

Procedure for imparting transverse orbital angular momentum by focusing spatiotemporally coupled ultrashort pulses

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A focusing system such as a single lens or a spherical mirror imparts intrinsic transverse orbital angular momentum (OAM) to spatiotemporal (ST) coupled fields, the ST intensity distribution of which presents ST covariance. This fact may greatly simplify the experimental setups used to date to impart transverse OAM. We evaluate analytically the imparted transverse OAM as a function of the focal length and the covariance. The focused fields with transverse OAM include elliptical ST vortices and rotating pulses without any ST phase singularity such as the “lighthouse” pulse. We provide closed-form, analytical expressions for these fields valid at any propagation distance from the focusing system, which are of interest in applications such as the interaction of these fields with matter. In general, focusing of ST coupled fields with intensity covariance generates mixed fields with ST vortices and rotating pulse fronts, where one or another feature dominates depending on the input field.

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I. INTRODUCTION

In the dynamic field of structured light, backed by major experimental achievements, orbital angular momentum (OAM) and vortex structures play a prominent role. After three decades devoted to spatial vortices and longitudinal OAM, research now expands to spatiotemporal (ST) optical vortices (STOVs) [1], featuring a line phase singularity transverse to the propagation direction, and to the associated transverse OAM. These STOVs are generated using standard, two-dimensional diffractive pulse shapers with either phase plates [2] or spatial light modulators [3,4] placed at the Fourier plane of $4f$ systems between two diffraction gratings. Recently, the use of metasurfaces has been proposed [5], and methods to impart or remove transverse OAM to already formed STOVs have been demonstrated [6].

Here, we show that a single focusing system such as a lens or a spherical mirror can impart transverse OAM to an OAM-free wave packet. The condition for the illuminating pulse to acquire transverse OAM is that its ST structure is coupled such that the intensity presents covariance between time and a transversal coordinate. The amount of transverse OAM per unit energy is evaluated as a function of the focal length and the covariance of the intensity.

With an input tilted pulse, the imparted transverse OAM produces a focusing pulse with a rotating pulse front without involving the formation of ST singularities. This is the “lighthouse” pulse previously known by its rotating wave fronts in time at the focal plane [7–9]. Partially coherent rotating pulses have been described, and its transverse OAM noticed

in Ref. [10]. Here, we provide a closed-form analytical expression of rotating pulses valid at any propagation distance for coherent illumination, which is the most frequent situation in light-matter interaction experiments [7–9,11], and evaluate its transverse OAM as a function of measurable parameters as the tilt parameter, transversal size, and focal length.

In a second example, focusing a pulse with n tilted line π steps in the phase between $n + 1$ tilted intensity lobes, and carrying no transverse OAM, produces a STOV of topological charge n at the focal plane. This illumination resembles the output from a $4f$ pulse shaper described in the experiments in Ref. [2], which also focuses to a STOV. Here, we point out that the $4f$ pulse shaper system is not necessary since the lens already endows the light with transverse OAM as long as the covariance is present, and the multilobe tilted structure can be generated by other means such as a spatial light modulator and a simple prism. The fact that the tilted multilobe structure of Hermite-Gauss shape focuses to a STOV has also been reported in Refs. [12,13], where the focused field is numerically evaluated at the focal plane for $n = 1$ and 2. Here, we obtain analytical expressions for the propagating pulse at any distance from the focusing system, and for arbitrary n , and point out that multilobe tilted structures with shapes other than Hermite-Gauss also focus into STOVs. In addition, we evaluate analytically the amount of transverse OAM of the focused pulse.

II. TRANSVERSE OAM IMPARTED BY A FOCUSING SYSTEM

We consider an optical pulse $E = \phi(x, y, t)e^{-i\omega_0 t}$ in free space that will be focused along the z direction under paraxial conditions and that comprises many optical oscillations owing to its narrow-band temporal frequency spectrum about

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a carrier frequency ω_0 . In this paper, we do not consider few-cycle, broadband pulses. For simplicity, ϕ is assumed to present ST couplings only in x and t , so that $\phi = \psi(x, t)Y(y)$. With this choice triple integrals for the physical magnitudes of interest factorize in double integrals in x and t multiplied by $\int |Y(y)|^2 dy$, which will not be written, and cancel out when considering the quotient of two triple integrals. With an adequate choice of the origin of time, one can get the temporal center of pulse packet to vanish, i.e., $t_c = \int |\psi|^2 t dx dt / \int |\psi|^2 dx dt = 0$ (all the integrals extend from $-\infty$ to $+\infty$). The origin of x can be chosen such that the transversal center of the pulse is zero, i.e., $x_c = \int |\psi|^2 x dx dt / \int |\psi|^2 dx dt = 0$.

Under the above paraxial and quasimonochromatic (many-cycle) conditions, the intrinsic OAM along the transverse y direction and the energy carried by the wave packet can be evaluated from [14]

$$J_y^{(i)} = -\frac{\varepsilon_0 c}{2k_0} \text{Im} \int \psi^* \partial_x \psi (t - t_c) dx dt, \quad (1)$$

with $t_c = 0$, and

$$W = \frac{\varepsilon_0 c}{2} \int |\psi|^2 dx dt, \quad (2)$$

where ε_0 is the free-space electric permittivity, c the speed of light in free space, and $k_0 = \omega_0/c$. The intrinsic transverse OAM is the OAM about a moving axis parallel to the y direction passing permanently through the pulse center, which is conserved on propagation [14]. Here, the center is that of the intensity distribution. In Ref. [15], the pulse center is also such that the intensity distribution, and the results for the intrinsic transverse OAM coincide with those in Ref. [14] (of course other choices for the pulse center are possible [16], as the center of the photon wave function, which leads to different values of the intrinsic OAM). It is assumed that the pulse to be focused does not carry any intrinsic transverse OAM so that the integral (1) with $t_c = 0$ is vanishes.

The focusing element may focus in x and y , or only in x , e.g., a cylindrical lens or mirror. Focusing in y adds nothing to the discussion, but simply makes $Y(y)$ to focus remaining decoupled. We then choose a cylindrical focusing element focusing only in x . Its center $x = 0$ is aligned with input pulse center, $x_c = 0$. Otherwise the focusing system would deviate the input pulse imparting an extrinsic OAM in which we are not interested. In ideal focusing, the primary effect of the focusing element is to impart a converging spherical wave front represented by the factor $e^{-ik_0 x^2/2f}$, where $f > 0$ is the focal length. The second effect, for ultrashort pulses, is to introduce a pulse-front curvature described by replacing t with $t - x^2/2cf$ in ψ . Thus, the field immediately after the focusing system is

$$\psi_f = \psi(x, t - x^2/2cf) e^{-ik_0 x^2/2f}, \quad (3)$$

or $\psi_f = \psi(x, t_f) e^{-ik_0 x^2/2f}$, where $t_f = t - x^2/2cf$.

To the purpose of evaluating the new intrinsic transverse OAM, and the OAM per unit energy (“per photon”), we first note from (3) that $W_f = W$, that $x_{c,f} = 0$, and that the temporal center is slightly shifted to $t_{c,f} = (\Delta x)^2/2cf$, where $(\Delta x)^2 = \int |\psi|^2 x^2 dx dt / \int |\psi|^2 dx dt$, as an effect of the pulse-front curvature. Using (1) with the focused field (3) and with

the new center $t_{c,f}$, performing the derivatives with the chain rule, changing the integration variable t to t_f , and taking into account that $x_{c,f} = 0$, one arrives at

$$J_{y,f}^{(i)} = -\frac{\varepsilon_0 c}{2k_0} \left\{ -\frac{\omega_0}{cf} \int |\psi|^2 x \left(t + \frac{x^2}{2cf} \right) dx dt \right. \\ \left. + \text{Im} \int \psi^* \left[\partial_x \psi - \partial_t \psi \left(\frac{x}{cf} \right) \right] \right. \\ \left. \times \left(t + \frac{x^2}{2cf} - t_{c,f} \right) dx dt \right\}, \quad (4)$$

where $\psi = \psi(x, t)$ and the subindex f in t_f is omitted at the end. No particular assumption for the input field has been made up to this point, except that the input pulse is aligned with the focusing system.

Considering focusing, the most common situation is collimated illumination. Writing $\psi = A e^{i\Phi}$, where A and Φ are the real amplitude and phase, $\text{Im}\{\psi^* \partial_x \psi\} = A^2 \partial_x \Phi = 0$ since the phase does not depend on x . Then, the input field does not indeed carry transverse OAM, and the three integral terms in the second row of (4) containing $\partial_x \psi$ vanish. For simplicity, and to focus on the phenomenon of interest, we will assume that the input field does not contain any temporal chirp. Similarly, $\text{Im}\{\psi^* \partial_t \psi\} = A^2 \partial_t \Phi = 0$, and the three integral terms with $\partial_t \psi$ also vanish. Since the phase is constant, the input field only can contain ST couplings in the amplitude. With these assumptions, only the first row in (4) remains, and when expressed per unit energy, we obtain the intrinsic transverse OAM imparted by the focusing system as

$$\frac{J_{y,f}^{(i)}}{W_f} = \frac{1}{f} \left[\frac{\int |\psi|^2 x t dx dt}{\int |\psi|^2 dx dt} + \frac{1}{2cf} \frac{\int |\psi|^2 x^3 dx dt}{\int |\psi|^2 dx dt} \right]. \quad (5)$$

The first term is the contribution from wave-front curvature and the second one from pulse-front curvature. The relevance of each term can be analyzed by introducing dimensionless variables $\xi = x/X_0$ and $\tau = t/t_0$, where X_0 and t_0 are characteristic half beam size and half pulse duration (as in the examples below). Then (5) can be expressed as $(X_0 t_0/f)(I_1 + \alpha I_2)$, where I_1 and I_2 are the two same quotients of integrals as in (5) but with variables ξ and τ , and $\alpha = (Z_R/f)/(\omega_0 t_0)$, with $Z_R = k_0 X_0^2/2$ the Rayleigh distance of the incident wave packet. The condition $Z_R/f \gg 1$ characterizes focusing without an appreciable focal shift [17,18] or large Fresnel number [19], and $\omega_0 t_0 \gg 1$ characterizes pulses with many oscillations or long duration. Henceforth, we will choose long enough duration for the desired focusing geometry (Z_R and f) such that $\alpha \ll 1$, whereby

$$\frac{J_{y,f}^{(i)}}{W_f} \simeq \frac{1}{f} \frac{\int |\psi|^2 x t dx dt}{\int |\psi|^2 dx dt}. \quad (6)$$

In the example of Fig. 2 with a focusing geometry with negligible focal shift ($Z_R/f = 20.8$), $\alpha = 0.167$ for the duration $t_0 = 50$ fs. In the example of Fig. 3 with a relevant focal shift ($Z_R/f = 1.33$), $\alpha = 0.011$ for the same duration $t_0 = 50$ fs (of course both examples are within the paraxial regime: respective divergence angles 0.4° and 0.057°). In addition, the second term in (5) is exactly zero for many fields with

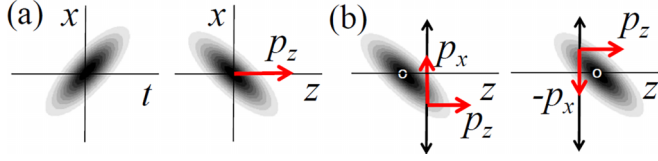


FIG. 1. (a) The pulse with positive covariance in (t, x) has negative covariance in (z, x) . (b) The lower and upper parts of the pulse are focused at different times, imparting opposite momenta p_x and $-p_x$, but the corresponding angular momenta with respect to the instantaneous pulse center (small circles) have the same sign.

ST couplings that preserve some symmetries, as in the two examples below, in which case (6) is exact.

Equation (6) is simple and looks particularly appealing conceptually. The quotient of integrals is the covariance of the intensity $|\psi|^2$ in the variables x and t . Focusing then imparts an intrinsic transverse OAM proportional to the power $1/f$ of the focusing system if ψ is a ST coupled field whose intensity distribution presents covariance. If the intensity covariates in the first and third (second and fourth) quadrants, the transverse OAM is positive (negative). A sketch of how the focusing system transmits the OAM is shown in Fig. 1. Let us illustrate this result with a couple of examples of interest in experiments.

III. EXAMPLES

A. Rotating pulse

Let the illumination be the tilted pulse [20,21]

$$\psi = \psi_x(x)\psi_t(t - px), \quad (7)$$

where p is the tilt parameter, and the functions $\psi_x(x)$ and $\psi_t(t)$ are taken real. Then the illumination (7) does not carry transverse OAM. It is readily seen that x_c and t_c of the tilted pulse (7) are zero if the mean values of $\psi_x(x)$ and $\psi_t(t)$ are zero as functions of their respective variables. After focusing, a simple calculus from (6) in which many integrals factorize and cancel out, results in

$$\frac{J_{y,f}^{(i)}}{W} \simeq \frac{p}{f} \frac{\int |\psi_x|^2 x^2 dx}{\int |\psi_x|^2 dx} = \frac{p}{f} (\Delta x)^2, \quad (8)$$

meaning that the tilted pulse acquires an intrinsic transverse OAM proportional to the tilt parameter and to the transversal width.

With the Gaussian tilted pulse $\psi = e^{-x^2/X_0^2} e^{-(t-px)^2/t_0^2}$, the intrinsic transverse OAM per unit energy after focusing is $J_{y,f}^{(i)}/W = pX_0^2/4f$. The second term in (5) due to pulse-front curvature vanishes in this case. The manifestation of this transverse OAM is an intensity pattern that rotates about a transverse y axis passing permanently through its center during propagation, as seen in Fig. 2. Here, the transverse OAM does not involve the formation of any ST phase singularity, hence the phase pattern is not shown.

Since pulse-front curvature does not contribute to the transverse OAM, we can evaluate the focused field neglecting it to obtain a field with the same OAM content. We take the Fresnel diffraction integral

$$\psi_f(x, t', z) = \sqrt{\frac{k_0}{2\pi iz}} \int dx' \psi(x', t') e^{-ik_0 x^2/2f} e^{ik_0(x-x')^2/2z} \quad (9)$$

($t' = t - z/c$ is the local time) as the solution of the paraxial wave equation $\partial_z \psi = (i/2k_0) \partial_{xx} \psi$ for paraxial, quasimonochromatic (many cycle, $t_0 \gg c/\omega_0$) pulses in the absence of material dispersion, for focusing the tilted pulse $\psi(x, t) = e^{-x^2/X_0^2} e^{-(t-px)^2/t_0^2}$ at $z = 0$. Use of integral 3.323.2 in Ref. [22] yields the expression

$$\psi_f(x, t', z) = \sqrt{\frac{q_{\text{eff}}}{z + q_{\text{eff}}}} e^{-t'^2/t_0^2} \exp\left[\frac{ik_0 x^2}{2(z + q_{\text{eff}})}\right] \times \exp\left[\frac{\left(\frac{2t'px}{t_0^2} + \frac{2i(t'p)^2 z}{k_0 t_0^4}\right) q_{\text{eff}}}{z + q_{\text{eff}}}\right], \quad (10)$$

where $1/q_{\text{eff}} = -1/f + 2i/k_0 X_{0,\text{eff}}^2$ is an initial, effective complex beam parameter, and $1/X_{0,\text{eff}}^2 = 1/X_0^2 + p^2/t_0^2$ is an initial, effective width. Figure 2 shows ST intensity patterns as the pulse focuses and beyond in a particular example with $p > 0$. At the focal plane, the tilt angle is always 90° , and at the far field, approaches zero. Note that the actual rotation in the z - x plane is counterclockwise, since $t' = t - z/c$, corresponding to the positive transverse OAM $pX_0^2/4f$ with $p > 0$.

Partially coherent rotating pulses have recently analyzed in Ref. [10], where a sophisticated scheme for their generation using a Fourier transform pulse shaper with a spatial light modulator at the Fourier plane is proposed, and its transverse OAM noticed. It follows from our analysis that a simple prism tilting the pulse and focusing produces the rotating pulse,

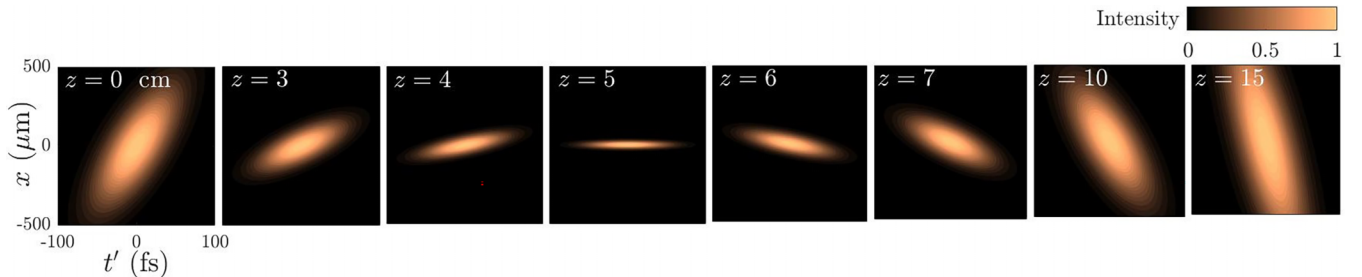


FIG. 2. Rotating pulse of carrier frequency $\omega_0 = 2.5$ rad/fs obtained by focusing the Gaussian tilted pulse $\psi = e^{-x^2/X_0^2} e^{-(t-px)^2/t_0^2}$ with $X_0 = 500$ μm , $t_0 = 50$ fs, tilt parameter $p = 0.1$ fs/ μm , with a focal length $f = 5$ cm. The intensity is normalized to the peak intensity at each distance.

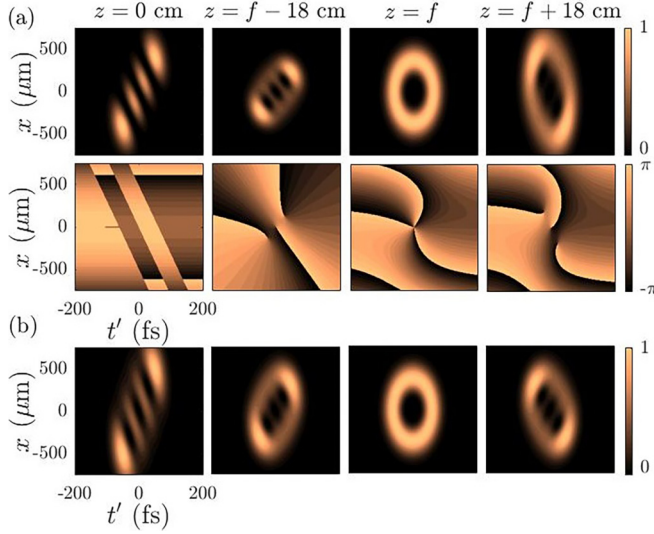


FIG. 3. (a) Elliptical STOV produced at the focal plane of a lens of focal length $f = 50$ cm with the illumination in (11) of carrier frequency $\omega_0 = 2.5$ rad/fs, $t_0 = 50$ fs, $X_0 = 0.4$ mm, and $n = 3$. Top: Intensity. Bottom: Phase of ψ . (b) Intensity of the STOV in Eq. (16) with $x_0 = 2f/k_0X_0 = 0.3$ mm. In (a) and (b) the intensity is normalized to the peak intensity at each distance.

whose transverse OAM can be controlled by the focal length, tilt parameter, and transversal size according to (8).

Also, the above focused tilted Gaussian pulse has previously been used in experiments because of its important properties at the focal plane, particularly in high-harmonic generation experiments for the formation of the so-called attosecond lighthouses [7–9]. The property that creates these lighthouses is a *rotating wave front in time* at the focal plane due to a transverse chirp at that plane. Instead, we stress here the rotating pulse front with propagation distance as a manifestation of its transverse OAM. In addition, (10) is an analytical expression valid at any propagation distance for such an interesting field that includes all above phenomena, namely, temporal wave-front rotation at the focal plane and pulse-front rotation on propagation, excluding only the pulse-front curvature accrued during the act of focusing.

B. Canonical STOV

Now let the illumination be

$$\psi(x, t) = e^{-\frac{t^2}{t_0^2}} e^{-\frac{x^2}{X_0^2}} \frac{1}{2^n} H_n\left(\frac{t}{t_0} \pm \frac{x}{X_0}\right), \quad (11)$$

where $H_n(\cdot)$ is the Hermite polynomial of order n . This illumination features $n + 1$ tilted intensity lobes in the t - x plane between the zeros of the Hermite polynomial, as in the top right-hand side of Fig. 3. The $1/2^n$ factor cancels the 2^n factor of the highest power term of the Hermite polynomial. This “preconditioned” illumination with $n = 1$ and $n = 2$ has been shown [12,13] to produce STOVs of topological charges $n = 1$ and $n = 2$ by numerical calculation of the field at the focal plane. Also, similar preconditioned tilted lobes are the output from a $4f$ pulse shaper with a spiral phase plate placed at the Fourier plane in one of the experiments in Ref. [2],

with this output also producing an elliptical STOV at the focal plane (far field) of a lens [2].

Below we provide an analytical description of the focused field at any distance from the focusing system producing elliptical STOVs of arbitrary charge n at the focal plane, and evaluate its transverse OAM. As discussed below, this focusing problem differs from the diffraction problem in Ref. [15] for topological charge $n = 1$ and in Ref. [23] for arbitrary n , where the diffraction of a prescribed elliptical STOV from a waist plane is studied.

Being real, the illumination (11) does not carry transverse OAM. After focusing, (6) [or (5) since the pulse-front curvature contribution to the transverse OAM vanishes] yields the result

$$\frac{J_{y,f}^{(i)}}{W} = \pm \frac{nX_0t_0}{4f}, \quad (12)$$

which depends only on the focal length, duration, and transversal size of (11), and will be better understood after examining the elliptical STOV at the focal plane. The change from zero to (12) shows that it is just focusing that imparts the transverse OAM for the formation of the STOV, and therefore preconditioning systems other than the $4f$ pulse shaper can create the STOV, for example, a spatial light modulator to create the lobes and a prism to tilt them.

Using again (9) with $\psi(x, t)$ in (11), the resulting integral can be identified, after some changes of variables, with integral 7.374.8 in Ref. [22], and the focused field written as

$$\begin{aligned} \psi_f(x, t', z) = & e^{-\frac{t'^2}{t_0^2}} \left[\frac{q_0}{q(z)} \right]^{\frac{1}{2}} e^{\frac{ik_0x^2}{2q(z)}} \left[\frac{q_0}{q(z)} \left(1 - \frac{z}{f} \right) \right]^{\frac{n}{2}} \\ & \times \frac{1}{2^n} H_n \left\{ \left[\frac{q(z)}{q_0} \frac{1}{\left(1 - \frac{z}{f} \right)} \right]^{\frac{1}{2}} \left(\frac{t'}{t_0} \pm \frac{x}{X_0} \frac{q_0}{q(z)} \right) \right\}, \end{aligned} \quad (13)$$

where $1/q_0 = -1/f + 2i/k_0X_0^2$ and $q(z) = q_0 + z$. Some ST intensity and phase profiles at different propagation distances are shown in Fig. 3(a) in a particular example. The $n \pi$ -step tilted lines at the zeros of the Hermite polynomial immediately form n single-charged ST punctual phase singularities that merge in a single n -charged elliptical vortex at the focal plane, which further splits into n unit-charged vortices up to the far field.

At the focal plane, $z = f$, the factor in the square bracket in the first row becomes zero and cancels all terms of the Hermite polynomial except for the highest power term. Equation (13) then reduces to

$$\psi_f(x, t', f) = e^{-\frac{t'^2}{t_0^2}} \left(\frac{-iX_0}{x_0} \right)^{\frac{1}{2}} e^{\frac{ik_0x^2}{2f}} e^{-\frac{x^2}{x_0^2}} \left(\frac{t'}{t_0} \mp i \frac{x}{x_0} \right)^n, \quad (14)$$

where $x_0 = 2f/k_0X_0$ is the focal Gaussian width. This equation indeed represents an elliptical STOV of topological charge n and ellipticity $\gamma = ct_0/x_0$. Following Ref. [14] or [15], the intrinsic transverse OAM of a STOV can be evaluated from the topological charge n at the plane where it is

elliptical as

$$\frac{J_y^{(i)}}{W} = \pm \frac{\gamma}{2} \frac{n}{\omega_0}. \quad (15)$$

Using that $x_0 = 2f/k_0X_0$, Eq. (15) is immediately seen to coincide with Eq. (12), that is, the transverse OAM of the elliptical STOV at the focal plane is that imparted by the lens.

IV. DISCUSSION

A. Focusing to versus diffraction of canonical STOVs

In the second example above we have deliberately chosen a loose focusing geometry in Fig. 3(a) with $Z_R/f = 1.33$ to make it evident that our focusing problem is different from the diffraction problem studied in Refs. [15,23], where collimated, elliptical STOVs are prescribed at a “focus” and then they are propagated freely, i.e., diffracted forwards and backwards.

Figure 3(b) shows the elliptical STOV of Ref. [23], or given by Eq. (16) below, of the same x_0 , t_0 , and n at the same axial position as in Fig. 3(a) for focusing, and their backwards and forwards propagated fields. In Fig. 3(a) for actual focusing, the focused field is not symmetric with respect to the focal plane, while in Fig. 3(b) the field diffracts symmetrically with respect to it. In Fig. 3(a) the waist [of the Gaussian beam enveloping the STOV in Eq. (13)] is located 18 cm before the focal plane because of the relevant focal shift $f\{(f^2/Z_R^2)/[1 + (f^2/Z_R^2)]\}$ [17,18], but in Fig. 3(b) the waist and the focal plane coincide. Indeed, the elliptical STOV in Fig. 3(a) contains the wave-front curvature factor $e^{ik_0x^2/2f}$ in Eq. (14), while the elliptical STOV in Fig. 3(b) does not.

The focusing problem we have addressed can only be approximated by the diffraction problem when $Z_R/f \gg 1$, i.e., when the focal shift is negligible [17–19]. Under this condition, Eq. (13) can readily be seen to approach Eq. (7) in Ref. [23] with any n , and in particular to Eq. (7) in Ref. [15] for $n = 1$. Specifically, Eq. (13) reduces to

$$\begin{aligned} \psi(x, t', z) \simeq & e^{-t'^2/t_0^2} \left(\frac{-f}{p(z)} \right)^{\frac{1}{2}} e^{\frac{ik_0x^2}{2p(z)}} \left(\frac{z}{p(z)} \right)^{\frac{n}{2}} \\ & \times \frac{1}{2^n} H_n \left[\left(\frac{p(z)}{z} \right)^{\frac{1}{2}} \left(\frac{t'}{t_0} \pm \frac{x}{x_0} \frac{z_R}{p(z)} \right) \right], \quad (16) \end{aligned}$$

where $p(z) = (z - f) - iz_R$, $z_R = k_0^2x_0^2/2$ is the Rayleigh distance of the focused pulse, and again $x_0 = 2f/k_0X_0$. Equation (16) is the same as Eq. (7) in Ref. [23] except for a global amplitude to match the actual amplitude at the focal plane. Thus, as already pointed out in Ref. [23], Eqs. (16), and (7) in Ref. [15] in particular, are approximations to focusing to canonical STOVs valid only for $Z_R/f \gg 1$, while our Eq. (13) is valid with any focusing geometry.

With regard to the intrinsic transverse OAM, we have demonstrated that it is imparted by the focusing system to the OAM-free illumination in Eq. (11). Instead, in the diffraction problems in Refs. [15,23], the transverse OAM is theoretically introduced when an elliptical STOV is prescribed at the waist or focus, as Ref. [15] demonstrates that an elliptical STOV of charge n carries the transverse OAM $(\pm\gamma/2)(n/\omega_0)$. There is no analysis of how it arrived there, and in particular of how a

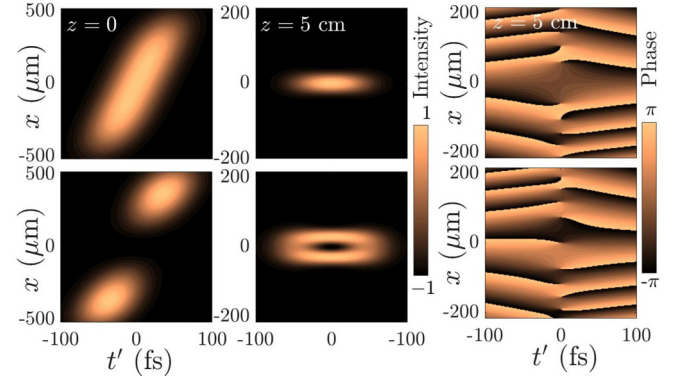


FIG. 4. Top: Rotating-type pulse with all parameters equal to those in Fig. 2, except that the input field $\psi = e^{-x^4/x_0^4} e^{-(t-px)^2/t_0^2}$ has a super-Gaussian transversal profile. Bottom: STOV-type pulse with all parameters equal to those in Fig. 2, except that the input field is $\psi = (x/x_0)e^{-x^4/x_0^4} e^{-(t-px)^2/t_0^2}$ with a zero-amplitude line at $x = 0$. Left: Intensity of illuminating field. Middle and right: Intensity and phase at the focal plane.

$4f$ pulse shaper (and the focusing lens afterwards) in Ref. [15] imparts the transverse OAM.

For completeness, we briefly consider the limit situation $f \rightarrow \infty$ in which the illumination in Eq. (11) is not focused but propagates freely undergoing diffraction. For $f \rightarrow \infty$ our Eq. (12) yields zero transverse OAM. This is in agreement with the second example in Ref. [23], Eqs. (9)–(13) and Fig. 2, where the free-space diffraction of the same initial field without focusing is studied, and its intrinsic transverse OAM is calculated according to Ref. [15] to be zero.

B. General fields with spatiotemporal intensity covariance

The two examples in Sec. III represent opposite situations where the imparted transverse OAM is manifested as a pure rotation of the pulse front without ST singularities and as a formation of a canonical, elliptical STOV with an ST phase singularity. Other illuminations with ST intensity covariance produce mixed fields where the first phenomenon or the second dominate. For illuminating fields with no zeros in the amplitude other than the Gaussian tilted pulse, a rotating-type pulse is generally produced, which may be accompanied by many ST vortices of very low intensity far off axis, as in Fig. 4 (top). A STOV-type pulse is produced when the amplitude has zero lines, as in Fig. 4 (bottom), where the central ST vortex is surrounded by an intense but imperfect ring, also accompanied by a myriad of low-intensity ST vortices far off axis, actually an infinite number of them at $t' = 0$. The last example demonstrates that focusing of multilobe tilted structures with shapes other than Hermite-Gauss shape also create STOVs.

C. Interaction with the focusing system

It is also of interest to examine the (total) transverse OAM. Contrary to the intrinsic part, its value depends on the particular transverse axis. When the transverse y axis (x, z) = 0 passing through the center of the focusing system is chosen,

the transverse OAM is given by [14]

$$J_y = \frac{\epsilon_0 z}{2k_0} \text{Im} \int \psi^* \partial_x \psi dx dt - \frac{\epsilon_0}{2} \int |\psi|^2 x dx dt. \quad (17)$$

It is obviously zero for an input collimated wave packet with $x_c = 0$ (irrespective of whether or not $z = 0$). A similar analysis under the same conditions that led to (5), leads now to the conclusion that the total transverse OAM continues to vanish after focusing. By conservation of the total angular momentum of the pulse-lens system, the pulse does not transmit angular momentum to the lens. Indeed, in Fig. 1 the recoil momenta opposite to p_x and $-p_x$ do not provide an angular momentum to the lens, and the small amounts of axial momenta transferred from p_z to the lens in the upper and lower parts provide opposite angular momenta and then zero net angular momentum. Focusing is then a zero-exchange interaction with regard the angular momentum in which the null OAM of the pulse is split into an intrinsic part and an opposite extrinsic part.

V. CONCLUSIONS

To conclude, ST couplings prove to be extremely useful for generating structured light carrying transverse OAM. This fact has also been demonstrated recently in Ref. [24], where arbitrarily oriented vortices are produced from longitudinal vortices with a spatial chirp. In this paper we have proposed a simple procedure to impart transverse OAM by focusing an ST coupled field, the ST intensity pattern of which presents covariance in space and time.

Except in Ref. [10], research on transverse OAM has focused on the ST phase singularities of STOVs. We have evaluated analytically the field of coherent, rotating pulses in

full space and time, which can be of interest for the analysis of their interaction with matter, e.g., in high-harmonic and attosecond pulse generation [7–9], and evaluated its transverse OAM as a function of the input transversal size, tilt parameter, and focal length that generate the rotating pulse.

STOVs can also be created by focusing specific profiles, where we now show that the covariance of the intensity in space and time is the key property which ensures the transverse OAM upon focusing. Hermite-Gauss, ST tilted multilobe profiles focus to a perfect STOV, whose focusing field is analytically described, and the transverse OAM is shown to be that imparted by the focusing system. Other multilobe profiles also focus to more or less perfect STOVs with the OAM imparted by the focusing system. To clarify these nuances, we also showed the focusing properties of cases in between the simple tilted pulse and the STOV that always contain transverse OAM but show varying levels of a vortex structure. We finally note that the magnitude of transverse OAM in the limit cases of the rotating pulse and STOV are, under similar focusing conditions and input duration and size, of the same order of magnitude, implying that pulses with transverse OAM but without ST phase singularities may be just as useful for future applications.

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