Metrology of weak quantum perturbations

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We consider quantum systems with a Hamiltonian containing a weak perturbation, i.e., $\boldsymbol{H} = \boldsymbol{H_0} + \boldsymbol{\lambda} \cdot \tilde{\boldsymbol{H}}$, $\boldsymbol{\lambda} = \{\lambda_1, \lambda_2, \dots, \}$, $\tilde{\boldsymbol{H}} = \{H_1, H_2, \dots, \}$, $|\boldsymbol{\lambda}| \ll 1$, and address situations where $\tilde{\boldsymbol{H}}$ is known but the values of the couplings $\boldsymbol{\lambda}$ are unknown and should be determined by performing measurements on the system. We consider two scenarios: in the first one we assume that measurements are performed on a given stationary state of the system, e.g., the ground state, whereas in the second one an initial state is prepared and then measured after evolution. In both cases, we look for the optimal measurements to estimate the couplings and evaluate the ultimate limits to precision. In particular, we derive general results for one and two couplings and analyze in detail some specific qubit models. Our results indicate that dynamical estimation schemes may provide enhanced precision upon a suitable choice of the initial preparation and the interaction time.

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I. INTRODUCTION

It is often the case that relevant physical phenomena correspond to weak perturbations to a stable unperturbed situation. This happens in a wide range of disciplines, ranging from applied mathematics [1] to biology [2] to chemistry [3] and physics [4], where the perturbation can represent a weak electric field interacting with an atom, causing a shift in its energy levels (Stark effect), or similarly a weak magnetic field (Zeeman effect). The same occurs in the presence of a weak energy modification to the Hamiltonian (Fermi's golden rule), or a weak gravitational force acting on two other physical bodies (tidal forces). In these situations, the nature of the perturbations is usually known, whereas the strengths of the perturbations are the quantities of interest. The Hamiltonian of those systems may be generally written as

$$\boldsymbol{H} = \boldsymbol{H_0} + \boldsymbol{\lambda} \cdot \tilde{\boldsymbol{H}}, \tag{1}$$

where $\boldsymbol{H_0}$ and $\tilde{\boldsymbol{H}} = \{H_1, H_2, \dots, \}$ are known Hamiltonian operators and $\boldsymbol{\lambda} = \{\lambda_1, \lambda_2, \dots, \}$ with $|\boldsymbol{\lambda}| \ll 1$ is a vector of small unknown coupling parameters whose values are unknown and should be determined by performing measurements on the system. To achieve this goal, there are two paradigmatic approaches, which will be referred to as static and dynamical estimation schemes throughout the paper. In the first one, the system may be prepared in a given stationary state, usually the ground state, which is measured to gain information about the value of the parameters. In a dynamical scenario, the system is instead prepared in a certain state, left to evolve for a given interaction time, and finally measured. In a dynamical estimation scheme, the initial state, as well as interaction time, may be optimized and thus the overall precision may be enhanced compared to a static scheme, though the practical implementation may be more challenging. In addition, the case of small perturbations to a given Hamiltonian $\boldsymbol{H_0}$, the Hamiltonian in Eq. (1) may also describe systems where the couplings have some target values $\bar{\boldsymbol{\lambda}}_0$ and the scope of the measurement is to monitor the system [5–7], i.e., to estimate possible deviations $\boldsymbol{\lambda} = \boldsymbol{\lambda}_0 - \bar{\boldsymbol{\lambda}}_0$ from those values.

A convenient framework to investigate the precision achievable by static and dynamical estimation schemes is that of quantum estimation theory [8–13], which provides a set of tools to determine the measurement that has to be performed on the system, i.e., to find the observable that is most sensitive to tiny variations of the parameters [14–25], and to optimize the initial preparation of the probe [26–35].

In particular, if the value of a single parameter is encoded in the family of quantum states $\{|\psi_\lambda\rangle\}$ (usually referred to as the quantum statistical model), one may prove that the ultimate precision achievable in estimating λ is obtained by measuring the observable L_λ, known as symmetric logarithmic derivative (SLD), which is the self-adjoint operator given by

$$L_\lambda = 2[|\partial_\lambda \psi_\lambda\rangle\langle\psi_\lambda| + |\psi_\lambda\rangle\langle\partial_\lambda \psi_\lambda|]. \tag{2}$$

Upon collecting the result of M repeated measurements on identical preparations of the system and suitably processing data (e.g., by MAXLIK [36–39] or Bayesian analysis [40,41]) the uncertainty in the determination of λ, i.e., the precision of the estimation scheme, is given by

$$\mathrm{Var}\lambda \simeq \frac{1}{M\,Q(\lambda)}, \tag{3}$$

where $Q(\lambda)$ is the so-called quantum Fisher information of the quantum statistical model $\{|\psi_\lambda\rangle\}$, i.e.,

$$Q_\lambda = 4[\langle\partial_\lambda\psi_\lambda||\partial_\lambda\psi_\lambda\rangle - |\langle\partial_\lambda\psi_\lambda||\psi_\lambda\rangle|^2] \tag{4}$$

(notice that $\langle\psi_\lambda||\partial_\lambda\psi_\lambda\rangle$ is a purely imaginary c-number, i.e., $\langle\psi_\lambda||\partial_\lambda\psi_\lambda\rangle^* = \langle\partial_\lambda\psi_\lambda||\psi_\lambda\rangle = -\langle\psi_\lambda||\partial_\lambda\psi_\lambda\rangle$).

The generalization to the estimation of more than one parameter can be obtained by introducing the so-called quantum Fisher information matrix (QFIM) $\boldsymbol{Q}$, which is a real symmetric $n \times n$ matrix with entries

$$Q_{\mu\nu} = 4[\mathrm{Re}\langle\partial_\mu\psi||\partial_\nu\psi\rangle + \langle\partial_\mu\psi||\psi\rangle\langle\partial_\nu\psi||\psi\rangle]. \tag{5}$$

The QFIM provides a bound on the covariance matrix (CM) of the estimates

$$\mathrm{Cov}(\lambda) \geqslant \frac{1}{M}\boldsymbol{Q}^{-1}. \tag{6}$$

This is a matrix inequality, and in general, it cannot be saturated. Physically, this corresponds to the unavoidable quantum noise that originate when the SLDs corresponding to different parameters do not commute [42]. In those cases, the total variance $\sum_\mu V(\lambda_\mu)$ (or a weighted combination of the CM elements) is a more interesting quantity to study, and since the μth diagonal entry of the covariance matrix is just the variance of the parameter λ_μ, the bound on the total variance is given as

$$\sum_\mu V(\lambda_\mu) \geqslant \frac{B}{M}, \quad B = \mathrm{Tr}\big[\boldsymbol{Q}^{-1}\big]. \tag{7}$$

The incompatibility between the parameters can be quantified by the so-called asymptotic incompatibility [43–46], also referred to as the quantumness of the quantum statistical model. This is defined as

$$R := \|i\,\boldsymbol{Q}^{-1}\boldsymbol{D}\|_\infty, \tag{8}$$

where $\|\boldsymbol{A}\|_\infty$ is the largest eigenvalue of the matrix $\boldsymbol{A}$ and

$$D_{\mu\nu} = -\frac{i}{2}\,\langle\psi_{\boldsymbol{\lambda}}|[L_\mu, L_\nu]|\psi_{\boldsymbol{\lambda}}\rangle = 4\mathrm{Im}\langle\partial_\mu\psi||\partial_\nu\psi\rangle \tag{9}$$

is the Uhlmann curvature of the statistical model. The quantity R is a real number in the range $0 \leqslant R \leqslant 1$ with the equality $R = 0$ satisfied for compatible parameters, i.e., when $\langle\psi_{\boldsymbol{\lambda}}|[L_\mu, L_\nu]|\psi_{\boldsymbol{\lambda}}\rangle = 0$. For just two parameters, one may write

$$R = \sqrt{\frac{\det\boldsymbol{D}}{\det\boldsymbol{Q}}}. \tag{10}$$

A tighter scalar bound, known as the Holevo-Cramer-Rao bound $\sum_\mu V(\lambda_\mu) \geqslant C_H/M$, with $C_H \geqslant B$, may also be derived (see [47] for details), and the quantumness R provides a bound to the normalized difference between C_H and B as follows:

$$\frac{C_H - B}{B} \leqslant R. \tag{11}$$

In the following sections, we aim at finding general formulas of Q for estimation problems involving the parameters of weakly perturbed systems in both the static and the dynamical estimation scenarios. In the case of two parameters both B and R will be investigated, and then we will analyze some specific models involving qubits, qutrits and harmonic oscillators, and where the Holevo-Cramer-Rao bound is known analytically, we check whether the inequality in Eq. (11) is tight. More precisely, Sec. II is devoted to static estimation schemes, with Sec. II A reporting general results and Secs. II B, II C, and II D devoted to specific models involving qubit, qutrit, and oscillatory systems, respectively. Section III is devoted to dynamical estimation schemes, with Sec. III A reporting general results and Secs. III B, III C, and III D discussing specific results for qubit, qutrit, and oscillatory systems, also comparing the performance of dynamical schemes to that of the corresponding static ones. Section IV closes the paper with some concluding remarks.

II. STATIC ESTIMATION OF WEAK PERTURBATIONS

In this section, we address estimation of weak perturbations in systems descibed by one- and two-parameter (time-independent) Hamiltonians of the form $H = H_0 + \lambda H_1$ and $H = H_0 + \lambda_1 H_1 + \lambda_2 H_2$. In particular, we assume that the system may be prepared in a given state (e.g., the ground state) and that repeated measurements may be performed on the system. We derive general expressions for the QFI Q and the quantumness R and discuss specific models involving qubit, qutrit, and oscillator systems.

A. General results for one and two parameters

Let us consider a system with Hamiltonian $H = H_0 + \lambda H_1$ where $\lambda \ll 1$. The nth eigenstate $|\psi_n\rangle$ of H may be obtained perturbatively to first-order in λ as follows:

$$|\psi_n\rangle = \big|\psi_n^0\big\rangle + \lambda\big|\psi_n^1\big\rangle + O(\lambda^2), \tag{12}$$

where $|\psi_n^0\rangle$ are eigenstates of H_0 and

$$\big|\psi_n^1\big\rangle = \sum_{m\neq n}\frac{\langle\psi_m^0|H_1|\psi_n^0\rangle}{E_n^0 - E_m^0}\big|\psi_m^0\big\rangle$$

is the first-order correction to the nth eigenstate. In general, $\langle\psi^1|\psi^1\rangle = N \neq 1$, and it is thus convenient to introduce the state $|\psi^1\rangle = \sqrt{N}|\phi^1\rangle$ and write the first-order corrected eigenstate $|\psi\rangle$ as a combination of two orthonormal states $|\psi^0\rangle$ and $|\phi^1\rangle$. The subscript n is omitted to simplify notation. The perturbed state and its derivative are thus given by

$$|\psi\rangle = |\psi^0\rangle + \lambda\sqrt{N}|\phi^1\rangle, \tag{13a}$$

$$|\partial_\lambda\psi\rangle = \sqrt{N}|\phi^1\rangle. \tag{13b}$$

According to Eq. (2), the SLD of this general model may be written, up to first order in λ as

$$L_\lambda = 2\sqrt{N}[|\psi^0\rangle\langle\phi^1| + |\phi^1\rangle\langle\psi^0| + 2\lambda\sqrt{N}|\phi^1\rangle\langle\phi^1|], \tag{14}$$

and the corresponding QFI as

$$Q(\lambda) = 4N + O(\lambda^2). \tag{15}$$

The QFI is independent on the perturbation (up to second order) and proportional to the norm of the first-order correction $|\psi^1\rangle$. This is a remarkably intuitive results, linking the estimability of a perturbation to its physical effect on the system. The same result may be also obtained expressing the QFI in terms of fidelity [48,49]. Notice also that the λ-dependent term in the SLD leads to negligible (second-order) contributions to the QFI and may be dropped. The optimal measurement is thus given by

$$L = 2\sqrt{N}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{16}$$

This expression makes it clear that the optimal measurements set coincides with the Pauli matrix σ_x over the basis $\{|\psi^0\rangle, |\phi^1\rangle\}$, i.e., a detection scheme that senses the coherence of the perturbed state in that basis.

Let us now address the case of systems with Hamiltonian of the form $H = H_0 + \lambda_1 H_1 + \lambda_2 H_2$ where H_1 and H_2 are in general noncommuting operators, $[H_1, H_2] \neq 0$. In this case the pertubations depend in a nontrivial way on two different parameters λ_1 and λ_2, which should be jointly estimated. For weak perturbations, the the nth eigenstate of H $|\psi_n\rangle$ may be written, in terms of the eigenbasis of H_0, as follows:

$$\begin{aligned} |\psi_n\rangle &= \big|\psi_n^0\big\rangle + \lambda_1 \sum_{m\neq n} \frac{\big\langle\psi_m^0\big|H_1\big|\psi_n^0\big\rangle}{E_n^0 - E_m^0}\big|\psi_m^0\big\rangle \\ &\quad + \lambda_2 \sum_{l\neq n} \frac{\big\langle\psi_l^0\big|H_2\big|\psi_n^0\big\rangle}{E_n^0 - E_l^0}\big|\psi_l^0\big\rangle \\ &= \big|\psi_n^0\big\rangle + \lambda_1\sqrt{N_1}\big|\phi_{n,1}^1\big\rangle + \lambda_2\sqrt{N_2}\big|\phi_{n,2}^1\big\rangle, \end{aligned} \tag{17}$$

where $|\phi_{n,\mu}^1\rangle$ are states, i.e., the normalized version of the first-order corrections $|\psi_{n,\mu}^1\rangle = \sqrt{N_\mu}|\phi_{n,\mu}^1\rangle$ having squared norms N_j (with $\mu = 1, 2$ the index of the parameter λ_μ). As we have done before, we drop the index n to simplify the notation. These states are not orthogonal one to each other but both are orthogonal to the unperturbed eigenspace of H_0, hence, we can express the perturbed state $|\psi\rangle$ in an orthonormal basis spanned by the triplet $\{|\psi^0\rangle, |j\rangle, |k\rangle\}$, with $\langle j|k\rangle = \delta_{jk}$. Upon writing the states $|\phi^l\rangle\}$ as

$$\begin{aligned} |\phi^1\rangle &= \cos\frac{\theta_1}{2}|j\rangle + \sin\frac{\theta_1}{2}|k\rangle, \\ |\phi^2\rangle &= e^{i\gamma}\cos\frac{\theta_2}{2}|j\rangle + e^{i(\gamma+\varphi)}\sin\frac{\theta_2}{2}|k\rangle, \end{aligned}$$

the perturbed state and its derivatives $|\partial_{\lambda_\mu}\psi\rangle = |\psi_\mu^1\rangle$ may be written as

$$\begin{aligned} |\psi\rangle &= |\psi^0\rangle + \left(\lambda_1\sqrt{N_1}\cos\frac{\theta_1}{2} + \lambda_2\sqrt{N_2}\mathrm{e}^{\mathrm{i}\gamma}\cos\frac{\theta_2}{2}\right)|j\rangle \\ &\quad + \left(\lambda_1\sqrt{N_1}\sin\frac{\theta_1}{2} + \lambda_2\sqrt{N_2}\mathrm{e}^{\mathrm{i}(\gamma+\varphi)}\sin\frac{\theta_2}{2}\right)|k\rangle, \end{aligned} \tag{18}$$

$$|\partial_{\lambda_1}\psi\rangle = \sqrt{N_1}\left(\cos\frac{\theta_1}{2}|j\rangle + \sin\frac{\theta_1}{2}|k\rangle\right), \tag{19}$$

$$|\partial_{\lambda_2}\psi\rangle = \sqrt{N_2}\,\mathrm{e}^{\mathrm{i}\gamma}\left(\cos\frac{\theta_2}{2}|j\rangle + \mathrm{e}^{\mathrm{i}\varphi}\sin\frac{\theta_2}{2}|k\rangle\right). \tag{20}$$

To quantify the orthogonality between the two perturbations, we consider the overlap $\omega = \langle\phi_1^1||\phi_2^1\rangle$ between the two first-order corrections, i.e.,

$$\omega = \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\mathrm{e}^{\mathrm{i}\gamma} + \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\mathrm{e}^{\mathrm{i}(\gamma+\varphi)}. \tag{21}$$

The SLD operators L_1 and L_2 for the two parameters λ_1 and λ_2 may be calculated according to Eq. (2). The explicit expressions are reported in Appendix A. The corresponding QFIM $\boldsymbol{Q}$ and Uhlmann curvature $\boldsymbol{D}$ are given by

$$\boldsymbol{Q} = \begin{pmatrix} 4N_1 & 4\sqrt{N_1N_2}\,\mathrm{Re}\omega \\ 4\sqrt{N_1N_2}\,\mathrm{Re}\omega & 4N_2 \end{pmatrix}, \tag{22}$$

$$\boldsymbol{D} = \begin{pmatrix} 0 & 4\sqrt{N_1N_2}\,\mathrm{Im}\omega \\ -4\sqrt{N_1N_2}\,\mathrm{Im}\omega & 0 \end{pmatrix}. \tag{23}$$

The ultimate bound B and the quantumness R thus read as follows:

$$B = \frac{N_1 + N_2}{4N_1N_2[1 - \mathrm{Re}^2\omega]}, \tag{24}$$

$$R = \sqrt{\frac{\mathrm{Im}^2\omega}{1 - \mathrm{Re}^2\omega}}. \tag{25}$$

As expected, the overlap between the perturbations is involved in all the quantities of interest. In particular, a real overlap ($\mathrm{Im}\omega = 0$) always provides maximum compatibility ($R = 0$) between the parameters to estimate. Moreover, if the overlap is zero (both $\mathrm{Re}\omega = 0$ and $\mathrm{Im}\omega = 0$), i.e., perturbations are orthogonal, the QFI matrix is diagonal, meaning that parameters are uncorrelated. On the other hand, if the overlap is a just a phase factor, we have $\mathrm{Re}^2\omega + \mathrm{Im}^2\omega = 1$, and thus $R = 1$, i.e., maximal incompatibility between the parameters. This may happen also when the dimension of the probing system is insufficient to estimate a certain number of parameters, as it will be illustrated in the next section by means of a qubit statistical model.

B. Qubit models

Let us consider a qubit system described by the orthonormal basis states $\{|0\rangle, |1\rangle\}$ of the unperturbed Hamiltonian $H_0 = \sigma_z$ with eigenenergies $E_0 = 1$ and $E_1 = -1$. The perturbed Hamiltonian is given by $H = \sigma_z + \lambda\sigma_x$, where σ_z and σ_x are standard Pauli matrices and λ is the small perturbation parameter that we want to estimate. The first-order perturbed ground state is given by

$$|\psi\rangle = |0\rangle + \frac{\lambda}{2}|1\rangle, \tag{26}$$

and the first-order corrected state is $|\psi^1\rangle = \frac{1}{2}|1\rangle$ with (squared) norm $N = 1/4$. The corresponding SLD is $L_\lambda = \sigma_x$ and the QFI is given by

$$Q = 1 + O(\lambda^2), \tag{27}$$

confirming the general results in Eqs. (15) and (2).

Let us now consider the more interesting case of a two-parameter perturbation, which highlights the issues arising from using an under-dimensioned (compared to the number of parameters) probe system. The perturbed Hamiltonian is

$$H = \sigma_z + \lambda_1\sigma_x + \lambda_2(\cos\alpha\,\sigma_x + \sin\alpha\,\sigma_y),$$

where λ_i (with $i = 1, 2$) are the perturbation parameters and α denotes a mixing angle which governs the orthogonality of the two perturbations. The first-order perturbed ground state of the system is given by

$$|\psi\rangle = |0\rangle + \tfrac{1}{2}(\lambda_1 + \lambda_2\,\mathrm{e}^{\mathrm{i}\alpha})|1\rangle. \tag{28}$$

Looking at the above equation, it is clear that the two perturbations cannot, in general, generate two orthogonal states where information about the two parameters is encoded [44]. In fact, the first-order corrected states corresponding to λ_1 and λ_2 are the same state except for a phase factor. In other words, the two perturbations leads to two degenerate states proportional to $|1\rangle$. Referring to the Bloch sphere representation introduced above, we have $\theta_1 = \theta_2 = 0$ and $\gamma = \alpha$. The overlap in Eq. (21) is given by $\omega = \mathrm{e}^{\mathrm{i}\alpha}$ and the QFIM displays off-diagonal elements. To make the two parameter compatible, a probe system with larger dimension should be necessarily employed (see also the next section).

C. Qutrit models

Let us consider a three-dimensional spin-1 system with a perturbed Hamiltonian given by $H = S_z + \lambda_1 S_x + \lambda_2(\cos\alpha\,S_x + \sin\alpha\,S_y)$, where $\{S_z, S_x, S_y\}$ denotes the irreducible representation of spin-1 operators in the z basis

$$S_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad S_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$
$$S_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \tag{29}$$

with $|m_s\rangle$, $m_1 = \{1, 0, -1\}$ being the standard eigenvectors and eigenvalues of S_z. This Hamiltonian is the direct generalization of that considered in the previous section, and a comparison will reveal the role of system dimension.

For the eigenstate $|\psi^0\rangle = |1, 0\rangle$, the first-order corrections are given by

$$|\psi_1^1\rangle = \frac{|1,-1\rangle - |1,1\rangle}{\sqrt{2}} = |\phi_1^1\rangle, \tag{30a}$$

$$|\psi_2^1\rangle = \frac{\mathrm{e}^{\mathrm{i}\alpha}|1,-1\rangle - \mathrm{e}^{-\mathrm{i}\alpha}|1,1\rangle}{\sqrt{2}} = |\phi_2^1\rangle, \tag{30b}$$

with squared norms given by $N_1 = N_2 = 1$. It is easy to see that these perturbation states live in two-level subsystem spanned by $|j\rangle = |1, 1\rangle$ and $|k\rangle = |1, -1\rangle$, and that they may be expressed as in Eq. (18) by setting $\theta_1 = \theta_2 = 3\pi/2$, $\gamma = -\alpha$, and $\varphi = 2\alpha$. The resulting overlap is real and given by $\omega = \cos\alpha$. In this case, the resulting QFIM $\boldsymbol{Q}$ and the ultimate bound B are

$$\boldsymbol{Q} = 4\begin{pmatrix} 1 & \cos\alpha \\ \cos\alpha & 1 \end{pmatrix}, \quad B = \frac{\csc^2\alpha}{2}, \tag{31a}$$

whereas the mean Uhlmann curvature is vanishing and the quantumness is zero $R = 0$. Moreover, the two perturbed states become orthogonal for $\alpha = \pi/2$, which corresponds to apply nonoverlapping perturbations. The QFIM becomes diagonal, meaning that the two parameters to be estimated are uncorrelated and the ultimate bound $B = 1/2$ is minimal and coincides with the Holevo bound C_H. Notice that perturbing a different eigenvector, say $|1, 1\rangle$, the situation is dramatically different since the two perturbations S_x and S_y generate the same first-order perturbed state $|1, 0\rangle$ and the resulting overlap is $\omega = \mathrm{e}^{\mathrm{i}\alpha}$.

Summarizing, a two-parameter perturbation cannot be suitably characterized (with maximum precision and compatibility) using a qubit system, whereas the use of a qutrit system allows one to achieve the ultimate limits to precision via a proper choice of the encoding Hamiltonian terms and of the initial unperturbed state.

D. Quantum anharmonic oscillator model

An interesting example of models which may be treated in our formalism that of a quantum oscillator weakly perturbed by anharmonic terms whose amplitudes are to be determined, e.g., because it may represent a resource [50]. The Hilbert space of the system is infinite dimensional and may offer an ideal playground to encode as much information as needed.

For the sake of simplicity we choose natural units ($\hbar = 1$) and set the frequency and the mass of the oscillator to one $m = \omega = 1$. We consider anharmonic perturbations to the harmonic potential such that the perturbed Hamiltonian reads

$$H = \tfrac{1}{2}(p^2 + x^2) + \epsilon_1 x^3 + \epsilon_2 x^4, \tag{32}$$

where we introduce the two anhamonicity parameters ϵ_1 and ϵ_2 as the unknown parameters to be estimated. Recalling that the number states $|n\rangle$ are eigenstates of the number operator N, i.e., $N|n\rangle = a^\dagger a|n\rangle = n|n\rangle$, with the ground state of the harmonic oscillator $|0\rangle$, satisfying $a|0\rangle = 0$, and the generic number state satisfying $|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle$. The perturbed ground state of the system may be obtained as in Eq. (17), where the two first-order corrections are given by

$$|\psi_1^1\rangle = -\frac{1}{2}\left(\frac{3}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{3}}|3\rangle\right), \tag{33a}$$

$$|\psi_2^1\rangle = -\frac{1}{2}\left(\frac{3}{\sqrt{2}}|2\rangle + \frac{1}{2}\sqrt{\frac{3}{2}}|4\rangle\right), \tag{33b}$$

with squared norms $N_1 = \frac{29}{24}$ and $N_2 = \frac{39}{32}$. The two perturbed states (33) are orthogonal and the same happens for any eigenstate $|n\rangle$ of the perturbed Hamiltonian. The QFIM (22) reads

$$\boldsymbol{Q} = 4\begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix}, \tag{34}$$

leading to $B = 466/1131 \simeq 0.41$. The Uhlmann curvature (23) vanishes, corresponding to a zero quantumness (25). By using a higher-order perturbation term in the Hamiltonian, say $H_2 = x^5$, perturbed states are no longer orthogonal, resulting in a real nonzero overlap ω (21). This leads to a QFIM with nonzero off-diagonal terms, meaning that the two parameters are no longer independent, but still compatible since the $\boldsymbol{D}$ matrix is null and the quantumness parameter

$R = 0$. In conclusion, preparing a quantum oscillator in its vacuum state is an efficient strategy to precisely sense the amplitude of anharmonic perturbations.

III. SENSING PERTURBATIONS BY DYNAMICAL PROBES

In this section, we address detection of weak perturbations by performing measurements on an evolved state $|\psi(t)\rangle = e^{-iHt}|\psi^0\rangle$ where $|\psi^0\rangle$ is a (nonstationary) given initial state and H is the perturbed Hamiltonian under investigation.

A. General results for one and two parameters

Let us start with a perturbation described by a single-parameter Hamiltonian. To obtain the QFI it is convenient to move into the interaction picture (with respect to the unperturbed Hamiltonian H_0), where the state vector is given by the unitary transformation $|\psi_I(t)\rangle = U_0^\dagger(t)|\psi(t)\rangle$, being $U_0(t) = e^{-iH_0t}$. The whole time evolution is expressed by

$$|\psi_I(t)\rangle = U_I(t)|\psi^0\rangle, \tag{35a}$$

$$U_I(t) = \mathcal{T}[\exp -i\lambda K(t)], \tag{35b}$$

$$K(t) = \int_0^t ds\, U_0^\dagger(s) H_1 U_0(s), \tag{35c}$$

where $\mathcal{T}[\ldots]$ denotes time-ordering and the operator $K(t) = K^\dagger(t)$ is hermitian. Up to first order in λ we have

$$U_I(t) \simeq \mathbb{I} - i\lambda K(t), \tag{36}$$

Going back to the Schrödinger picture the evolved state and its derivative with respect to the unknown parameter may be written as

$$|\psi_\lambda(t)\rangle = U_0(t)[\mathbb{I} - i\lambda K(t)]|\psi^0\rangle, \tag{37a}$$

$$|\partial_\lambda \psi_\lambda(t)\rangle = -iU_0(t)K(t)|\psi^0\rangle. \tag{37b}$$

The leading-order behavior corresponds to a λ-independent (zeroth-order) expression of the QFI

$$Q(t) = 4[\langle\psi^0|K^2(t)|\psi^0\rangle - \langle\psi^0|K(t)|\psi^0\rangle^2]. \tag{38}$$

Despite the fact that it may appear as a rough approximation, this expression of the QFI allows us to grab the main features of the dynamical case and to compare results with those obtained in the static one. The QFI in Eq. (38) depends on time and is independent of λ. In other words, the evolution introduces a time dependence, whereas it does not affect the covariant nature of the estimation problem.

Analogously, in the case of a two-parameter Hamiltonian $H = H_0 + \lambda_1 H_1 + \lambda_2 H_2$, the time evolution operator in the interaction picture can be approximated at first order as $U_I(t) \simeq \mathbb{I} - i\int_0^t ds\, U_0^\dagger(s)(\lambda_1 H_1 + \lambda_2 H_2)U_0$. Upon introducing the operators

$$K_1(t) = \int_0^t ds\, U_0^\dagger(s) H_1 U_0(s), \tag{39a}$$

$$K_2(t) = \int_0^t ds\, U_0^\dagger(s) H_2 U_0(s), \tag{39b}$$

the leading order of the elements of the QFI matrix may be evaluated as follows:

$$\begin{aligned}
Q_{11} &= 4[\langle\psi^0|K_1^2|\psi^0\rangle - \langle\psi^0|K_1|\psi^0\rangle^2],\\
Q_{12} &= 4[\mathrm{Re}(\langle\psi^0|K_1K_2|\psi^0\rangle) - \langle\psi^0|K_1|\psi^0\rangle\langle\psi^0|K_2|\psi^0\rangle],\\
Q_{21} &= 4[\mathrm{Re}(\langle\psi^0|K_2K_1|\psi^0\rangle) - \langle\psi^0|K_2|\psi^0\rangle\langle\psi^0|K_1|\psi^0\rangle],\\
Q_{22} &= 4[\langle\psi^0|K_2^2|\psi^0\rangle - \langle\psi^0|K_2|\psi^0\rangle^2],
\end{aligned} \tag{40}$$

where we omit the time dependence. The matrix elements of the Uhlmann curvature are given by

$$D_{12} = 4\mathrm{Im}(\langle\psi^0|K_1K_2|\psi^0\rangle) = -D_{21}, \tag{41}$$

and the quantumness parameter R reads as follows:

$$R = \frac{4|\mathrm{Im}\langle\psi^0|K_1K_2|\psi^0\rangle|}{\sqrt{\det \boldsymbol{Q}}}. \tag{42}$$

Now that the general framework has been set, in the following we reexamine some of the examples of the previous sections to compare the performance of static and dynamical estimation schemes.

B. Qubit models

Let us consider a single qubit, initially prepared in the generic state $|\psi^0\rangle = \cos(\frac{\theta}{2})|0\rangle + e^{i\phi}\sin(\frac{\theta}{2})|1\rangle$. We first consider a single-parameter perturbation. The system evolves according to the unitary $\mathrm{U} = \exp(-itH)$ where t is the time parameter and $H = \sigma_z + \lambda\sigma_x$ is the perturbed Hamiltonian with λ small. Using Eqs. (35c) and (38) we have

$$K(t) = e^{it}\sin t|0\rangle\langle 1| + e^{-it}\sin t|1\rangle\langle 0|, \tag{43a}$$

$$Q(t) = 4\sin^2 t[1 - \cos^2(t+\phi)\sin^2(\theta)]. \tag{43b}$$

To compare this result with the QFI obtained in the static case, we set $|0\rangle$ as the initial (unperturbed) state at $t = 0$, i.e., $\theta = 0$. The dynamical QFI is given by $Q(t) = 4\sin^2 t$ and achieves a maximum at $t = \pi/2$, where it four times greater than the corresponding static QFI.

C. Qutrit models

We consider the same spin-1 system as in Sec. II C and the same Hamiltonian. To compare results with the static scenario, we set the initial state to $|\psi^0\rangle = |1, 0\rangle$, the QFIM and bound will read

$$\boldsymbol{Q} = 16\sin^2\frac{t}{2}\begin{pmatrix} 1 & \cos\alpha \\ \cos\alpha & 1 \end{pmatrix}, \quad \boldsymbol{D} = \boldsymbol{0}, \tag{44}$$

$$\boldsymbol{B} = \left(8\sin^2 t/2\sin^2\alpha\right)^{-1}, \quad \boldsymbol{R} = 0. \tag{45}$$

The D matrix and the R parameter vanish, i.e., we have compatibility between the two parameters. The QFI is maximal (B is minimal) for orthogonal perturbations $\alpha = \pi/2$ and for $t = \pi$ the QFIM is diagonal and maximal. As it happens with qubits, in the dynamical scenario the bound is improved by a factor of 4.

D. Anharmonic oscillator

We consider the same system of Sec. II D, prepare the oscillator in the unperturbed ground state and let it evolve

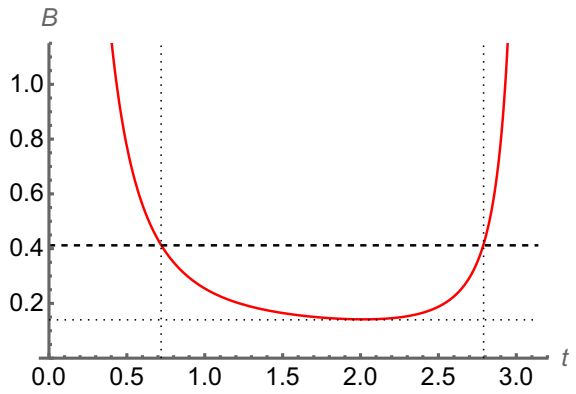

FIG. 1. B bound on the total variance for the joint estimation of the anharmonicity parameters as a function of the interaction time. The red solid line is the dynamical bound and the dashed black line denotes the static one.

according to the perturbed Hamiltonian. Using Eqs. (39a) and (39b) we evaluate the QFIM, which is a diagonal matrix with entries (see Appendix B for details)

$$Q_{11} = \tfrac{29}{3} - 9\cos t - \tfrac{2}{3}\cos 3t, \tag{46}$$

$$Q_{22} = 3\,(7+\cos 2t)\,\sin^2 t, \tag{47}$$

$$Q_{12} = Q_{21} = 0, \tag{48}$$

whereas the quantumness R vanishes.

In Fig. 1 we show the bound B as a function of time (B is a periodic function) compared to the static bound. As is apparent from the plot, the dynamical scheme beats the static one in the range $t \in (0.721, 2.79)$. The absolute minimum is obtained for $t \simeq 2.0$, where we have $B \simeq 0.14$, clearly lower than the corresponding static value. We conclude that preparing the oscillator in the unperturbed ground state and performing measurements after a moderate time evolution is an effective way to reveal the presence of anharmonic perturbations and to estimate their amplitudes.

IV. CONCLUSION

In this paper, we addressed the estimation of weak quantum perturbations analyzing two estimation scenarios: a static one, where the parameters are inferred by performing measurements on a stationary state, and a dynamical one, where the system is prepared in a suitably optimized initial state and measurements are performed after a given interaction time, which itself may be optimized to enhance precision.

We found general formulas for the relevant quantities to assess precision (i.e., the SLD, the QFIM, the scalar bound B on the total variance, and the quantumness R) up to the leading order in the perturbation parameters, and analyzed in some details few quantum statistical models involving qubit, qutrit, and oscillatory systems.

Our results indicate that dynamical estimation schemes generally improve precision, although only for specific preparations of the system and values of the interaction time. Ultimately, the choice between one scheme and the other does depend on the specific features of the involved system, on the experimental difficulties related to the preparation of the initial state, and on the modulation of the interaction time. Our results provide solid tools to compare the two approaches in a generic situations.

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APPENDIX A: EXPLICIT EXPRESSIONS OF THE SLDs FOR A TWO-PARAMETER PERTURBATIONS

Starting from the perturbed state in Eq. (18) and its derivatives in Eqs. (19) and (20), the matrix elements $\alpha_{jk} = [L_1]_{jk}$ of the SLD operator relative to the parameter λ_1 read

$$\alpha_{11} = 0, \tag{A1}$$

$$\alpha_{22} = 4\big(\lambda_1 N_1 c_1^2 + \lambda_2\sqrt{N_1N_2}c_1c_2\cos\gamma\big), \tag{A2}$$

$$\alpha_{33} = 4\big(\lambda_1 N_1 s_1^2 + \lambda_2\sqrt{N_1N_2}s_1s_2\cos(\gamma+\varphi)\big), \tag{A3}$$

$$\alpha_{12} = \alpha_{21} = 2\sqrt{N_1}c_1, \tag{A4}$$

$$\alpha_{13} = \alpha_{31} = 2\sqrt{N_1}s_1, \tag{A5}$$

$$\alpha_{23} = \alpha_{32}^* = 4\lambda_1 N_1 c_1 s_1 + 2\lambda_2\sqrt{N_1N_2}\big(c_1s_2\mathrm{e}^{-\mathrm{i}(\gamma+\varphi)} + c_2s_1\mathrm{e}^{\mathrm{i}\gamma}\big), \tag{A6}$$

whereas $\beta_{jk} = [L_2]_{jk}$, i.e., those of the SLD operator relative to the parameter λ_2 are given by

$$\beta_{11} = 0, \tag{A7}$$

$$\beta_{22} = 4\big(\lambda_2 N_2 c_2^2 + \lambda_1\sqrt{N_1N_2}c_1c_2\cos\gamma\big), \tag{A8}$$

$$\beta_{33} = 4\big(\lambda_2 N_2 s_2^2 + \lambda_1\sqrt{N_1N_2}s_1s_2\cos(\gamma+\varphi)\big), \tag{A9}$$

$$\beta_{12} = \beta_{21} = 2\sqrt{N_2}c_2\mathrm{e}^{-\mathrm{i}\gamma}, \tag{A10}$$

$$\beta_{13} = \beta_{31}^* = 2\sqrt{N_2}s_2\mathrm{e}^{-\mathrm{i}(\gamma+\varphi)}, \tag{A11}$$

$$\beta_{23} = \beta_{32}^* = 4\lambda_2 N_2 c_2 s_2\mathrm{e}^{-\mathrm{i}\varphi} + 2\lambda_1\sqrt{N_1N_2}\big(c_1s_2\mathrm{e}^{-\mathrm{i}(\gamma+\varphi)} + c_2s_1\mathrm{e}^{\mathrm{i}\gamma}\big), \tag{A12}$$

where N_j is the squared norm of the perturbation vector $|\psi_{n,j}^1\rangle$, $c_j = \cos\frac{\theta_j}{2}$, and $s_j = \sin\frac{\theta_j}{2}$, with $j = 1, 2$.

APPENDIX B: K_1 AND K_2 FOR THE ANHARMONIC OSCILLATOR

In this Appendix, we present the explicit expressions of K_1 and K_2 in Eqs. (39a) and (39b) and their use in evaluating the elements of the QFIM. The calculations are tedious but straightforward, upon writing the nonlinear Hamiltonians in

normal order as follows [51–53]:

$$x^n = \frac{1}{2^{n/2}}(a+a^\dagger)^n = \frac{n!}{2^{n/2}}\sum_{k=0}^{[n/2]}\sum_{l=0}^{n-2k}\frac{a^{\dagger l}a^{n-2k-l}}{2^k\,k!\,l!\,(n-2k-l)!}, \tag{B1}$$

where $[n]$ denotes the integer part of n. We also use the fact that for a generic function $f(a,a^\dagger)$ of the bosonic operators one has

$$e^{iya^\dagger a}f(a,a^\dagger)e^{-iya^\dagger a} = f(ae^{-iy},a^\dagger e^{iy}). \tag{B2}$$

We thus have

$$K_1 = \sum_{k=0}^{1}\sum_{l=0}^{3-2k}\int_0^t dy\, e^{-iy(3-2k-2l)} \times \frac{3!}{2^{3/2}}\frac{a^{\dagger l}a^{3-2k-l}}{2^k\,k!\,l!\,(3-2k-l)!}, \tag{B3}$$

$$= \sum_{k=0}^{1}\sum_{l=0}^{3-2k} e^{-i\frac{t}{2}(2k+2l-3)}\frac{\sin\left[\frac{t}{2}(2k+2l-3)\right]}{\frac{1}{2}(2k+2l-3)} \times \frac{3!}{2^{3/2}}\frac{a^{\dagger l}a^{3-2k-l}}{2^k\,k!\,l!\,(3-2k-l)!}, \tag{B4}$$

and

$$K_2 = \sum_{k=0}^{2}\sum_{l=0}^{4-2k}\int_0^t dy\, e^{-iy(4-2k-2l)} \times \frac{4!}{2^2}\frac{a^{\dagger l}a^{4-2k-l}}{2^k\,k!\,l!\,(4-2k-l)!}, \tag{B5}$$

$$= \sum_{k=0}^{2}\sum_{l=0}^{4-2k} e^{-i\frac{t}{2}(2k+2l-4)}\frac{\sin\left[\frac{t}{2}(2k+2l-4)\right]}{\frac{1}{2}(2k+2l-4)} \times \frac{4!}{2^2}\frac{a^{\dagger l}a^{4-2k-l}}{2^k\,k!\,l!\,(4-2k-l)!}. \tag{B6}$$

If we take the unperturbed ground states (the vacuum state of the harmonic oscillator) we have $\langle 0|K_1|0\rangle = 0$ and $\langle 0|K_2|0\rangle = \frac{3}{4}t$.

To calculate the expectation values $\langle 0|K_1^2|0\rangle$, $\langle 0|K_2^2|0\rangle$, and $\langle 0|K_1K_2|0\rangle$ and evaluate the QFIM using Eq. (40) we need to calculate expectations values of the form $\langle 0|a^{\dagger l'}a^{n'-2k'-l'}a^{\dagger l}a^{n-2k-l}|0\rangle$. In particular, to calculate $\langle 0|K_1K_2|0\rangle$, we need

$$\begin{aligned}&\langle 0|a^{\dagger l'}a^{n'-2k'-l'}a^{\dagger l}a^{n-2k-l}|0\rangle\\ &= \delta_{l',0}\delta_{l,n-2k}\langle 0|a^{n'-2k'}a^{\dagger n-2k}|0\rangle\\ &= \delta_{l',0}\delta_{l,n-2k}\delta_{k',k+\frac{n'-n}{2}}\sqrt{(n'-2k')!(n-2k)!}\\ &= 0 \quad \text{if } n' = n\pm 1.\end{aligned} \tag{B7}$$

We conclude that $\langle 0|K_1K_2|0\rangle = 0$ and the same happens for the quantumness R. To calculate the diagonal elements of the QFIM we use

$$\langle 0|a^{\dagger l'}a^{n-2k'-l'}a^{\dagger l}a^{n-2k-l}|0\rangle = (n-2k)!\,\delta_{l',0}\delta_{l,n-2k}\delta_{k,k'},$$

such that

$$\begin{aligned}Q_{11} &= 4\,\langle 0|K_1^2|0\rangle\\ &= 4\left(\frac{3!}{2^{3/2-1}}\right)^2\sum_{k=0}^{1}\frac{\sin^2\left[\frac{t}{2}(3-2k)\right]}{(3-2k)^2\,2^{2k}\,(k!)^2\,(3-2k)!}\\ &= \frac{29}{3} - 9\cos t - \frac{2}{3}\cos 3t,\end{aligned} \tag{B8}$$

and

$$\begin{aligned}Q_{22} &= 4\left(\langle 0|K_2^2|0\rangle - \langle 0|K_2|0\rangle^2\right)\\ &= 4\left\{\left(\frac{4!}{2^{4/2-1}}\right)^2\left[\sum_{k=0}^{1}\frac{\sin^2\left[\frac{t}{2}(4-2k)\right]}{(4-2k)^2\,2^{2k}\,(k!)^2\,(4-2k)!}\right.\right.\\ &\quad\left.\left. + \lim_{k\to 2}\frac{\sin^2\left[\frac{t}{2}(4-2k)\right]}{(4-2k)^2\,2^{2k}\,(k!)^2\,(4-2k)!}\right] - \left(\frac{3}{4}t\right)^2\right\}\\ &= 3\,(7+\cos 2t)\,\sin^2 t.\end{aligned} \tag{B9}$$

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