# Hardy-type paradoxes for an arbitrary symmetric bipartite Bell scenario

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As with a Bell inequality, Hardy's paradox manifests a contradiction between the prediction given by quantum theory and local hidden-variable theories. In this work, we give two generalizations of Hardy's arguments for manifesting such a paradox to an *arbitrary*, but symmetric, Bell scenario involving two observers. Our constructions recover that of Meng *et al.* [Phys. Rev. A **98**, 062103 (2018)] and that first discussed by Cabello [Phys. Rev. A **65**, 032108 (2002)] as special cases. Among the two constructions, one can be naturally interpreted as a demonstration of the failure of the transitivity of implications (FTI). Moreover, one of their special cases is equivalent to a ladder-proof-type argument for Hardy's paradox. Through a suitably generalized notion of success probability called degree of success, we provide evidence showing that the FTI-based formulation exhibits a higher degree of success compared with all other existing proposals. Moreover, this advantage seems to persist even if we allow imperfections in realizing the zero-probability constraints in such paradoxes. Explicit quantum strategies realizing several of these proofs of nonlocality without inequalities are provided.

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## I. INTRODUCTION

In the thought-provoking paper by Einstein *et al.* [1], the strong correlations between measurement outcomes have led them to suspect that quantum theory could be somehow completed (with additional variables). This was eventually shown to be untenable by Bell [2], who proved that *no* local hidden-variable (LHV) models can reproduce all quantum-mechanical predictions. In particular, he demonstrated how, with the help of so-called Bell inequalities, one can experimentally falsify the predictions of LHV models. Today, we know that Bell nonlocality not only opens the door to answer fundamental questions in physics but also serves as an important resource for device-independent quantum information [3,4].

Interestingly, Bell inequalities are not the only way to manifest Bell nonlocality [3]. Indeed, Greenberger, Horne, and Zeilinger (GHZ) [5] showed in their seminal work that a logical contradiction can be demonstrated between the quantum mechanical prediction on a four-qubit GHZ state and that of *any* deterministic LHV model (DLHVM). Soon after, such a contradiction was also provided for a three-qubit GHZ state [6] and a two-qubit singlet state [7]. This last construction, in particular, was adapted to give the well-known Peres-Mermin game [8] for showing quantum pseudotelepathy.

A common feature of these logical proofs is that they rely strongly on the perfect correlation of *maximally entangled* states. In contrast, Hardy [9] provided a different type of logical proof of "nonlocality without inequality" for a *partially entangled* two-qubit state. In Hardy's proof, a contradiction comes about only when a certain event is observed, see Fig. 1. The probability at which this event occurs is thus commonly called the *success probability*, as it facilitates (initiates) the chain of logical reasoning in Hardy's arguments.

Hardy's original proof was soon generalized to cater for certain bipartite quantum states of *arbitrary* local Hilbert space dimension [10], an arbitrary number of qubit systems [11] (see also [12,13]), an arbitrary *partially entangled* two-qubit state [14], and later to an experimental scenario involving an arbitrary number of binary-outcome measurement settings [15]. In the meantime, Stapp's reformulation [16] of Hardy's argument (which leads to the so-called Hardy paradox) made clear [17] that the paradox can also be interpreted as the failure of the transitivity implications (FTI), thereby demonstrating Bell nonlocality.

Several years later, relaxations of Hardy's original formulation were also proposed. For example, motivated by Kar's observation [18] that *no* mixed two-qubit entangled states exhibit Hardy's paradox, Liang and Li [19,20] generalized Hardy's argument by relaxing one of the equality constraints to an inequality constraint (see also Ref. [21]). Indeed, their construction allowed them to demonstrate a Hardy-type logical contradiction for certain mixed two-qubit states via a generalized notion of success probability, called degree of success in Ref. [22]. Subsequently, Kunkri *et al.* [23] showed that this generalization could give a higher degree of success compared to the original formulation in Ref. [14]. A brief discussion of a further generalization from Cabello and that of Liang-Li to a scenario with an arbitrary number of measurement settings was subsequently given in Ref. [24].

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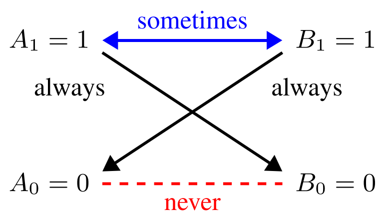
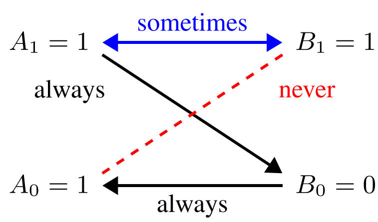



(a) Hardy's argument




(b) Stapp's argument

FIG. 1. Schematic representation of (a) Hardy's argument and (b) Stapp's reformulation [16] for demonstrating the inadequacy of DLHVMs in reproducing quantum predictions. Here, $A_x$ and $B_y$ represent, respectively, the outcome (0 or 1) observed by Alice and Bob when she performs the $x$th measurement and he performs the $y$th measurement. In a DLHVM, the implications (black arrows) and the forbidden event (red) imply that the event $A_1 = B_1 = 1$ is also forbidden, yet quantum theory defies this implication.

Indeed, a noticeably higher success probability (or degree of success) can be obtained if we are willing to consider a Bell scenario with more measurement settings [15,24], outcomes [25], or both [26]. As with Ref. [26], in this work, we propose two generalizations of Hardy's arguments applicable to an arbitrary (symmetric) bipartite Bell scenario, which recovers, respectively, that of Ref. [26] and that discussed by Refs. [20,23,24] as a special case. We provide evidence showing that the one that can be interpreted as a demonstration of FTI leads to a degree of success higher than all those offered by other existing proposals, even in the presence of imperfections.

## II. HARDY'S PARADOX AND ITS GENERALIZATION IN THE SIMPLEST BELL SCENARIO

### A. Hardy's original formulation

Consider the simplest Clauser-Horne-Shimony-Holt (CHSH) Bell scenario, i.e., one in which two observers each perform two binary-outcome measurements. Let $x$ and $y$ represent, respectively, Alice's and Bob's settings (also called inputs) while $a$ and $b$ are their outcomes (also called outputs). Moreover, let $A_x(B_y) = 0,\ 1$ denotes the outcome of Alice (Bob) when given input $x$ $(y) = 0, 1$. The probability distribution $\{P(a, b|x, y) = P(A_x, B_y)\}$ admissible in LHV models can be described by convex mixtures of local deterministic strategies $\{A_x = f_A(x, \lambda), B_y = f_B(y, \lambda)\}$, where $f_A$ $(f_B)$ is a deterministic function of the input $x$ $(y)$ and LHV $\lambda$. The Hardy paradox of Ref. [14] is encapsulated by

$$\begin{aligned} &P(0,0|0,0) = 0, \quad P(1,1|0,1) = 0, \\ &P(1,1|1,0) = 0, \quad P(1,1|1,1) = q > 0. \end{aligned} \tag{1}$$

For DLHVMs, the equality constraints of Eq. (1) imply $P(1, 1|1, 1) = 0$, which contradicts the inequality constraint of Eq. (1). In other words, together with the equality constraints, the occurrence of the event $x = y = a = b = 1$ contradicts the prediction of *any* DLHVM. Consequently, the quantity $q \equiv P(1, 1|1, 1)$ is also known as the *success probability*. In quantum theory, it is known [27] that the maximal attainable success probability is $\frac{5\sqrt{5}-11}{2} \approx 9.02\%$. Finally, note that a general LHV model can always be seen mathematically as a convex mixture of DLHVM. Thus, the observation of Eq. (1) also rules out a general LHV model.

### B. Generalization due to Cabello-Liang-Li

Cabello's [21] relaxation of Hardy's argument, originally proposed for a tripartite scenario and subsequently applied in the bipartite scenario by Liang and Li [20], takes the form

$$\begin{aligned} &P(0,0|0,0) = p, \quad P(1,1|0,1) = 0, \\ &P(1,1|1,0) = 0, \quad P(1,1|1,1) = q. \end{aligned} \tag{2}$$

Hereafter, we refer to this as the Cabello-Liang-Li (CLL) argument. Compared with Eq. (1), we see that in this argument, $P(0, 0|0, 0)$ is allowed to take a nonzero value. From Fig. 1(a), we see that for any DLHVM, if the event $x = y = a = b = 1$ occurs, so must the event $x = y = a = b = 0$. However, there exist other local deterministic strategies (e.g., one where $a = b = 0$ regardless of $x$ and $y$) where the latter event occurs while the former does not. Thus, for the prediction of a general LHV model, we must have $p \equiv P(0, 0|0, 0) \geqslant q \equiv P(1, 1|1, 1)$. In other words, one may take the positive value of the quantity

$$q - p = P(1, 1|1, 1) - P(0, 0|0, 0) \tag{3}$$

as a witness for successfully demonstrating a logical contradiction based on such an argument. In Ref. [23], the authors refer to $q - p$ as the success probability of such an argument. However, since $q - p$ is the difference between two conditional probabilities, we shall follow Ref. [22] and refer to $q - p$ instead as the *degree of success* (DS). In Ref. [23], the authors showed that this DS can reach $\sim$10.79%.

### C. Our generalization based on FTI

Now, let us revisit Eq. (1) and see how a nonzero value of $q$ can be understood as a failure of the transitivity of implications (FTI), thanks to Stapp's formulation of Hardy's paradox in Ref. [16]. To this end, note from the two zero constraints on the left of Eq. (1) that they entail not only the inferences of Fig. 1(a), but also those of Fig. 1(b), i.e.,

$$A_1 = 1 \Rightarrow B_0 = 0 \Rightarrow A_0 = 1. \tag{4}$$

Thus, if $q > 0$, meaning that the event $A_1 = 1$ and $B_1 = 1$ can simultaneously occur with some nonzero probability, and *if* implications are *transitive* (as in classical logic), it must be the case that, at least *sometimes*, $B_1 = 1 \Rightarrow A_0 = 1$. However, this contradicts the remaining zero constraint in Eq. (1), thus manifesting an FTI.

Using this reformulation, we now provide a different relaxation of Hardy's paradox via

$$\begin{aligned} &P(0,0|0,0) = 0, \quad P(1,1|0,1) = r, \\ &P(1,1|1,0) = 0, \quad P(1,1|1,1) = q. \end{aligned} \tag{5}$$

From Fig. 1(b), we see that for any DLHVMs, if the event $x = y = a = b = 1$ occurs, so must the event for $x = 0$ and $y = a = b = 1$. However, there are local deterministic strategies where the converse does not hold. Thus, for a general LHV model (obtained by averaging the deterministic ones), we must have $r \equiv P(1, 1|0, 1) \geqslant q \equiv P(1, 1|1, 1)$. Hence, in analogy with the CLL argument, we refer to

$$q - r = P(1, 1|1, 1) - P(1, 1|0, 1) \tag{6}$$

as the DS of such an argument. In Ref. [28], it was shown that in quantum theory, the largest value of $q - r$ is $\frac{1}{8} = 12.5\%$, attainable by performing projective measurements on a two-qubit pure state and higher than that achievable with Eq. (2).

#### 1. Maximal degree of success for two-qubit pure states

In fact, the general two-qubit *pure* state and observables satisfying the zero-probability constraints of Eq. (5) are [28]

$$|\psi\rangle = \sin\theta(\cos\alpha\,|0\rangle - \sin\alpha\,|1\rangle)|1\rangle + \cos\theta\,|1\rangle|0\rangle\,, \tag{7a}$$

$$A_0 = \sigma_z, \quad A_1 = \cos 2\alpha\,\sigma_z - \sin 2\alpha\,\sigma_x,$$
$$B_0 = \sigma_z, \quad B_1 = \cos 2\beta\,\sigma_z - \sin 2\beta\,\sigma_x, \tag{7b}$$

where $\theta, \alpha, \beta \in [0, \pi]$. From here, the corresponding DS of Eqs. (6) and (7), as a function of the parameters $\theta$, $\alpha$, and $\beta$, can be shown to be

$$P_{\text{succ}}(\theta, \alpha, \beta) = \tfrac{1}{2}\sin\alpha[(\cos 2\beta\cos 2\theta - 1)\sin\alpha + \sin 2\beta\sin 2\theta]. \tag{8}$$

Naturally, one may wonder which entangled two-qubit pure state gives the largest value of $P_{\text{succ}}$. To this end, note that the entanglement of the two-qubit state of Eq. (7a), as measured according to the concurrence [29], is

$$C(|\psi\rangle) = |\sin 2\theta\cos\alpha|. \tag{9}$$

Using variational techniques, the largest DS that we have found for given concurrence $C$ is

$$P^*_{\text{succ}}(C) = \frac{\sqrt{1 - C^2}}{2}(1 - \sqrt{1 - C^2}), \tag{10}$$

for which $\theta = \beta \in \{\frac{\pi}{4}, \frac{3\pi}{4}\}$. From Fig. 2, it is clear that for any given concurrence, this DS is always larger than that from the CLL argument, which, in turn, is larger than that from Hardy's original formulation. For completeness, we include in Appendix A the parametric form of the maximal DS as a function of the concurrence for the CLL argument and Hardy's original formulation.

#### 2. FTI argument with imperfections

Evidently, imperfections in any realistic experimental scenario make it essentially impossible to realize the zero-probability equality constraints in all these different formulations. To understand the impact of these imperfections, we now *relax* Eq. (5) and consider

$$P(0, 0|0, 0) \leqslant \epsilon, \quad P(1, 1|0, 1) = r,$$
$$P(1, 1|1, 0) \leqslant \epsilon, \quad P(1, 1|1, 1) = q, \tag{11}$$

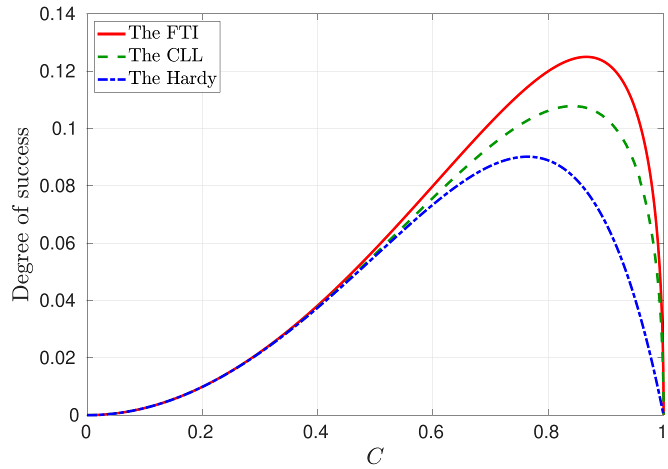



FIG. 2. The maximal DS (which reduces to the success probability in the Hardy argument) in demonstrating a proof of nonlocality without inequality for given concurrence [29]. From top to bottom, we plot $q - r$ of Eq. (5) for our FTI argument (red, solid), $q - p$ of Eq. (2) for CLL's argument (green, dashed, see Ref. [23]), and $q$ of Eq. (1) for Hardy's argument (blue, dashed-dotted, see Ref. [14]). Note that the entangled state that gives the largest DS differs from one formulation to the other.

where $\epsilon$ is the tolerance from a deviation of the zero-probability equality constraints.[1] Then, the maximal DS allowed in an LHV theory satisfies

$$q - r - 2\epsilon = P(1, 1|1, 1) - P(1, 1|0, 1) - 2\epsilon \overset{\mathcal{L}}{\leqslant} 0, \tag{12}$$

which follows from the following rewriting [22,27] of the Clauser-Horne Bell inequality [30]:

$$P(1, 1|1, 1) - P(1, 1|1, 0) - P(1, 1|0, 1) - P(0, 0|0, 0) \overset{\mathcal{L}}{\leqslant} 0. \tag{13}$$

From Eq. (12), we see that when an $\epsilon$ deviation from the zero-probability constraints is allowed, it is expedient to consider, instead, $q - r - 2\epsilon$ as the *generalized* DS. That is, whenever this quantity is nonzero, we again find a contradiction with the prediction given by all LHV theories.

For the CLL argument, a similar discussion has been made in Ref. [22], giving rise to a generalized DS of $q - p - 2\epsilon$, see Eq. (2). In Fig. 3, we show the corresponding maximal generalized DS (MGDS), i.e., the largest value of $q - r - 2\epsilon$ from Eq. (11) and $q - p - 2\epsilon$ from Eq. (2) when the tolerance $\epsilon \in (0, 0.5)$. Our results clearly show that for any amount of tolerance in this range, the FTI argument generally gives a somewhat higher MGDS compared to the CLL one. Moreover, as we see from Fig. 3, the MGDS for both arguments increases when $\epsilon$ increases from zero up to some critical value. Qualitatively, we can appreciate this by noting that when the zero-probability constraints are slightly relaxed, the

[1]Note that $\epsilon$ should *not* be understood as an uncertainty in the estimation of the conditional probability; otherwise, similar tolerance should also be included in the expression for $P(1, 1|0, 1)$ and $P(1, 1|1, 1)$.

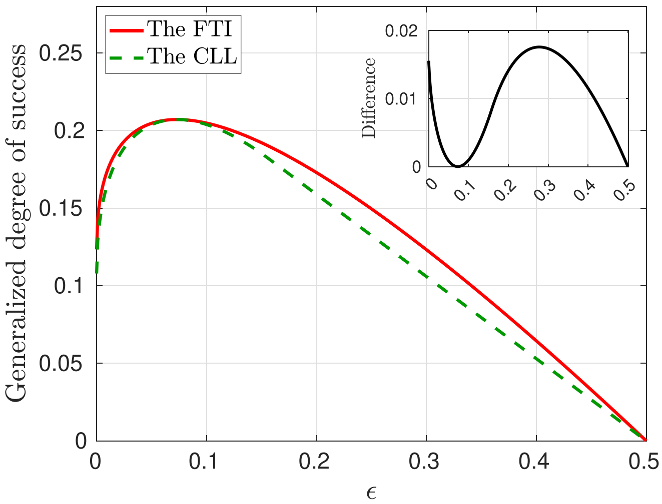



FIG. 3. MGDS as a function of the deviation $\epsilon$ from the zero-probability constraints. The top curve (red, solid) shows a lower bound on the MGDS for our FTI argument (obtained by optimizing over two-qubit states and projective measurements); the middle curve (green, dashed) shows an upper bound on the MGDS for CLL's argument (see also Ref. [22]) obtained using level 3 of the SDP hierarchy described in Ref. [31]. The inset highlights the difference between the two MGDS (provided by FTI's MGDS less the CLL's), which is always positive within a numerical precision of $10^{-7}$. Note that two probabilities coincide when $\epsilon \approx 0.0730$.

range of available nonlocal quantum strategies also increases. Nonetheless, when $\epsilon$ is sufficiently large, we see from Eq. (12) that a nonzero generalized DS must involve a $P(1,1|1,1)$ close to unity and a $P(1,1|0,1)$ close to 0, i.e., the corresponding $\vec{P}$ must become close to that producible by a DLHVM, thereby resulting in a decrease in MGDS.

Note that for our relaxed FTI argument of Eq. (11), upper bounds on the MGDS (from level 3 of the semidefinite programming (SDP) hierarchy introduced in Ref. [31]) coincide with the lower bounds (based on a two-qubit pure state with rank-1 projective measurements) to within a numerical precision of $10^{-7}$. However, for the CLL argument, if we consider only qubit strategies, then as already noted in Ref. [22, Fig. 1], there appears to be a gap between the MGDS achievable with such strategies and the SDP upper bound (for $\epsilon_1 \lesssim \epsilon < \epsilon_2$, where $\epsilon_1 = 0.158, \epsilon_2 = 0.5$). Upon closer inspection, we find that for every $\epsilon$ in this interval, the SDP upper bound on MGDS is attainable to within the same precision by considering an appropriate convex mixture of the qubit strategy for $\epsilon = \epsilon_1$ and $\epsilon = \epsilon_2$, or equivalently, a ququart strategy obtained from their direct sum. That is, we can saturate the SDP upper bound for $\epsilon = p\epsilon_1 + (1-p)\epsilon_2$ by mixing the quantum strategies for $\epsilon = \epsilon_1$ and $\epsilon = \epsilon_2$, respectively, with weight $p$ and $1-p$ while fulfilling all other constraints, cf. Eq. (2) with tolerance $\epsilon$.

## III. GENERALIZATION OF HARDY'S PROOF BEYOND THE SIMPLEST BELL SCENARIO

Having understood how Hardy and Hardy-type paradoxes work in the simplest Bell scenario, the time is now ripe to discuss their generalization to more complex Bell scenarios. In this section, we propose, respectively, a generalization of both the Hardy-type paradox of CLL, Eq. (2), and that based on FTI, Eq. (5), to an *arbitrary* bipartite $k$-input $d$-output Bell scenario, i.e., one in which both parties have a choice over $k$ alternative $d$-outcome measurements.

### A. Generalization of CLL Hardy-type paradox

Specifically, for the CLL Hardy-type paradox, the following conditions on the joint conditional probabilities:

$$P(A_{k-1} < B_{k-1}) = q, \text{ if } k \in \text{odd},$$
$$P(A_{k-1} > B_{k-1}) = q, \text{ if } k \in \text{even}, \tag{14a}$$
$$P(A_i > B_{i-1}) = 0, \ \forall\, i \in \text{odd} \cap \{1, \ldots, k-1\}, \tag{14b}$$
$$P(A_i < B_{i-1}) = 0, \ \forall\, i \in \text{even} \cap \{1, \ldots, k-1\}, \tag{14c}$$
$$P(A_{i-1} > B_i) = 0, \ \forall\, i \in \text{odd} \cap \{1, \ldots, k-1\}, \tag{14d}$$
$$P(A_{i-1} < B_i) = 0, \ \forall\, i \in \text{even} \cap \{1, \ldots, k-1\}, \tag{14e}$$
$$P(A_0 < B_0) = p, \tag{14f}$$

together with $q > p$ define our generalization of this paradox, where the outcomes $A_x, B_y$ may take $d$ possible values, say, from $\{0, 1, \ldots, d-1\}$. For the special case of $p = 0$, one obtains a generalization of the original Hardy paradox to an *arbitrary* bipartite $k$-input $d$-output Bell scenario. If we further set $d = 2$, then the construction reduces to one equivalent (under relabeling of inputs and outputs) to the ladder proof of nonlocality [15]. If, instead, we take $d = 2$ in Eq. (14) without setting $p = 0$, one obtains the argument briefly discussed in Ref. [24]. All these relations are summarized in Fig. 4.

To see that the constraints of Eq. (14) with $q - p > 0$ indeed constitute a proof of nonlocality without inequality, we begin by restricting our attention to a DLHVM where the measurement outcomes take definite values, denoted by $\{A_i = s_i^A\}$ and $\{B_i = s_i^B\}$, where $i \in \{0, 1, \ldots, k-1\}$. We depict the logical structure behind this argument schematically in Fig. 5. Let us now consider the case of even and odd $k$ separately, starting with odd $k$. Then, in order for a DLHVM to reproduce Eq. (14a), i.e., $P(A_{k-1} < B_{k-1}) = q > 0$, the model must produce events $\{A_i = s_i^A\}$ and $\{B_i = s_i^B\}$ such that $s_{k-1}^A < s_{k-1}^B$. Similarly, the other constraints of Eq. (14) imply constraints on the relationship between $\{s_i^A\}$ and $\{s_i^B\}$, where $i \in \{0, 1, \ldots, k-1\}$. For example, together with the conditions of Eq. (14c) and Eq. (14e) for $i = k-1$, i.e., $P(A_{k-1} < B_{k-2}) = 0$ and $P(A_{k-2} < B_{k-1}) = 0$, we get

$$s_{k-2}^B \leqslant s_{k-1}^A < s_{k-1}^B \leqslant s_{k-2}^A. \tag{15}$$

By considering the other zero-probability constraints one at a time for the remaining $i$, we arrive at

$$s_0^A \leqslant \cdots \leqslant s_{k-2}^B \leqslant s_{k-1}^A < s_{k-1}^B \leqslant s_{k-2}^A \leqslant \cdots \leqslant s_0^B. \tag{16}$$

This means that for any DLHVMs that give $s_{k-1}^A < s_{k-1}^B$, the constraints of Eq. (16) imply that they must also give $s_0^A < s_0^B$. However, there can be other DLHVMs where $s_0^A < s_0^B$ holds even though $s_{k-1}^A \not< s_{k-1}^B$. Thus, for a general LHV model, the conditions of Eq. (16) imply that $p \geqslant q$. In other words, a nonzero value of the DS $q - p$ witness Bell nonlocality without resorting to a Bell inequality.

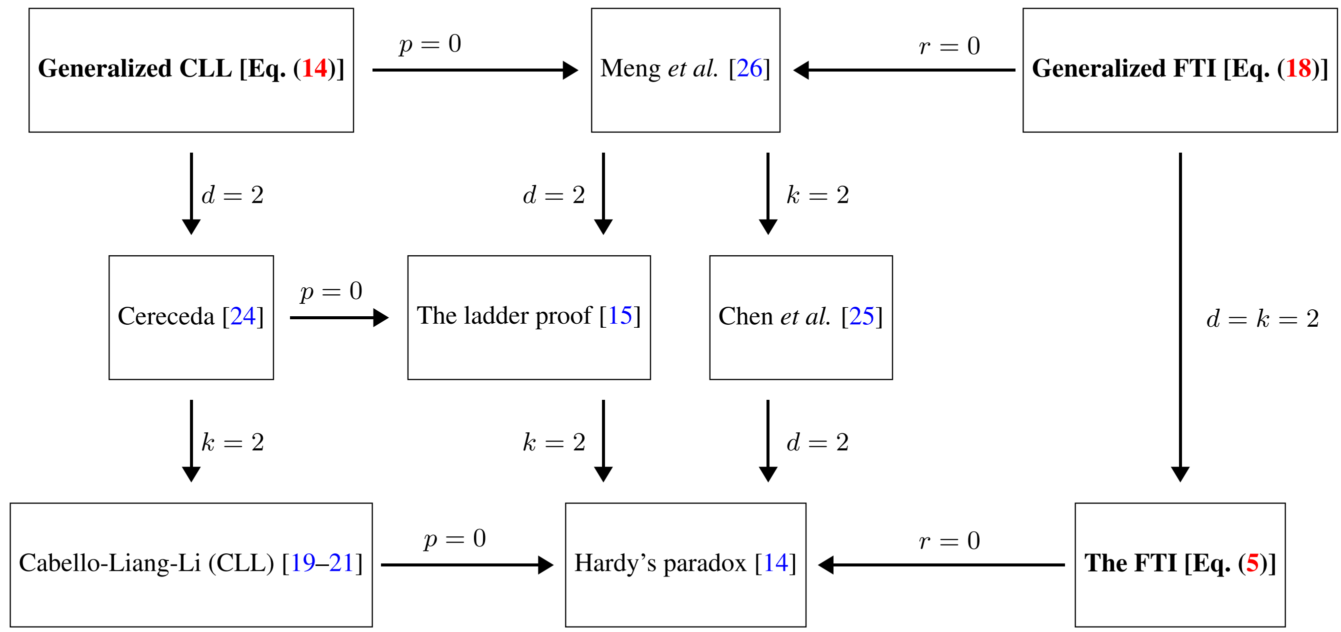



FIG. 4. Summary of the relationships between the various Hardy-type paradoxes discussed in this work. Our new constructions are printed in boldface.

Similarly, for even $k$, starting from $P(A_{k-1} > B_{k-1}) = q > 0$ and considering the other zero-probability constraints leads to, for any DLHVMs,

$$s_0^A \leqslant \cdots \leqslant s_{k-2}^A \leqslant s_{k-1}^B < s_{k-1}^A \leqslant s_{k-2}^B \leqslant \cdots \leqslant s_0^B. \quad (17)$$

Again, this observation implies that $p \geqslant q \Leftrightarrow p - q \geqslant 0$ for any LHV model for all $k \geqslant 2$ and $d \geqslant 2$.

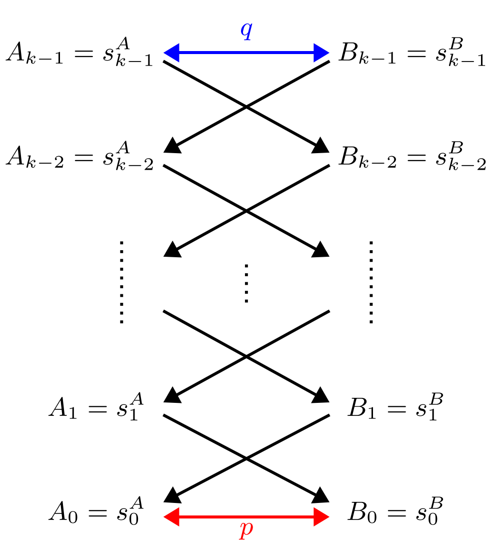



FIG. 5. Logical structure of the generalized CLL arguments in the $k$-input $d$-output Bell scenario. The generalization of Hardy's original argument to these cases is recovered by setting $p = 0$.

### B. Generalization of the FTI-based Hardy-type paradox

Next, let us describe our generalization of the FTI-based Hardy-type paradox from Eq. (5), which consists of the following conditions:

$$P(A_{k-1} < B_{k-1}) = q, \quad (18a)$$

$$P(A_i < B_{i-1}) = 0, \ \forall\, i \in \{1, \dots, k-1\}, \quad (18b)$$

$$P(B_{i-1} < A_{i-1}) = 0, \ \forall\, i \in \{1, \dots, k-1\}, \quad (18c)$$

$$P(A_0 < B_{k-1}) = r, \quad (18d)$$

and the requirement of $q > r$. The special case of $r = 0$, which can be seen as a generalization of Stapp's argument [16], has been proposed and discussed in Ref. [26]. To recover Eq. (5) from Eq. (18), one sets $k = d = 2$ and applies the relabeling $A_i = 0 \leftrightarrow A_i = 1$ for all $i \in \{0, 1\}$. In Fig. 6, we depict schematically the logical structure of this paradox.

As with our explanation to Eq. (14), for any DLHVM satisfying $P(A_{k-1} < B_{k-1}) = q > 0$, the model must produce events $\{A_i = s_i^A\}$ and $\{B_i = s_i^B\}$ such that $s_{k-1}^A < s_{k-1}^B$. At the same time, the other inequality constraints from Eq. (18) imply $s_{k-2}^B \leqslant s_{k-1}^A$, $s_{k-2}^A \leqslant s_{k-2}^B$, etc., leading to

$$s_0^A \leqslant s_0^B \leqslant \cdots \leqslant s_{k-2}^A \leqslant s_{k-2}^B \leqslant s_{k-1}^A < s_{k-1}^B, \quad (19)$$

which implies $s_0^A < s_{k-1}^B$. This means that with the zero-probability constraints, a DLHVM equipped with a strategy giving $s_{k-1}^A < s_{k-1}^B$ must also give $s_0^A < s_{k-1}^B$. Again, other DLHVMs may give $s_0^A < s_{k-1}^B$ even though $s_{k-1}^A \not< s_{k-1}^B$. Thus, from Eq. (18d), we conclude that for a general LHV model (obtained by averaging over local deterministic strategies), we must have $r \geqslant q \Leftrightarrow r - q \geqslant 0$ for all $k, d \geqslant 2$.

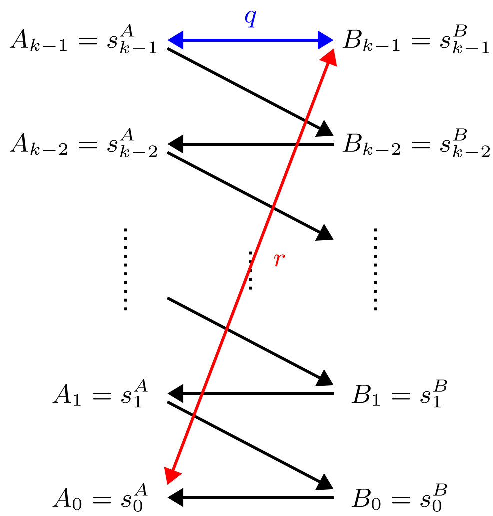



FIG. 6. Logical structure of our FTI arguments in the $k$-input $d$-output Bell scenario. Generalized Stapp's arguments as introduced in Ref. [26] are recovered by setting $r = 0$.

### C. Proof of equivalence of generalized Stapp's proof and generalized ladder proof of nonlocality

Interestingly, the authors of Ref. [26] proved that in the $k$-input 2-output scenario, the generalized Stapp's argument, Eq. (18) with $r = 0$, and the ladder proof of nonlocality, cf. Eq. (14) with $p = 0$, are equivalent. In other words, these two sets of conditions can be obtained from each other via an appropriate relabeling of inputs and outputs. In what follows, we show that this equivalence also holds for arbitrary $k, d \geqslant 2$.

*Theorem III.1.* For any symmetric bipartite Bell scenario, the set of conditions given in Eq. (14) with $p = 0$ is equivalent to the set of conditions of Eq. (18) with $r = 0$.

*Proof.* Let us rewrite Alice's and Bob's measurement outcomes in Eq. (14), respectively, as $A'_{x'}$ and $B'_{y'}$, and let

$$\begin{aligned}\mathscr{L}_{1k-} &:= \left\{1, 2, \ldots, \left\lfloor \tfrac{k}{2} \right\rfloor - 1\right\}, \quad \mathscr{L}_{1k} := \mathscr{L}_{1k-} \cup \left\{\left\lfloor \tfrac{k}{2} \right\rfloor\right\},\\ \mathscr{L}_{0k-} &:= \{0\} \cup \mathscr{L}_{1k-}, \quad \mathscr{L}_{0k} := \mathscr{L}_{0k-} \cup \left\{\left\lfloor \tfrac{k}{2} \right\rfloor\right\}.\end{aligned} \tag{20}$$

For *odd* $k$, one may verify that the following relabeling,

$$\begin{aligned}A'_{x'} &= \begin{cases} A_{\ell-1}, & x' = k - 2\ell \ \wedge\ \ell \in \mathscr{L}_{1k} \\ A_{k-1-\ell}, & x' = k - 1 - 2\ell \ \wedge\ \ell \in \mathscr{L}_{0k} \end{cases},\\ B'_{y'} &= \begin{cases} B_{k-1}, & y' = k - 1 \\ B_{k-1-\ell}, & y' = k - 2\ell \ \wedge\ \ell \in \mathscr{L}_{1k} \\ B_{\ell-1}, & y' = k - 1 - 2\ell \ \wedge\ \ell \in \mathscr{L}_{1k} \end{cases},\end{aligned} \tag{21}$$

transforms the $2k$ conditions of Eq. (14) to those of Eq. (18). To see this, note that under this transformation, the condition of Eq. (14a) stays as

$$P(A'_{k-1} < B'_{k-1}) = P(A_{k-1} < B_{k-1}) = q, \tag{22}$$

which is Eq. (18a). With the transformation, the conditions of Eq. (14b), Eq. (14d), Eq. (14c), and Eq. (14f) with $p = 0$, respectively, become the requirements that each of the following probabilities vanish:

$$\begin{aligned}&P(A'_{k-2\ell} > B'_{k-2\ell-1}) = P(A_{\ell-1} > B_{\ell-1}),\ \ \ell \in \mathscr{L}_{1k},\\ &P(A'_{k-2\ell-1} > B'_{k-2\ell}) = P(A_{k-1-\ell} > B_{k-1-\ell}),\ \ \ell \in \mathscr{L}_{1k};\\ &P(A'_{k-2\ell-1} < B'_{k-2\ell-2}) = P(A_{k-1-\ell} < B_{k-2-\ell}),\ \ \ell \in \mathscr{L}_{0k-};\\ &P(A'_0 < B'_0) = P(A_{\frac{k-1}{2}} < B_{\frac{k-1}{2}-1}).\end{aligned} \tag{23}$$

In addition, the conditions of Eq. (14e) become

$$\begin{aligned}&P(A'_{k-2\ell-2} < B'_{k-2\ell-1}) = P(A_\ell < B_{\ell-1}),\ \ \ell \in \mathscr{L}_{1k-} \text{ and}\\ &P(A'_{k-2} < B'_{k-1}) = P(A_0 < B_{k-1}),\end{aligned} \tag{24}$$

To summarize, the requirement that the probabilities in the first two lines of Eq. (23) vanish is identical to the condition of Eq. (18b), the requirement that the probabilities in the last two lines of Eq. (23) and the first line of Eq. (24) vanish is identical to the condition of Eq. (18c), and the requirement that the probability in the last line of Eq. (24) vanishes is identical to the condition of Eq. (18d).

Similarly, for *even* $k$, one can verify that the following relabeling,

$$\begin{aligned}A'_{x'} &= \begin{cases} d - 1 - A_{\ell-1}, & x' = k - 2\ell \ \wedge\ \ell \in \mathscr{L}_{1k} \\ d - 1 - A_{k-1-\ell}, & x' = k - 1 - 2\ell \ \wedge\ \ell \in \mathscr{L}_{0k} \end{cases},\\ B'_{y'} &= \begin{cases} d - 1 - B_{k-1}, & y' = k - 1 \\ d - 1 - B_{k-1-\ell}, & y' = k - 2\ell \ \wedge\ \ell \in \mathscr{L}_{1k} \\ d - 1 - B_{\ell-1}, & y' = k - 1 - 2\ell \ \wedge\ \ell \in \mathscr{L}_{1k} \end{cases},\end{aligned} \tag{25}$$

transforms the $2k$ conditions of Eq. (14) to those of Eq. (18). ■

For completeness, we show in Table I bounds on the maximal DS found for different Hardy-type paradoxes in several $k$-input 2-output and $k$-input 3-output Bell scenarios; analogous results for a larger number of outputs but with $k$ set to 2 are shown in Table II. One thing worth noticing is that for all these numerical results, we observe that the DS from our FTI arguments is always higher than that obtained from all these other proposals.

## IV. DISCUSSION

Hardy and Hardy-type paradoxes are fascinating proofs of Bell nonlocality without resorting to Bell inequalities. Aside from fundamental interests (see, e.g., Refs. [28,35–37]), they are also known to be relevant in the task of randomness amplification [38] (see, e.g., Refs. [39,40]). In this work, we propose a Hardy-type paradox that can be naturally understood via the FTI, cf. Refs. [16,17].

As with the Hardy-type paradoxes formulated by CLL [20,21], we show that a DS generalizing the notion of success probability—whose nonnegative value witnesses Bell-nonlocality—may be introduced for the FTI-based Hardy-type paradox. In the simplest Bell scenario with two inputs and two outputs, we show that the new FTI-based formulations give the highest DS among all existing (i.e., Hardy, CLL, and FTI-based) formulations. Moreover, this advantage—in the form of a generalized DS—persists even when the zero-probability constraints required in all these formulations are relaxed.

Then, we provide—as with Ref. [26] for the original Hardy paradox—a generalization of the FTI-based formulation and the CLL-type formulation, to symmetric Bell scenarios involving an *arbitrary* number of inputs and outputs. In turn, this allows us to show that a ladder-type, cf. Ref. [15], and

TABLE I. Comparison of the best upper bound (UB) and the best lower bound (LB) found on the DS for three different Hardy and Hardy-type paradoxes beyond the CHSH scenarios with $k \geqslant 2$ inputs. From top to bottom, we list the results for the Hardy paradox given by the ladder proof of nonlocality without inequality [15] (see also Ref. [26]), generalized CLL's argument due to Ref. [24], our FTI-based formulation [Eq. (18)] for $d = 2$, generalized Hardy paradox given by Meng *et al.* [26], our generalized CLL formulation [Eq. (14)] for $d = 3$, and our FTI-based formulation [Eq. (18)] for $d = 3$. See also Fig. 4 for the relationship between all these paradoxes. The UBs were obtained by considering level 1 (or higher) of the SDP hierarchy introduced in Ref. [31]. Not all upper bounds reported here were obtained using the higher-level SDP hierarchy because some of these higher-level computations, due to numerical issues, resulted in worse upper bounds. The LBs, obtained numerically, may be attained using the strategies provided in Ref. [32] (see also Appendix B).

| $d$ | Type ($k$) | | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 2 | Boschi [15] | UB | 0.090 20 | 0.17459 | 0.23132 | 0.27095 | 0.29999 |
| | | LB | 0.090 17* | 0.174 55 | 0.231 26 | 0.270 88 | 0.299 95 |
| 2 | Cereceda [24] | UB | 0.107 85 | 0.185 23 | 0.238 01 | 0.275 46 | 0.303 27 |
| | | LB | 0.107 81* | 0.185 19 | 0.237 96 | 0.275 42 | 0.303 21 |
| 2 | FTI-based | UB | 0.125 01 | 0.207 13 | 0.259 76 | 0.295 79 | 0.321 92 |
| | | LB | 0.125* | 0.207 11 | 0.259 73 | 0.295 76 | 0.321 90 |
| 3 | Meng [26] | UB | 0.141 94 | 0.267 82 | 0.348 23 | 0.401 96 | |
| | | LB | 0.141 33 | 0.267 79 | 0.348 16 | 0.401 84 | |
| 3 | CLL-type | UB | 0.167 94 | 0.282 72 | 0.357 06 | 0.407 63 | |
| | | LB | 0.167 91 | 0.282 65 | 0.356 98 | 0.407 53 | |
| 3 | FTI-based | UB | 0.193 13 | 0.312 30 | 0.384 67 | 0.432 25 | |
| | | LB | 0.193 09 | 0.312 26 | 0.384 60 | 0.432 16 | |

an FTI-based proof of nonlocality without inequality are equivalent for an arbitrary symmetric Bell scenario, thereby generalizing the result of Ref. [26] beyond the binary-outcome Bell scenarios.

For several simple Bell scenarios, we further observe (see Tables I and II) numerically that our FTI-based generalizations provide the largest value of the DS. We do not currently have any concrete physical intuition behind this observation but it will be interesting to develop one in the future. Another natural question left open from the current work is to determine if this trend continues to hold for an arbitrary, symmetric Bell scenario. Also worth noting is that within each type of logical argument, the largest values of DS found appear to increase monotonically when one increases either the number of inputs $k$ or the number of outputs $d$ involved—an analytic proof of this observation will be more than welcome.

On the other hand, given the close connection found [28,41,42] between the optimizing strategy for a Hardy paradox and its self-testing [43] property, it would also be interesting to see if the optimizing correlations found for these new generalizations are also self-testing (and non-exposed [36]). From an application perspective, one may also be interested in the potential of such correlations for device-independent applications, especially in randomness amplification [38], and proofs of Bell-nonlocality in the presence of measurement independence [35].

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### APPENDIX A: DEGREE OF SUCCESS VS CONCURRENCE

For Hardy's argument, we can again take Eq. (7) but now with the constraint [28]

$$\tan\theta \sin\alpha = \tan\beta. \tag{A1}$$

TABLE II. Comparison of the UB and the LB found on the DS for the Hardy paradox due to Ref. [25], our generalized CLL formulation [Eq. (14)], and our FTI-based formulation [Eq. (18)] for Bell scenarios with two inputs and $d \geqslant 2$ outputs. See also Fig. 4 for the relationship between all these paradoxes. Most of the UBs were obtained by considering level 1 of the SDP hierarchy introduced in Ref. [31] but those marked with $^{\ddagger}$ were obtained by considering level 2 of the SDP hierarchy introduced in Refs. [33,34] (see also the caption of Table I). The LBs, obtained numerically, may be attained using the strategies provided in Ref. [32] (see also Appendix B). Entries marked with * are known to be tight quantum bound.

| Type ($d$) | | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| Chen [25] | UB | 0.090 20 | 0.141 94 | 0.176 59$^{\ddagger}$ | 0.203 17$^{\ddagger}$ | 0.224 41$^{\ddagger}$ | 0.241 96$^{\ddagger}$ |
| | LB | 0.090 17* | 0.141 33 | 0.176 56 | 0.203 06 | 0.224 24 | 0.241 75 |
| CLL-type | UB | 0.107 85 | 0.167 94 | 0.208 90$^{\ddagger}$ | 0.239 59$^{\ddagger}$ | 0.263 92$^{\ddagger}$ | 0.283 95$^{\ddagger}$ |
| | LB | 0.107 81* | 0.167 91 | 0.208 83 | 0.239 48 | 0.263 78 | 0.283 78 |
| FTI-based | UB | 0.125 01 | 0.193 13 | 0.238 44 | 0.271 76$^{\ddagger}$ | 0.297 82$^{\ddagger}$ | 0.319 04 |
| | LB | 0.125* | 0.193 09 | 0.238 39 | 0.271 75 | 0.297 73 | 0.318 80 |

Hence, we again have the concurrence given by Eq. (9). Moreover, the success probability of Eq. (1) is

$$P_{\text{succ}}(\theta, \alpha, \beta) = (\cos\theta \cos\alpha \sin\beta)^2, \tag{A2}$$

subjected to Eq. (A1). Rewriting $P_{\text{succ}}$ in terms of $C$ and $\beta$ and using variational techniques, the largest DS is obtained for $\cos^2\beta = \frac{C}{2-C}$, giving

$$P^*_{\text{succ}}(C) = \frac{C^2(1-C)}{(2-C)^2}. \tag{A3}$$

Similarly, for the CLL argument using the state and observables given in Ref. [28, Eqs. (40, 41)], we have

$$C(|\psi\rangle) = |\cos^2\phi \sin 2\theta| \tag{A4}$$

and the degree of success

$$P_{\text{succ}}(\phi, \theta, \alpha, \beta) = (\cos\phi \sin\theta \cos\alpha)^2 - (\sin\phi \cos\beta + \cos\phi \sin\theta \sin\beta)^2, \tag{A5}$$

with constraint $(\tan\phi + \cos\theta \tan\alpha)\tan\beta = \sin\theta$. To obtain the maximal DS for a given concurrence $C$, we may take $\alpha = \frac{\pi}{2} - \beta$ and use the constraint to eliminate $\phi$ and $\theta$, thus arriving at

$$P_{\text{succ}}(C, \beta) = (C-1)\cos^4\beta + \sqrt{2}\cos\beta \sin\beta \times \sqrt{(1-C)\cos^2\beta[1+C+(C-1)\cos 2\beta]} \times \sin\left[\frac{1}{2}\sin^{-1}\left(\frac{C\sec^2\beta}{C+\tan^2\beta}\right)\right]. \tag{A6}$$

Using, e.g., MATHEMATICA, we can numerically optimize $P_{\text{succ}}(C, \beta)$ for fixed values of $C$ and verify that the resulting plot matches the green dashed line in Fig. 2.

TABLE III. Summary of the largest zero-probability-constraint violation for each optimal strategy [32] used to give the best lower bound on the DS presented in Tables I and II.

| $(k, d)$ | Hardy | CLL | FTI |
|---|---|---|---|
| (3, 2) | $2.4057 \times 10^{-14}$ | $1.1211 \times 10^{-14}$ | $9.7925 \times 10^{-16}$ |
| (4, 2) | $3.7182 \times 10^{-14}$ | $1.5057 \times 10^{-15}$ | $7.1356 \times 10^{-16}$ |
| (5, 2) | $1.2657 \times 10^{-15}$ | $3.2069 \times 10^{-10}$ | $1.5557 \times 10^{-12}$ |
| (6, 2) | $2.2401 \times 10^{-15}$ | $1.6756 \times 10^{-11}$ | $4.0152 \times 10^{-12}$ |
| (2, 3) | $2.8391 \times 10^{-15}$ | $1.2214 \times 10^{-16}$ | $2.9997 \times 10^{-16}$ |
| (3, 3) | $2.4822 \times 10^{-9}$ | $4.9471 \times 10^{-16}$ | $3.8448 \times 10^{-10}$ |
| (4, 3) | $1.6744 \times 10^{-11}$ | $2.0745 \times 10^{-14}$ | $4.8225 \times 10^{-14}$ |
| (5, 3) | $9.9972 \times 10^{-14}$ | $4.7729 \times 10^{-14}$ | $4.8075 \times 10^{-12}$ |
| (2, 4) | $8.0796 \times 10^{-8}$ | $1.8599 \times 10^{-16}$ | $3.2093 \times 10^{-16}$ |
| (2, 5) | $3.5385 \times 10^{-10}$ | $2.3190 \times 10^{-14}$ | $5.0784 \times 10^{-8}$ |
| (2, 6) | $4.3552 \times 10^{-10}$ | $4.7699 \times 10^{-11}$ | $7.8787 \times 10^{-11}$ |
| (2, 7) | $1.2624 \times 10^{-7}$ | $2.3396 \times 10^{-12}$ | $5.9839 \times 10^{-8}$ |

## APPENDIX B: QUANTUM STRATEGIES

In this Appendix, we give some further information about the quantum strategies that reproduce our best lower bound (LB) on the various DS shown in Tables I and II. The actual quantum strategies for each case are available in Ref. [32]. For convenience, we refer to the various generalizations of Hardy [14] due to Boschi *et al.* [15], Chen *et al.* [25], and Meng *et al.* [26] as Hardy paradoxes. Indeed, in all three cases, the DS is exactly the success probability of observing such a paradox.

Next, notice that the best quantum strategies we have found for the Hardy, CLL-type, and FTI-based arguments for a $k$-input, $d$-output Bell scenario, denoted by $(k, d)$, are always attained by performing real rank-1 projective measurements on a real pure quantum state $|\psi\rangle$ residing in the two-qudit Hilbert space $\mathbb{R}^d \otimes \mathbb{R}^d$. However, due to numerical imprecisions, the zero-probability constraints are *not* always strictly enforced in all these optimal strategies found. For completeness, we list in Table III the largest deviation found among all the zero-probability constraints for each of these "optimal" strategies.


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