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Genuine multipartite entanglement from a thermodynamic perspective

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Among the various types of quantum entanglement, the most fascinating and interesting one is genuine entanglement undoubtedly. In this paper, the unique genuine decomposition theorem with respect to any pure state is given and the relations between battery capacity gap and ergtropic gap are studied. By exploiting the battery capacity of multipartite systems, we introduce a vector-valued genuine measure compared with other existing measures and find that the one is superior to some other existing measures in confirming whether the deterministic LOCC transformation between two multipartite pure states is prohibited. Through some proper constructions, one can derive many GME measures. We also give the explicit expression of bipartite battery capacity gap with respect to mixed states by the convex roof extension and investigate some monogamy relations related to the battery capacity gap.

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I. INTRODUCTION

Entanglement as exceedingly fascinating feature in quantum mechanics has provided key resources in quantum information over the past 30 years [1–3]. For various reasons, entanglement measures are needed for quantifying the resource. Entanglement measures for bi-partite systems turn out to be all equivalent to each other [4–7]. However, when considering multipartite entanglement, the problem becomes dramatically complicated. Even when considering the three-qubit scenario, that is also the case. In order to explicitly write down the generalized Schmidt decomposition of any three-qubit pure state $|\psi\rangle$, one needs to use five free parameters [8]. This boils down to a mathematical problem on a five-dimensional manifold possibly being extremely complicated. Among all entangled states, the most wonderful and important ones are the genuine multipartite entangled (GME) states first considered in the seminal papers [9,10]. We also recognize the importance of genuine multipartite entanglement from the fact that the presence of such entanglement is a necessary and sufficient condition for the success of quantum teleportation in a three-qubit system that Alice, Bob, and Charlie share a genuinely entangled state.

In order to faithfully quantify entanglement resources appearing in teleportation protocols, Ma *et al.* [11] identified two criteria: (a) the measure Δ satisfies the condition that, for any biseparable state $|\psi\rangle$, $\Delta(|\psi\rangle) = 0$, and in this case, we say the measure Δ to be genuine; (b) if for any nonseparable state $|\psi\rangle$, $\Delta(|\psi\rangle) > 0$, and in this case, we call the measure Δ to be faithful. So far, although many entanglement measures have been proposed, unfortunately very few of them are GME. For example, the ones appearing in the literature [12–20] fail to meet criterion (a) and the ones appearing in the literature [21–25] violate criterion (b).

Very recently, quantum thermodynamics has become such a thriving field that there is hope of bridging the gap between quantum mechanics and thermodynamics. In fact, there are many similarities between thermodynamics and quantum entanglement [26]. Instead of capturing entanglement of bipartite systems through entropy, Puliyil *et al.* [26] and Yang *et al.* [27] used the measures defined in terms of thermodynamic quantities called ergotropic gap and battery capacity gap, respectively.

In this paper, we present the genuinely unique decomposition theorem of multipartite pure states with a benefit that we can have a better understanding of the structure of the entangled pure state. We also compare and discuss the relations and differences between ergotropic gap and battery capacity gap under certain conditions. After defining a corresponding entanglement of formation measure with respect to general mixed states by the convex roof extension, an explicit formula of bipartite battery capacity gap is shown and we also study the monogamy relation meaning that multipartite entanglement fails to be freely shared between subsystems. The monogamy relation is a quite surprising phenomenon regarded as the key resource in quantum information. Finally, we introduce a vector-valued measure for capturing the multipartite entanglement and illustrate its superiority. We also show the relation between real-valued measures and the proposed vector-valued measure.

II. GENUINE UNIQUE DECOMPOSITION THEOREM OF MULTIPARTITE PURE STATES

We first present the following theorem of the uniqueness of the decomposition of general pure states into genuinely entangled states. The conclusion of this theorem provides us with a clearer understanding of the structure of multipartite pure states.

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Theorem II.1. If $|\phi\rangle$ is a multipartite pure state, then there is a genuine unique decomposition of $|\phi\rangle$:

$$|\phi\rangle = |\phi\rangle_{\Omega_1} \otimes |\phi\rangle_{\Omega_2} \otimes \cdots \otimes |\phi\rangle_{\Omega_k},$$

where Ω_j, $j = 1, \cdots, k$, are disjoint partition of $\{\Lambda_1, \Lambda_2, \cdots, \Lambda_n\}$, that is to say,

$$\Omega_i \bigcap \Omega_j = \emptyset \;\&\; \bigcup_{j=1}^{k} \Omega_j = \{\Lambda_1, \cdots, \Lambda_n\}$$

and $|\phi\rangle_{\Omega_j}$, $j = 1, \cdots, k$, are genuinely entangled pure states. This decomposition is unique, up to the order of these genuinely entangled pure states and phase factors.

Proof. In the following proof, we will show a proof by contradiction.

Firstly, assume that the disjoint partition is fixed, namely, these subsets $\Omega_1, \Omega_2, \cdots, \Omega_k$ is fixed up to their order. We will show, in this case, the decomposition is unique up to the order of genuinely entangled pure states and phase factors. Indeed, if $|\phi\rangle = |\phi\rangle_{\Omega_1} \otimes |\phi\rangle_{\Omega_2} \otimes \cdots \otimes |\phi\rangle_{\Omega_k} = |\phi\rangle'_{\Omega_1} \otimes |\phi\rangle'_{\Omega_2} \otimes \cdots \otimes |\phi\rangle'_{\Omega_k}$, in view of $\langle\phi|\phi\rangle = 1$, we have $\langle\phi|\phi\rangle'_{\Omega_1} \times \langle\phi|\phi\rangle'_{\Omega_2} \times \cdots \times \langle\phi|\phi\rangle'_{\Omega_k} = 1$. Invoking $|\langle\phi|\phi\rangle'_{\Omega_j}| \leqslant 1$, $j = 1, \ldots, k$, it is obvious that $|\langle\phi|\phi\rangle'_{\Omega_j}| = 1$, $j = 1, \ldots, k$, which results in $|\phi\rangle_{\Omega_j} = e^{i\theta_j}|\phi\rangle'_{\Omega_j}$, $j = 1, \ldots, k$.

In what follows, suppose that there are two kind of disjoint partitions Ω_j and $\hat{\Omega}_l$, then there exists $A_i \in \Omega_j \bigcap \hat{\Omega}_l$ with $\Omega_j \neq \hat{\Omega}_l$. We divide the case into two subcases: (i) $\hat{\Omega}_l \subset \Omega_j$ or $X_j \subset \hat{\Omega}_l$; (ii) $\hat{\Omega}_l \nsubseteq \Omega_j$ & $\Omega_j \nsubseteq \hat{\Omega}_l$ & $\Omega_j \bigcap \hat{\Omega}_l \neq \emptyset$. For convenience, set $B = \Omega_j$ and $D = \hat{\Omega}_l$. In the first subcase (i), without loss of generality, we assume $B \subset D$. We can write $|\phi\rangle$ as $|\phi\rangle = |\phi\rangle_B \otimes |\phi\rangle_{B^c}$ and $|\phi\rangle = |\phi\rangle_D \otimes |\phi\rangle_{D^c}$. According to $|\phi\rangle = |\phi\rangle_B \otimes |\phi\rangle_{B^c}$, tracing over B^c, we deduce

$$\rho_B = \mathrm{Tr}_{B^c}(|\phi\rangle_B\langle\phi| \otimes |\phi\rangle_{B^c}\langle\phi|) = |\phi\rangle_B\langle\phi|. \tag{1}$$

However, from $|\phi\rangle = |\phi\rangle_D \otimes |\phi\rangle_{D^c}$, after tracing over B^c, we derive $\rho_B = \mathrm{Tr}_{B^c}(|\phi\rangle_D\langle\phi| \otimes |\phi\rangle_{D^c}\langle\phi|) = \mathrm{Tr}_{B^c \bigcap D}(|\phi\rangle_D\langle\phi|) = \sum_{i=1}^{s} \lambda_i^2 |\phi^i\rangle_B\langle\phi^i|$, with $\langle\phi^i|\phi^j\rangle_B = 0$, for all $i \neq j$. Since $|\phi\rangle_D$ is genuinely entangled, it turns out that the Schmidt number $s > 1$. This yields a contradiction. In the second subcase (ii),

$$\begin{aligned}\rho_B &= \mathrm{Tr}_{B^c}(|\phi\rangle_D\langle\phi| \otimes |\phi\rangle_{D^c}\langle\phi|)\\ &= \mathrm{Tr}_{D/(B\bigcap D)}(|\phi\rangle_D\langle\phi|) \otimes \mathrm{Tr}_{(B\bigcup D)^c}(|\phi\rangle_{D^c}\langle\phi|)\\ &= \sum_t \mu_t |\eta^t\rangle_{B\bigcap D}\langle\eta^t| \otimes \sum_s \lambda_s |\xi^s\rangle_{B/(B\bigcap D)}\langle\xi^s|\\ &= \sum_{t,s} \mu_t\lambda_s |\eta^t\xi^s\rangle_B\langle\eta^t\xi^s|,\end{aligned} \tag{2}$$

where $|\phi\rangle_D = \sum_{t=1}^{q} \mu_t |\eta^t\rangle_{B\bigcap D}|\eta^t\rangle_{B/(B\bigcap D)}$ and $|\phi\rangle_{D^c} = \sum_{s=1}^{p} \lambda_s |\xi^s\rangle_{B/(B\bigcap D)}|\xi^s\rangle_{(B\bigcup D)^c}$ are the Schmidt decompositions with respect to $|\phi\rangle_D$ and $|\phi\rangle_{D^c}$, respectively. It is clear that the above equalities (1) and (2) yield a contradiction. Hence, we complete this proof. ■

According to the unique decomposition theorem above, if the genuine unique decomposition of a pure state $|\phi\rangle$ is expressed as

$$|\phi\rangle = |\phi\rangle_{\Omega_1} \otimes |\phi\rangle_{\Omega_2} \otimes \cdots \otimes |\phi\rangle_{\Omega_k},$$

then it can be seen readily that the pure state $|\phi\rangle$ is k separable and not $k+1$-separable [26], and the partition is also unique.

Example II.1. Consider the pure state $|\phi\rangle_{ABCD} = \frac{1}{2\sqrt{2}}(|0000\rangle + |0110\rangle + |1000\rangle + |1110\rangle + |0001\rangle + |0111\rangle + |1001\rangle + |1111\rangle)$. By a simple calculation, we find that the pure state $|\phi\rangle$ can be decomposed as the tensor product of three genuinely entangled pure states:

$$|\phi\rangle = |\phi\rangle_A \otimes |\phi\rangle_{BC} \otimes |\phi\rangle_D,$$

where $|\phi\rangle_A = \frac{1}{\sqrt{2}}(|0\rangle_A + |1\rangle_A)$, $|\phi\rangle_B = \frac{1}{\sqrt{2}}(|0\rangle_B + |1\rangle_B)$, and $|\phi\rangle_{BC} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. The theorem above ensures the uniqueness of such decomposition.

III. THE BATTERY CAPACITY GAP AND THE ERGOTROPIC GAP

The study of work extraction from an isolated quantum system under a cyclic Hamiltonian process dates back to late 1970's ([28,29]). Assume that A is a n-dimensional quantum system initialized in the state $\hat{\rho} \in \mathcal{D}(\mathbb{C}^d)$ and forced to evolve in time by external modulations of its Hamiltonian $\hat{H} = \sum_{j=0}^{d-1} \epsilon_j |\epsilon_j\rangle\langle\epsilon_j|$, with $|\epsilon_j\rangle$ beingan eigenstate corresponding to energy eigenvalue ϵ_j, satisfying $\epsilon_j \leqslant \epsilon_{j+1}$, $j = 0, \cdots, d-1$, and the minimum energy eigenvalue $\epsilon_0 = 0$ and $\epsilon_1 > 0$. We denote by $\mathrm{Tr}(\hat{\rho}\hat{H})$ the mean internal energy of $\hat{\rho}$, and $\hat{U}$ the element of the unitary group $\mathrm{U}(d)$. The average amount of work $W_{\hat{U}}(\hat{\rho})$ extracted in this process can be calculated through the following formula [28,29]:

$$W_{\hat{U}}(\hat{\rho}) := \mathrm{Tr}(\hat{\rho}\hat{H}) - \mathrm{Tr}(\hat{U}\hat{\rho}\hat{U}^\dagger\hat{H}).$$

We denote by $W_e(\hat{\rho})$ the optimal average amount of work(called ergotropy):

$$W_e(\hat{\rho}) := \max_{\hat{U}} W_{\hat{U}}(\hat{\rho}) = W_{\hat{U}^\downarrow}(\hat{\rho}) = \mathrm{Tr}(\hat{\rho}\hat{H}) - \mathrm{Tr}(\hat{\rho}^\downarrow\hat{H}),$$

where $\hat{U}^\downarrow$ is the optimal unitary, and denote by $\hat{\rho}^\downarrow$ the passive state with $\mathrm{Tr}(\hat{\rho}^\downarrow\hat{H})$ fulfilling minimum mean energy [30]. Likewise, we can give the definition of anti-ergotropy as the minimum average amount of work:

$$W_{ae}(\hat{\rho}) := \min_{\hat{U}} W_{\hat{U}}(\hat{\rho}) = W_{\hat{U}^\uparrow}(\hat{\rho}) = \mathrm{Tr}(\hat{\rho}\hat{H}) - \mathrm{Tr}(\hat{\rho}^\uparrow\hat{H}),$$

where now $\hat{U}^\uparrow$ is its optimal unitary, and denote by $\hat{\rho}^\uparrow$ the active state with $\mathrm{Tr}(\hat{\rho}^\downarrow\hat{H})$ fulfilling maximum mean energy. Clearly, $W_{ae}(\hat{\rho}) \leqslant 0$, and the absolute value $|W_{ae}(\hat{\rho})|$ indicates how much energy is needed to fully charge the quantum battery. Let $|\lambda_j^\downarrow\rangle(|\lambda_j^\uparrow\rangle)$ be the eigenstate of $\hat{\rho}$ with respect to the energy eigenvalue $\lambda_j^\downarrow(\lambda_j^\uparrow)$ with $\lambda_j^\downarrow \geqslant \lambda_{j+1}^\downarrow(\lambda_j^\uparrow \leqslant \lambda_{j+1}^\uparrow)$. We can express them explicitly as [28–31]:

$$\hat{U}^{\downarrow(\uparrow)} = \sum_j |\epsilon_j^{\uparrow(\downarrow)}\rangle\langle\lambda_j^{\downarrow(\uparrow)}|, \quad \hat{\rho}^{\downarrow(\uparrow)} = \sum_j \lambda_j^{\downarrow(\uparrow)} |\epsilon_j^{\uparrow(\downarrow)}\rangle\langle\epsilon_j^{\uparrow(\downarrow)}|.$$

Next, we define the battery capacity of the system as follows [27]:

$$\mathcal{C}(\hat{\rho}) := \mathrm{Tr}(\hat{\rho}^\uparrow\hat{H}) - \mathrm{Tr}(\hat{\rho}^\downarrow\hat{H}). \tag{3}$$

The battery capacity is the difference between the maximum mean energy allowed by the system and the minimum mean energy, and equals the difference between the ergotropy of the state $\hat{\rho}$ and its antiergotropy :

$$\mathcal{C}(\hat{\rho}) = W_e(\hat{\rho}) - W_{ae}(\hat{\rho}).$$

It can be verified readily that $\mathcal{C}(\hat{\rho}) = \mathcal{C}(\hat{U}\hat{\rho}\hat{U}^\dagger)$ and so, $\mathcal{C}(\hat{\rho})$ is a unitarily invariant. In fact, $\mathcal{C}(\hat{\rho})$ is determined completely by the energy eigenvalues of the Hamiltonian $\hat{H}$ and the eigenvalues of the state $\hat{\rho}$. Referring to Ref. [27], we show an explicit formula expressing the relation between the battery capacity and these eigenvalues:

$$\mathcal{C}(\hat{\rho}) = \sum_{j=0}^{d-1} \epsilon_j^\uparrow(\lambda_j^\uparrow - \lambda_j^\downarrow) = \sum_{j=0}^{d-1} \lambda_j^\uparrow(\epsilon_j^\uparrow - \epsilon_j^\downarrow).$$

With respect to the quantum battery, such as the quantum system thermally isolated but mechanically coupled to a work source or load, the battery capacity $\mathcal{C}(\hat{\rho})$ is the optimal average amount of work which can be transfered during any thermodynamic cycle where the battery evolves as a unitary transformation.

Recently, research on ergotropy work of multipartite quantum systems has aroused great interest [21,32–45]. For composite quantum systems consisting of several local Hamiltonians, the entangled state can allow global work storage exceeding the sum of its local parts. This leads to the entanglement measure based on energy for multipartite systems. Indeed, such as for $\hat{\rho}_{\Lambda\Lambda'} \in \mathcal{D}(\mathbb{C}^{d_\Lambda} \otimes \mathbb{C}^{d_{\Lambda'}})$, the bipartite battery capacity gap $\Delta_{\Lambda|\Lambda'}(\hat{\rho}_{\Lambda\Lambda'})$, which is the difference between local battery capacity $\mathcal{C}^l_{\Lambda|\Lambda'}(\hat{\rho}_{\Lambda\Lambda'})$ and global battery capacity $\mathcal{C}^g_{\Lambda|\Lambda'}(\hat{\rho}_{\Lambda\Lambda'})$, turns out to be local operations and classical communication (LOCC) monotone, and an entanglement measure [35,37]. We explicitly express the bipartite battery capacity gap $\Delta_{\Lambda|\Lambda'}(\hat{\rho}_{\Lambda\Lambda'})$ as [27]:

$$\Delta_{\Lambda|\Lambda'}(\hat{\rho}_{\Lambda\Lambda'}) := \mathcal{C}^g_{\Lambda|\Lambda'}(\hat{\rho}_{\Lambda\Lambda'}) - \mathcal{C}^l_{\Lambda|\Lambda'}(\hat{\rho}_{\Lambda\Lambda'}).$$

Similarly, we can also define bipartite ergotropic gap $\Delta^e_{\Lambda|\Lambda'}(\hat{\rho}_{\Lambda\Lambda'})$ as [26]:

$$\Delta^e_{\Lambda|\Lambda'}(\hat{\rho}_{\Lambda\Lambda'}) := \mathcal{W}^g_{\Lambda|\Lambda'}(\hat{\rho}_{\Lambda\Lambda'}) - \mathcal{W}^l_{\Lambda|\Lambda'}(\hat{\rho}_{\Lambda\Lambda'}),$$

where $\mathcal{W}^l_{\Lambda|\Lambda'}(\hat{\rho}_{\Lambda\Lambda'})$ and $\mathcal{W}^g_{\Lambda|\Lambda'}(\hat{\rho}_{\Lambda\Lambda'})$ are global and local ergotropic works extracted respectively through the application of product unitary $\hat{U}_\Lambda \otimes \hat{U}_{\Lambda'}$ and joint unitary $\hat{U}_{\Lambda\Lambda'}$.

For multipartite systems different subgroups of the parties can come together and accordingly different type of battery capacity can be extracted from the system. For a n-party system we can define fully separable battery capacity gap for a pure state $|\phi\rangle_{\Lambda_1\cdots\Lambda_n} \in \otimes_{i=1}^n \mathbb{C}^{d_i}$ [27]:

$$\Delta_{\Lambda_1|\cdots|\Lambda_n}(|\phi\rangle) := \mathcal{C}^g_{\Lambda_1|\cdots|\Lambda_n}(|\phi\rangle) - \mathcal{C}^l_{\Lambda_1|\cdots|\Lambda_n}(|\phi\rangle), \tag{4}$$

which is the difference between global battery capacity $\mathcal{C}^g_{\Lambda_1|\cdots|\Lambda_n}(|\phi\rangle)$ obtained through applying joint unitary to the whole system and fully local battery capacity $\mathcal{C}^l_{\Lambda_1|\cdots|\Lambda_n}(|\phi\rangle)$ obtained through local unitaries on the respective subsystems. Similarly, we show the definition of fully separable ergotropic gap [27]:

$$\Delta^e_{\Lambda_1|\cdots|\Lambda_n}(|\phi\rangle) := \mathcal{W}^g_{\Lambda_1|\cdots|\Lambda_n}(|\phi\rangle) - \mathcal{W}^l_{\Lambda_1|\cdots|\Lambda_n}(|\phi\rangle). \tag{5}$$

For a system governed by the Hamiltonian, which is assumed to be total interaction free global Hamiltonian throughout the article, that is, $\hat{H} = \sum_{i=1}^n \tilde{H}_{\Lambda_i}$ ($\tilde{H}_{\Lambda_i} = I_{d_1} \otimes \cdots \otimes I_{d_{i-1}} \otimes \hat{H}_{\Lambda_i} \otimes I_{d_{i+1}} \otimes \cdots \otimes I_{d_n}$), suppose the maximum spectrums of the Hamiltonian $\hat{H}_{\Lambda_s}, s = 1, \cdots, n$, are $E_s, s = 1, \cdots, n$, respectively. It is clear that the maximum spectrum E of the global Hamiltonian $\hat{H}$ is equal to $\sum_{s=1}^n E_s$, and therefore $\mathcal{C}^g_{\Lambda_1|\cdots|\Lambda_n}(|\phi\rangle) = \sum_{s=1}^n E_s$. Hence, in view of (4), the fully separable battery capacity gap can be expression as

$$\begin{aligned}\Delta_{\Lambda_1|\cdots|\Lambda_n}(|\phi\rangle) &= \mathcal{C}^g_{\Lambda_1|\cdots|\Lambda_n}(|\phi\rangle) - \mathcal{C}^l_{\Lambda_1|\cdots|\Lambda_n}(|\phi\rangle)\\ &= \sum_{i=1}^n \big(E_i - \mathrm{Tr}\big(\hat{\rho}^\uparrow_{\Lambda_i}\hat{H}_{\Lambda_i}\big)\\ &\quad + \mathrm{Tr}\big(\hat{\rho}^\downarrow_{\Lambda_i}\hat{H}_{\Lambda_i}\big)\big).\end{aligned} \tag{6}$$

In order to calculate the fully separable battery capacity gap $\Delta_{\Lambda_1|\cdots|\Lambda_n}(|\phi\rangle)$ and the fully separable ergotropic gap $\Delta^e_{\Lambda_1|\cdots|\Lambda_n}(|\phi\rangle)$, we apply the genuine decomposition of the entangled state.

Theorem III.1. Let Ω_i, $i = 1, \cdots, k$, be disjoint partition of $\{\Lambda_1, \Lambda_2, \cdots, \Lambda_n\}$, and $|\phi\rangle_{\Omega_i}$, $i = 1, \cdots, k$, be genuinely entangled states, then $\Delta_{\Lambda_1|\cdots|\Lambda_n}(|\phi\rangle) = \sum_{i=1}^k \Delta_{\Lambda_{i_1}|\cdots|\Lambda_{i_{c_i}}}(|\phi\rangle_{\Omega_i})$, where $\Omega_i = \{\Lambda_{i_1}, \cdots, \Lambda_{i_{c_i}}\}$.

Proof. For $\forall\Lambda \in \Omega_i$, we have $\hat{\rho}_\Lambda = \mathrm{Tr}_{\Lambda^c}(|\phi\rangle\langle\phi|) = \mathrm{Tr}_{\Lambda^c\bigcap\Omega_i}(|\phi\rangle_{\Omega_i}\langle\phi|)$. Owing to (6) and

$$\begin{aligned}\Delta_{\Lambda_{i_1}|\cdots|\Lambda_{i_{c_i}}}(|\phi\rangle_{\Omega_i}) &= \sum_{j=1}^{c_i} \big(E_{i_j} - \mathrm{Tr}\big(\hat{\rho}^\uparrow_{\Lambda_{i_j}}\hat{H}_{\Lambda_{i_j}}\big)\\ &\quad + \mathrm{Tr}\big(\hat{\rho}^\downarrow_{\Lambda_{i_j}}\hat{H}_{\Lambda_{i_j}}\big)\big),\end{aligned}$$

one can see

$$\Delta_{\Lambda_1|\cdots|\Lambda_n}(|\phi\rangle) = \sum_{i=1}^k \Delta_{\Lambda_{i_1}|\cdots|\Lambda_{i_{c_i}}}\big(|\phi\rangle_{\Omega_i}\big).$$

■

Theorem III.2. Let Ω_i, $i = 1, \cdots, k$, be disjoint partition of $\{\Lambda_1, \Lambda_2, \cdots, \Lambda_n\}$, and $|\phi\rangle_{\Omega_i}$, $i = 1, \cdots, k$, be genuinely entangled states, then $\Delta^e_{\Lambda_1|\cdots|\Lambda_n}(|\phi\rangle) = \sum_{i=1}^k \Delta^e_{\Lambda_{i_1}|\cdots|\Lambda_{i_{c_i}}}(|\phi\rangle_{\Omega_i})$, where $\Omega_i = \{\Lambda_{i_1}, \cdots, \Lambda_{i_{c_i}}\}$.

Proof. The proof is similar to Theorem III.1. ■

From Theorem III.1 and Theorem III.2, one find that when $n > 2$, both $\Delta_{\Lambda_1|\cdots|\Lambda_n}$ and $\Delta^e_{\Lambda_1|\cdots|\Lambda_n}$ fail to be GME measures. Next, we give two simple examples illustrating their possibly being independent.

Example III.1. We consider two pure states $|\phi\rangle = \sqrt{0.5}|00\rangle + \sqrt{0.4}|11\rangle + \sqrt{0.1}|22\rangle$ and $|\phi\rangle' = \sqrt{0.54}|00\rangle + \sqrt{0.34}|11\rangle + \sqrt{0.12}|22\rangle$. Let the global Hamiltonian be $\hat{H} = \hat{H}_\Lambda \otimes I + I \otimes \hat{H}_{\Lambda'}$ where $\hat{H}_\Lambda = \hat{H}_{\Lambda'} = |1\rangle\langle 1| + 3|2\rangle\langle 2|$. The marginals of $|\phi\rangle'$ and $|\phi\rangle'$ are $\hat{\rho}_\Lambda = 0.5|0\rangle\langle 0| + 0.4|1\rangle\langle 1| + 0.1|2\rangle\langle 2|$ and $\hat{\rho}'_\Lambda = 0.54|0\rangle\langle 0| + 0.34|1\rangle\langle 1| + 0.12|2\rangle\langle 2|$, respectively. It can be seen easily that $\Delta^e_{\Lambda|\Lambda'}(|\phi\rangle) = \Delta^e_{\Lambda|\Lambda'}(|\phi\rangle') = 1.4$, $\Delta_{\Lambda|\Lambda'}(|\phi\rangle') = 3.48$ and $\Delta_{\Lambda|\Lambda'}(|\phi\rangle) = 3.6$.

In this case, the ergotropic gap fails to distinguish these two pure states. However, $\Delta_{\Lambda|\Lambda'}(|\phi\rangle') < \Delta_{\Lambda|\Lambda'}(|\phi\rangle)$, means that any deterministic LOCC transformation $|\phi\rangle' \mapsto |\phi\rangle$ is prohibited.

Example III.2. Consider two pure states $|\phi\rangle = \sqrt{0.5}|00\rangle + \sqrt{0.3}|11\rangle + \sqrt{0.2}|22\rangle$ and $|\phi\rangle' = \sqrt{0.45}|00\rangle + \sqrt{0.4}|11\rangle + \sqrt{0.15}|22\rangle$. Let the global Hamiltonian be $\hat{H} = \hat{H}_\Lambda \otimes I + I \otimes \hat{H}_{\Lambda'}$ where $\hat{H}_\Lambda = \hat{H}_{\Lambda'} = |1\rangle\langle 1| + 3|2\rangle\langle 2|$. The marginals of $|\phi\rangle'$ and $|\phi\rangle'$ are $\hat{\rho}_\Lambda = 0.5|0\rangle\langle 0| + 0.3|1\rangle\langle 1| + 0.2|2\rangle\langle 2|$ and $\hat{\rho}'_\Lambda = 0.45|0\rangle\langle 0| + 0.4|1\rangle\langle 1| + 0.15|2\rangle\langle 2|$, respectively. It can be verified easily that $\Delta^e_{\Lambda|\Lambda'}(|\phi\rangle) = 1.8$, $\Delta^e_{\Lambda|\Lambda'}(|\phi\rangle') = 1.7$, $\Delta_{\Lambda|\Lambda'}(|\phi\rangle') = \Delta_{\Lambda|\Lambda'}(|\phi\rangle) = 4.2$. In this case, the battery capacity gap fails to distinguish these two pure states. However, $\Delta^e_{\Lambda|\Lambda'}(|\phi\rangle) > \Delta^e_{\Lambda|\Lambda'}(|\phi\rangle')$, means that any deterministic LOCC transformation $|\phi\rangle' \mapsto |\phi\rangle$ is prohibited.

In Example III.4, $\Delta^e_{\Lambda|\Lambda'}$ provides more information, but by contrast in Example III.3, $\Delta_{\Lambda|\Lambda'}$ provides more information. We also remark that, in a Hamiltonian with equispaced energy levels, the ergotropic gap and fully separable battery capacity gap are equivalent:

Theorem III.3. Let the Hamiltonian for the i^{th} subsystem be $\hat{H}_{\Lambda_i} = \sum_{j=0}^{d_i-1} j\epsilon_i|\epsilon_j\rangle\langle\epsilon_j|$. Then, for any pure state $|\phi\rangle$, $\Delta_{\Lambda_1|\cdots|\Lambda_n}(|\phi\rangle) = 2\Delta^e_{\Lambda_1|\cdots|\Lambda_n}(|\phi\rangle)$.

Proof. Set the passive state as

$$\hat{\rho}^{\downarrow}_{\Lambda_i} = \sum_{j=0}^{d_i-1} \lambda_j^{\downarrow i}|\epsilon_j\rangle\langle\epsilon_j|$$

and the active state as

$$\hat{\rho}^{\uparrow}_{\Lambda_i} = \sum_{j=0}^{d_i-1} \lambda_j^{\uparrow i}|\epsilon_j\rangle\langle\epsilon_j|.$$

Clearly, we have $\lambda_j^{\downarrow i} = \lambda_{d_i-1-j}^{\uparrow i}$ and $\lambda_j^{\uparrow i} = \lambda_{d_i-1-j}^{\downarrow i}$. From these two equalities above, we deduce

$$\begin{aligned}(d_i-1)\epsilon_i - \mathrm{Tr}\hat{\rho}^{\uparrow}_{\Lambda_i}\hat{H} &= \sum_{j=0}^{d_i-1}\lambda_j^{\uparrow i}((d_i-1)\epsilon_i - j\epsilon_i)\\ &= \sum_{j=0}^{d_i-1}\lambda_{d_i-1-j}^{\downarrow i}(d_i-1-j)\epsilon_i\\ &= \sum_{j=0}^{d_i-1}\lambda_j^{\downarrow i}\times j\epsilon_i = \mathrm{Tr}(\hat{\rho}^{\downarrow}_{\Lambda_i}\hat{H}),\\ i &= 1,\cdots,n.\end{aligned}$$

Adding up the above set of equations, we can easily obtain

$$\Delta_{\Lambda_1|\cdots|\Lambda_n}(|\phi\rangle) = 2\Delta^e_{\Lambda_1|\cdots|\Lambda_n}(|\phi\rangle).$$

■

Corollary III.1. Assume the Hamiltonian $\hat{H}_{\Lambda_i}$ for the i^{th} subsystem of a given n-qubit system is $\hat{H}_{\Lambda_i} = |1\rangle_{\Lambda_i}\langle 1|$. Then, $\Delta_{\Lambda_1|\cdots|\Lambda_n}(|\phi\rangle) = 2\Delta^e_{\Lambda_1|\cdots|\Lambda_n}(|\phi\rangle)$.

IV. MONOGAMY RELATION IN A *N*-QUBIT SYSTEM

A very important and surprising one among numerous phenomena with respect to multipartite entanglement, the monogamy relation indicates that entanglement resources cannot be freely shared between subsystems, and even is so basic as the no-cloning theorem [21,39–43].

We first express bipartite concurrence $C_{\Lambda|\Lambda'}(|\phi\rangle)$ of a pure bipartite state $|\phi\rangle \in \mathfrak{H}_\Lambda \otimes \mathfrak{H}_{\Lambda'}$ as [38]:

$$C_{\Lambda|\Lambda'}(|\phi\rangle) = \sqrt{2\left(1 - \mathrm{Tr}\left(\hat{\rho}^2_\Lambda\right)\right)},$$

where $\hat{\rho}_\Lambda = \mathrm{Tr}_{\Lambda'}(|\phi\rangle\langle\phi|)$. In the above expression about bipartite concurrence, we need to emphasize that $\mathfrak{H}_\Lambda$ and $\mathfrak{H}_{\Lambda'}$ may be arbitrary finite-dimensional Hilbert spaces.

Throughout this section, we assume that $\hat{H}$ is total interaction free global Hamiltonian of a *n*-qubit system and the Hamiltonian $\hat{H}_{\Lambda_i}$ for the i^{th} subsystem of the given *n*-qubit system is $\hat{H}_{\Lambda_i} = |1\rangle_{\Lambda_i}\langle 1|$. And, in what follows, we assume the Hilbert spaces $\mathfrak{H}_\Lambda$ and $\mathfrak{H}_{\Lambda'}$ both are two-dimensional with the Hamiltonian for Λ and Λ' subsystems being $\hat{H}_\Lambda = |1\rangle_\Lambda\langle 1|$ and $\hat{H}_{\Lambda'} = |1\rangle_{\Lambda'}\langle 1|$, respectively. It can be verified easily by the definitions of $\Delta_{\Lambda|\Lambda_1\cdots\Lambda_{n-1}}$ and $C_{\Lambda|\Lambda_1\cdots\Lambda_{n-1}}$ that for any *n*-qubit pure state $|\phi\rangle \in \mathfrak{H}_\Lambda \otimes \mathfrak{H}_{\Lambda_1} \otimes \mathfrak{H}_{\Lambda_2} \otimes \cdots \otimes \mathfrak{H}_{\Lambda_{n-1}}$,

$$\begin{aligned}\Delta_{\Lambda|\Lambda_1\cdots\Lambda_{n-1}}(|\phi\rangle) &= 2\Delta^e_{\Lambda|\Lambda_1\cdots\Lambda_{n-1}}(|\phi\rangle)\\ &= 2\left(1 - \sqrt{1 - C^2_{\Lambda|\Lambda_1\cdots\Lambda_{n-1}}(|\phi\rangle)}\right)\end{aligned} \quad (7)$$

and for any 2-qubit pure state $|\psi\rangle \in \mathfrak{H}_\Lambda \otimes \mathfrak{H}_{\Lambda'}$,

$$\begin{aligned}\Delta_{\Lambda|\Lambda'}(|\psi\rangle) &= 2\Delta^e_{\Lambda|\Lambda'}(|\psi\rangle)\\ &= 2\left(1 - \sqrt{1 - C^2_{\Lambda|\Lambda'}(|\psi\rangle)}\right).\end{aligned} \quad (8)$$

We define a corresponding entanglement of formation measure still denoted by $C_{\Lambda|\Lambda'}$ with respect to mixed states $\hat{\rho}_{\Lambda\Lambda'}$ by the following convex roof extension:

$$C_{\Lambda|\Lambda'}(\hat{\rho}_{\Lambda\Lambda'}) := \min_{\{p_j,|\phi_j\rangle\}} \sum_{j=1}^n p_j C_{\Lambda|\Lambda_i}(|\phi_j\rangle),$$

where the minimum is taken over all pure state decompositions of $\hat{\rho}_{\Lambda\Lambda'} := \sum_{j=1}^n p_j|\phi_j\rangle\langle\phi_j|$. Refering to [46], there exactly exists an optimal decomposition $\{p_j, |\phi_j\rangle\}$ of $\hat{\rho}_{\Lambda\Lambda'}$ with $C_{\Lambda|\Lambda'}(|\phi_j\rangle) = C_{\Lambda|\Lambda'}(\hat{\rho}_{\Lambda\Lambda'})$. Let $h(x) = 2(1 - \sqrt{1-x^2})$, which is an increase monotonic and convex function. In view of (8), we can write $\Delta_{\Lambda|\Lambda'}(|\phi\rangle) = h(C_{\Lambda|\Lambda'}(|\phi\rangle))$. For bipartite mixed state $\hat{\rho}_{\Lambda\Lambda'}$, still denote by $\Delta_{\Lambda|\Lambda'}$ the entanglement of formation with respect to bipartite separable battery capacity gap $\Delta_{\Lambda|\Lambda'}$. Using an optimal convex decomposition $\{q_k, |\varphi_k\rangle\}$ for $\Delta_{\Lambda|\Lambda'}(\hat{\rho}_{\Lambda\Lambda'})$, we have:

$$\begin{aligned}\Delta_{\Lambda|\Lambda'}(\hat{\rho}_{\Lambda\Lambda'}) &= \sum_k q_k\Delta_{\Lambda|\Lambda'}(|\varphi_k\rangle)\\ &= \sum_k q_k h(C_{\Lambda|\Lambda'}(|\varphi_k\rangle))\\ &\leqslant \sum_j p_j h(C_{\Lambda|\Lambda'}(|\phi_j\rangle))\\ &= \sum_j p_j h(C_{\Lambda|\Lambda'}(\hat{\rho}_{\Lambda\Lambda'}))\\ &= h(C_{\Lambda|\Lambda'}(\hat{\rho}_{\Lambda\Lambda'})).\end{aligned} \quad (9)$$

We can also obtain:

$$\begin{aligned}\Delta_{\Lambda|\Lambda'}(\hat{\rho}_{\Lambda\Lambda'}) &= \sum_k q_k \Delta_{\Lambda|\Lambda'}(|\varphi_k\rangle)\\ &= \sum_k q_k h(C_{\Lambda|\Lambda'}(|\varphi_k\rangle))\\ &\geqslant h(\sum_k q_k C_{\Lambda|\Lambda'}(|\varphi_k\rangle))\\ &\geqslant h(\sum_j p_j C_{\Lambda|\Lambda'}(|\phi_j\rangle))\\ &= h(C_{\Lambda|\Lambda'}(\hat{\rho}_{\Lambda\Lambda'})), \end{aligned} \tag{10}$$

where the first and second inequality are due to convexity and monotonicity of the function g, respectively. So, we derive $\Delta_{\Lambda|\Lambda'}(\hat{\rho}_{\Lambda\Lambda'}) = h(C_{\Lambda|\Lambda'}(\hat{\rho}_{\Lambda\Lambda'}))$ and $\Delta_{\Lambda|\Lambda'}(\hat{\rho}_{\Lambda\Lambda'}) = \sum_i p_j h(C_{\Lambda|\Lambda'}(|\phi_j\rangle))$.

Theorem IV.1. For any n-qubit pure state $|\phi\rangle \in \mathfrak{H}_\Lambda \otimes \mathfrak{H}_{\Lambda_1} \otimes \mathfrak{H}_{\Lambda_2} \otimes \cdots \otimes \mathfrak{H}_{\Lambda_{n-1}}$ and all $\alpha \geqslant 1$, the following monogamy relation holds:

$$\Delta^\alpha_{\Lambda|\Lambda_1\cdots\Lambda_{n-1}}(|\phi\rangle) \geqslant \Delta^\alpha_{\Lambda|\Lambda_1}(|\phi\rangle) + \cdots + \Delta^\alpha_{\Lambda|\Lambda_{n-1}}(|\phi\rangle).$$

Proof. For convenience, we introduce a function

$$g(x) = 2(1-\sqrt{1-x}),\ x \in [0,1].$$

It follows by the definition of the bipartite separable battery capacity gap that

$$\Delta_{\Lambda|\Lambda_1\cdots\Lambda_{n-1}}(|\phi\rangle) = g\big(C^2_{\Lambda|\Lambda_1\cdots\Lambda_{n-1}}(|\phi\rangle)\big).$$

After a straightforward calculation, we derive

$$\frac{dg^\alpha(x)}{dx} = \frac{\alpha 2^{\alpha-1}(1-\sqrt{1-x})^{\alpha-1}}{\sqrt{1-x}}.$$

It turns out to be increasing function iff $\alpha \geqslant 1$. Therefore, for all $\alpha \geqslant 1$, $g^\alpha(x)$ is convex. Noting that $g^\alpha(0) = 0$, for all $\alpha \geqslant 1$, one can easily see that $g^\alpha(x+y) \geqslant g^\alpha(x) + g^\alpha(y)$. So, $g^\alpha(x_1 + x_2 + \cdots + x_n) \geqslant g^\alpha(x_1) + g^\alpha(x_2) + \cdots + g^\alpha(x_n)$. We now let $C^\alpha_{\Lambda|\Lambda_j}(|\phi\rangle_{\Lambda\Lambda_1\cdots\Lambda_{n-1}})$ stand for $C^\alpha_{\Lambda|\Lambda_j}(\mathrm{Tr}_{\Lambda_1\cdots\Lambda_{j-1}\Lambda_{j+1}\cdots\Lambda_{n-1}}(|\phi\rangle_{\Lambda\Lambda_1\cdots\Lambda_{n-1}}))$. Invoking $C^2_{\Lambda|\Lambda_1\cdots\Lambda_{n-1}}(|\phi\rangle) \geqslant C^2_{\Lambda|\Lambda_1}(|\phi\rangle) + \cdots + C^2_{\Lambda|\Lambda_{n-1}}(|\phi\rangle)$ [47] and the monotonicity of the function g, we get

$$\begin{aligned} & g^\alpha\big(C^2_{\Lambda|\Lambda_1\cdots\Lambda_{n-1}}(|\phi\rangle)\big)\\ &\quad \geqslant g^\alpha\big(C^2_{\Lambda|\Lambda_1}(|\phi\rangle) + \cdots + C^2_{\Lambda|\Lambda_{n-1}}(|\phi\rangle)\big)\\ &\quad \geqslant g^\alpha\big(C^2_{\Lambda|\Lambda_1}(|\phi\rangle)\big) + \cdots + g^\alpha\big(C^2_{\Lambda|\Lambda_{n-1}}(|\phi\rangle)\big). \end{aligned} \tag{11}$$

Since

$$\Delta_{\Lambda|\Lambda_j}(|\phi\rangle) = g\big(C^2_{\Lambda|\Lambda_j}(|\phi\rangle)\big)$$

and

$$\Delta_{\Lambda|\Lambda_1\cdots\Lambda_{n-1}}(|\phi\rangle) = g\big(C^2_{\Lambda|\Lambda_1\cdots\Lambda_{n-1}}(|\phi\rangle)\big),$$

one can see that

$$\begin{aligned} & \Delta^\alpha_{\Lambda|\Lambda_1\cdots\Lambda_{n-1}}(|\phi\rangle)\\ &\quad \geqslant g^\alpha\big(C^2_{\Lambda|\Lambda_1}(|\phi\rangle)\big) + \cdots + g^\alpha\big(C^2_{\Lambda|\Lambda_{n-1}}(|\phi\rangle)\big)\\ &\quad = \Delta^\alpha_{\Lambda|\Lambda_1}(|\phi\rangle) + \cdots + \Delta^\alpha_{\Lambda|\Lambda_{n-1}}(|\phi\rangle). \end{aligned} \tag{12}$$

The proof is completed. ■

Corollary IV.1. For any n-qubit pure state $|\phi\rangle \in \mathfrak{H}_{\Lambda_1} \otimes \mathfrak{H}_{\Lambda_2} \otimes \cdots \otimes \mathfrak{H}_{\Lambda_n}$, the following relation holds:

$$\Delta_{\Lambda_1|\cdots|\Lambda_n}(|\phi\rangle) \geqslant \sum_{i<j} \Delta_{\Lambda_i|\Lambda_j}(|\phi\rangle). \tag{13}$$

Proof. By a direct calculation, we can easily obtain

$$\Delta_{\Lambda_i|\Lambda_i^c}(|\phi\rangle) = 2\big(E_i - \mathrm{Tr}\big(\hat{\rho}^\uparrow_{\Lambda_i}\hat{H}_{\Lambda_i}\big) + \mathrm{Tr}\big(\hat{\rho}^\downarrow_{\Lambda_i}\hat{H}_{\Lambda_i}\big)\big).$$

From Eq. (6), it follows that

$$\Delta_{\Lambda_1|\cdots|\Lambda_n}(|\phi\rangle) = \frac{1}{2}\sum_i \Delta_{\Lambda_i|\Lambda_i^c}(|\phi\rangle).$$

Therefore, in view of Theorem IV.1, one sees

$$\Delta_{\Lambda_i|\Lambda_i^c}(|\phi\rangle) \geqslant \sum_j \Delta_{\Lambda_i|\Lambda_j}(|\phi\rangle).$$

Obviously, $\Delta_{\Lambda_j|\Lambda_i}(|\phi\rangle) = \Delta_{\Lambda_i|\Lambda_j}(|\phi\rangle)$. Hence, by these inequalities above, we have

$$\Delta_{\Lambda_1|\cdots|\Lambda_n}(|\phi\rangle) \geqslant \sum_{i<j} \Delta_{\Lambda_i|\Lambda_j}(|\phi\rangle).$$

■

We readily have a similar version of the theorem with respect to the ergotropic gap Δ^e.

Theorem IV.2. For any n-qubit pure state $|\phi\rangle \in \mathfrak{H}_\Lambda \otimes \mathfrak{H}_{\Lambda_1} \otimes \mathfrak{H}_{\Lambda_2} \otimes \cdots \otimes \mathfrak{H}_{\Lambda_{n-1}}$, and all $\alpha \geqslant 1$, the following monogamy relation holds: $(\Delta^e_{\Lambda|\Lambda_1\cdots\Lambda_{n-1}}(|\phi\rangle))^\alpha \geqslant (\Delta^e_{\Lambda|\Lambda_1}(|\phi\rangle))^\alpha + \cdots + (\Delta^e_{\Lambda|\Lambda_{n-1}}(|\phi\rangle))^\alpha$.

Corollary IV.2. For any n-qubit pure state $|\phi\rangle \in \mathfrak{H}_{\Lambda_1} \otimes \mathfrak{H}_{\Lambda_2} \otimes \cdots \otimes \mathfrak{H}_{\Lambda_n}$, the following relation holds:

$$\Delta^e_{\Lambda_1|\cdots|\Lambda_n}(|\phi\rangle) \geqslant \sum_{i<j} \Delta^e_{\Lambda_i|\Lambda_j}(|\phi\rangle). \tag{14}$$

Regarding entanglement as a resource originating from some kind of special connection between these subsystems, the last corollary seems to be saying that the overall connection of the system is greater than the sum of all possible connections between any two subsystems. Next, we show a simple example illustrating that the inequality (13) is tight by finding a state such that the bound is saturated.

Example IV.1. In this example, we consider the following simple three-qubit pure state:

$$|\phi\rangle = \frac{1}{\sqrt{2}}|000\rangle + \frac{1}{\sqrt{2}}|110\rangle.$$

Let the global Hamiltonian be $\hat{H} = \hat{H}_A \otimes I_2 \otimes I_2 + I_2 \otimes \hat{H}_B \otimes I_2 + I_2 \otimes I_2 \otimes \hat{H}_C$ where $\hat{H}_A = \hat{H}_B = \hat{H}_C = |1\rangle\langle 1|$. One can see easily that

$$\Delta_{A|B|C}(|\phi\rangle) = 2,\ \Delta_{A|B}(|\phi\rangle) = 2,$$

$$\Delta_{A|C}(|\phi\rangle) = 0,\ \Delta_{B|C}(|\phi\rangle) = 0.$$

Therefore,

$$\Delta_{A|B|C}(|\phi\rangle) = \Delta_{A|B}(|\hat{\rho}_{AB}\rangle) + \Delta_{A|C}(|\hat{\rho}_{AC}\rangle) + \Delta_{B|C}(|\hat{\rho}_{BC}\rangle).$$

V. THE VECTOR-VALUED MEASURE FUNCTION OF MULTIPARTITE ENTANGLEMENT

So far, all entanglement measures are real-valued [12–25]. Based on the LOCC monotonicity of $\Delta_{X|X^c}$ [27], we introduce a vector-valued measure.

For any multipartite entangled pure state $|\phi\rangle$, we define the vector-valued measure:

$$\overrightarrow{\Delta}(|\phi\rangle) := \big(\Delta_{X_1|X_1^c}(|\phi\rangle), \cdots, \Delta_{X_N|X_N^c}(|\phi\rangle)\big),$$

where $\{(X_1, \cdots, X_N)\}$ is the collection of all nonempty proper subsets of the set $\{\Lambda_1, \cdots, \Lambda_n\}$ ordered by some means. Here, in order to avoiding repetition, we assume that arbitrary two sets X_i and X_j belonging to the collection satisfies that $X_i \neq X_j$ and $X_i \neq X_j^C$ whenever $i \neq j$.

In this section, the Hamiltonian $\hat{H_{\Lambda_i}} = \sum_{j=0}^{d_i-1} \epsilon_j^i |\epsilon_j^i\rangle\langle\epsilon_j^i|$, with $|\epsilon_j^i\rangle$ being eigenstate corresponding to energy eigenvalue ϵ_j^i, satisfying $\epsilon_j^i \leqslant \epsilon_{j+1}^i$, $j = 0, \cdots, d_i - 1$, and the minimum energy eigenvalue $\epsilon_0^i = 0$ and $\epsilon_1^i > 0$. This ensures that if $\Delta_{X|X^c}(|\phi\rangle) = 0$, then the state $|\phi\rangle$ must be biseparable with respect to the partition $\{X, X^c\}$.

Let R^N be partially ordered by the convex cone R_+^N, where $R_+^N = \{(x_1, \cdots, x_N) \in R^N | x_i \geqslant 0,\ i = 1, \cdots, N\}$, that is, letting $x = (x_1, \cdots, x_N)$, $y = (y_1, \cdots, y_N)$, we say $x \succeq y$ iff $x - y \in R_+^N$, which is equivalent to $x_i \geqslant y_i$, $i = 1, \cdots, N$. We call a vector-valued measure of multipartite entanglement $\overrightarrow{\Gamma}$ to be genuine if, for any biseparable entangled pure state $|\phi\rangle$, there exists a zero entry of the vector $\overrightarrow{\Gamma}(|\phi\rangle)$, faithful if, for any genuinely entangled pure state $|\phi\rangle$, none of the entries of the vector $\overrightarrow{\Gamma}(|\phi\rangle)$ are equal to zero, and LOCC monotony if, for any deterministic LOCC transformation $|\phi\rangle' \mapsto |\phi\rangle$, one must have $\overrightarrow{\Gamma}(|\phi\rangle') \succeq \overrightarrow{\Gamma}(|\phi\rangle)$. In this sense, the vector-valued measure $\overrightarrow{\Delta}$ is genuine, faithful, and LOCC monotone.

Example V.1. Consider $|\phi\rangle = \frac{1}{\sqrt{2}}|000\rangle + \frac{1}{2}|101\rangle + \frac{1}{2}|110\rangle$ and $|\phi\rangle' = \frac{1}{2}|000\rangle + \frac{1}{\sqrt{2}}|101\rangle + \frac{1}{2}|110\rangle$. Let the global Hamiltonian be $\hat{H} = \hat{H}_A \otimes I_2 \otimes I_2 + I_2 \otimes \hat{H}_B \otimes I_2 + I_2 \otimes I_2 \otimes \hat{H}_C$ where $\hat{H}_A = \hat{H}_B = \hat{H}_C = |1\rangle\langle 1|$. It can be seen readily that $\Delta_{\text{avg}}^G(|\phi\rangle) = \Delta_{\text{avg}}^G(|\phi\rangle')$, $\Delta_{\min}^G(|\phi\rangle) = \Delta_{\min}^G(|\phi\rangle')$, $\Delta_V^G(|\phi\rangle) = \Delta_V^G(|\phi\rangle')$, $\Delta_F^G(|\phi\rangle) = \Delta_F^G(|\phi\rangle')$ [26]. In this case, these four measures above, take same value for both these states and hence they remain silent to compare their entanglement. In fact, only studying entanglement based on these measure, will result in a loss of a lot of information. If we choose to apply the vector-valued measure to measure the entanglement, one gets, respectively,

$$\begin{aligned}\overrightarrow{\Delta}(|\phi\rangle) &= (\Delta_{A|BC}(|\phi\rangle),\ \Delta_{B|CA}(|\phi\rangle),\\ &\Delta_{C|AB}(|\phi\rangle)) = (2, 1, 1),\\ \overrightarrow{\Delta}(|\phi\rangle') &= (\Delta_{A|BC}(|\phi\rangle'),\ \Delta_{B|CA}(|\phi\rangle'),\\ &\Delta_{C|AB}(|\phi\rangle')) = (1, 1, 2).\end{aligned} \tag{15}$$

Observing that $\overrightarrow{\Delta}(|\phi\rangle) - \overrightarrow{\Delta}(|\phi\rangle') \notin R_+^3$ and $\overrightarrow{\Delta}(|\phi\rangle)' - \overrightarrow{\Delta}(|\phi\rangle) \notin R_+^3$, hence, both deterministic transformations $|\phi\rangle' \to |\phi\rangle$ and $|\phi\rangle \to |\phi\rangle'$ are prohibited under any LOCC. Unlike bipartite systems, entanglement in multipartite systems should consider the specific structure of the entanglement. Hence, vector-valued measures perhaps are better candidates.

Example V.2. Consider the GHZ state and W state, respectively,

$$\begin{aligned}|\text{GHZ}\rangle &= \frac{1}{\sqrt{2}}|000\rangle + \frac{1}{\sqrt{2}}|111\rangle,\\ |\text{W}\rangle &= \frac{1}{\sqrt{3}}|100\rangle + \frac{1}{\sqrt{3}}|010\rangle + \frac{1}{\sqrt{3}}|001\rangle.\end{aligned} \tag{16}$$

One can easily see that, respectively,

$$\begin{aligned}\overrightarrow{\Delta}(|\text{GHZ}\rangle) &= (\Delta_{A|BC}(|\text{GHZ}\rangle), \Delta_{B|CA}(|\text{GHZ}\rangle),\\ &\Delta_{C|AB}(|\text{GHZ}\rangle)) = (2, 2, 2),\\ \overrightarrow{\Delta}(|\text{W}\rangle) &= (\Delta_{A|BC}(|\text{W}\rangle), \Delta_{B|CA}(|\text{W}\rangle),\\ &\Delta_{C|AB}(|\text{W}\rangle)) = \left(\tfrac{4}{3}, \tfrac{4}{3}, \tfrac{4}{3}\right).\end{aligned} \tag{17}$$

Based on the two equations above, one can see $\overrightarrow{\Delta}(|\text{GHZ}\rangle) \succ \overrightarrow{\Delta}(|\text{W}\rangle)$, which means that the vector-valued measure $\overrightarrow{\Delta}$ is a proper measure distinguish these two states [48].

For any n-dimensional vector $\vec{v} = (v_1, v_2, \cdots, v_N)$, set, respectively:

$$\prod \vec{v} = \prod_{i=1}^{n} v_i,$$

$$f(\vec{v}) = f(v_1, v_2, \cdots, v_N),$$

where f is a function. The function f is said to be

A1. genuine if it satisfies the condition that $f(\vec{v})$ must be equal to zero whenever $\prod \vec{v} = 0$.

A2. faithful if it satisfies the condition that $f(\vec{v})$ must be greater than zero whenever $\prod \vec{v} \neq 0$.

A3. L monotone if it satisfies the condition that the inequality $f(\vec{u}) \geqslant f(\vec{v})$ must holds whenever $\vec{u} \succeq \vec{v}$.

Here, we need to pay attention to the domain of f. For instance, in the case of three-qubit, the triangle inequalities of the biseparable battery capacity gaps must be satisfied [26]:

$$\Delta_{X|YZ}(|\phi\rangle) \leqslant \Delta_{Y|ZX}(|\phi\rangle) + \Delta_{Z|XY}(|\phi\rangle),$$

where $X, Y, Z \in \{A, B, C\}$. If a function f satisfies these three properties above, we can construct GME measure function Γ in the following way:

$$\Gamma(|\phi\rangle) = f(\overrightarrow{\Delta}(|\phi\rangle)) = f\big(\Delta_{X_1|X_1^c}(|\phi\rangle), \cdots, \Delta_{X_N|X_N^c}(|\phi\rangle)\big).$$

Theorem V.1. If f is a function, defined in the range of $\overrightarrow{\Delta}$, and satisfies the above three conditions A1, A2, and A3, then the measure Γ is genuine, faithful, and LOCC monotone.

Proof. For any pure state $|\phi\rangle$, if $\prod \overrightarrow{\Delta}(|\phi\rangle) = 0$, then by the definition of $\overrightarrow{\Delta}$, there exists a proper subset X of the set $\{\Lambda_1, \cdots, \Lambda_n\}$ such that $\Delta_{X|X^c}(|\phi\rangle)) = 0$. This means that the state $|\phi\rangle$ is biseparable. If condition A1 is satisfied, then

$\Gamma(|\phi\rangle) = f(\overrightarrow{\Delta}(|\phi\rangle)) = 0$ for any bi-separable pure state $|\phi\rangle$. Therefore, the measure Γ is genuine.

If $\prod \overrightarrow{\Delta}(|\phi\rangle) \neq 0$, then the state $|\phi\rangle$ is genuinely entangled state. Assume that condition A2 is satisfied, then $f(\overrightarrow{\Delta}(|\phi\rangle)) > 0$ for any genuinely entangled state $|\phi\rangle$. So, the measure Γ is faithful.

If there is a deterministic LOCC transformation $|\phi\rangle' \mapsto |\phi\rangle$, one always have $\overrightarrow{\Delta}(|\phi\rangle') \succeq \overrightarrow{\Delta}(|\phi\rangle)$. Assume that condition A3 is satisfied, then $f(\overrightarrow{\Delta}(|\phi\rangle')) \geqslant f(\overrightarrow{\Delta}(|\phi\rangle))$. So, the measure Γ is LOCC monotone. ■

If Γ and Θ are two GME measures constructed by the above method which both are genuine, faithful, and LOCC monotone, then the product $\Gamma \cdot \Theta$ is also GME measure satisfying these properties. In Ref. [27], Yang *et al.*, inspired by [26], introduced four GME measures: the minimum of the biseparable battery capacity gap, the average biseparable capacity gap, the battery capacity fill and the battery capacity volume. All these measurements can be seen as constructed in this way. In 2023, Ge *et al.* [49] gave a unified proof indicating Puliyil's version of ergotropic fill was LOCC monotone. In fact, ergotropic fill is a GME measure. The ergotropic fill actually can likewise be viewed as constructed in the manner of the theorem V.3 [26]. It turns out that the vector-valued measure is superior to any real-valued measure constructed in the manner of Theorem V.3 in confirming whether the deterministic LOCC transformation between two multipartite pure states be prohibited. In this section, we only consider the vector-valued measure based on the bipartite battery capacity gap. Similarly, we can also construct a vector-valued measure based on concurrence or other bipartite measures.

Example V.3. Assume that $\mathfrak{S}_N$ is the symmetric group on N elements. For the vector $\vec{v} = (v_1, v_2, \cdots, v_N)$ and $\sigma \in \mathfrak{S}_N$, we set,

$$\sigma(\vec{v}) := (v_{\sigma(1)}, v_{\sigma(2)}, \cdots, v_{\sigma(N)}).$$

Clearly, the measure set as

$$\Theta(|\phi\rangle) = \langle \vec{v}, \sigma(\vec{v})\rangle \times \prod \overrightarrow{\Delta}(|\phi\rangle)$$

is a GME measure. For example, letting $\sigma \in \mathfrak{S}_N$ be a cycle:

$$\sigma = (1\ \ 2\ \ 3) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \tag{18}$$

one obtains

$$\begin{aligned}\Theta(|\phi\rangle) = {} & \Delta_{A|BC}(|\phi\rangle)\Delta_{B|CA}(|\phi\rangle)\Delta_{C|AB}(|\phi\rangle) \\ & \times (\Delta_{A|BC}(|\phi\rangle)\Delta_{B|CA}(|\phi\rangle) \\ & + \Delta_{B|CA}(|\phi\rangle)\Delta_{C|AB}(|\phi\rangle) \\ & + \Delta_{C|AB}(|\phi\rangle))\Delta_{A|BC}(|\phi\rangle)).\end{aligned} \tag{19}$$

In this way, many smooth GME measures can be obtained, and it is important to compare them with existing GME measures. We will pursue these efforts in the future.

VI. CONCLUSIONS

In this paper, we present the genuine unique decomposition theorem of multipartite pure states. The conclusion of this theorem provides us with a clearer understanding of the structure of multipartite pure states. We also compare and analyze the relations between ergotropic gap and battery capacity gap. It turns out to be that these two measures may be independent, and hence may obtain more information by considering both these measures. By the convex roof extension, we explicitly express the formula of the entanglement of formation associated with the measure $\Delta_{\Lambda|\Lambda'}$ with respect to general bipartite mixed states. Then, we find that the entanglement of formation measure fulfils the monogamy relation and the new relations reported in Corollary IV.2 and Corollary IV.4, which means the overall connection of the system is greater than the sum of all possible connections between any two subsystems. Finally, we introduce the vector-valued measure for capturing the multipartite entanglement, illustrate its superiority and show the relation between existing real-valued measures and the proposed vector-valued measure.

As for the future, in the light of Theorem V.3, finding good functions employed to construct genuine measurements worth considering, and deeper relations between ergotropic gap and battery capacity gap are also needed.

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