Many-body entanglement and spectral clusters in the extended hard-core bosonic Hatano-Nelson model

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We study many-body entanglements and spectra of the extended bosonic Hatano-Nelson model in the hardcore limit. We show that the system undergoes a phase transition from a gapless phase to a charge-density-wave phase accompanied by a parity-time transition in the first-excited state. The phase transition is characterized by the crossing of the ground-state biorthogonal-order parameter and the sudden change of the first-excited-state entanglement entropy. The gapless phase is verified by the logarithmic scaling of the ground-state entanglement entropy with the central charge c = 1. Furthermore, we show that all energy spectral clusters would form ellipses in strong nearest-neighbor interactions, for which we establish a universal scaling law. The lengths of the major and minor axes are shown to obey power laws with respect to the nearest-neighbor interaction. The exact expressions are derived for the numbers of energy levels on the outermost elliptic ring of each clusters.

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I. INTRODUCTION

Quantum entanglement, which is a central concept of quantum mechanics, has broad potential applications, such as quantum teleportation [1], quantum computation [2,3], and quantum sensing [4] in quantum technology and quantum information science [2,5]. A notable application of the quantum entanglement for a many-body system is to characterize the equilibrium phase transition [6,7], where the entanglement entropy has been shown to display a logarithmic divergence in a one-dimensional model [8]. Quantum entanglement is a crucial tool in understanding quantum phases of Hermitian many-body systems.

Non-Hermitian systems are of particular interest because of many unique phenomena that have no counterparts in Hermitian systems [9,10]. It is known that non-Hermitian skin effects [11-25] and exceptional points [26-37] are two fascinating phenomena in non-Hermitian systems. Recently, the entanglement entropy has been generalized to non-Hermitian systems for understanding phase transitions. For instance, the biorthogonal entanglement entropy has been introduced to describe the equilibrium phase transitions [38–44], where the central charge has been found to be negative at an exceptional point from the entanglement entropy under specific treatments [38,41,42]. The negative central charge has been argued to be described by the nonunitary conformal field theory [38,41]. The understanding of the entanglement entropy at an exceptional point remains an open question.

Many-body physics of non-Hermitain systems is another interesting topic, in which rich unique phenomena such as many-body skin effects [45–52], many-body edge bursts

[53], and entanglement phase transitions [54–57] have been explored in recent years. Concerns for many-body spectra [48,49,52] might be at the core of the researches as a direct extension of the single-particle physics. Recently, studies show that interactions can induce parity-time (\mathcal{PT}) transitions [49] and spectral structures [48,49] in the interacting fermionic Hatano-Nelson (FHN) model under periodic boundary conditions (PBCs).

In order to further understand phase transitions and spectral structures induced by the interaction, we instead study the extended interacting bosonic Hatano-Nelson (BHN) model with a hard-core constraint, which is inequivalent to the FHN model under PBCs even in the thermodynamic limit [52]. We show that the BHN model undergoes a phase transition in the ground state accompanied by a \mathcal{PT} transition in the first-excited state. The ground-state phase transition can be described by the biorthogonal-order parameter. The central charge c = 1 for the gapless phase is derived from both the biorthogonal entanglement entropy and the self-norm entanglement entropy. For the first-excited phase transition, we find that the biorthogonal and self-norm entanglement entropies display a sudden change near the critical point. In addition, we study the many-body energy spectra by increasing the interaction, where we find that the energy spectral clusters form ellipses in strong nearest-neighbor interactions. We establish the universal laws of spectral clusters and show that the structure of an ellipse is dependent on the interaction, the filling, and the number of clusters.

This paper is organized as follows. In Sec. II, we introduce the bosonic Hatano-Nelson model. In Sec. III, we study the entanglement entropy of the ground state and the first-excited state. In Sec. IV, we discuss the properties of spectral clusters in strong nearest-neighbor interactions. In Sec. V, we summarize our results.

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II. MODEL

In this paper, we study the one-dimensional extended BHN model, which is a non-Hermitian system with a nearestneighbor nonreciprocal hopping. The Hamiltonian of the extended BHN model can be written as

$$H = \sum_{l=1}^{L} \left[(t+\gamma)b_{l}^{\dagger}b_{l+1} + (t-\gamma)b_{l+1}^{\dagger}b_{l} + \frac{U}{2}n_{l}(n_{l}-1) + Vn_{l}n_{l+1} \right],$$
(1)

where $b_l^{\dagger}(b_l)$ is the creation (annihilation) operator of a boson at the lattice site *l*, and $n_l = b_l^{\dagger} b_l$ is the bosonic number operator. *L* is the length of the chain. The real parameters *t* and γ denote the reciprocal and nonreciprocal hopping coefficients of bosons between two nearest-neighboring sites, respectively. The coupling coefficient *U* is the on-site interaction, and *V* is the nearest-neighbor interaction between two bosons in adjacent lattice sites. The PBCs are imposed by $b_{L+1} = b_1$.

Because it is extremely difficult to diagonalize a huge non-Hermitian matrix, numerical simulations of soft-core bosons are beyond the scope of the exact-diagonalization method. In the following, we study the system with the exact diagonalization in PBCs and merely consider the system with a hard-core constraint $(U \rightarrow \infty)$. The extended BHN model in the hard-core limit is given by

$$H = \sum_{l} \left[(t+\gamma) b_{l}^{\dagger} b_{l+1} + (t-\gamma) b_{l+1}^{\dagger} b_{l} + V n_{l} n_{l+1} \right], \quad (2)$$

with the constraint $n_l = \{0, 1\}$. The BHN model in the hardcore limit is equivalent to the FHN model in open boundary conditions (OBCs) [49], but is different from the FHN model in PBCs [52]. The eigenvalues of the BHN model in Eq. (2) are always real for $\gamma < t$ in OBCs as the Hamiltonian can be mapped to a Hermitian Hamiltonian under a site-dependent similarity transform [52]. In the following, we set t = 1 during the simulations and study the extended hard-core BHN model in Eq. (2) under PBCs as we are interested in the properties of the complex-valued energy spectra.

III. ENTANGLEMENT ENTROPY

In the half filling, the extended hard-core BHN can be mapped to the spin-1/2 chain,

$$\hat{H} = \sum_{l} \left[\frac{t}{2} \left(\sigma_{l}^{x} \sigma_{l+1}^{x} + \sigma_{l}^{y} \sigma_{l+1}^{y} \right) + \frac{V}{4} \sigma_{l}^{z} \sigma_{l+1}^{z} \right. \\ \left. + i \frac{\gamma}{2} \left(\sigma_{l}^{x} \sigma_{l+1}^{y} - \sigma_{l}^{y} \sigma_{l+1}^{x} \right) - \frac{V}{2} \sigma_{l}^{z} + \frac{V}{4} \right], \qquad (3)$$

by performing the following transformations between the bosonic operators and the spin operators,

$$\sigma_l^+ = b_l, \tag{4}$$

$$\sigma_l^- = b_l^\dagger,\tag{5}$$

$$\sigma_l^z = 1 - 2b_l^{\dagger} b_l, \tag{6}$$

where $\sigma_l^+ = (\sigma_l^x + i\sigma_l^y)/2$ and $\sigma_l^- = (\sigma_l^x - i\sigma_l^y)/2$ are the raising and lowering operators, and σ_l^x , σ_l^y , and σ_l^z are Pauli matrices.

In the case of $\gamma = 0$, the model in Eq. (3) is the well-known (Hermitian) *XXZ* model, which undergoes the Berezinskii-Kosterlitz-Thouless transition between the doubly degenerated antiferromagnetic phase and the gapless *XY* phase. In the bosonic language, two such ground states are named the charge-density-wave (CDW) phase and the gapless phase. In order to describe the phase transition, we introduce the entanglement entropy between a part *A* and a part *B*,

$$S_A = -\mathrm{Tr}_A(\rho_A \ln \rho_A),\tag{7}$$

where $\rho_A = \text{Tr}_B(|\psi_j\rangle\langle\psi_j|)$, and $|\psi_j\rangle$ is the *j*th eigenstate of the BHN model. In the following, we choose the subsystems $A = \{1, \dots, L/2\}$ and $B = \{L/2 + 1, \dots, L\}$. For a Hermitian system under PBCs, it is shown that the entanglement entropy scales logarithmically as [58],

$$S_{L/2} \propto \frac{c}{3} \ln(L) \tag{8}$$

at the critical point of a finite-size chain. The central charge c = 1 for the XXZ model that can be described by the conformal field theory (CFT) [58].

When $\gamma \neq 0$, the model in Eq. (3) is the generalized *XXZ* model with a complex Dzyaloshinskii-Moriya interaction (DMI), which is a non-Hermitian system. Although the Hamiltonian in Eq. (3) is non-Hermitian, the ground-state energy (the state with the lowest real part of eigenvalues) is real for any γ . To verify whether the entanglement can describe the phase transition of the non-Hermitian *XXZ* chain, we compute both the biorthogonal entanglement entropy $S_{L/2}^{RR}$ and the self-norm entanglement entropy $S_{L/2}^{RR}$ of the ground state. The biorthogonal and self-norm reduced density matriices



FIG. 1. Many-body phase diagram of the half-filled hard-core BHN model with respect to γ and V at t = 1 up to L = 24 lattice sites in PBCs. (a) The phase diagram obtained from the change of the first-excited energy as shown in panels (c) and (d), in which the red solid line denotes critical points in the thermodynamic limit derived by extrapolations. (b) The biorthogonal correlation function $C_{1,L/2}$ at $\gamma = 0.6$, where the crossing point denotes the critical point. (c) The four lowest real parts of eigenenergies at $\gamma = 0.6$. (d) The imaginary parts of the four eigenenergies shown in panel (c).



FIG. 2. Half-chain entanglement entropy of the BHN model at t = 1 and $\gamma = 0.6$ in PBCs. (a)–(c) The biorthogonal entanglement entropy in the half filling, the self-norm entanglement entropy of the ground states with a quarter of the filling as the function of *V*. (d)–(f) The scaling of the biorthogonal entanglement entropy from panel (a), the self-norm entanglement entropy from panel (b) at V = 2 and V = 4, and the biorthogonal entanglement entropy from panel (c) at V = 2 and V = 7.

are defined by the combination of the biorthogonal eigenstates $\rho_A^{RL} = \text{Tr}_B(|\psi_0^R\rangle\langle\psi_0^L|)$ and only the right density matrix $\rho_A^{RR} = \text{Tr}_B(|\psi_0^R\rangle\langle\psi_0^R|)$, respectively [38,44]. The global phase diagram of the non-Hermitian *XXZ* that is obtained by the sudden change of the first-excited energy from a real value to a complex value [cf. Figs. 1(c) and 1(d)] is demonstrated in Fig. 1(a), where we find that the complex DMI enlarges the gapless regime compared to the original Hermitian *XXZ* model (The critical value $V_c > 2$ for a nonzero γ). The CDW phase is characterized by the CDW order parameter $O_{\text{CDW}} =$ $\lim_{r\to\infty} C_{1,r}$, where $C_{1,r} = (1 - 2n_1)(1 - 2n_r)$ [see Fig. 1(b)]. The gapless phase is described by both the biorthogonal entanglement entropy and the self-norm entanglement entropy of the ground state [see Figs. 2(a), 2(b), 2(d), and 2(e)], with a central charge c = 1 as in the Hermitian *XXZ* model.

The BHN model shown in Eq. (2) [or in Eq. (3)] has a \mathcal{PT} symmetry, the system thus exists either real eigenenergies or complex eigenenergies in conjugated pairs. Consequently, the BHN model can in principle exhibit a \mathcal{PT} transition between the \mathcal{PT} symmetric phase and the \mathcal{PT} broken phase. A \mathcal{PT} transition in the first-excited state has recently been discussed in Ref. [49] for the FHN model. The \mathcal{PT} transition in the first-excited state has recently been discussed in Ref. [49]. In the BHN model is confirmed and presented in Fig. 1(a). In the following, we instead investigate the properties of quantum entanglement in the first-excited state for such a \mathcal{PT} transition.





FIG. 3. Entanglement entropy of the first-excited states at L = 24 at t = 1 and $\gamma = 0.6$ in PBCs. (a) The biorthogonal entanglement entropy with respect to V. (b) The self-norm entanglement entropy as the function of V.

To achieve it, we calculate both the biorthogonal entanglement entropy $S_{L/2}^{RL}$ and the self-norm entanglement entropy $S_{L/2}^{RR}$ for the first-excited states. We note that the first-excited states are doubly degenerated in the \mathcal{PT} broken regime, which seems to bring additional problems to compute the entanglement entropy. However, thanks to the \mathcal{PT} symmetry, the first-excited states can be distinguished according to the imaginary part of the eigenvalues. In the following, we perform the calculations based on the first-excited state with the negative imaginary part of the energy. We find that both $S_{L/2}^{RL}$ and $S_{L/2}^{RR}$ exhibit a sudden change when varying the interaction as shown in Fig. 3, indicating a phase transition happens. We point out that it is impossible to extract the critical exponents using either $S_{L/2}^{RL}$ or $S_{L/2}^{RR}$. The study of exceptional points using the finite-size scaling theory of the entanglement entropy is open. Our results indicate that the entanglement entropy can serve as a valid quantity for finding critical points.

Finally, we briefly discuss the properties of the entanglement entropy when the system is away from the half filling. In the case of the non-half filling, the ground phases in the whole regime of γ are gapless as the system cannot form a CDW phase. Consequently, the entanglement entropy of the ground phases should exhibit a logarithmic scaling with the central charge c = 1. We verify this argument by the performing scaling of the biorthogonal entanglement entropy as shown in Figs. 2(e) and 2(f).

IV. SPECTRAL CLUSTERS

In this section, we turn to the energy spectra to investigate the effects of the interaction. It is easy to show that the singleparticle energy spectrum which can be exactly obtained for the BHN model is a closed curve in the case of noninteractions [49,52]. However, the spectral structure of the BHN model is too complicated to study under finite interactions. Recently, it was found interestingly that the energy spectrum of the FHN model away from the half filling is separated into spectral clusters [cf. Figs. 4(a)-4(c)] when the interaction strength V is larger than the range of each cluster [49]. It is worthwhile to note that energy spectra can also form clusters even in the half filling, which is not discussed in Ref. [49]. The spectral clusters in the half filling would be located only in the real-energy axis as energies are always real under strong



FIG. 4. The spectral clusters of the BHN model with L = 10 and N = 3 at t = 1 and $\gamma = 0.2$. (a)–(c) Spectral clusters for $n_s = 1, 2$, and 3 and V = 20. (d)–(f) Spectral clusters for $n_s = 1, 2$, and 3 and V = 200.

interactions [cf. Figs. 1(c) and 1(d)]. In the following, we study the BHN model for arbitrary filling under strong interactions to investigate universal properties of energy spectral structures.

It can be seen from Figs. 4(a)-4(c) that the energy spectrum is symmetrically distributed in the complex plane [49]. Assume the system has N particle numbers, the energy spectrum would form N clusters. The central position of each cluster E_c is around $(n_s - 1)V$. Here, we use n_s , with $n_s = 1, ..., N$, to label these clusters individually. As the system has a particlehole symmetry, the properties of clusters for L - N particles are the same as those for N particles. Let us first study the properties of the central positions E_c of these energy clusters. We show that the central positions of these clusters are not exactly located at $(n_s - 1)V$. Instead, they are [49] at

 $E_c = (n_s - 1)V + \varepsilon,$

with

$$\varepsilon \approx C(t^2 - \gamma^2). \tag{10}$$

It was shown that C = 2/V for the cluster $n_s = N$ in Ref. [49]. However, we argue that this coefficient is not always valid for other clusters. For example, we find that C = 2/V for the cluster $n_s = N$, but C = -L/V for the cluster $n_s = 1$ in half filling [cf. Figs. 5(a) and 5(b)]. Moreover, we find that *C* is indeed a quadratic function with respect to n_s for other cases. The coefficient *C* and the relation $n_s - N$ is found to satisfy the following quadratic function,

$$C = -\frac{4}{V(L/2 - 1)}(n_s - L/2)^2 - \frac{2(L/2 - 3)}{V(L/2 - 1)}(n_s - L/2) + \frac{2}{V},$$
 (11)

in half filling. This argument is verified by numerical simulation using L = 12 at V = 40 and V = 200 [cf. Figs. 5(c) and 5(d)]. For instance, we find that C = 0.05 = 2/V when $n_s = N = L/2$ and that C = -0.3 = -L/V when $n_s = 1$ at V = 40, which are consistent with Eq. (11).

Next, we turn to study the shapes of clusters. Interestingly, we find that all spectral clusters will form perfect ellipses if the



FIG. 5. The energy shift in the half-filled BHN model with L = 12 at t = 1 in PBCs. (a) and (b) The energy shift ε for the cluster $n_s = N$ and $n_s = 1$ with respect to γ at V = 40, respectively. (c) and (d) The coefficients C in ε as the function of n_s for V = 40 and V = 200 at $\gamma = 0.2$, respectively.

interaction V is strong enough [cf. Figs. 4(d)-4(f)]. Let us first discuss the rightmost cluster ($n_s = N$) which is the simplest case as shown in Fig. 4(f). We fit the data in Fig. 4(f), finding that the function is an elliptic curve,

$$\frac{(x-x_0)^2}{a^2} + \frac{y^2}{b^2} = 1,$$
(12)

with the fitting parameters $x_0 \approx 400.0096$, $a \approx 5.6001 \times 10^{-5}$, and $b \approx 3.0401 \times 10^{-5}$. Here, 2a and 2b are the lengths of the major and minor axes of the ellipse, and $x_0 = E_c$ is the position of the center of the ellipse, which satisfies $x_0 \approx (n_s - 1)V + 2(t^2 - \gamma^2)/V$.



FIG. 6. The scaling parameters of the major and minor axes at t = 1 and $\gamma = 0.2$. (a) The scaling of major and minor axes of the ellipse for the cluster $n_s = 3$ in the BHN model with L = 10 and N = 3 particles. (b) and (c) The exponent p_a and coefficients C_a and C_b with respect to N for cluster $n_s = N$ up to L = 14. (d) The exponent p_a as the function L for cluster $n_s = 1$. The star symbols in panels (a)–(c) denote the data in the half filling N = L/2.

(9)

We are surprised to find that the major axis 2a and the minor axis 2b obey a universal scaling law [cf. Fig. 6], which is given by

$$2a = C_a V^{p_a} + V_{0a}, (13)$$

$$2b = C_b V^{p_b} + V_{0b}, (14)$$

where C_a , C_b , p_a , p_b , V_{0a} , and V_{0b} are fitting parameters. To obtain the accurate values of C_a , C_b , p_a , and p_b , we fit the major and minor axes 2a and 2b with respect to V of the rightmost cluster for L = 8, 10, 12, and 14 and $n_s = N$. The results are shown in Figs. 6(a)-6(c) and Table I [see the Appendix for details], where we find that, when the system is away from the half filling, C_a , C_b , p_a , and p_b are given as

$$C_a = 2[(t+\gamma)^N + (t-\gamma)^N], \quad p_a = 1 - N,$$
(15)

$$C_b = 2[(t+\gamma)^N - (t-\gamma)^N], \quad p_b = 1 - N.$$
 (16)

The above scaling exponents can be obtained from the perturbation theory [49]. When the system is in the half filling, the eigenenergies are alway real for large interactions [49], and thus only C_a and p_a need to be derived, which are given by

$$C_a = 4[(t + \gamma)^N + (t - \gamma)^N], \quad p_a = 1 - L/2.$$
 (17)

We note that $V_{0a} = V_{0b} = 0$ for an arbitrary filling.

To verify whether the above scaling exponents p_a and p_b obtained for $n_s = N$ are also valid for other clusters, we study the spectral properties of the first cluster ($n_s = 1$). It can be seen from Fig. 4(d) that the energy spectrum forms parallel ellipses for the first cluster. The major axis 2*a* and the minor axis 2*b* of the outermost ellipse are numerically fitted by using Eqs. (13) and (14). The parameters C_a , C_b , p_a , p_b , V_{0a} , and V_{0b} are shown in Table II [see the Appendix for details], where we find that p_a and p_b remain,

$$p_a = p_b = 1 - L/2, \tag{18}$$

for the half-filled system. While if the system is away from the half filling, p_a and p_b satisfy

$$p_a = p_b = -2 \tag{19}$$

for all N and n_s , except for the single-particle case (N = 1), where the interaction is useless [cf. Fig. 6(d)].

For more general cases, that is, $2 \le n_s \le N - 1$, it seems that the exponents p_a and p_b satisfy

$$p_a = p_b = 1 - n_s \tag{20}$$

for arbitrary fillings. However, we have to note that there are several exceptions where p_a and p_b are not consistent with this scaling rule in Eq. (20). The reason for this may be that the clusters for $2 \le n_s \le N - 1$ have complex structures [see Figs. 4(b) and 4(e)], and one has to increase the interaction V to have a perfect ellipse. However, when V is too large, the fitting functions V^{p_a} and V^{p_b} become exponential small, which we think may lead to the instability of the fitting. We note that other possible relations between p_a , p_b , and n_s cannot be ruled out. This problem is left for future study.

Finally, we investigate the numbers of energy levels on a single elliptic ring. It is found that the energy levels distribute

symmetrically in the elliptic lines, in which the number of levels N_q is related to the chain length *L*. Strictly speaking, we find that the number of levels N_q on the outermost elliptic ring is

$$N_q = L, \tag{21}$$

for the clusters $n_s = 1$ and $n_s = N$ [see Figs. 4(d) and 4(f)]. For other clusters $2 \le n_s \le N - 1$, the number of energy levels N_q is given by

$$N_q = (n_s + L/2 - N - 1)L,$$
(22)

dependent on L, N, and n_s . For example, $N_q = 30$ for L = 10, N = 3, and $n_s = 2$ as shown in Fig. 4(e). Besides, the energy values E on the outermost ellipse can be derived as

$$E(q) = E_c + a\cos(q) + ib\sin(q), \qquad (23)$$

with $q = 2n\pi/N_q$, and $n = 0, 1, ..., N_q - 1$. The energy spectrum in Eq. (23) is similar to the single-particle energy spectrum, indicating that one may analyze the many-body physics using the concepts or techniques from the singleparticle picture. In addition, universal properties of manybody systems under intermediate interactions are completely unknown so far, which is beyond the scope of this work and left for future research.

V. CONCLUSION

We have studied the entanglement entropy and spectral clusters of the extended hard-core BHN model in PBCs. We show that the extended hard-core BHN model undergoes a phase transition in the ground state that is accompanied by a \mathcal{PT} transition in the first-excited state discussed in the FHN model [49]. The phase transition can be described by the CDW order parameter of the ground state and the biorthogonal and self-norm entanglement entropies of the first-excited state.

Furthermore, we study the properties of the energy eigenvalues of each cluster led by the nearest-neighbor interaction, showing that the shape of each cluster is an elliptic function when the interaction is strong enough. We explore the universal scaling laws for the size of the ellipses with respect to the interactions and derive the corresponding scaling exponents. Finally, we analyze the number of energy levels of each ellipse and find a universal expression for the number of levels in the outermost circles of clusters.

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APPENDIX: NUMERICAL DATA FOR $n_s = N$ AND $n_s = 1$

In this Appendix, we display the data for $n_s = N$ in Table I and the data for $n_s = 1$ in Table II, respectively. Please see below for details.

TABLE I. The fitting parameters of the major and minor axes 2*a* and 2*b* with respect to V from V = 40 to V = 100 with the step dV = 5 for the cluster $n_s = N$, with L = 8, 10, 12, and 14, t = 1, and $\gamma = 0.2$. We note that V = 20 to V = 60 with the step dV = 5 is used for L = 14 and N = 7 to fit as V^{P_a} is too small when V is large.

L	Ν	1 - N	C_a	p_a	C_b	p_b
8	1	-0	4.000	-0.000	0.800	-0.000
8	2	-1	4.160	-1.000	1.600	-1.000
8	3	-2	4.505	-2.001	2.427	-2.000
8	4	-3	9.981	-3.001	_	_
10	1	-0	4.000	-0.000	0.800	-0.000
10	2	-1	4.160	-1.000	1.600	-1.000
10	3	-2	4.491	-2.001	2.438	-2.001
10	4	-3	5.007	-3.002	3.334	-3.001
10	5	-4	11.35	-4.002	-	_
12	1	-0	4.000	-0.000	0.800	-0.000
12	2	-1	4.160	-1.000	1.600	-1.000
12	3	-2	4.491	-2.001	2.438	-2.001
12	4	-3	4.991	-3.001	3.344	-3.001
12	5	-4	5.692	-4.002	4.335	-4.001
12	6	-5	13.13	-5.002	_	_
14	1	-0	4.000	-0.000	0.807	-0.000
14	2	-1	4.160	-1.000	1.600	-1.000
14	3	-2	4.491	-2.001	2.438	-2.001
14	4	-3	5.991	-3.001	3.344	-3.001
14	5	-4	5.673	-4.002	4.353	-4.002
14	6	-5	6.587	-5.003	5.507	-5.002
14	7	-6	15.70	-6.008	_	-

TABLE II. The fitting parameters of the major and minor axes 2*a* and 2*b* with respect to V from V = 40 to V = 100 with the step dV = 5 for the cluster $n_s = 1$, with L = 8, 10, 12, and 14, t = 1, and $\gamma = 0.2$.

L	Ν	1 - N	C_a	p_a	V_{0a}	C_b	p_b	V_{0b}
8	1	-0	0.000	-0.000	4.000	0.0000	0.000	0.800
8	2	-1	-3.539	-2.005	6.928	0.0277	-2.000	1.386
8	3	-2	-11.07	-1.989	6.472	-2.9880	-2.002	1.294
8	4	-3	95.66	-2.992	0.000	_	_	-
10	1	-0	0.000	0.000	4.000	0.000	0.000	0.800
10	2	-1	-1.527	-2.006	7.391	0.0119	-2.000	1.478
10	3	-2	-8.934	-1.999	8.988	-0.5884	-2.001	1.798
10	4	-3	-11.56	-1.999	6.928	-5.605	-2.007	1.386
10	5	-4	376.57	-3.990	0.000	_	_	-
12	1	-0	-0.000	-0.000	4.000	0.000	-0.000	0.800
12	2	-1	-0.7764	-2.007	7.608	0.006	-2.000	1.522
12	3	-2	-5.260	-2.003	10.13	-0.1479	-2.000	2.026
12	4	-3	-13.84	-1.996	10.45	-1.995	-2.006	2.091
12	5	-4	-11.05	-2.000	7.208	-7.248	-1.999	1.442
12	6	-5	1551.37	-4.989	0.000	_	_	-
14	1	-0	-0.000	-0.000	4.000	-0.000	-0.000	0.800
14	2	-1	-0.4431	-2.007	7.727	0.00345	-2.000	1.545
14	3	-2	-3.137	-2.005	10.73	-0.0434	-1.999	2.146
14	4	-3	-10.23	-2.000	12.31	-0.706	-2.003	2.462
14	5	-4	-17.46	-1.995	11.52	-3.876	-2.005	2.304
14	6	-5	-10.41	-2.001	7.391	-8.433	-1.993	1.478
14	7	-6	6591.26	-5.987	0.000	_	_	_

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