Weyl channels for multipartite systems

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Quantum channels, a subset of quantum maps, describe the unitary and nonunitary evolution of quantum systems. We study a generalization of the concept of Pauli maps to the case of multipartite high-dimensional quantum systems through the use of the Weyl operators. The condition for such maps to be valid quantum channels, i.e., complete positivity, is derived in terms of Fourier transform matrices. From these conditions, we find the extreme points of this set of channels and identify an elegant algebraic structure nested *within* them. In turn, this allows us to expand upon the concept of "component erasing channels" introduced in earlier work by the authors. We show that these channels are completely characterized by elements drawn of finite cyclic groups. An algorithmic construction for such channels is presented and the smallest subsets of erasing channels which generate the whole set are determined.

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I. INTRODUCTION

The description of open quantum systems [1,2] serves a twofold purpose. First, it lies at the core of the measurement problem [3,4], thus bearing a fundamental interest. On the other hand, it describes quantum systems where the inevitable interaction with an environment is taken into account [5]. For most implementations of quantum devices it is crucial to understand and control such unwanted interaction. In both cases, a natural language for such a description is that of quantum channels, which have been subject of intense research [6].

The properties of a quantum channel dictate the characteristics of the associated quantum dynamics. In the realm of qubits, the set of quantum channels has been explored and thanks to a better understanding of its geometry, several physical properties of the set, such as divisibility [7–9], non-Markovianity [10,11], channel capacity [12], among others have been unraveled. In a previous paper [13] we proposed and studied a class of channels acting on multiqubit systems that either erased or preserved the Pauli components of the state. These are the so-called Pauli component erasing (PCE) maps, which are an important subset of the Pauli maps. We found that every PCE channel corresponds uniquely to a vector subspace of a discrete vector space. Such channels can be associated with measurements and asymptotic Lindbladian evolution.

Moreover, most of the applications in the field of quantum information have been built upon qubits. Nevertheless, many real-world realizations of quantum systems have more than two levels that can be used to provide an important technical advantage. Such advantage is indeed employed to develop several important tasks like quantum cryptography [14,15], quantum computation [16–18], violation of Bell inequalities [19], randomness generation [20], and quantum communications [21], among others. For this reason, the study of high-dimensional and multiparticle systems is of relevance.

In this article, we introduce the concept of Weyl channels for systems composed of many particles, allowing each of these to be of different dimensions. We begin defining these channels in Sec. II as diagonal channels in the basis of multiparticle Weyl matrices, which are tensor products of the well-known Weyl matrices. Moving forward, we proceed to diagonalize the Choi-Jamiołkowski matrix, revealing a linear relationship between the eigenvalues and those of the channel. From this, we find two significant properties of the set of Weyl channels: (1) its extreme points in Sec. III, and (2) a subgroup structure of all Weyl channels in Sec. IV. Then, in Sec. V we extend the notion of *component erasing* channels by introducing the Weyl erasing channels. Given its semigroup property, we describe the generator subset by means of the aforementioned algebraic structure of Weyl channels. Finally, we wrap up and conclude in Sec. VI.

II. WEYL CHANNELS

A well-known generalization of the Pauli matrices to arbitrary d-dimensional Hilbert spaces was introduced by Weyl [22] and involves the following unitary matrices [23]:

$$U(m,n) = \sum_{k=0}^{d-1} \omega^{mk} |k\rangle\langle k+n|. \tag{1}$$

Here we introduce the notation we use throughout: ω is the primitive dth root of unity $\exp(2\pi i/d)$. All arithmetical operations over latin indices are taken over modulo d.

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We will further be mainly concerned with systems of N qudits, for which we introduce the following standard notations:

$$U(\vec{m},\vec{n}) = \bigotimes_{\alpha=1}^{N} U(m_\alpha, n_\alpha). \tag{2}$$

Greek indices will always run over a range from 1 to N, and the arithmetic operations over them will always be the usual ones. When the range is not specified, it will be from 1 to N.

We now write, for example,

$$U(\vec{m},\vec{n}) = \sum_{\vec{k}} \omega^{\vec{m}\cdot\vec{k}} |\vec{k}\rangle\langle\vec{k}+\vec{n}|, \tag{3}$$

the notational conventions being self-explanatory. Note further that all our results can routinely be extended to the more complicated case in which the different particles in the N-particle system have different dimensions d_α. The "vectors" $\vec{m}$ are then replaced by lists of integers, with $0 \leqslant m_\alpha \leqslant d_\alpha - 1$. Whereas this complicates the notation considerably, no points of essential interest are thereby introduced. We thus leave it to the interested reader to develop these issues. When nontrivial points arise in this respect, we explicitly point this out.

These unitary matrices satisfy certain elementary properties:

$$\begin{aligned} \mathrm{Tr} U(m,n)^\dagger U(m',n') &= d\delta_{mm'}\delta_{nn'}, \\ U(m,n)U(m',n') &= \omega^{m'n} U(m+m',n+n'), \\ U(m,n)U(m',n') &= \omega^{m'n-mn'} U(m',n')U(m,n), \\ U(m,n)^\dagger &= \omega^{mn} U(-m,-n), \end{aligned} \tag{4}$$

as well, of course, as their vectorial equivalents.

We now define Weyl maps and the corresponding channels: any density matrix on the space of N qudits; that is, on $(\mathbb{C}^d)^{\otimes N}$, can be expressed as

$$\rho = \frac{1}{d^N} \sum_{\vec{m},\vec{n}} \alpha(\vec{m},\vec{n}) U(\vec{m},\vec{n}), \tag{5}$$

where $\alpha(\vec{m},\vec{n})$ satisfies $\alpha(\vec{m},\vec{n}) = \omega^{\vec{m}\cdot\vec{n}}\alpha^*(-\vec{m},-\vec{n})$ in order for ρ to satisfy the condition of Hermiticity. More intricate conditions need to be satisfied in order to yield a positive matrix, but we shall not be concerned with these. A Weyl map is now defined as follows:

$$\rho \to \rho' = \mathcal{E}[\rho] = \frac{1}{d^N} \sum_{\vec{m},\vec{n}} \tau(\vec{m},\vec{n})\alpha(\vec{m},\vec{n}) U(\vec{m},\vec{n}). \tag{6}$$

Here the $\tau(\vec{m},\vec{n})$ are complex numbers, whereas ρ is the density matrix given in (5). In other words, if the $U(\vec{m},\vec{n})$ are viewed as generators of the vector space of all Hermitian matrices, the Weyl maps act *diagonally* on this set. Such maps have been studied in the context of decoherence quantum circuits [24], non-Markovianity [25], and quantum communications [26], and it has been found that some invariants can be used to transfer quantum information in a robust manner [21].

We now wish to find the conditions necessary and sufficient for $\mathcal{E}$ to be a quantum channel; that is, to be trace- and Hermiticity-preserving, as well as completely positive. For the former two conditions, we require

$$\tau(\vec{m},\vec{n}) = \tau(-\vec{m},-\vec{n})^\star, \tag{7a}$$

$$\tau(0,0) = 1. \tag{7b}$$

To verify complete positivity, we must check the circumstances under which the Choi–Jamiołkowski matrix, given by

$$\mathcal{D} = \frac{1}{d^N} \sum_{\vec{m},\vec{n}} \tau(\vec{m},\vec{n}) U(\vec{m},\vec{n}) \otimes U(\vec{m},\vec{n})^\star, \tag{8}$$

is positive semidefinite. Interestingly, Eq. (8) is the corresponding Choi–Jamiołkowski matrix, even if $U(\vec{m},\vec{n})$ are not Weyl operators, but an arbitrary basis of Hilbert-Schmidt space, as shown in Appendix A. To specify the criteria for matrix in Eq. (8) to be positive semidefinite, we evaluate its eigenvalues $\lambda(\vec{m},\vec{n})$. This is easily done after noticing that the various elements of the sum, namely, the $U(\vec{m},\vec{n}) \otimes U(\vec{m},\vec{n})^\star$ all commute for arbitrary values of $\vec{m}$ and $\vec{n}$, as readily follows from (4):

$$\begin{aligned} &[U(m,n)\otimes U(m,n)^\star][U(m',n')\otimes U(m',n')^\star] \\ &\quad= [U(m,n)U(m',n')] \otimes [U(-m,n)U(-m',n')] \\ &\quad= [U(m+m',n+n')] \otimes [U(-(m+m'),n+n')] \end{aligned} \tag{9}$$

The symmetry of the final expression proves the claim, and the extension to the case of arbitrary N is straightforward.

It now remains to determine the eigenvalues of $U(\vec{m},\vec{n})$, which given its tensor product structure [see Eq. (2)] can be reduced to the single-qudit case of $U(m,n)$. These can be calculated directly studying the recursion relation that follows from the eigenvalue equation for the Weyl operators, see Appendix B. One can then readily see that the eigenvalues $\mu(r,s)$ of $U(m,n)\otimes U(m,n)^\star$ take the form

$$\mu(r,s) = \omega^{mr-ns}, \tag{10}$$

where r and s are arbitrary integers modulo d that serve as labels for the eigenvalue. The degeneracy pattern of these eigenvalues is complicated, but since our focus is on the positivity of $\mathcal{D}$, we do not need to consider these details.

The set of eigenvalues of $U(\vec{m},\vec{n})\otimes U(\vec{m},\vec{n})^*$ is then given by

$$\mu(\vec{r},\vec{s}) = \omega^{\vec{m}\cdot\vec{r}-\vec{n}\cdot\vec{s}}. \tag{11}$$

The condition for the positive semidefiniteness of $\mathcal{D}$ is thus that, for all $\vec{r}$ and $\vec{s}$,

$$d^{-N} \sum_{\vec{m},\vec{n}} \tau(\vec{m},\vec{n})\omega^{\vec{m}\cdot\vec{r}-\vec{n}\cdot\vec{s}} = \lambda(\vec{r},\vec{s}) \geqslant 0. \tag{12}$$

Note that condition (7a) on $\tau(\vec{m},\vec{n})$ straightforwardly shows that the left-hand side of (12) is real, so that the inequality is meaningful.

The $\lambda(\vec{r},\vec{s})$ are the eigenvalues of $\mathcal{D}$. They can also be used to characterize the Weyl channel $\mathcal{E}$. Inverting the relation (12)

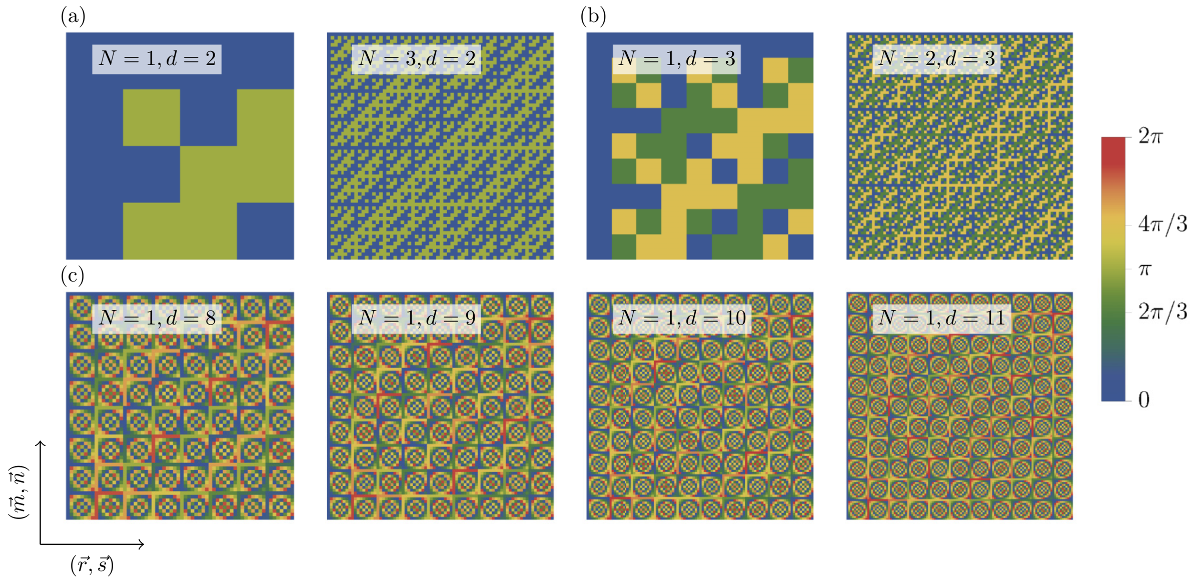

FIG. 1. Visualization of the argument of the matrix elements $\bigotimes_\alpha [F_\alpha \otimes F_\alpha^*](\vec{m}, \vec{n}; \vec{r}, \vec{s}\,)$ for different dimensions and number of particles; rows and columns are indexed by the double indices $(\vec{m}, \vec{n})$ and $(\vec{r}, \vec{s})$, respectively. This matrix map the $\tau(\vec{m}, \vec{n})$ of a Weyl map to the eigenvalues $\lambda(\vec{r}, \vec{s}\,)$ of its Choi-Jamiołkowski matrix, see Eqs. (12) and (14). We show plots for systems of (a) qubits, (b) qutrits, and (c) single-qudits. Notice that not only the total dimension is relevant, but also the number of particles; for instance, compare $N = 3$, $d = 2$ with $N = 1$, $d = 8$.

we get

$$\tau(\vec{m}, \vec{n}) = d^{-N} \sum_{\vec{r}, \vec{s}} \lambda(\vec{r}, \vec{s}\,) \omega^{-\vec{m}\cdot\vec{r}+\vec{n}\cdot\vec{s}}, \tag{13a}$$

$$\sum_{\vec{r}, \vec{s}} \lambda(\vec{r}, \vec{s}\,) = d^N. \tag{13b}$$

Here (13b) follows from $\mathrm{Tr}\mathcal{D} = d^N$, which is a consequence of (7b) and (8).

From (8) and (10) follows that $\tau(\vec{m}, \vec{n})$ and $\lambda(\vec{r}, \vec{s}\,)$ are connected by the following *linear* relationship:

$$\tau(\vec{m}, \vec{n}) = \sum_{\vec{r}, \vec{s}} \bigotimes_\alpha [F_\alpha \otimes F_\alpha^\star](\vec{m}, \vec{n}; \vec{r}, \vec{s}\,) \lambda(\vec{r}, \vec{s}\,), \tag{14}$$

where F_α is the quantum Fourier transform matrix for dimension d_α in the general case, and of dimension d in the case we shall generally study (see Fig. 1).

We have therefore obtained a full characterization of Weyl channels: choosing arbitrary $\lambda(\vec{r}, \vec{s}\,)$ that are positive and add up to d^N, the $\tau(\vec{m}, \vec{n})$ given by (13a) define a Weyl channel.

It is important to highlight that the set of channels introduced in the present work are different from other kinds of generalization of Pauli channels introduced previously [27–29]. For instance, in Refs. [28,29] the *generalized Pauli channels constant on axes* have been studied. Those are Weyl channels for single-particle systems with the restriction that all $\tau(\vec{m}, \vec{n})$ are equal for those indices $(\vec{m}, \vec{n})$ in the same nontrivial subgroup of $\mathbb{Z}_d \oplus \mathbb{Z}_d$. Complete positivity conditions in terms of a single inequality have been provided, which can be verified with (12).

Let us conclude by contrasting the Weyl channels with the well-known Pauli channels in a comprehensive manner. To facilitate this comparison, let us examine a system comprising two qubits. The Pauli and Weyl channels in question operate in a diagonal fashion within two distinct bases of the space encompassing bounded operators, acting upon a Hilbert space of dimension four. While the former manipulates tensor products of Pauli matrices, the latter engages with the basis of Weyl matrices in a 4×4 matrix space. It is important to note that Weyl matrices generally lack a tensor product structure. Another distinguishing characteristic is the order of these matrices: Pauli matrices exhibit an order of two, meaning that squaring them yields the identity matrix. In contrast, 4×4 Weyl matrices typically possess an order of four, with the exception of cases like $U(2, 2)$, $U(0, 2)$, and $U(2, 0)$, which have an order of two. To further illustrate this disparity, Fig. 1 provides a visual comparison between the plots of $\bigotimes_\alpha [F_\alpha \otimes F_\alpha^*](\vec{m}, \vec{n}; \vec{r}, \vec{s}\,)$ for two scenarios: one with $N = 3$ and $d = 2$ in Fig. 1(a), and the other with $N = 1$ and $d = 8$ in Fig. 1(c). This comparison allows us to discern that the Pauli and Weyl channels exhibit distinct behaviors in a system of dimension eight.

III. SET OF EXTREME POINTS

The set of Weyl channels is clearly convex, since equations (13) imply that any Weyl channel is given by a convex sum of channels of the form

$$\tau_{\vec{r}_0, \vec{s}_0}(\vec{m}, \vec{n}) = \omega^{-\vec{m}\cdot\vec{r}_0+\vec{n}\cdot\vec{s}_0}, \tag{15}$$

where $\vec{r}_0, \vec{s}_0$ are fixed vectors whose elements are integer numbers modulo d.

Furthermore, we can see that the set of Weyl channels is in fact a $(d^{2N} - 1)$-dimensional simplex. Recall that all eigenvalues $\lambda(\vec{r}, \vec{s}\,)$ of the Choi–Jamiołkowski matrix of a Weyl channel must be non-negative, and sum up to d^N [see Eq. (13b)]. The set $\lambda(\vec{r}, \vec{s}\,)$ is thus the standard $(d^{2N} - 1)$-dimensional simplex. Since the connection (12) between the

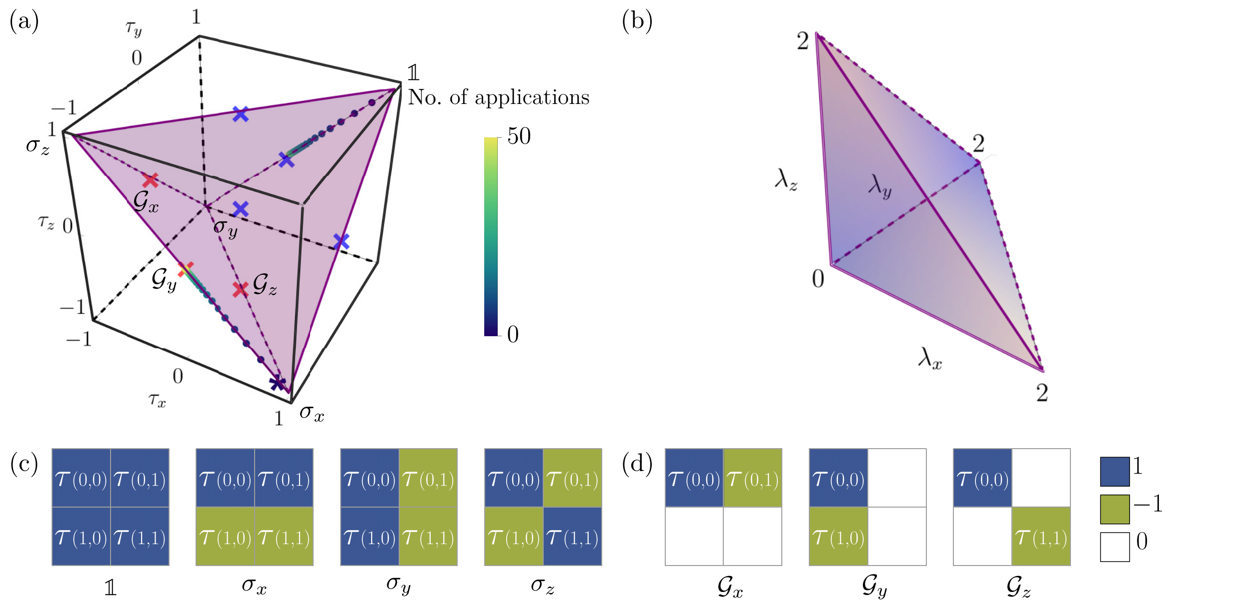

FIG. 2. Simplexes of (a) the $\tau(m,n)$ of all single-qubit Weyl channels, in which we identify $\tau(0,1)=\tau_x$, $\tau(1,0)=\tau_y$, and $\tau(1,1)=\tau_z$, as usual, and similarly for λs; and of (b) the eigenvalues $\lambda(m,n)$ of the corresponding Choi-Jamiołkowski matrix. Additionally, we depict in panels (c) and (d) the extreme points of panel (a) and Weyl erasing generators, respectively.

λs and the τs is linear and invertible, then the set of all τs is also a $(d^{2N}-1)$-dimensional simplex. Note, however, that the τs are complex, so they are actually part of a bigger $2d^{2N}$-dimensional real vector space. Nonetheless, conditions (7) [which are automatically satisfied by the formulas (12)] additionally limit the τs so that the number of degrees of freedom is back to $d^{2N}-1$.

Moreover, the extreme points of the simplex of Weyl channels are given by the d^{2N} channels of equation (15). This is because the extreme points of the λ's simplex are clearly

$$\lambda(\vec{r},\vec{s}\,)=d^N\delta_{\vec{r},\vec{r}_0}\delta_{\vec{s},\vec{s}_0}. \tag{16}$$

Therefore, those of the set of Weyl channels are given by applying the transformation (13a) to these extreme points, obtaining as a result the channels of Eq. (15). In fact, these channels are the only Weyl channels with the property that, for all $\vec{m},\vec{n}$, $|\tau(\vec{m},\vec{n})|=1$, as shown in the following theorem.

Theorem 1. A Weyl channel is an extreme point of the set of Weyl channels if and only if $|\tau(\vec{m},\vec{n})|=1$ for all $\vec{m},\vec{n}$. *Proof.* We have already proved that extreme points are of the form (15), and therefore satisfy that $|\tau(\vec{m},\vec{n})|=1$, so we only need to prove the converse. Equation (13a) says that

$$\tau(\vec{m},\vec{n})=d^{-N}\sum_{\vec{r},\vec{s}}\lambda(\vec{r},\vec{s}\,)\omega^{-\vec{m}\cdot\vec{r}+\vec{n}\cdot\vec{s}}. \tag{17}$$

Recall that $d^{-N}\lambda(\vec{r},\vec{s}\,)$ are non-negative and add up to 1. It follows from the triangle inequality that, if the sum on the right-hand side of (17) has more than one term, then $|\tau(\vec{m},\vec{n})|<1$. Thus, a Weyl channel with $|\tau(\vec{m},\vec{n})|=1$ must have all $\lambda(\vec{r},\vec{s}\,)$ equal to zero except one, say $\lambda(\vec{r}_0,\vec{s}_0)$. In other words, if the Choi–Jamiołkowski matrix of a Weyl channel has only one eigenvalue $\lambda(\vec{r}_0,\vec{s}_0)$ different from zero, then that Weyl channel is an extreme point. ■

The simplest case that illustrates the result of this theorem are the single-qubit Weyl quantum channels. Given that the Weyl operators for $d=2$ reduce to the Pauli operators, the extremal points for these are the vertices of the well-known tetrahedron of qubit quantum channels [30], which we illustrate in Fig. 2(a).

We can now characterize in greater detail these extreme points. For a given value of $\vec{r}_0$, $\vec{s}_0$, the effect of the channel given by (15) on a Weyl matrix $U(\vec{m},\vec{n})$ is

$$\begin{aligned}\mathcal{E}_{\vec{r}_0,\vec{s}_0}[U(\vec{m},\vec{n})]&=\omega^{-\vec{r}_0\cdot\vec{m}+\vec{s}_0\cdot\vec{n}}U(\vec{m},\vec{n})\\&=U(\vec{s}_0,\vec{r}_0)U(\vec{m},\vec{n})U(\vec{s}_0,\vec{r}_0)^{\dagger}.\end{aligned} \tag{18}$$

We see therefore that the extreme points of the set of all Weyl channels are unitary channels. Since all Weyl channels are convex combinations of the extreme points, it immediately follows that all the Weyl channels are simply random unitary channels, constructed from the Weyl unitaries. In fact, from this point of view, $\lambda(\vec{r},\vec{s}\,)/d^N$ can be interpreted physically as the probability of applying $U(\vec{r},\vec{s}\,)$ to the system, since a Weyl channel such as that in (13a) is a convex sum of the extreme points $\tau_{\vec{r},\vec{s}}$ with weights $\lambda(\vec{r},\vec{s}\,)/d^N$.

IV. A MATHEMATICAL STRUCTURE WITHIN WEYL CHANNELS

In this section, we focus on a subset of Weyl channels with physical relevance and mathematical beauty. We consider the Weyl channels, which, when iterated infinitely, converge to channels that completely erase, preserve, or introduce phases to the projections of the density matrix onto the Weyl operator basis. Our main results include the characterization of the group property of this particular subset and a method to determine these channels. Specifically, we show that the corresponding channels can be obtained by identifying all subgroups of $\mathbb{Z}_d^{\oplus N} \oplus \mathbb{Z}_d^{\oplus N}$ and their homomorphisms to $\mathbb{Z}_d$.

A. Subgroup property of Weyl channels

Theorem 2. Let $\tau(\vec{m},\vec{n})$ and $\tau(\vec{m}',\vec{n}')$ have both norm 1. Then so does $\tau(\vec{m}+\vec{m}',\vec{n}+\vec{n}')$ and additionally

$$\tau(\vec{m}+\vec{m}',\vec{n}+\vec{n}') = \tau(\vec{m},\vec{n})\tau(.,\vec{m}')\vec{n}' \tag{19}$$

Proof. From (13) follows that, quite generally, $\tau(\vec{m},\vec{n})$ are convex combinations of complex numbers of the form ω^k, with k an integer. Nonetheless, the only such convex combinations having norm 1 are themselves numbers of the form ω^k, with k an integer, therefore $\tau(\vec{m},\vec{n})$ and $\tau(\vec{m}',\vec{n}')$ are, under the hypotheses of the theorem, of the form ω^k and $\omega^{k'}$, respectively. Similarly for $\tau(\vec{m}',\vec{n}')$, which we take to be equal to $\omega^{k'}$.

Setting $\tau(\vec{m},\vec{n}) = \omega^k$ in (13a) and conveniently rewriting the equation results in

$$\omega^k = d^{-N}\sum_l \omega^l \sum_{\vec{r},\vec{s}} \lambda(\vec{r},\vec{s})\delta_{-\vec{m}\cdot\vec{r}+\vec{n}\cdot\vec{s},l}, \tag{20}$$

and a similar expression for $\omega^{k'}$, replacing $\vec{m}$ and $\vec{n}$ with their primed versions. Since $\lambda(\vec{r},\vec{s})$ are positive and sum up to d^N, it follows that the right-hand side is a convex sum of ω^l. For it to equal ω^k, an extreme point, only the term with $l = k$ can be different from zero. It follows that $\lambda(\vec{r},\vec{s}) = 0$, whenever $-\vec{m}\cdot\vec{r}+\vec{n}\cdot\vec{s} \neq k$ or $-\vec{m}'\cdot\vec{r}+\vec{n}'\cdot\vec{s} \neq k'$. From this follows straightforwardly that again $\lambda(\vec{r},\vec{s}) = 0$ whenever

$$-(\vec{m}+\vec{m}')\cdot\vec{r}+(\vec{n}+\vec{n}')\cdot\vec{s} \neq (k+k'), \tag{21}$$

which implies

$$\tau(\vec{m}+\vec{m}',\vec{n}+\vec{n}') = \omega^{k+k'} = \tau(\vec{m},\vec{n})\tau(\vec{m}',\vec{n}'). \tag{22}$$

■

This means that the set of all $(\vec{m},\vec{n})$, such that $\tau(\vec{m},\vec{n})$ has norm 1, form an additive subgroup of the Abelian group

$$\mathcal{G} = \mathbb{Z}_d^{\oplus N} \oplus \mathbb{Z}_d^{\oplus N} \tag{23}$$

with respect to vector addition modulo d. Note that this is one case where a significant difference arises when the N particles have different dimensions d_α; the vectors $(\vec{m},\vec{n})$ would then belong to the group

$$\mathcal{G} = \left(\bigoplus_{\alpha=1}^{N}\mathbb{Z}_{d_\alpha}\right) \oplus \left(\bigoplus_{\alpha=1}^{N}\mathbb{Z}_{d_\alpha}\right). \tag{24}$$

In other words, the set $\mathcal{H} \subseteq \mathcal{G}$ on which $\tau(\vec{m},\vec{n})$ has norm 1 forms a subgroup of $\mathcal{G}$ and τ can be seen as a homomorphism from $\mathcal{H}$ to $\mathbb{Z}_d$.

To determine all $\tau(\vec{m},\vec{n})$ of Weyl channels satisfying Eq. (19), we must proceed in two steps:

(1) Determine all subgroups $\mathcal{H} \subseteq \mathcal{G}$.

(2) Determine all homomorphisms from $\mathcal{H}$ to $\mathbb{Z}_d$.

In the following section we present an algorithm to determine all subgroups $\mathcal{H}$ of $\mathcal{G}$ and homomorphisms from $\mathcal{H}$ to $\mathbb{Z}_d$.

We wish to remark that, for a given quantum channel, only those $\tau(\vec{m},\vec{n})$ which have norm 1 need to satisfy Eq. (19), while the other coefficients are only restricted by the complete positivity condition, Eq. (12). Therefore, Eq. (19) is not a constraint on the simplex of all possible Weyl channels, but only a relationship that the $\tau(\vec{m},\vec{n})$ with norm 1 of a given channel must have as a consequence of the complete positivity condition.

B. Weyl channels

To determine all $\tau(\vec{m},\vec{n})$ of Weyl channels satisfying Eq. (19) we proceed in two steps. The first step involves identifying all subgroups of $\mathcal{G}$ [cf. Eq. (24)]. We begin by stating two relevant facts about finite Abelian groups and then discuss how to find the subgroups for the more general case of an Abelian group $\mathcal{G}$, which encompasses the majority of our discussion in this section. After that, we describe the second step, which is how to determine all phases of $\tau(\vec{m},\vec{n})$ of a Weyl channel by determining all homomorphisms from a subgroup to the roots of unity. We direct the reader's attention to Appendix E where several examples of the algorithm presented in this section are studied. Pursuing it may enhance comprehension of the contents of this section.

Whenever p and q are coprime, the group $\mathbb{Z}_{pq}$ is isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}_q$. Therefore, we may use the prime decomposition of d_α to separate each $\mathbb{Z}_{d_\alpha}$ in Eq. (24) as a sum of cyclic groups of prime power order. We apply this procedure to all values of α in equation (24). Subsequently, we organize the terms corresponding to distinct prime factors. As a result, the group $\mathcal{G}$ can be expressed as the direct sum:

$$\mathcal{G} = \bigoplus_p \mathcal{G}_p, \tag{25}$$

where each term $\mathcal{G}_p$ corresponds to a prime factor p present in the factorization of any d_α, and is given by

$$\mathcal{G}_p = \bigoplus_{\eta=1}^{r} \mathbb{Z}_{p^{M_\eta}}. \tag{26}$$

Here, M_η is arranged in nonincreasing order, and r represents the count of nonzero values in the set M_η. For example, let us consider a system composed by a ququart ($d_1 = 4$) and a qubit ($d_2 = 2$). The group associated in Eq. (24) for this system is

$$\mathcal{G} = \mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2. \tag{27}$$

To put this group $\mathcal{G}$ in the form of Eq. (25) we see that Eq. (27) is the direct product of only cyclic groups of order that are a power of two; that is, $\mathcal{G} = \mathcal{G}_2$. Moreover, $\mathcal{G}$ is already of the form of Eq. (26), so $M_1 = M_2 = 2$ and $M_3 = M_4 = 1$.

We associate each group $\mathcal{G}_p$ with a corresponding sequence $\overline{M} = M_1 \cdots M_r$, which is a *partition* of $M = \sum_\alpha M_\alpha$. Therefore, we refer to $\mathcal{G}_p$ as a group of type $\overline{M}$. Furthermore, for any partition of M there exists an Abelian group of order p^M that is unique up to an isomorphism [31]. If one has a subgroup $\mathcal{H}_p$ of $\mathcal{G}_p$, the corresponding partition, let us call it $\overline{N}$, satisfies $N_\alpha \leqslant M_\alpha$. On the other hand, once the choice of the nonincreasing order for the partition of the group $\mathcal{G}_p$, the corresponding partitions for the subgroups $\mathcal{H}_p$ inherit a well-defined order from the group, and the corresponding partitions cannot therefore be taken in nonincreasing order.

Another important fact about finite Abelian groups is that they all have a basis; that is, they can be generated by the integer combinations of a set of elements. In our particular case, a simple way of choosing a basis is by picking a generating element for each cyclic group in Eq. (26). We denote them $\vec{e}_\alpha$, and therefore an arbitrary $h \in \mathcal{H}$ can be *uniquely* expressed as

$$h = \sum_{\alpha=1}^{r} n_\alpha \vec{e}_\alpha, \tag{28}$$

where $n_\alpha \in \mathbb{Z}_{p^{M_\alpha}}$, and the multiplication of a group element by an integer m is defined as the addition of the group element to itself repeated m times. The number r of elements in the basis is independent of the choice of basis and it is known as the group's rank r.

The general idea for finding all subgroups of $\mathcal{G}_p$ is to determine a subset of subgroups such that, upon applying all automorphisms $T : \mathcal{G}_p \mapsto \mathcal{G}_p$, all others are found. We say that two subgroups of $\mathcal{G}_p$ are T-isomorphic when there is an automorphism T mapping one to the other. Then, to find the subgroups of $\mathcal{G}_p$ we first determine any subset with the maximum number of subgroups that are not T-isomorphic. We call these "representative subgroups." By definition, applying all automorphisms T (which we describe how to find in Appendix C) to the representative subgroups all other subgroups of $\mathcal{G}_p$ are found.

Note the difference between the concept of isomorphism for the subgroups and the concept of T-isomorphism. The latter depends not only on the group structure of the subgroup $\mathcal{H}$, but also on the way in which it is embedded in the group $\mathcal{G}_p$. For instance, we can embed the group $\mathbb{Z}_2$ in the group $\mathbb{Z}_{2^2} \oplus \mathbb{Z}_2$ either as a subgroup of the first summand or as a subgroup of the second. In other words, the partition $\overline{M}$ describing the full group is $\overline{M} = 2\,1$ and the subgroup $\mathbb{Z}_2$ can be embedded with a partition $0\,1$ as well as $1\,0$. The two subgroups, being both isomorphic to $\mathbb{Z}_2$, are abstractly isomorphic, but that isomorphism cannot be extended to an isomorphism of $\mathbb{Z}_4 \oplus \mathbb{Z}_2$.

All subgroups of $\mathcal{G}_p$ are found applying all its automorphisms T to the subgroups generated by the bases

$$\mathcal{B} = \{p^{s_1}\vec{e}_1, \dots, p^{s_r}\vec{e}_r\}, \quad 0 \leqslant s_\alpha \leqslant M_\alpha. \tag{29}$$

Nevertheless, more than one different selection $\mathbb{S} = \{s_\alpha\}$ may determine two bases of subgroups T-isomorphic, in other words, that are connected by an automorphism of $\mathcal{G}_p$. For example, consider a group $\mathcal{G}_p$ of type $\overline{M} = 2\,2\,1\,1$. The partitions $\mathbb{S} = 0\,1\,0\,1$ and $\mathbb{S}' = 1\,0\,1\,1$ determine bases of T-isomorphic subgroups, because the automorphism defined as $T(\vec{e}_1) = \vec{e}_2$, $T(\vec{e}_2) = \vec{e}_1$, $T(\vec{e}_3) = \vec{e}_4$, and $T(\vec{e}_4) = \vec{e}_3$ maps one to the other. From each of these T-isomorphic sets of subgroups we can pick an arbitrary element, which will be called *the representative subgroup*.

To find the representative subgroups we need a criterion to determine when two bases $\mathcal{B}$ of the form (29) generate T-isomorphic groups. Let us denote by $\tilde{M}_1, \dots, \tilde{M}_q$ the q different values in the sequence of numbers in $\overline{M}$ (for instance, if $\bar{M} = 2211$, then $q = 2$ and $\tilde{M}_1 = 2, \tilde{M}_2 = 1$). Furthermore, we define the subset $S_j = \{s_\alpha, \forall\, \alpha : M_\alpha = \tilde{M}_j\}$ of $\mathbb{S}$; that is, S_j is the subset of $\mathbb{S}$ formed by all the s_α whose α correspond to the indices of the M_α that are equal to $\tilde{M}_j$. Then, the criterion is the following: two different sets $\mathbb{S}$ and $\mathbb{S}'$ determine bases of T-isomorphic subgroups whenever their corresponding subsets S_j and S'_j are the same for all j.

We are ready to describe the complete algorithm to determine all the subgroups of a given group $\mathcal{G}$. First, decompose $\mathcal{G}$ as a sum of prime power order groups $\mathcal{G}_p$. For every $\mathcal{G}_p$, find all sets $\mathbb{S} = \{s_\alpha\}$ and discriminate between them to find the only ones that determine representative subgroups. Then apply to them all automorphisms T of $\mathcal{G}_p$, so all subgroups of $\mathcal{G}_p$ will be found, albeit with repetitions. A description of the group of automorphisms of an arbitrary Abelian group $\mathcal{G}$ is provided in Ref. [32], and the technique is summarized for the sake of completeness in Appendix C. Finally, to find the subgroups of $\mathcal{G}$ apply the direct sum between all different subgroups of each $\mathcal{G}_p$.

Furthermore, a way to count the total number of subgroups of $\mathcal{G}_p$ is already known in the literature. Since any Abelian group of prime power order can only have subgroups that are also of prime power order, subgroups of order p^L, with $L < M$, can also be characterized by a partition $\overline{L}$ of L. It is shown in Refs. [31,33] that necessary and sufficient conditions for the partition $\overline{L}$ to correspond to a possible subgroup of the group determined by the partition $\overline{M}$ of M are

$$L_\alpha = 0 \; (\alpha > r), \tag{30a}$$

$$L_\alpha \leqslant M_\alpha, \tag{30b}$$

$$L_\alpha \geqslant L_{\alpha+1}. \tag{30c}$$

An expression for the number of different subgroups of type $\overline{L}$ is already known in the literature. For that matter, we refer the reader to Appendix F.

To fully determine the coefficients $\tau(\vec{m}, \vec{n})$ with norm 1, we interpret them as a function that maps $\mathcal{H}$ to the group of roots of unity ω^j. We consider $\tau(\vec{m}, \vec{n}) = \omega^{\phi(\vec{m},\vec{n})}$, thus we are looking for all homomorphisms $\phi : \bigoplus_{M_\alpha} \mathbb{Z}_{p^{M_\alpha}} \mapsto \mathbb{Z}_{p^{M_1}}$. To determine one of such functions uniquely, it is sufficient to specify the values of ϕ on a basis of $\mathcal{H}$, as described in Appendix D.

Note that all the above remarks greatly simplify when d is a prime number. In that case the group $\mathcal{G}$ is additionally a vector space. The set of subgroups can then be described as the set of vector subspaces using the usual techniques of linear algebra. All the partitions described above then reduce to partitions of the type where M_α is either one or zero, and the partition is fully characterized by the number of its nonzero elements, which correspond to the subspace's *dimension*. Finally, the homomorphism τ can be described as a linear map from the vector space $\mathcal{G}$ to the field $\mathbb{Z}_d$, which is once more straightforwardly described in terms of linear algebra.

V. WEYL ERASING CHANNELS

A. Generalities

In this section we focus on a particular class of Weyl channels; those for which $|\tau(\vec{m}', \vec{n}')| = 0$ or 1. In other words, we discuss Weyl channels that completely erase, preserve or introduce specific phases to the projections of the density matrix of a system of qudits onto the Weyl-matrix basis. We refer to this subset of Weyl channels as Weyl erasing channels.

Weyl erasing channels are an interesting subset of Weyl channels because they arise from the composition of one or more of these channels an infinite number of times. For instance, the infinite composition of any Weyl channel for which all $|\tau(\vec{m},\vec{n})| < 1$, except $\tau(\vec{0},\vec{0}) = 1$, results in the completely depolarizing channel. In the general case, the repeated application of a Weyl channel may not converge to a single Weyl erasing channel, however it oscillates between two or more such channels. For example, applying many times the single-qubit Weyl channel depicted with an asterisk in Fig. 2(a) asymptotically oscillates between the channel collapsing the Bloch sphere onto the y axis and other channel collapsing it onto the y axis while reflecting it across the x-z plane. This oscillation continues indefinitely between $\vec{\tau} = (1, 0, 1, 0)$ and $\vec{\tau}' = (1, 0, -1, 0)$. Alternatively, given a time independent evolution that gives rise to a Weyl channel, will have similar behavior as the discrete case.

In the following, we derive a Kraus representation of Weyl erasing channels that will provide insight into the physical implementation of these channels. We then focus on deriving an expression of the eigenvalues of the Choi-Jamiołkowski matrix of Weyl erasing channels exclusively, as we already have an expression for all Weyl channels [cf. (12)]. For this, we begin presenting an algorithm that uses the mathematical machinery developed in Sec. IV B to find all $\tau(\vec{m},\vec{n})$ of any Weyl erasing channel:

(1) Find all sets of indices $\{(\vec{m},\vec{n}) : |\tau(\vec{m},\vec{n})| = 1\}$ by determining all subgroups $\mathcal{H} \subset \mathcal{G}$, with $\mathcal{G}$ that in (25).

(2) Find the values of $\tau(\vec{m},\vec{n}) = \omega^{\phi(\vec{m},\vec{n})}$ for all $(\vec{m},\vec{n}) \in \mathcal{H}$ by determining all homomorphisms $\phi : \mathcal{H} \mapsto \bigoplus_p \mathbb{Z}_{p^{M_1(p)}}$, with $M_1(p)$ such that $\mathbb{Z}_{p^{M_1(p)}}$ denotes the largest order cyclic group for every $\mathcal{G}_p$ in (25).

(3) Assign $\tau(\vec{m}',\vec{n}') = 0$ for all $(\vec{m}',\vec{n}') \notin \mathcal{H}$.

While an exhaustive enumeration of all Weyl erasing channels for a large number N of particles is not practical, the construction provides insights into the mathematical structure of this set. In fact, we show examples of Weyl erasing channels for a single-qubit, a four-level system and a system composed by both of them in Figs. 4–8. We remark that the algorithms to determine $\tau(\vec{m},\vec{n})$ of Weyl and Weyl erasing channels are both the same up until determining all homomorphisms. Then, for the former one may assign any value to the $\tau(\vec{m}',\vec{n}') \notin \mathcal{H}$ as long as they keep the channel completely positive, whereas for the latter one must assign the value zero to all $\tau(\vec{m}',\vec{n}') \notin \mathcal{H}$.

Let us now evaluate the eigenvalues of the Choi matrix of a Weyl erasing channel. To be consistent with the previous section, we consider Weyl erasing channels of a system of N particles, each with dimension a power of p, so the group in question is $\mathcal{G}_p$ [see Eq. (26)]. The group's rank is then $r = 2N$. Recall these channels have $\tau(\vec{m},\vec{n}) = 0$ for all $(\vec{m},\vec{n}) \notin \mathcal{H}$, thus, we find from (17) that

$$\lambda(\vec{r},\vec{s}\,) = d^{-N} \sum_{(\vec{m},\vec{n})\in\mathcal{H}} \tau(\vec{m},\vec{n}) \prod_{\alpha=1}^{N} \omega_\alpha^{m_\alpha r_\alpha - n_\alpha s_\alpha}, \tag{31}$$

where $\omega_\alpha = \exp(2\pi i/p^{M_{2\alpha}})$. Since the composition of two Weyl channels is simply to multiply both sets of $\tau(\vec{m},\vec{n})$, we can see that $\tau(\vec{m},\vec{n})$ of Weyl erasing channels are those of an extreme Weyl channel [cf. (15)] for all $(\vec{m},\vec{n}) \in \mathcal{H}$, and $\tau(\vec{m}',\vec{n}') = 0$ for all $(\vec{m}',\vec{n}') \notin \mathcal{H}$. Hence, substituting

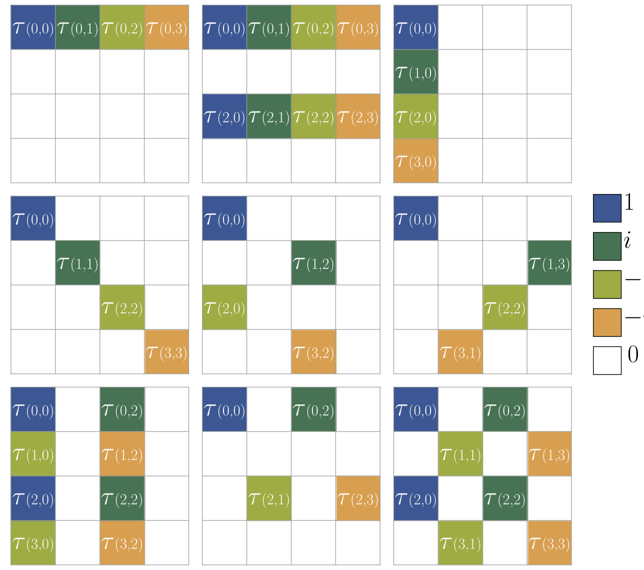

FIG. 3. Weyl erasing generators for a single-particle of dimension $d = 4$. Concatenating in all possible ways the corresponding channels one obtains all the Weyl quantum channels of a four-level system with $|\tau(m, n)| = 0, 1$.

$\tau(\vec{m},\vec{n})$, and considering $\omega_{p^\alpha}^k = \omega_{p^\beta}^{p^{(\alpha-\beta)k}}$, $\alpha > \beta$, we can write

$$\lambda(\vec{r},\vec{s}\,) = \sum_{(\vec{m},\vec{n})\in\mathcal{H}} \omega_{p^{M_1}}^{f(\vec{m},\vec{n})}, \tag{32}$$

where $f(\vec{m},\vec{n}) = \sum_{\alpha=1}^{N} p^{M_1 - M_{2\alpha-1}}[m_\alpha(r_\alpha - r_{0,\alpha}) - n_\alpha(s_\alpha - s_{0,\alpha})]$. To evaluate this expression we note that f is a homomorphism, $f : \mathcal{G}_p \mapsto \mathbb{Z}_{p^{M_2}}$. Therefore, the sum evaluates to zero unless f maps $\mathcal{H}$ to the trivial group:

$$\lambda(\vec{r},\vec{s}\,) = \begin{cases} |\mathcal{H}|, & f(\vec{m},\vec{n} = 0) \text{ for all } (\vec{m},\vec{n}) \in \mathcal{H} \\ 0, & \text{otherwise}, \end{cases} \tag{33}$$

where $|\mathcal{H}|$ is defined as the number of elements of $\mathcal{H}$. Furthermore, let us consider the group $\mathcal{H}^\perp = \{(\vec{m},\vec{n}) : f(\vec{m},\vec{n}) = 0\}$. The only nonzero $\lambda(\vec{r},\vec{s}\,)$ are those with indices $(\vec{r},\vec{s}\,)$ such that $(\vec{r} - \vec{r}_0, \vec{s} - \vec{s}_0) \in \mathcal{H}^\perp$.

Finally, having obtained the eigenvalues of the Choi-Jamiołkowski matrix of Weyl erasing channels we describe their canonical Kraus representation. Recall that that Weyl

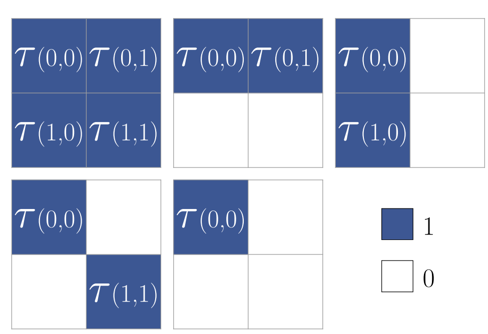

FIG. 4. Single-qubit Weyl erasing channels with $\tau(m, n) = 0, 1$. These are completely characterized by the sets $\{(m, n) : \tau(m, n) = 1\}$, which are the subgroups of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

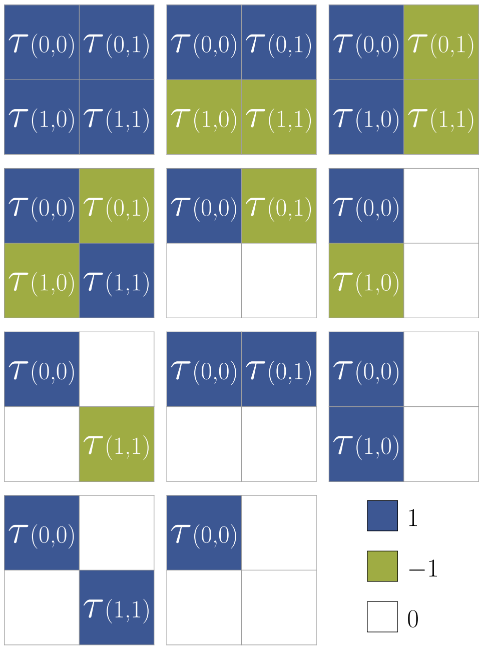

FIG. 5. Single-qubit Weyl erasing channels with $|\tau(m,n)| = 0, 1$. These are completely characterized by two elements: (i) the sets $\{(m,n) : |\tau(m,n)| = 1\}$, which are the subgroups of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, and (ii) all homomorphisms $\phi : \mathbb{Z}_2 \oplus \mathbb{Z}_2 \mapsto \mathbb{Z}_2$.

matrices are the Kraus operators of Weyl channels with probabilities equal to $\lambda(\vec{r},\vec{s}\,)$ [cf. Eq. (18)]. It then follows that Kraus operators of Weyl erasing channels are the subset of Weyl matrices $U(\vec{r},\vec{s}\,)$, with $(\vec{r}-\vec{r}_0, \vec{s}-\vec{s}_0) \in \mathcal{H}^{\perp}$, each with probability $|\mathcal{H}^{\perp}|/|\mathcal{G}|$.

B. Generators

In the following, we investigate the smallest subset of Weyl erasing channels which, under composition, generate the whole set. For the sake of simplicity, we start by finding the generators of Weyl channels with $\tau(\vec{m},\vec{n}) = 0$ or 1, as these are Weyl channels characterized only by subgroups of $\mathcal{G}$. Subsequently, we move to the most general Weyl erasing channels that either preserve, erase, or introduce phases to the density matrix.

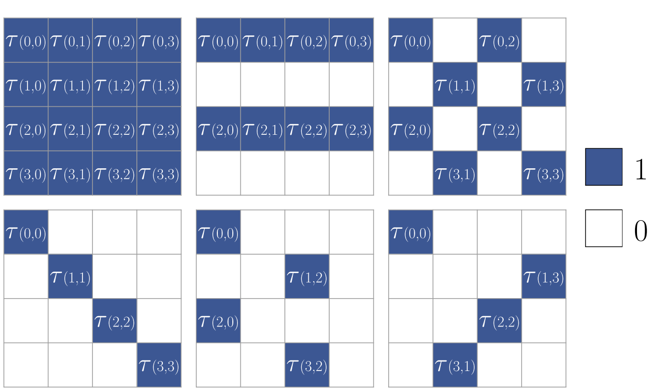

FIG. 6. Some four-level system Weyl erasing channels with $\tau(m,n) = 0, 1$. Each of those is completely characterized by a set $\{(m,n) : \tau(m,n) = 1\}$, which is a subgroup of $\mathbb{Z}_4 \oplus \mathbb{Z}_4$.

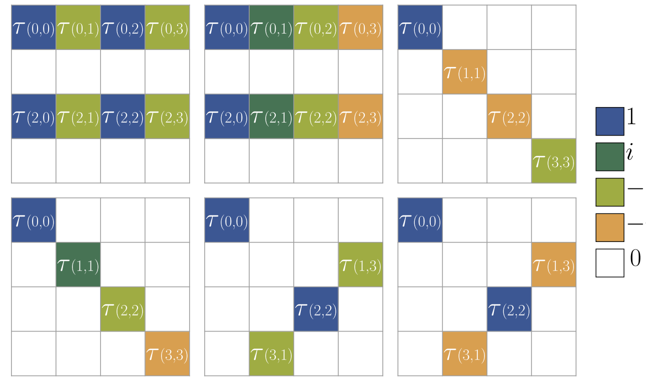

FIG. 7. Some Weyl erasing channels, with $d = 4$, $N = 1$, and $|\tau(m,n)| = 0, 1$. Each of those is completely characterized by (1) a set $\{(m,n) : |\tau(m,n)| = 1\}$, which is a subgroup of $\mathbb{Z}_4 \oplus \mathbb{Z}_4$, and (2) a homomorphism $\phi : \mathbb{Z}_4 \oplus \mathbb{Z}_4 \mapsto \mathbb{Z}_4$.

We determine those subgroups that are indecomposable, in the sense that they cannot be generated as the nontrivial composition of two Weyl channels. We call these the generator subgroups. We consider once again the group $\mathcal{G}$ shown in Eq. (25), thus, we first discuss how to determine the generator subgroups of $\mathcal{G}_p$, and, from those, determine the generator subgroups of $\mathcal{G}$. Similarly to what we did in Sec. IV B, the generator subgroups V_p of $\mathcal{G}_p$ can be found constructing representative generator subgroups $V_p^{\star}$ and applying all automorphisms of $\mathcal{G}_p$ to them.

We claim that a representative subgroup is a generator V_p of $\mathcal{G}_p$ if and only if its basis is of the form

$$\mathcal{B}_{V_p} = \{\vec{e}_1, \dots, \vec{e}_{j-1}, p^{s_j}\vec{e}_j, \vec{e}_{j+1}, \dots, \vec{e}_r\}, \quad 1 \leqslant s_j \leqslant M_j. \tag{34}$$

This is verified as follows: Consider a subgroup $\mathcal{H}_p$ with basis (29) such that its set of values $\mathbb{S} = \{s_\alpha\}_\alpha$ has two (or more) values $s_\beta, s_\gamma \neq 0$. That is, $\mathcal{H}_p$ has the basis $\mathcal{B}_{\mathcal{H}_p} = \{\vec{e}_1, \dots, \vec{e}_{\beta-1}, p^{s_\beta}\vec{e}_\beta, \dots, \vec{e}_{\gamma-1}, p^{s_\gamma}\vec{e}_\gamma, \dots, \vec{e}_r\}$. Then, $\mathcal{H}_p$ can be expressed as the nontrivial intersection of the group g'_p with basis $\mathcal{B}_{V'_p} = \{\vec{e}_1, \dots, \vec{e}_{\beta-1}, p^{s_\beta}\vec{e}_\beta, \dots, \vec{e}_r\}$ and another group V''_p with basis $\mathcal{B}_{V''_p} = \{\vec{e}_1, \dots, \vec{e}_{\gamma-1}, p^{s_\gamma}\vec{e}_\gamma, \dots, \vec{e}_r\}$. Now, we check that a group $\mathcal{H}_p$ with a basis of the form (34) cannot be expressed as an intersection of two subgroups containing $\mathcal{H}_p$*strictly*. Note that groups $\mathcal{H}'_p$ satisfying $\mathcal{H}_p \subsetneq \mathcal{H}'_p \subset \mathcal{G}_p$ must have a basis of the form $\{\vec{e}_1, \dots, p^{s'_j}\vec{e}_j, \dots, \vec{e}_r\}$, with $s'_j < s_j$. Therefore, if we have two such groups $\mathcal{H}'_p$ and $\mathcal{H}''_p$, then the group arising from intersection $\mathcal{H}'_p \cap \mathcal{H}''_p$ has a basis $\{\vec{e}_1, \dots, p^{\max(s'_j, s''_j)}\vec{e}_j, \dots, \vec{e}_r\}$, which does not generate $\mathcal{H}_p$ because the integer span of $p^{s_{j'}}\vec{e}_{j'}$ or $p^{s_{j''}}\vec{e}_{j''}$ are strictly larger than the integer span of $p^{s_j}\vec{e}_j$.

Finally, to find all generator subgroups V of $\mathcal{G} = \bigoplus_p \mathcal{G}_p$ we proceed as follows: We begin by finding the generator subgroups of $\mathcal{G}_p$. Then, the generator subgroups V are the subgroups of the form

$$V = \bigoplus_p H_p, \tag{35}$$

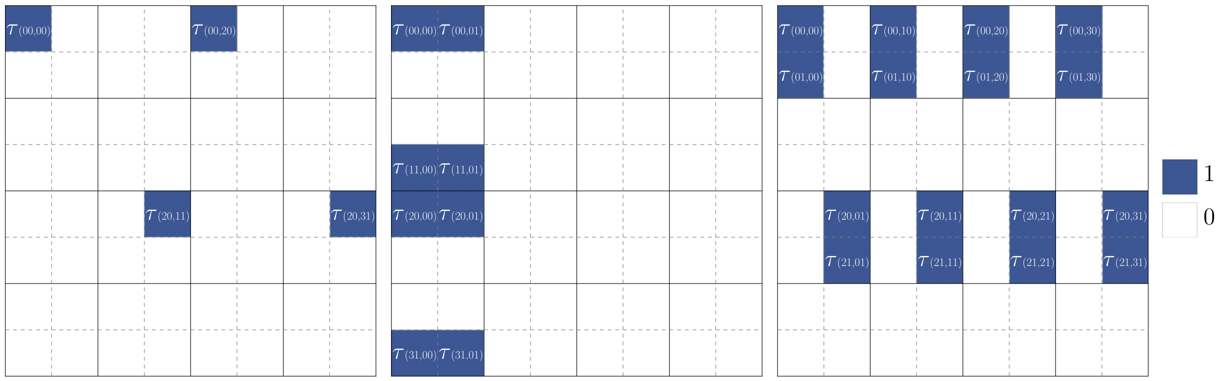

FIG. 8. Some Weyl erasing channels of a system with a four-level particle and a qubit, with $\tau(\vec{m},\vec{n}) = 0, 1$. Recall that (m_α, n_α) corresponds to the αth particle.

where $H_p = \mathcal{G}_p$ for all p except for one $p = p'$, for which $H_{p'} = V_{p'}$. To see this, notice that V is a generator since any other subgroup $\mathcal{H}'$ strictly containing it must also have $H'_p = \mathcal{G}_p$ for $p \neq p'$ and $H'_{p'} \subsetneq V_{p'}$. Therefore, for two such subgroups $\mathcal{H}'$ and $\mathcal{H}''$ to generate V, $H'_{p'}$ and $H''_{p'}$ should generate $V_{p'}$, which is impossible since $V_{p'}$ is a generator of $\mathcal{G}_{p'}$. On the other hand, any subgroup of $\mathcal{G}$ that is composed as the sum of groups $\mathcal{G}_p$ except for two (or more) primes p' and p'' in which the sum is of any corresponding generator subgroup, may be generated by two subgroups $\mathcal{H}$ of the form (35) with suitable generators $H_{p'} = V_{p'}$ and $H_{p''} = V_{p''}$.

To summarize up to this point: to determine the generators of Weyl erasing channels with $\tau(\vec{m},\vec{n}) = 0, 1$ one is only required to determine the generator subgroups of the corresponding group in which the indices $(\vec{m},\vec{n})$ of $\tau(\vec{m},\vec{n}) = 1$ live. To determine the generators of Weyl erasing channels that also introduce phases one must still find generator subgroups, and a suitable homomorphism for each of them as well.

More specifically, a Weyl erasing generator with $|\tau(\vec{m},\vec{n})| = 0, 1$ is completely characterized by a generator subgroup of $\mathcal{G}$ and a homomorphism $\phi : \mathcal{G} \mapsto \bigoplus \mathbb{Z}_{p^{M_2(p)}}$, with $\mathbb{Z}_{p^{M_2(p)}}$ is the largest order cyclic group for every $\mathcal{G}_p$. The composition of two channels with subgroups $\mathcal{H}_1$ and $\mathcal{H}_2$ and homomorphisms ϕ_1 and ϕ_2 yields a channel corresponding to $\mathcal{H}_1 \cup \mathcal{H}_2$ and homomorphism $\phi_1 + \phi_2$. Therefore, for every generator subgroup V, if we define a basis ϕ_V^α of the space of homomorphisms, then the set of channels corresponding to each generator subgroup V, together with any element ϕ_V^α mapping V to the whole group $\mathbb{Z}_{p^{M_2(p)}}$, form a set of generators. For instance, we show a set of Weyl erasing generators of a four-level system in Fig. 3.

To avoid misunderstandings, note that the "generators" for channels with $\tau(\vec{m},\vec{n}) = 0, 1$, those characterized completely by a generator subgroup (and one may say that also by a homomorphism $\phi = 0$), are not generators of the whole set of Weyl erasing channels $[|\tau(\vec{m},\vec{n})| = 0, 1]$. The former channels can always be obtained through iterated composition of channels with a nonzero homomorphism. For example, the single-qubit Weyl erasing channels depicted with a blue point in Fig. 2(a), except for the one located at $\vec{\tau} = (0, 0, 0)$, are Weyl generator channels with $\tau(m, n) = 0, 1$. However, they can be obtained from composing two times each of the three Weyl generator channels depicted with a red point. Those are the generators of all Weyl erasing channels of a single-qubit.

VI. CONCLUSIONS

In this paper, we explored a class of quantum channels of multipartite systems with different-dimensional particles. Our focus was on extending the study of Pauli and Weyl channels; the former have been studied for systems of many qubits and the latter for single d-level systems.

We begin by introducing the multiparticle Weyl operators $U(\vec{m},\vec{n})$ to define a Weyl map as a diagonal map in this basis; its eigenvalues are denoted by $\tau(\vec{m},\vec{n})$. We derived the constraints on $\tau(\vec{m},\vec{n})$ for a Weyl map to preserve the trace and Hermiticity of the density matrix, as well as to be completely positive. For the latter, we diagonalized its Choi-Jamiołkowski matrix and found the linear relationship between these eigenvalues and $\tau(\vec{m},\vec{n})$.

Several features of Weyl channels emerged from our study. We identified the extreme points of the set, which correspond to Weyl operators, highlighting the random unitary nature of Weyl channels. Additionally, we established a subgroup structure within a Weyl channel, showing that the indices $(\vec{m},\vec{n})$ of all $|\tau_{\vec{m},\vec{n}}| = 1$ is a subgroup of the direct product of groups $\mathbb{Z}_{d_\alpha} \oplus \mathbb{Z}_{d_\alpha}$, where d_α represents the dimension of the αth particle.

Furthermore, we introduced Weyl erasing channels, which are Weyl channels that either preserve, erase, or introduce phases to the Weyl operator basis. These extend the concept of *component-erasing* channels. Given that all channels of this type exhibit the subgroup structure, with the remaining $\tau_{\vec{m},\vec{n}}$ set to zero, we were able to find the smallest subset which generate, under composition, the whole set.

Our work contributes to the understanding of many-body quantum channels, and, moreover, to the dynamics of d-level systems, which have growing importance for both theoretical and practical purposes. We have provided a useful characterization of a unitary generalization of Pauli channels that extends to the most general scenario of multipartite and d-dimensional systems. We believe this to be of theoretical importance for the description of d-level gates, Simultaneously, the subset of Weyl erasing channels hold significance for the study of errors beyond Pauli errors. A major result of this work is the identification of the set of generators of the Weyl erasing channels. A clear understanding of their structure is sufficient to characterize the totality of the Weyl erasing channels, since all simply arise via composition of generators. Our results do not, unfortunately, lead to a completely transparent characterization of those generators, so that this is left to future research. Moreover, the implications of the existence of a group structure in the dynamics of the corresponding channel are to be explored. Future research directions include investigating divisibility, non-Markovianity, and the subset of entanglement-breaking channels among other properties of the Weyl channel set. Finally, the findings of this article might be used in applications over quantum metrology, calculation of channel capacities and other dynamical properties where optimization over the set of channels is required.

ACKNOWLEDGMENTS

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APPENDIX A: COMPUTATION OF THE CHOI–Jamiołkowski MATRIX

This Appendix demonstrates that the Choi-Jamiołkowski matrix of any diagonal map $\mathcal{E}$ takes the form

$$\mathcal{D} = \frac{1}{d^N} \sum_{\vec{m},\vec{n}} \tau(\vec{m},\vec{n}) U(\vec{m},\vec{n}) \otimes U(\vec{m},\vec{n})^{\star}, \tag{A1}$$

whenever $U(\vec{m},\vec{n})$ form an orthogonal basis of unitaries [28].

The Choi–Jamiołkowski matrix of a quantum map $\mathcal{E}$ is defined as follows:

$$\mathcal{D} = \frac{1}{d^N} \sum_{\vec{k},\vec{l}} \mathcal{E}[|\vec{k}\rangle\langle\vec{l}\,|] \otimes (|\vec{k}\rangle\langle\vec{l}\,|). \tag{A2}$$

We may now express $|\vec{k}\rangle\langle\vec{l}\,|$ in terms of any orthogonal basis of unitaries $U(\vec{m},\vec{n})$

$$|\vec{k}\rangle\langle\vec{l}\,| = \frac{1}{d^N} \sum_{\vec{m},\vec{n}} \mathrm{Tr}(U^\dagger(\vec{m},\vec{n})|\vec{k}\rangle\langle\vec{l}\,|) U(\vec{m},\vec{n}). \tag{A3}$$

Substituting this in the expression (A2) for $\mathcal{D}$ yields

$$\frac{1}{d^N} \sum_{\vec{k},\vec{l}} \mathcal{E}(|\vec{k}\rangle\langle\vec{l}\,|) \otimes |\vec{k}\rangle\langle\vec{l}\,|$$

$$= \frac{1}{d^{2N}} \sum_{\vec{k},\vec{l},\vec{m},\vec{m}',\vec{n},\vec{n}'} \mathcal{E}[\mathrm{Tr}(U^\dagger(\vec{m},\vec{n})|\vec{k}\rangle\langle\vec{l}\,|) U(\vec{m},\vec{n})]$$

$$\otimes [\mathrm{Tr}(U^\dagger(\vec{m}',\vec{n}')|\vec{k}\rangle\langle\vec{l}\,|) U(\vec{m}',\vec{n}')] \tag{A4a}$$

$$= \frac{1}{d^{2N}} \sum_{\vec{k},\vec{l},\vec{m},\vec{m}',\vec{n},\vec{n}'} \mathrm{Tr}(U^\dagger(\vec{m},\vec{n})|\vec{k}\rangle\langle\vec{l}\,|) \mathrm{Tr}(U(\vec{m}',\vec{n}')|\vec{l}\,\rangle\langle\vec{k}|)$$

$$\times \tau(\vec{m},\vec{n}) U(\vec{m},\vec{n}) \otimes U(\vec{m}',\vec{n}')^*, \tag{A4b}$$

where we have used the complex conjugate of (A3{}{}{}). Then, using the definition of trace it follows

$$= \frac{1}{d^{2N}} \sum_{\vec{k},\vec{l},\vec{m},\vec{m}',\vec{n},\vec{n}'} \langle l|U^\dagger(\vec{m},\vec{n})|k\rangle\langle k|U(\vec{m}',\vec{n}')|l\rangle \tau(\vec{m},\vec{n})$$

$$\times U(\vec{m},\vec{n}) \otimes U(\vec{m}',\vec{n}')^* \tag{A4c}$$

$$= \frac{1}{d^N} \sum_{\vec{m},\vec{n}} \tau(\vec{m},\vec{n}) U(\vec{m},\vec{n}) \otimes U(\vec{m},\vec{n})^*. \tag{A4d}$$

This expression can also be obtained also using a recently result of Siewert, in which he derives an expression for the maximally entangled state in terms of an arbitrary orthogonal basis [34].

APPENDIX B: EIGENVALUES OF $U(m, n)$ AND OF $U(m, n) \otimes U(m, n)^*$

We find the eigenvalues of $U(m,n)$. Since it is unitary, we express the eigenvalues as ω^c, with $c \in \mathbb{R}$. Let us consider an eigenvector $|\phi\rangle = \sum_r \phi(r)|r\rangle$ with eigenvalue $\xi = \omega^c$. The eigenvalue equation for $U(m,n)$ leads to the following relation:

$$\phi(r+n) = \omega^{-mr}\omega^c\phi(r). \tag{B1}$$

Starting with an arbitrary index r and applying this recursion equation $l-1$ times, we obtain

$$\phi(r+nl) = \omega^{-lmr - \frac{1}{2}l(l-1)mn + cl}\phi(r). \tag{B2}$$

In the particular case in which $l = l' := \frac{d}{\gcd(d,n)}$ we may use that $l'n$ is a multiple of d, so

$$\phi(r) = \omega^{-l'mr - \frac{1}{2}l'(l'-1)mn + cl'}\phi(r), \tag{B3}$$

which implies that [for values of r such that $\phi(r) \neq 0$]

$$-l'mr - \tfrac{1}{2}l'(l'-1)mn + cl' = sd \tag{B4}$$

for some integer s. Therefore,

$$c = \frac{sd}{l'} + \tfrac{1}{2}(l'-1)mn + mr = \gcd(d,n)s + mr$$

$$+ \tfrac{1}{2}(l'-1)mn. \tag{B5}$$

So we conclude that all eigenvalues of $U(m,n)$ necessarily have the form

$$\omega^{\gcd(d,n)s + mr + \frac{1}{2}(l'-1)mn} \tag{B6}$$

for s and r integers.

Furthermore, taken modulo d, the set $\{\gcd(d,n)s + mr \mid s, r \in \mathbb{Z}_d\}$ is equivalent to $\{ns + mr \mid s, r \in \mathbb{Z}_d\}$ which is also equivalent to $\{\gcd(m,n)k \mid k \in \mathbb{Z}_d\}$. Therefore, the d eigenvalues of $U(m,n)$ are

$$\xi = \omega^{\gcd(m,n)k + \frac{1}{2}(l'-1)mn}. \tag{B7}$$

From this, it is straightforward that the eigenvalues of $U(m,n)\otimes U(m,n)^*$ are

$$\mu(r,s)=\omega^{\gcd(m,n)k-\gcd(m,n)h}. \tag{B8}$$

This set is equivalent to

$$\mu(r,s)=\omega^{mr-ns}, \tag{B9}$$

where r, s are integers modulo d.

APPENDIX C: AUTOMORPHISMS OF FINITE ABELIAN GROUPS

In the following we describe the bijective homomorphisms T of an arbitrary Abelian group. Without loss of generality we limit ourselves to groups that are the direct sum of groups of the type $\mathbb{Z}_{p^M}$, specifically,

$$\mathcal{G}=\bigoplus_{\alpha=1}^{r}\mathbb{Z}_{p^{M_\alpha}}. \tag{C1}$$

To fix notations, we work with a fixed basis $\vec{e}_\alpha$, $1\leqslant\alpha\leqslant r$, where r is the *rank* of $\mathcal{G}$. The map T is therefore uniquely determined by the values of $T\vec{e}_\alpha$. Since $\vec{e}_\alpha$ is a basis, we can write

$$T\vec{e}_\alpha=\sum_{\beta=1}^{r}t_{\alpha\beta}\vec{e}_\beta. \tag{C2}$$

The $t_{\alpha\beta}$ are then uniquely determined, if we view them as homomorphisms from $\mathbb{Z}_{p^{M_\beta}}$ to $\mathbb{Z}_{p^{M_\alpha}}$. Since such homomorphisms can always be expressed through the multiplication by some appropriate number, the expression given in (C2) is meaningful.

Now let us specify more precisely the range of variation of the $t_{\alpha\beta}$. We distinguish two cases:

(1) $M_\alpha\leqslant M_\beta$: in this case any number modulo p^{M_α} will do, and two different such numbers provide different homomorphisms.

(2) $M_\alpha>M_\beta$: in this case, the number needs to be a multiple of $p^{M_\alpha-M_\beta}$, since otherwise it is not possible to define the map. In that case, we may describe $t_{\alpha\beta}$ as $p^{M_\alpha-M_\beta}\tau_{\alpha\beta}$, where $\tau_{\alpha\beta}$ is an arbitrary number modulo p^{M_β}.

Consider the matrix T in greater detail, and just as in the main text, let us denote by $\tilde{M}_1,\dots,\tilde{M}_q$ the *distinct* values of M_α in *strictly decreasing order*. We define ν_α to be the number of times $\tilde{M}_\alpha$ appears repeated in the original series. This defines a division of the T matrix in *blocks* of size $\nu_\alpha\times\nu_\beta$, where $1\leqslant\alpha,\beta\leqslant q$.

We first take the elements $t_{\alpha\beta}$ modulo p. As a consequence of the observation (2), all blocks with $\alpha<\beta$ are filled with zeros, whereas all other blocks have arbitrary entries. It thus follows that the matrix is invertible modulo p if and only if all the diagonal blocks are invertible. The number of invertible $\nu_\alpha\times\nu_\alpha$ matrices modulo p is given by

$$I_\alpha=\prod_{\beta=1}^{\nu_\alpha}(p^{\nu_\alpha}-p^{\beta-1}). \tag{C3}$$

One sees this by observing that we may first choose an arbitrary nonzero vector of length ν_α in $p^{\nu_\alpha}-1$ different ways, then chose a second vector independent from the first, and so on.

All the other entries in the blocks below the diagonal; that is, the $t_{\alpha\beta}$ with $\alpha>\beta$, can be chosen arbitrarily. If we thus define

$$K_0=\sum_{1\leqslant\beta<\alpha\leqslant q}\nu_\alpha\nu_\beta, \tag{C4}$$

then the total number of possible forms of the matrix T modulo p is

$$N(p)=p^{K_0}\prod_{\alpha=1}^{q}I_\alpha. \tag{C5}$$

We now need to work out the number of ways this can be extended to the full matrix, where the entries have the full range of variation specified above. Note first that the condition of invertibility carries over automatically upon extension, as the inverse matrix of T modulo p can be extended uniquely to the inverse of the extended matrix.

To the entries on or below the diagonal; that is, with $t_{\alpha\beta}$ such that $\alpha\geqslant\beta$, we can add any number of the form $p\tau_{\alpha\beta}$, where $\tau_{\alpha\beta}$ is an arbitrary number taken modulo $p^{\tilde{M}_\alpha-1}$. So the number of possibilities of extending these blocks is given by p^{K_1}, where

$$K_1=\sum_{1\leqslant\beta\leqslant\alpha\leqslant q}(\tilde{M}_\alpha-1)\nu_\alpha\nu_\beta. \tag{C6}$$

For the blocks above the diagonal; that is, the blocks with $t_{\alpha\beta}$ such that $\alpha<\beta$, they are of the form $p^{\tilde{M}_\alpha-\tilde{M}_\beta}\tau_{\alpha\beta}$, with $\tau_{\alpha\beta}$ a number modulo $p^{\tilde{M}_\beta}$, so that the total number of ways of extending the blocks above the diagonal is p^{K_2}, with

$$K_2=\sum_{1\leqslant\alpha<\beta\leqslant q}\tilde{M}_\beta\nu_\alpha\nu_\beta. \tag{C7}$$

The final result for the total number of automorphisms is thus given by

$$N_{tot}(M_1,\dots,M_r)=p^{K_0+K_1+K_2}\prod_{\alpha=1}^{q}I_\alpha. \tag{C8}$$

APPENDIX D: HOMOMORPHISMS FROM $\mathcal{H}$ TO THE CYCLIC GROUP $\mathbb{Z}_d$

Here we describe the set of homomorphisms ϕ from an Abelian group of the form $\mathcal{H}=\bigoplus_{\alpha=1}^{r}\mathbb{Z}_{p^{M_\alpha}}$ to the cyclic group $\mathbb{Z}_{p^{M_1}}$. As always, the numbers M_α are ordered in decreasing order.

We may as always choose a basis $\vec{e}_\alpha$ of $\mathcal{H}$, each having order p^{M_α}. The homomorphism ϕ is then uniquely determined by a set of homomorphisms ϕ_α from the cyclic groups $\mathbb{Z}_{p^{M_\alpha}}$ to $\mathbb{Z}_{p^{M_1}}$.

Whenever $M_1=M_\alpha$, ϕ_α simply reduces to multiplication by an arbitrary r_α number modulo p^{M_1}. On the other hand, if $M_\alpha<M_1$, then ϕ_α is given by the multiplication by a number of the form $p^{M_1-M_\alpha}r_\alpha$, where r_α is an arbitrary number modulo p^{M_α}.

If we therefore define ν as the number of $M_\alpha = M_1$, so that $M_\nu \geqslant M_1$ but $M_{\nu+1} < M_1$, and $\nu = 0$ if $M_\alpha < M_1$ for all α, then ϕ can be expressed as follows:

$$\phi\left(\sum_\alpha c_\alpha \vec{e}_\alpha\right) = \vec{\phi}\cdot\vec{c} := \sum_\alpha \phi_\alpha c_\alpha, \tag{D1a}$$

$$\phi_\alpha = \begin{cases} p^{M_1 - M_\alpha} s_\alpha & (\alpha \geqslant \nu) \\ t_\alpha & (\alpha < \nu), \end{cases} \tag{D1b}$$

where s_α and t_α are numbers modulo p^{M_α} and p^{M_1}, respectively.

The total number of such homomorphisms is therefore given by p^K with

$$K = \sum_{\alpha=1}^{\nu} M_\alpha + M_1(r - \nu). \tag{D2}$$

APPENDIX E: EXAMPLES

To illustrate the application of the mathematical tools presented in the main text, we provide detailed examples of how to identify the Weyl channels as is described in Sec. IV. Remember that said channels are characterized by a subgroup $\mathcal{H}$ of the group of indices $\mathcal{G}$ (which corresponds to the indices whose τs have norm 1) and a homomorphism (which gives the phases to each τ). The examples will be arranged in increasing generality, starting with a system of one qudit with prime dimension and ending with the most general case of many qudits of arbitrary dimensions.

1. Single particle with prime dimension

Here we show how to follow the algorithm described in Sec. IV for the case of one qudit with prime dimension $d = p$. In this case, the group of indices for the τs is simply $\mathbb{Z}_p \oplus \mathbb{Z}_p$ and we search for all its subgroups and then all homomorphisms to $\mathbb{Z}_p$. While we do this in general, we simultaneously show the specific case for $p = 2$ (a single qubit).

We start determining the types of subgroups of $\mathbb{Z}_p \oplus \mathbb{Z}_p$ and a representative subgroup of each type. For that, we take the following steps:

(1) We select a basis for $\mathbb{Z}_p \oplus \mathbb{Z}_p$, the simplest would be $\{\vec{e}_1, \vec{e}_2\} := \{(1, 0), (0, 1)\}$.

(2) Define M_α as the number such that p^{M_α} is the order of $\vec{e}_\alpha$. In this case, $M_1 = M_2 = 1$ and therefore the partition for the group is $\bar{M} = M_1 M_2 = 11$.

(3) Find all the sets $\mathbb{S} = \{s_\alpha\}$ with $0 \leqslant s_\alpha \leqslant M_\alpha$. In this case, there are four said sets: $\{0, 0\}, \{0, 1\}, \{1, 0\}, \{1, 1\}$.

(4) For each set, we define the basis $\mathcal{B} = \{p^{s_1}\vec{e}_1, p^{s_2}\vec{e}_2\}$, and therefore get the following bases:

$$\mathbb{S} = \{0, 0\} \rightarrow \mathcal{B} = \{p^0\vec{e}_1, p^0\vec{e}_2\} = \{\vec{e}_1, \vec{e}_2\},$$
$$\mathbb{S} = \{0, 1\} \rightarrow \mathcal{B} = \{p^0\vec{e}_1, p^1\vec{e}_2\} = \{\vec{e}_1\}, \tag{E1}$$

$$\mathbb{S} = \{1, 0\} \rightarrow \mathcal{B} = \{p^1\vec{e}_1, p^0\vec{e}_2\} = \{\vec{e}_2\},$$
$$\mathbb{S} = \{0, 0\} \rightarrow \mathcal{B} = \{p^0\vec{e}_1, p^0\vec{e}_2\} = \{\}. \tag{E2}$$

Notice that $p\vec{e}_\alpha = (0, 0)$, which does not contribute to the basis.

(5) Now we only keep bases that are not T-isomorphic, so as to avoid unnecessary redundancies when applying automorphisms in the next step.

As mentioned in the main text, we do so by first defining the sequence of numbers $\tilde{M}_1, \ldots, \tilde{M}_q$ given by the q different values in the sequence of numbers in $\bar{M}$. In this case, as $\bar{M} = 11$, there is only one number in said sequence, being $\tilde{M}_1 = 1$. Then, we define the subsets $S_j = \{s_\alpha, \forall\, \alpha : M_\alpha = \tilde{M}_j\}$. In this case, we only have one such subset, which happens to be the whole set, $S_1 = \{s_1, s_2\}$. As described in the main text, two different bases of those described in the previous step are T-isomorphic if all their sets S_j are the same. In this case, this means that the bases constructed from $\mathbb{S} = \{1, 0\}$ and $\mathbb{S} = \{0, 1\}$ are T-isomorphic. Therefore, we may only keep one of those bases, let us say we keep $\{0, 1\}$ and discard the other one, so that the representative bases are

$$\{\vec{e}_1, \vec{e}_2\},\ \{\vec{e}_1\},\ \{\}. \tag{E3}$$

The subgroups generated by these bases are the "representative subgroups." Then, to find all possible subgroups, we need to find all automorphisms of $\mathbb{Z}_p \oplus \mathbb{Z}_p$ and apply them to these representative groups, so that we can obtain all subgroups of each type starting from the representatives.

As shown in Eq. (C2), automorphisms of $\mathbb{Z}_p \oplus \mathbb{Z}_p$ are determined by a matrix $t_{\alpha\beta}$ with dimensions $r \times r$, where r is the number of elements in the basis of the group. Therefore, in this case the automorphisms are characterized by 2×2 matrices, where each entry $t_{\alpha\beta}$ can be a number modulo p.

Furthermore, these entries are constrained by the conditions given in Appendix C, and since in this case $M_1 = M_2$, all $t_{\alpha\beta}$ fall into case 1 of the aforementioned conditions. This implies that all $t_{\alpha\beta}$ are numbers modulo $p^{M_\alpha} = p$. This gives a total of p^4 possible matrices, but we need to only keep those that are invertible. For example, for the especial case of one qubit, we construct all 2×2 matrices such that all entries $t_{\alpha\beta}$ are numbers modulo 2, and out of these 16 matrices, only six of them are invertible:

$$T_1 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},\quad T_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},\quad T_3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{E4}$$

$$T_4 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},\quad T_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\quad T_6 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{E5}$$

Therefore, these matrices represent the six possible automorphisms of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

To find all the subgroups, we simply apply all automorphisms on each representative subgroup found in Eq. (E3). For the case of one qubit, the result would be as follows:

(1) $\{\vec{e}_1, \vec{e}_2\}$: The group generated by this basis is the whole group, and when we apply any automorphism, we always get back the whole group because automorphisms are invertible. Therefore, the only group here is the whole group $\mathbb{Z}_p \oplus \mathbb{Z}_p$.

(2) $\{\vec{e}_1\}$: For this basis, the representative subgroup it generates is $\{(0, 0), (1, 0)\}$. As before, we apply all the automorphisms to this subgroup. We can see an example for the case of one qubit, when applying T_1 to the element $(1, 0)$, we get as a result $(0, 1)$, so applying T_1 to the subgroup gives as a result $\{(0, 0), (0, 1)\}$. Similarly, when using the other automorphisms in the case of one qubit, we get the following

subgroups (excluding repetitions):

$$\{(0,0),(1,0)\},\quad \{(0,0),(0,1)\},\quad \{(0,0),(1,1)\}. \tag{E6}$$

(3) {}: In this case, the representative subgroup is the trivial $\{(0,0)\}$ and applying any automorphism leaves this subgroup intact.

Therefore, we have found all subgroups of $\mathbb{Z}_p \oplus \mathbb{Z}_p$. For the case of $p = 2$, they are the complete group, the subgroups obtained in Eq. (E6) and the trivial subgroup $\{(0,0)\}$, see Fig. 4. That is, up until this point we have found the sets of indices $\{m, n\}$ for which $\tau(m, n)$ can have norm 1 in a Weyl channel.

Now we find all homomorphisms $\phi : \mathbb{Z}_p \oplus \mathbb{Z}_p \to \mathbb{Z}_p$. A homomorphism is characterized by its value in each element in the basis, $\phi_\alpha := \phi(\vec{e}_\alpha)$. To know the possible values of ϕ_α, we first define ν like in Appendix D, as the number such that $M_\nu \geqslant n$ but $M_{\nu+1} < n$ (where n is the exponent of the co-domain of ϕ, in this case the co-domain is $\mathbb{Z}_p$, so that $n = 1$ and we can see that $\nu = 2$). The possible values of ϕ_α are given by the cases in Eq. (D1b):

(1) ϕ_1: Since $\alpha = 1 < \nu = 2$, we have the second case and therefore ϕ_1 is a number modulo $p^n = p$.

(2) ϕ_2: Since $\alpha = 2 \geqslant \nu = 2$, we have the first case, so that $\phi_2 = p^{n-M_2} s_s = s_2$ with s_2 a number modulo $p^{M_2} = p$. Therefore, ϕ_2 is also a number modulo p.

To find all possible homomorphisms, we determine all possible pairs ϕ_1, ϕ_2. Since ϕ_1, ϕ_2 can be any number modulo p, we have a total of p^2 homomorphisms. In the special case of one qubit, they are $\phi_1 = \phi_2 = 0$; $\phi_1 = 0, \phi_2 = 1$; $\phi_1 = 1, \phi_2 = 0$; and $\phi_1 = \phi_2 = 1$. We show in Fig. 5 all Weyl erasing channels of a single qubit.

2. Single particle with $d = p^n$

Now we generalize to the case of a single particle with $d = p^n$. To have a concrete example to show, we consider a particle with $d = 2^2 = 4$.

(1) We select a basis of $\mathbb{Z}_{p^n} \oplus Z_{p^n}$, for example, $\{\vec{e}_1, \vec{e}_2\} = \{(1,0),(0,1)\}$.

(2) Define M_α as the number such that p^{M_α} is the order of $\vec{e}_\alpha$. In this case, $M_1 = M_2 = n$, so the partition of the group is $\bar{M} = M_1 M_2 = nn$. For the special case of a four-level system, the partition is $\bar{M} = 22$.

(3) Find all the sets $\mathbb{S} = \{s_\alpha\}$ with $0 \leqslant s_\alpha \leqslant M_\alpha$. For the case of a four-level system, there are nine said sets: $\{0,0\}$, $\{0,1\}, \ldots, \{2,1\}$, $\{2,2\}$.

(4) For each set, we define a basis $\mathcal{B} = \{p^{s_1}\vec{e}_1, p^{s_2}\vec{e}_2\}$. For example, for a four-level system, the bases are

$$\begin{aligned}&\mathbb{S} = \{0,0\} \to \mathcal{B} = \{\vec{e}_1, \vec{e}_2\},\\ &\mathbb{S} = \{0,1\} \to \mathcal{B} = \{\vec{e}_1, 2\vec{e}_2\},\ \mathbb{S} = \{0,2\} \to \mathcal{B} = \{\vec{e}_1\},\end{aligned} \tag{E7}$$

$$\begin{aligned}&\mathbb{S} = \{1,0\} \to \mathcal{B} = \{2\vec{e}_1, \vec{e}_2\},\\ &\mathbb{S} = \{1,1\} \to \mathcal{B} = \{2\vec{e}_1, 2\vec{e}_2\},\ \mathbb{S} = \{1,2\} \to \mathcal{B} = \{2\vec{e}_1\},\end{aligned} \tag{E8}$$

$$\begin{aligned}&\mathbb{S} = \{2,0\} \to \mathcal{B} = \{\vec{e}_2\},\\ &\mathbb{S} = \{2,1\} \to \mathcal{B} = \{2\vec{e}_2\},\ \mathbb{S} = \{2,2\} \to \mathcal{B} = \{\}.\end{aligned} \tag{E9}$$

(5) We define the sequence of numbers $\tilde{M}_1, \ldots, \tilde{M}_q$ given by the q different values in the sequence $\bar{M}$. In this case, we only have $\tilde{M}_1 = n$. Then, we define the subsets $S_j = \{s_\alpha, \forall\, \alpha : M_\alpha = \tilde{M}_j\}$; in this case we only have $S_1 = \{s_1, s_2\}$. As said before, different bases are T-isomorphic if all sets S_j are the same, and we only need to keep one of them. Therefore, for the case of one four-level system, we only have to keep the following bases, which generate the representative subgroups:

$$\{\vec{e}_1, \vec{e}_2\},\ \{\vec{e}_1, 2\vec{e}_2\},\ \{\vec{e}_1\},\ \{2\vec{e}_1, 2\vec{e}_2\},\ \{2\vec{e}_1\},\ \{\}.$$

Once again, automorphisms are characterized by 2×2 matrices $t_{\alpha\beta}$. Since $M_1 = M_2 = n$, all $t_{\alpha\beta}$ fall into the first case of Appendix C, which implies that all $t_{\alpha\beta}$ are numbers modulo $p^{M_\alpha} = p^n$. This gives a total of p^{4n} possible matrices, of which we only keep those that are invertible (have nonzero determinant modulo p). For example, in the case of a four-level system, there are 96 such matrices.

Find all subgroups. As before, to find all subgroups of $\mathbb{Z}_d \oplus \mathbb{Z}_d$, we apply all automorphisms to each of the representative subgroups found in the first step and omit duplicates. As always, these subgroups describe the indexes $\tau(m, n)$ which can have norm 1. We show in Fig. 6 some Weyl erasing channels of a four-level system that are completely characterized by subgroups of $\mathbb{Z}_4 \oplus \mathbb{Z}_4$.

We find all homomorphisms $\phi : \mathbb{Z}_{p^n} \oplus \mathbb{Z}_{p^n} \to \mathbb{Z}_{p^n}$. As for the last case, the homomorphism is characterized by two values $\phi_1 = \phi(\vec{e}_1)$, $\phi_2 = \phi(\vec{e}_2)$. Using Appendix D, we find that ϕ_1 and ϕ_2 are both numbers modulo p^n. To find all possible homomorphisms, we determine all possible pairs of ϕ_1, ϕ_2 which gives a total total of p^{2n} homomorphisms. For the case of a four-level system, the 16 homomorphisms are given by all pairs of numbers ϕ_1, ϕ_2 modulo 4. We show in Fig. 7 some Weyl erasing channels for a four-level system.

3. Single particle with arbitrary dimension

Now we consider a single particle with arbitrary dimension d, which can be written with its prime factorization as $d = \prod_{i=1}^{K} p_i^{n_i}$. In this case, what we have done in the last examples does not apply, since it only applies for groups of the form $\mathbb{Z}_{p^{M_1}} \oplus \mathbb{Z}_{p^{M_2}} \oplus \cdots \oplus \mathbb{Z}_{p^{M_K}}$ (notice that all the groups in the sum are powers of the same prime).

However, we can still find the subgroups of $\mathbb{Z}_d \oplus \mathbb{Z}_d$. To do it, we use the fact that $\mathbb{Z}_{pq} \simeq \mathbb{Z}_p \oplus \mathbb{Z}_q$ whenever p and q are coprime. Therefore, $\mathbb{Z}_d \simeq \mathbb{Z}_{p_1^{n_1}} \oplus \cdots \oplus \mathbb{Z}_{p_K^{n_K}}$, and after reordering we have that

$$\mathbb{Z}_d \oplus \mathbb{Z}_d \simeq \bigoplus_{i=1}^{K} \mathbb{Z}_{p_i^{n_i}} \oplus \mathbb{Z}_{p_i^{n_i}}. \tag{E10}$$

Furthermore, it is a well known fact that subgroups of $F_1 \oplus F_2$ with F_1 and F_2 groups of coprime orders, are obtained as Cartesian products of subgroups of F_1 with subgroups of F_2. Therefore, because of the decomposition of Eq. (E10), we can find the subgroups of $\mathbb{Z}_d \oplus \mathbb{Z}_d$ by obtaining all the subgroups of each $\mathbb{Z}_{p_i^{n_i}} \oplus \mathbb{Z}_{p_i^{n_i}}$ (which can be done as in the last example) and then taking all their possible Cartesian products.

To find the homomorphisms $\phi : \mathbb{Z}_d \oplus \mathbb{Z}_d \to \mathbb{Z}_d$, we picture the ϕ as going from $\bigoplus_{i=1}^{K} \mathbb{Z}_{p_i^{n_i}} \oplus \mathbb{Z}_{p_i^{n_i}}$ to $\bigoplus_{i=1}^{K} \mathbb{Z}_{p_i^{n_i}}$. Any such homomorphism can be written as the direct sums of homomorphisms $\phi_i : \mathbb{Z}_{p_i^{n_i}} \oplus \mathbb{Z}_{p_i^{n_i}} \to \mathbb{Z}_{p_i^{n_i}}$, which we obtained in the last example. Therefore, by constructing all such direct sums, we obtain all the homomorphisms we were looking for.

For example, if $d = 12$ all we need to do is find all the subgroups and homomorphisms of $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ and of $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ and then take Cartesian products of these subgroups and the direct sum of the homomorphisms.

4. *N* particles of dimension of prime power dimension

Now we consider a system consisting of N particles, each with dimension p^{n_i} for $i = 1, \dots, N$, ordered such that $n_1 \geqslant n_2 \geqslant \dots \geqslant n_N$ (notice that the prime p is the same for all particles). In this case, the problem is to find all the subgroups of $\mathcal{G} = \mathbb{Z}_{p^{n_1}} \oplus \mathbb{Z}_{p^{n_1}} \oplus \dots \oplus \mathbb{Z}_{p^{n_N}} \oplus \mathbb{Z}_{p^{n_N}}$ and homomorphisms from $\mathcal{G}$ to $\mathbb{Z}_{p^{n_1}}$. As an example, we develop a system of one qubit and one four-level system.

Similarly to the other examples, to find the representative subgroups we take the following steps:

(1) Select a basis of $\mathcal{G}$. For example, in the case of a qubit and a four-level system, the group is $\mathcal{G} = \mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, and we can choose a basis $\{\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4\}$, with $\vec{e}_1 = (1, 0, 0, 0)$, $\vec{e}_2 = (0, 1, 0, 0)$, $\vec{e}_3 = (0, 0, 1, 0)$, $\vec{e}_4 = (0, 0, 0, 1)$, where the first two entries add mod 4 and the last two add mod 2.

(2) Next, we find the partition of $\mathcal{G}$. For the qubit and four-level system, the orders of $\vec{e}_1$ and $\vec{e}_2$ are four and the orders of $\vec{e}_3$, $\vec{e}_4$ are two, so that the partition of the group is $\bar{M} = M_1M_2M_3M_4 = 2\,2\,1\,1$.

(3) We find all the sets $\mathbb{S} = \{s_\alpha\}$ with $0 \leqslant s_\alpha \leqslant M_\alpha$, in this case there are 36 said sets.

(4) For each set $\mathbb{S}$, we define the basis $\mathbb{B} = \{p^{s_1}\vec{e}_1, p^{s_2}\vec{e}_2, p^{s_3}\vec{e}_3, p^{s_4}\vec{e}_4\}$.

(5) As before, some of the bases created this way are redundant, since they are T-isomorphic. To eliminate this redundancy, we first define $\tilde{M}_1, \dots, \tilde{M}_q$ given by the q different values of numbers in $\bar{M}$. In the example of a four-level system and a qubit, we have that $\tilde{M}_1 = 2$, $\tilde{M}_2 = 1$. Then, we define the sets $S_j = \{s_\alpha, \forall\, \alpha : M_\alpha = \tilde{M}_j\}$, which in this case are $S_1 = \{s_1, s_2\}$ and $S_2 = \{s_3, s_4\}$. Finally, bases are T-isomorphic if their corresponding sets S_j are equal. For example, the bases that come from the sets $\mathbb{S} = \{2, 1, 1, 0\}$ and $\mathbb{S}' = \{1, 2, 0, 1\}$ are T-isomorphic, since $S_1 = S_1' = \{2, 1\}$ and $S_2 = S_2' = \{1, 0\}$. Therefore, after eliminating redundant bases and keeping only one of each batch, we get the following 18 bases:

$$\mathbb{S} = \{0,0,0,0\} \to \mathcal{B} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4\}, \quad \mathbb{S} = \{0,0,0,1\} \to \mathcal{B} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}, \quad \mathbb{S} = \{0,0,1,1\} \to \mathcal{B} = \{\vec{e}_1, \vec{e}_2\},$$

$$\mathbb{S} = \{0,1,0,0\} \to \mathcal{B} = \{\vec{e}_1, 2\vec{e}_2, \vec{e}_3, \vec{e}_4\}, \quad \mathbb{S} = \{0,1,0,1\} \to \mathcal{B} = \{\vec{e}_1, 2\vec{e}_2, \vec{e}_3\}, \quad \mathbb{S} = \{0,1,1,1\} \to \mathcal{B} = \{\vec{e}_1, 2\vec{e}_2\},$$

$$\mathbb{S} = \{0,2,0,0\} \to \mathcal{B} = \{\vec{e}_1, \vec{e}_3, \vec{e}_4\}, \quad \mathbb{S} = \{0,2,0,1\} \to \mathcal{B} = \{\vec{e}_1, \vec{e}_4\}, \quad \mathbb{S} = \{0,2,1,1\} \to \mathcal{B} = \{\vec{e}_1\},$$

$$\mathbb{S} = \{1,1,0,0\} \to \mathcal{B} = \{2\vec{e}_1, 2\vec{e}_2, \vec{e}_3, \vec{e}_4\}, \quad \mathbb{S} = \{1,1,0,1\} \to \mathcal{B} = \{2\vec{e}_1, 2\vec{e}_2, \vec{e}_3\}, \quad \mathbb{S} = \{1,1,1,1\} \to \mathcal{B} = \{2\vec{e}_1, 2\vec{e}_2\},$$

$$\mathbb{S} = \{2,1,0,0\} \to \mathcal{B} = \{2\vec{e}_2, \vec{e}_3, \vec{e}_4\}, \quad \mathbb{S} = \{2,1,0,1\} \to \mathcal{B} = \{2\vec{e}_2, \vec{e}_3\}, \quad \mathbb{S} = \{2,1,1,1\} : \{2\vec{e}_2\},$$

$$\mathbb{S} = \{2,2,0,0\} \to \mathcal{B} = \{\vec{e}_3, \vec{e}_4\}, \quad \mathbb{S} = \{2,2,0,1\} \to \mathcal{B} = \{\vec{e}_3\}, \quad \mathbb{S} = \{2,2,0,0\} \to \mathcal{B} = \{\}.$$

As in the other cases, these bases form the representative subgroups of the group.

As before, the automorphisms are described by matrices $t_{\alpha\beta}$. For the special case of a qubit and four-level system, the matrices are of dimensions 4×4 (because there are four elements in the basis) and the conditions on the entries $t_{\alpha\beta}$ can be found using the cases described in Appendix C, which lead to the following:

(1) t_{11}: $M_1 = M_1$ so that t_{11} is a number modulo $p^{M_1} = 2^2 = 4$.

(2) t_{12}: $M_1 = M_2$ so that t_{12} is a number modulo $p^{M_1} = 2^2 = 4$.

(3) t_{13}: $M_1 > M_3$ so that $t_{13} = p^{M_1 - M_3}\tau_{13} = 2\tau_{13}$ with τ_{13} a number modulo $p^{M_3} = 2$. Therefore, the possible values are 0 and 2.

(4) The same can be done for the rest of the values, and we find that $t_{11}, t_{12}, t_{21}, t_{22} \in \{0, 1, 2, 3\}$; $t_{13}, t_{14}, t_{23}, t_{24} \in \{0, 2\}$ and $t_{31}, t_{32}, t_{33}, t_{34}, t_{41}, t_{42}, t_{43}, t_{44} \in \{0, 1\}$.

Then, running through all possible matrices with these entries and keeping only the invertible ones, we find 147 456 matrices.

As before, to find all subgroups, we apply these automorphisms to every representative subgroup and discard repetitions. This procedure gives us the 249 subgroups of $\mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, some of which are shown in Fig. 8.

Finally, we find the homomorphisms $\phi : \mathcal{G} \to \mathbb{Z}_{p^{n_1}}$. For the case of a qubit and a four-level system, we need the homomorphisms $\phi : \mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to \mathbb{Z}_4$. As before, we need to follow the procedure mentioned in Appendix D. In this case, $n = 2$ and therefore $\nu = 2$. The homomorphisms ϕ are characterized by the values in the basis $\phi_\alpha = \phi(\vec{e}_\alpha)$ which have to follow the conditions of Eq. (D1b), that lead to the following:

(1) ϕ_1: Since $\alpha = 1 < 2 = \nu$, we are in the second case of Eq. (D1b), thus ϕ_1 is a number modulo $p^n = 4$.

(2) ϕ_2: Since $\alpha = 2 = 2 = \nu$, we are in the first case of Eq. (D1b), thus $\phi_2 = p^{n-M_2}s_2 = s_2$ with s_2 a number modulo $p^{M_2} = 4$.

(3) ϕ_3 : Since $\alpha = 3 > 2$, $\phi_3 = p^{n-M_3}s_3 = 2s_3$ with s_3 a number modulo $p^{M_3} = 2$, so that $\phi_3 = 0, 2$.

(4) ϕ_4 : Equivalently to ϕ_3 we find that $\phi_4 = 0, 2$.

Therefore, the homomorphisms for a qubit and a four-level system are given by the four numbers ϕ_1, ϕ_2, ϕ_3, ϕ_4 with

$\phi_1, \phi_2 \in \{0, 1, 2, 3\}$ and $\phi_3, \phi_4 \in \{0, 2\}$ for a total of 64 possibilities.

5. Most general case

In the most general case we have N particles, each with arbitrary dimension d_i and so the group under consideration is $\mathcal{G} = \bigoplus_{i=1}^{N} \mathbb{Z}_{d_i} \oplus \mathbb{Z}_{d_i}$.

Then, in this direct sum, we can first separate each $\mathbb{Z}_{d_i}$ as a sum of cyclic groups of prime power orders, such as it was done in Sec. E 3 of this Appendix. Then, having written $\mathcal{G}$ as a direct sum of cyclic groups with prime power order, we collect together the cyclic groups of order that is a power of two, then cyclic groups of order power of three, five, seven, and so on for each prime.

After this, we can find the subgroups and homomorphisms of each of these collections as it was done in Sec. E 4. Finally, the subgroups of $\mathcal{G}$ can be found as Cartesian products of subgroups of different collections.

APPENDIX F: NUMBER OF SUBGROUPS PER TYPE $\overline{L}$

An expression for the number of different subgroups of type $\overline{L}$ is already known in the literature. To introduce this expression we first need to consider the Ferrers graph of $\overline{L}$, that is, L squares of which the first L_1 are in the first row, the next L_2 in the second, and so on. Then, the conjugate partition $\overline{L}'$ is defined as the Ferrers graph of $\overline{L}$ obtained by inverting rows and columns. Similarly, the partition $\overline{M}'$ is defined as the conjugate partition of $\overline{L}$. The number of subgroups $\mathcal{H}$ of type $\overline{L}$ of $\mathcal{G}_p$ is given by

$$\prod_{\alpha\geqslant 1} p^{M'_{\alpha+1}(L'_\alpha - M'_\alpha)} \begin{bmatrix} L'_\alpha - M'_{\alpha+1} \\ M'_\alpha - M'_{\alpha+1} \end{bmatrix}_p , \tag{F1}$$

where the symbol

$$\begin{bmatrix} n \\ m \end{bmatrix}_p = \prod_{s=1}^{m} \frac{p^{n-s+1} - 1}{p^{m-s+1} - 1} \tag{F2}$$

denotes the number of vector subspaces of dimension m in a vector space of dimension n over the field $\mathbb{Z}_p$. The proof is rather intricate, and we refer the reader to the relevant literature, such as Refs. [33,35]. However, the key fact is that the number of subgroups obtained by our algorithm can be compared with (F1) to check that all subgroups of a given partition $\overline{L}$ have been found.

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