} = 30 \text{ dB}$ , the difference between  $3\sigma_{\text{sq}}^2$  and  $4\sigma_{\text{sq}}^2$  is negligible, which causes a nearly perfect overlapping of the dotted and solid curves in Fig. 11(b).

Since the alternating placement of GKP stabilizer measurements has the best performance, we have assumed this option for the one-way half-teleportation protocol throughout the main text of this paper.

# APPENDIX D: COST COMPARISON WITH KNOWN SCHEMES

In this Appendix, we compare the costs of a quantum repeater employing a concatenation of GKP qudits with a polynomial code with another scheme where we consider MM qudits concatenated with the same qudit code (QPYC) as in our approach [51]. First we consider the cost function

$$C = \frac{2n\log_2(D)}{L_0R},$$

where *n* is the overall number of used qudits and *R* denotes the achieved secret-key rate as given in Ref. [52, Eq. 27]. The factor  $\log_2(D)$  describes that abstract qudits of dimension *D* require more resources. As we are discussing schemes based on different photonic encodings, it is also interesting to consider the number of required optical modes leading to the

$$\begin{array}{c} \sigma_{\mathrm{in},p}^{2} & \overline{\mathcal{L}(\eta^{-1})} & \overline{\mathcal{L}(\eta)} & \sigma_{\mathrm{in},p}^{2} + 1 - \eta & \sigma_{\mathrm{in},p}^{2} + 1 - \eta & p \\ \sigma_{\mathrm{in},q}^{2} & \overline{\mathcal{L}(\eta^{-1})} & \overline{\mathcal{L}(\eta)} & \sigma_{\mathrm{sq}}^{2} + 1 - \eta & \sigma_{\mathrm{sq}}^{2} + 1 - \eta \\ & |\mp\rangle & \sigma_{\mathrm{sq}}^{2} & \sigma_{\mathrm{sq}}^{2} + \sigma_{\mathrm{in},p}^{2} + 1 - \eta & \overline{\mathcal{L}(\eta)} & \overline{\mathcal{L}(\eta)} & \overline{\mathcal{L}(\eta)} & \overline{\mathcal{L}(\eta)} & \overline{\mathcal{L}(\eta^{-1})} & \overline{\mathcal{L}(\eta$$

FIG. 10. Propagation of Gaussian errors for the half-teleportation protocol with additional GKP stabilizer measurements at *alternating* placements. Because of periodic boundary conditions, we have  $\sigma_{in,p}^2 = \sigma_{out,p}^2 = \sigma_{sq}^2$  and  $\sigma_{in,q}^2 = \sigma_{out,q}^2 = 2\sigma_{sq}^2$ . Therefore, it turns out that the variance of *q* errors reaching all *p* measurements is given by  $\sigma_{sq}^2 + \sigma_{in,q}^2 + 1 - \eta = 3\sigma_{sq}^2 + 1 - \eta$ . In addition to these continuous displacement errors, a discrete Pauli error channel  $\mathcal{P}_{sq}(Z, \sigma_{GKP}^2)$  leads to lower-level logical errors on every *X* measurement, where  $\sigma_{GKP}^2 = 2\sigma_{sq}^2 + \sigma_{in,p}^2 + 1 - \eta = 3\sigma_{sq}^2 + 1 - \eta$  is the variance of errors reaching and altering lower-level GKP stabilizer measurements.



FIG. 11. Lower bound on the SKR per logical channel use,  $\log_2(D) - H(\mathcal{P})$ , normalized by the number N of repeater stations for the one-way half-teleportation protocol and various placements of lower-lever GKP stabilizer measurements. We plot SKR/N as a function of the repeater spacing  $L_0$  for a repeater line of total length  $L = NL_0 = 2000$  km, coupling efficiencies  $\eta_c \eta_{det} = 99.9\%$ , and squeezing levels of (a)  $s_{GKP} = 20$  dB or (b)  $s_{GKP} = 30$  dB.

cost

$$C' = \frac{2nm}{L_0 R},$$

where *m* is the number of required optical modes per qudit [see Eq. (24) in Ref. [51]]. Since our schemes also consider QPYC as the high-level code it is of large interest to compare it with the MM approach. As it can be seen in Fig. 12, depending on the considered cost function MM is either better than GKP or vice versa. When considering function *C* our schemes are worse than the MM scheme even for large squeezing values as 30 dB, but the cost of the MM scheme is almost the same as the two-way teleportation protocol. However, with respect to cost function *C'* all three GKP schemes outperform the MM one at 30 dB and at 20 dB also the two-way teleportation scheme outperforms the MM one. The cost function considering the number of used modes benefits the GKP encoding, as in the GKP encoding by increasing the qudit dimension *D* one squeezes more information into the same physical Hilbert



(b) FIG. 12. Comparison of repeater costs between our proposed schemes and proposed schemes employing multimode encodings concatenated with polynomial codes for different cost functions. In (a) we consider a cost function where we consider the qudit dimension to contribute to the cost, while in (b) we consider the number of used modes instead of the qudit dimension.

6000

 $L[\mathrm{km}]$ 

8000

10000

4000

40

20

2000

space, while in the case of the MM encoding by increasing D we also introduce new modes such that the dimension of the physical Hilbert space is also increased. One might also consider quantum Reed-Solomon codes as discussed in Ref. [52]. These codes have been shown to reduce the overall cost in combination with the MM encoding, as they allow for the encoding of multiple logical qudits while still attaining the quantum Singleton bound. We think that a similar effect will also occur in a combination with a GKP encoding which is beyond the scope of this paper.

However, this comparison of costs between different platforms should be taken with a grain of salt. Ultimately, the real-world cost is given by SKR per spent dollar which might be proportional to these cost functions, but different platforms might have different proportionality constants. Furthermore, in this model there is no actual cost assigned to the state quality as, e.g., the cost of squeezing as it is hard to quantify.

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