Witnessing quantum coherence with prior knowledge of observables

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Quantum coherence is the key resource in quantum technologies, including faster computing, secure communication, and advanced sensing. Its quantification and detection are, therefore, paramount within the context of quantum information processing. Having certain prior knowledge of the observables may enhance the efficiency of coherence detection. In this work, we posit that the trace of the observables is a known quantity. Our investigation confirms that this assumption indeed extends the scope of coherence-detection capabilities. Utilizing this prior knowledge of the trace of the observables, we establish a series of coherence-detection criteria. We investigate the detection capabilities of these coherence criteria from diverse perspectives and ultimately ascertain the existence of four distinct and inequivalent criteria. These findings contribute to the deepening of our understanding of coherence-detection methodologies, thereby potentially opening new avenues for advancements in quantum technologies.

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I. INTRODUCTION

Quantum coherence [1,2] is a fundamental phenomenon in quantum mechanics, enabling quantum systems to exist in superpositions of multiple states simultaneously. It plays a pivotal role in various quantum technologies, from quantum computing [3] and quantum metrology [4,5] to nanoscale thermodynamics [6–9] and energy transport in biological systems [10–12]. As such, coherence represents a critical resource for various quantum information-processing tasks.

Significant strides have been made toward the measurement of quantum coherence based on numerous innovative approaches [13–43]. Quantum resource theory [14–16,22,30] has garnered significant attention, proving instrumental in advancing our understanding of coherence. Under the quantum resource theory framework, various coherence monotones and measures have been introduced, including the relative entropy of coherence, the ℓ_1 norm of coherence, the coherence of formation, the geometric measure of coherence, and the robustness of coherence.

Coherence witnesses, like their entanglement-witness [44–48] counterparts, have emerged as potent tools for coherence detection in experimental settings [49–52] and coherence quantification in theoretical contexts. In contrast to conventional methods that rely on state tomography, coherence witnesses can directly identify the coherent states [53–55]. It is worth noting that Napoli *et al.* [15] made significant contributions to estimating the lower bound on the robustness of coherence by utilizing the coherence witnesses, emphasizing the importance of this concept in deepening our understanding of quantum coherence. More recently, Wang *et al.* [54,55] conducted a relatively comprehensive exploration of coherence witnesses. Nevertheless, research on coherence witnesses remains relatively limited. In practical applications, we often possess valuable prior knowledge about the observables involved. This naturally raises an important question: can such prior knowledge enhance the accuracy and effectiveness of coherence detection?

In this work, we systematically investigate coherence witnesses within the framework of prior knowledge about the trace of observables. We demonstrate that knowing the exact value of the trace for a given observable indeed enhances our ability to detect coherent states. Moreover, by leveraging varying degrees of prior knowledge about the trace of observables, we establish a series of criteria for detecting coherence. Then, we explore the properties of these criteria, such as completeness, finite completeness, finite intersection, and inclusion, which assist us in singling out four classes of inequivalent coherence criteria.

The rest of this paper is organized as follows. In Sec. II, we provide an overview of coherence witnesses and unveil a series of coherence criteria under varying degrees of prior knowledge concerning the traces of observables. In Sec. III, we analyze these coherence criteria from various perspectives, including completeness, finite completeness, finite intersection, and inclusion. Finally, in Sec. IV we summarize our findings and outline potential avenues for future research.

II. COHERENCE WITNESSES WITH PRIOR KNOWLEDGE OF OBSERVABLES

We denote by $\mathcal{H}$ a d-dimensional Hilbert space with computational basis $\mathcal{B} := \{|i\rangle,\ i = 1, 2, \dots, d\}$, $\mathbb{H}$ the set of all

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$d \times d$ Hermitian matrices, and $\mathbb{D}$ the set of all density matrices (self-adjoint positive-semidefinite matrices with trace 1). Let Δ be the set of incoherent states with respect to the basis $\mathcal{B}$; namely, Δ comprises all density matrices of the form

$$\delta = \sum_{i=1}^{d} \delta_i |i\rangle\langle i|. \tag{1}$$

Hence, all the states within $\mathbb{D} \setminus \Delta$ are referred to as coherent states.

By definition the set Δ of incoherent states is convex and compact. From the Hahn-Banach theorem [56], a hyperplane which separates an arbitrary given coherent state from the set of incoherent states must exist. A coherence witness is a Hermitian operator $W \in \mathbb{H}$ such that (1) $\mathrm{Tr}(W\delta) \geqslant 0$ for all incoherent states $\delta \in \Delta$ and (2) a coherent state ρ such that $\mathrm{Tr}(W\rho) < 0$ exists. The first condition implies that all the diagonals of W are non-negative, while the second implies that W has some negative eigenvalue. We denote by $\mathbb{H}_{\geqslant}$ the set of $d \times d$ Hermitian matrices with non-negative diagonal elements and Λ_- the set with some negative eigenvalue. The general coherence witnesses are then generally given by $\mathbb{H}_{\geqslant} \cap \Lambda_-$.

In addition to the above coherent witnesses, there are other kinds of coherent witnesses defined by strengthening condition 1 to $\mathrm{Tr}(W\delta) = 0$ for all $\delta \in \Delta$ but relaxing condition 2 to $\mathrm{Tr}(W\rho) \neq 0$ for some state ρ [15,52,54]. Here, the strengthened condition implies that all the diagonals of W must be zero, while the relaxed condition allows for detecting more coherent states by this witness. In other words, prior knowledge about the observables may enable one to detect the coherence better.

For the usual coherence witness $W \in \mathbb{H}_{\geqslant} \cap \Lambda_-$, the condition $\mathrm{Tr}(W\delta) = 0$ for all incoherent states $\delta \in \Delta$ is equivalent to $\mathrm{Tr}[W] = 0$. This motivates us to define the set of all coherence witnesses with the same trace. Therefore, for each real number $R \geqslant 0$, we introduce the set

$$\mathbb{W}_R := \{W \in \mathbb{H}_{\geqslant} \,\big|\, \mathrm{Tr}[W] = R\}. \tag{2}$$

Note that the set of observables with zero trace $\mathbb{W}_0$ is just the set of all $d \times d$ Hermitian matrices with zero diagonal entries. For example, in a qubit system, i.e., $d = 2$, $\mathbb{W}_0 = \{a\sigma_1 + b\sigma_2 \mid a, b \in \mathbb{R}\}$. Here and in the following, σ_i, where $i = 1, 2, 3$, represents the Pauli matrices defined as

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The prior knowledge of the trace of the witness can enhance the ability to detect coherence. For fixed $R > 0$ we have, for any $W = (w_{ij}) \in \mathbb{W}_R$ and $\delta = \sum_{i=1}^{d} \delta_i |i\rangle\langle i| \in \Delta$,

$$0 \leqslant \mathrm{Tr}[W\delta] = \sum_{i=1}^{d} w_{ii}\delta_i \leqslant \sum_{i=1}^{d} w_{ii} = R.$$

Therefore, if $W = (w_{ij})$ is previously known in the set $\mathbb{W}_R$, then either $\mathrm{Tr}[W\rho] < 0$ or $\mathrm{Tr}[W\rho] > R$ implies the coherence of ρ. Moreover, some $W = (w_{ij}) \in \mathbb{W}_R$ and density matrix ρ such that $\mathrm{Tr}[W\rho] > R$ do exist. In fact, for any $0 < r < R$, let

$$W = (R - r)|1\rangle\langle 1| + \sqrt{rR}(|1\rangle\langle 2| + |2\rangle\langle 1|) + r|2\rangle\langle 2|$$

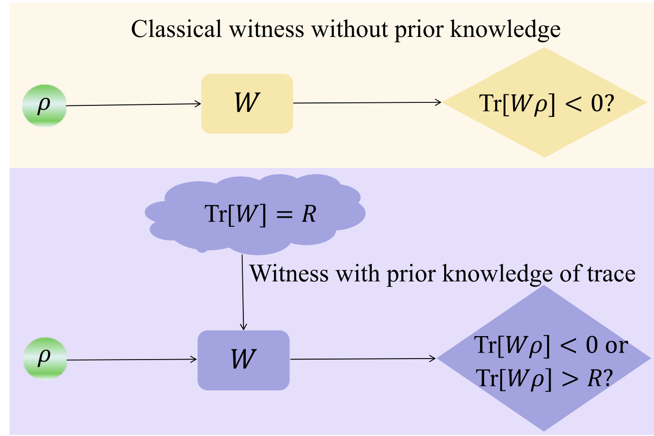

FIG. 1. Classical coherence witness vs a coherence witness with prior knowledge of the trace.

and $W = \lambda_+|v_+\rangle\langle v_+| + \lambda_-|v_-\rangle\langle v_-|$ be its spectral decomposition. As $\lambda_+\lambda_- = (R - r)r - (\sqrt{rR})^2 = -r^2 < 0$, we can always assume $\lambda_+ > 0$ and $\lambda_- < 0$. For $\rho = |v_+\rangle\langle v_+|$ we have

$$\begin{aligned}\mathrm{Tr}[W\rho] &= \lambda_+ = (\lambda_+ + \lambda_-) - \lambda_- \\ &= \mathrm{Tr}[W] - \lambda_- = R - \lambda_- > R.\end{aligned}$$

As a consequence, the prior knowledge of the trace of the observables indeed extends the scope of coherent-state detections (see Fig. 1).

However, if we just know that the observables cannot be traceless, i.e., such observables are chosen from the set $\mathbb{W}_> := \{W \in \mathbb{H}_{\geqslant} \mid \mathrm{Tr}[W] > 0\}$, we find that such little prior knowledge of the trace of observables cannot help us to enhance the coherence detection. That is, to ensure that the witness W detects the coherence of ρ, we still need to observe the violation of $\mathrm{Tr}[W\rho] \geqslant 0$ as usual. To prove this, we need to show only that

$$\{\mathrm{Tr}[W\delta] \,\big|\, W \in \mathbb{W}_>, \delta \in \Delta\} = \{r \in \mathbb{R} \mid r \geqslant 0\};$$

that is, for any real number $r \geqslant 0$, $W \in \mathbb{W}_>$ and $\delta \in \Delta$ always exist such that $\mathrm{Tr}[W\delta] = r$. If $r > 0$, choosing any $W \in \mathbb{W}_>$ with $\mathrm{Tr}[W] = dr$ and $\delta = \frac{\mathbb{I}_d}{d}$, we have

$$\mathrm{Tr}[W\delta] = \frac{1}{d}\mathrm{Tr}[W] = r.$$

If $r = 0$, setting $W = |1\rangle\langle 1| + |1\rangle\langle 2| + |2\rangle\langle 1|$ and $\rho = |2\rangle\langle 2|$, we have $\mathrm{Tr}[W\rho] = 0 = r$.

Throughout this paper, we set $\mathcal{R} := \{r \in \mathbb{R} | r \geqslant 0\} \cup \{>, \geqslant\}$. For each $R \in \mathcal{R}$, we set $I_R := \{\mathrm{Tr}[W\delta] \mid W \in \mathbb{W}_R, \delta \in \Delta\}$ and $D_R := \{\mathrm{Tr}[W\rho] \mid W \in \mathbb{W}_R, \rho \in \mathbb{D}\}$. For each $R \in \mathcal{R}$, a corresponding coherence-witness criterion can be formulated as follows.

Coherence criterion: $(\mathbb{W}_R, D_R, I_R)$. For any $\rho \in \mathbb{D}$, if there exists some $W \in \mathbb{W}_R$ such that $\mathrm{Tr}[W\rho] \in D_R \setminus I_R$, then ρ is a coherent state.

For any $R \in \mathcal{R}$, we can immediately verify that $D_R = \mathbb{R}$. From the above discussion, the set I_R is classified into following three classes (see Fig. 2): (1) $I_R = \{r \in \mathbb{R} | 0 \leqslant r \leqslant R\}$ when R is a positive real number, (2) $I_R = \{r \in \mathbb{R} \mid r \geqslant 0\}$ when R is either $\geqslant$ or $>$, and (3) $I_R = \{0\}$ when $R = 0$.

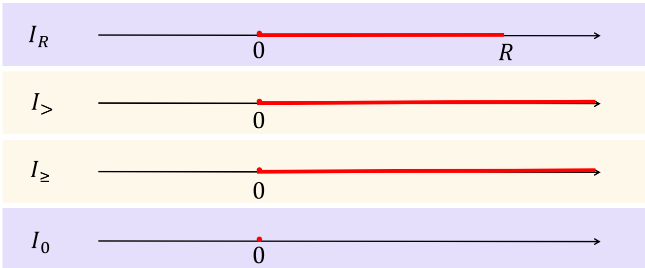

FIG. 2. An intuitive view of the range of I_R for $R \in \{r \in \mathbb{R} | r \geqslant 0\} \cup \{>, \geqslant\}$.

To classify coherence criteria we define $\mathbb{C}_R[W]$ as the set of all coherent states that can be witnessed by W in the setting $(\mathbb{W}_R,\ D_R,\ I_R)$, that is, $\mathbb{C}_R[W] = \{\rho \in \mathbb{D} \mid \mathrm{Tr}[W\rho] \in D_R \setminus I_R\}$. Moreover, we denote $\mathbb{W}_R^c := \{W \in \mathbb{W}_R \mid \mathbb{C}_R[W] \neq \emptyset\}$, i.e., the set of all nontrivial coherence witnesses of the coherence criterion $(\mathbb{W}_R, D_R, I_R)$. We find that $\mathbb{W}_R^c = \mathbb{W}_R \cap \Lambda_-$ (see the proof in the Appendix).

For $W \in \mathbb{H}_\geqslant$, if we set $R = \mathrm{Tr}[W]$, then W belongs to $\mathbb{W}_R$ and $\mathbb{W}_\geqslant$. We find that

$$\mathbb{C}_R[W] = \mathbb{C}_\geqslant[W] \cup \mathbb{C}_\geqslant[R\mathbb{I}_d - W].$$

It is easy to check that $\mathbb{C}_\geqslant[W]$ is a convex set; i.e., if $\mathrm{Tr}[W\rho] < 0$ and $\mathrm{Tr}[W\sigma] < 0$, then $\mathrm{Tr}[W(t\rho + (1-t)\sigma)] < 0$ for $t \in [0, 1]$. However, if $\mathbb{C}_\geqslant[W] \neq \emptyset$ and $\mathbb{C}_\geqslant[R\mathbb{I}_d - W] \neq \emptyset$, the set $\mathbb{C}_R[W]$ is not a convex set. In this case, $\mathbb{C}_R[W]$ is a disjoint union of two convex sets (see cases A and C in Fig. 3 for a qubit system). In fact, for any $\rho \in \mathbb{C}_\geqslant[W]$ and $\sigma \in \mathbb{C}_\geqslant[R\mathbb{I}_d - W]$, we must have $\mathrm{Tr}[W\rho] < 0$ and $\mathrm{Tr}[W\sigma] > R \geqslant 0$. Therefore, there must be some $t_* \in (0, 1)$ such that $\mathrm{Tr}\{W[t_*\rho + (1-t_*)\sigma]\} = 0$, which indicates that $t_*\rho + (1 - t_*)\sigma \notin \mathbb{C}_R[W]$.

We say that two coherence criteria $(\mathbb{W}_{R_1}, D_{R_1}, I_{R_1})$ and $(\mathbb{W}_{R_2}, D_{R_2}, I_{R_2})$ are equivalent if there is a bijection $F : \mathbb{W}_{R_1}^c \to \mathbb{W}_{R_2}^c$ such that $\mathbb{C}_{R_1}[W] = \mathbb{C}_{R_2}[F(W)]$ for any $W \in \mathbb{W}_{R_1}^c$. It is an intuition that the coherence criteria arising from knowing different values of traces may be almost the same in detecting the coherence. For any positive real number R_1 or R_2, the two criteria $(\mathbb{W}_{R_1}, D_{R_1}, I_{R_1})$ and $(\mathbb{W}_{R_2}, D_{R_2}, I_{R_2})$ are equivalent in the above sense. In fact, we can define a map F from $\mathbb{W}_{R_1}^c$ to $\mathbb{W}_{R_2}^c$ by sending $W \in \mathbb{W}_{R_1}^c$ to $\frac{R_2}{R_1}W$. Clearly, $\frac{R_2}{R_1}W \in \mathbb{W}_{R_2}^c$ as W has non-negative diagonals and $\mathrm{Tr}[\frac{R_2}{R_1}W] = \frac{R_2}{R_1}\mathrm{Tr}[W] = \frac{R_2}{R_1}R_1 = R_2$. It is easy to verify that it is a bijection. For any $\rho \in \mathbb{D}$, $0 \leqslant \mathrm{Tr}[W\rho] \leqslant R_1$ is equivalent to $0 \leqslant \mathrm{Tr}[\frac{R_2}{R_1}W\rho] \leqslant R_2$. Therefore, we always have $\mathbb{C}_{R_1}[W] = \mathbb{C}_{R_2}[F(W)]$.

However, to fully classify the previously mentioned coherence criteria, we need to study more properties of these criteria. After that, we will go back and tackle this problem.

III. PROPERTIES AND CLASSIFICATION OF COHERENCE CRITERIA

For each $R \in \mathcal{R}$, we have established the coherence criterion $(\mathbb{W}_R,\ D_R,\ I_R)$. It is natural to ask whether these criteria are complete in the sense that they detect all the coherent states, namely, the following relation holds:

$$\mathbb{D}\setminus\Delta = \bigcup_{W\in\mathbb{W}_R} \mathbb{C}_R[W].$$

Concerning the completeness of our coherence criteria, we have the following conclusion.

Theorem 1. Completeness. For each $R \in \mathcal{R}$, the coherence criterion $(\mathbb{W}_R, D_R, I_R)$ is complete.

Proof. Suppose ρ is a coherent state. Without loss of generality, we assume $\rho_{mn} = \langle m|\rho|n\rangle \neq 0$, where $m \neq n$. We need to show that some $W \in \mathbb{W}_R$ such that $\mathrm{Tr}[W\rho] \in D_R \setminus I_R$ always exist. As $R \in \mathcal{R}$, the interval $(-\infty, 0) \subseteq D_R \setminus I_R$. It is sufficient to prove that some $W \in \mathbb{W}_R$ such that $\mathrm{Tr}[W\rho] < 0$ always exist. We prove the theorem according to following four cases.

(1) $R = 0$. We define $W_{j,k}^{\mathrm{Re}} := (|j\rangle\langle k| + |k\rangle\langle j|)/2$ and $W_{j,k}^{\mathrm{Im}} := \mathrm{i}(|j\rangle\langle k| - |k\rangle\langle j|)/2$ for each $1 \leqslant j < k \leqslant d$. Note that we always have $\mathrm{Tr}[W_{j,k}^{\mathrm{Re}}\rho] = \mathrm{Re}(\rho_{jk})$ and $\mathrm{Tr}[W_{j,k}^{\mathrm{Im}}\rho] = \mathrm{Im}(\rho_{jk})$, where $\rho_{jk} = \langle j|\rho|k\rangle$, $\mathrm{Re}(z) = \alpha$, and $\mathrm{Im}(z) = \beta$ for any complex number $z = \alpha + \beta\mathrm{i} \in \mathbb{C}$. Clearly, $\mathcal{W}_0 := \{W_{j,k}^{\mathrm{Re}}, W_{j,k}^{\mathrm{Im}} \mid 1 \leqslant j < k \leqslant d\} \subseteq \mathbb{W}_0^c$. As $\rho_{mn} \neq 0$, either $\mathrm{Tr}[W_{m,n}^{\mathrm{Re}}\rho] \neq 0$ or $\mathrm{Tr}[W_{m,n}^{\mathrm{Im}}\rho] \neq 0$. That is, ρ is in $\mathbb{C}_0[W_{m,n}^{\mathrm{Re}}]$ or $\mathbb{C}_0[W_{m,n}^{\mathrm{Im}}]$. Therefore,

$$\mathbb{D}\setminus\Delta = \bigcup_{W\in\mathcal{W}_0} \mathbb{C}_0[W]. \tag{3}$$

Without loss of generality, we assume that $\mathrm{Tr}[W_{m,n}^{\mathrm{Re}}\rho] \neq 0$. Since $\mathrm{Tr}[W_{m,n}^{\mathrm{Re}}\rho]$ is a real number, $\mathrm{Tr}[W_{m,n}^{\mathrm{Re}}\rho]$ or $\mathrm{Tr}[-W_{m,n}^{\mathrm{Re}}\rho]$ is negative, and both $\pm W_{m,n}^{\mathrm{Re}}$ are in $\mathbb{W}_0$. Hence, some $W \in \mathbb{W}_0$ such that $\mathrm{Tr}[W\rho] < 0$ always exists.

(2) R is $\geqslant$. Denote $\mathcal{W}_\geqslant := \{\pm W_{j,k}^{\mathrm{Re}}, \pm W_{j,k}^{\mathrm{Im}} \mid 1 \leqslant j < k \leqslant d\}$. By definition, we have $\mathcal{W}_\geqslant \subseteq \mathbb{W}_\geqslant^c$. Similar to the argument in case 1, some $W \in \mathcal{W}_\geqslant$ such that $\mathrm{Tr}[W\rho] < 0$ always exists. Therefore,

$$\mathbb{D}\setminus\Delta = \bigcup_{W\in\mathcal{W}_\geqslant} \mathbb{C}_\geqslant[W]. \tag{4}$$

(3) R is $>$. From case 2, there is some $W \in \mathcal{W}_\geqslant$ such that $\mathrm{Tr}[W\rho] < 0$. Note that the diagonals of W are all zeros. For each $\epsilon > 0$, set $W_\epsilon = W + \epsilon\mathbb{I}_d$. Clearly, W_ϵ has positive diagonals, and $\mathrm{Tr}[W_\epsilon] = d\epsilon > 0$. Hence, $W_\epsilon \in \mathbb{W}_>$. Specially, choosing $\epsilon = -\frac{\mathrm{Tr}[W\rho]}{2}$, we have

$$\begin{aligned}\mathrm{Tr}[W_\epsilon\rho] &= \mathrm{Tr}[W\rho] + \epsilon\mathrm{Tr}[\rho]\\ &= \mathrm{Tr}[W\rho] + \epsilon = \frac{\mathrm{Tr}[W\rho]}{2} < 0.\end{aligned}$$

(4) R is a positive real number. From case 3, there is some $W_\epsilon \in \mathcal{W}_>$ such that $\mathrm{Tr}[W_\epsilon\rho] < 0$. Set $W_R = \frac{R}{\mathrm{Tr}[W_\epsilon]}W_\epsilon$. Clearly, $W_R \in \mathbb{W}_R$ and

$$\mathrm{Tr}[W_R\rho] = \frac{R}{\mathrm{Tr}[W_\epsilon]}\mathrm{Tr}[W_\epsilon\rho] < 0.$$

The above four cases together complete the proof. ■

So each coherence criterion presented here is complete. As a consequence, we can restate the coherence criteria by the following four classes.

(A) For $(\mathbb{W}_R, D_R, I_R)$, with R being a positive real number, a state $\rho \in \mathbb{D}$ is coherent if and only if $W \in \mathbb{W}_R$ such that $\mathrm{Tr}[W\rho] < 0$ or $\mathrm{Tr}[W\rho] > R$ exists.

(B_1) For $(\mathbb{W}_>, D_>, I_>)$, a state $\rho \in \mathbb{D}$ is coherent if and only if there exists $W \in \mathbb{W}_>$ such that $\mathrm{Tr}[W\rho] < 0$.

(B_2) For $(\mathbb{W}_\geqslant, D_\geqslant, I_\geqslant)$, a state $\rho \in \mathbb{D}$ is coherent if and only if $W \in \mathbb{W}_\geqslant$ such that $\mathrm{Tr}[W\rho] < 0$ exists.

(C) For $(\mathbb{W}_0, D_0, I_0)$, a state $\rho \in \mathbb{D}$ is coherent if and only if $W \in \mathbb{W}_0$ such that $\mathrm{Tr}[W\rho] \neq 0$ exists.

Criteria B_1 and B_2 coincide with the standard criterion, while criteria A and C differ due to the incorporation of prior knowledge of the trace of observables. Generally, given a Hermitian matrix W with $R = \mathrm{Tr}[W]$, we always have

$$\mathbb{C}_\geqslant[W] \subseteq \mathbb{C}_\geqslant[W] \cup \mathbb{C}_\geqslant[R\mathbb{I}_d - W] = \mathbb{C}_R[W].$$

The prior knowledge of the trace of W could extend the scope of coherent-state detections. Now we proceed to apply these criteria to a qubit system as an example, highlighting the benefits of detecting coherence with prior knowledge of the trace of observables.

Example 1. For a qubit system, i.e., when $d = 2$, prior knowledge of the exact value R of the trace of the observable W can double the efficiency of observing coherent states compared to the standard criterion. More precisely, $\mathbb{C}_R[W]$ is a disjoint union of $\mathbb{C}_\geqslant[W]$ and $\mathbb{C}_\geqslant[R\mathbb{I}_2 - W]$, and there is a one-to-one correspondence between $\mathbb{C}_\geqslant[W]$ and $\mathbb{C}_\geqslant[R\mathbb{I}_2 - W]$, given by (x, y, z) mapped to $(-x, -y, -z)$.

In fact, let $W = (R\mathbb{I}_2 + a\sigma_1 + b\sigma_2 + c\sigma_3)/2$ and $\rho = (\mathbb{I}_2 + x\sigma_1 + y\sigma_2 + z\sigma_3)/2$. Note that W is an effective coherent witness; i.e., W belongs to $\mathbb{H}_\geqslant \cap \Lambda_-$ if and only if $|c| \leqslant R < \sqrt{a^2 + b^2 + c^2}$. And ρ is a density matrix if and only if $x^2 + y^2 + z^2 \leqslant 1$. Thus, one can recognize the points within the unit ball centered at the origin as the elements of density matrices. The set of all incoherent states is $\Delta = \{(\mathbb{I}_2 + z\sigma_3)/2 \mid -1 \leqslant z \leqslant 1\}$. Moreover, $\mathrm{Tr}[W\rho] = (R + ax + by + cz)/2$. Therefore, the condition $\mathrm{Tr}[W\rho] < 0$ is equivalent to $ax + by + cz < -R$, and the condition $\mathrm{Tr}[W\rho] < 0$ or $\mathrm{Tr}[W\rho] > R$ is equivalent to $ax + by + cz < -R$ or $ax + by + cz > R$.

If $R > 0$ and we have no information about the specific value of R, then we can employ only criterion B_1 or B_2 (i.e., the standard coherence criterion) for this witness. Therefore, W is capable of detecting only the coherence of ρ whose corresponding coordinates (x, y, z) satisfy $ax + by + cz < -R$ (see cases B_1 and B_2 in Fig. 3). On the contrary, if we have prior knowledge of the value of R, we can apply criterion A to this witness, enabling the detection of states with coordinates (x, y, z) satisfying either $ax + by + cz < -R$ or $ax + by + cz > R$ (see case A in Fig. 3).

If $R = 0$ (in this context, $c = 0$ since $|c| \leqslant R$) and we have no specific information about the value of R, then we can apply only criterion B_2 to this witness. Consequently, W is capable of detecting the coherence of ρ whose corresponding coordinates (x, y, z) satisfy $ax + by < 0$. On the contrary, if we are aware beforehand that $R = 0$, then we can apply criterion C to this witness, allowing the detection of states with coordinates (x, y, z) satisfying either $ax + by < 0$ or $ax + by > 0$ (see case C in Fig. 3).

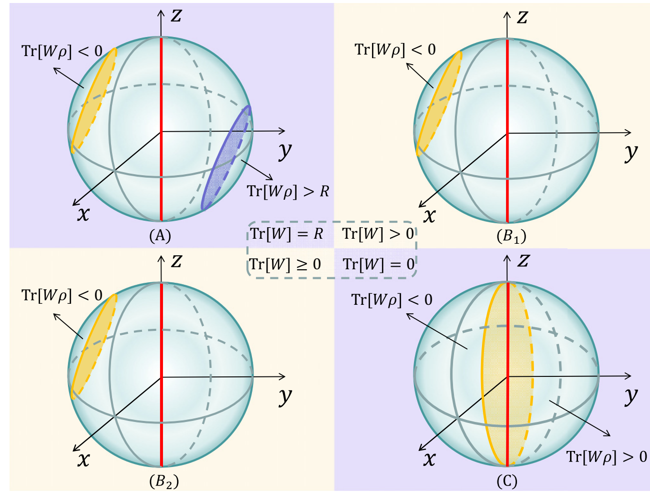

FIG. 3. An intuitive view of the detection range of each type of coherence witness for the qubit system. Here, the red line on the z axis represents the set of incoherent states. The orange disk represents the states ρ with $\mathrm{Tr}[W\rho] = 0$. The purple disk represents the coherent states ρ with $\mathrm{Tr}[W\rho] = R$.

Therefore, prior knowledge of the trace of the observable can double the efficiency of observing coherent states.

A coherence criterion $(\mathbb{W}_R, D_R, I_R)$ is called finitely completable if all the coherent states can be detected by a finite set of coherent witnesses in $\mathbb{W}_R^c$. That is, a finite set $\{W_i\}_{i=1}^n \subseteq \mathbb{W}_R^c$ such that

$$\mathbb{D} \setminus \Delta = \bigcup_{i=1}^{n} \mathbb{C}_R[W_i]$$

exists. Otherwise, we call it finitely incompletable.

Theorem 2. Characterization of finite completeness. Let R be an element in $\mathcal{R}$. The coherence criterion $(\mathbb{W}_R, D_R, I_R)$ is finitely completable if and only if R is 0 or $\geqslant$.

Proof. From Eqs. (3) and (4) and the definition of finitely completable, both coherence criteria $(\mathbb{W}_0, D_0, I_0)$ and $(\mathbb{W}_\geqslant, D_\geqslant, I_\geqslant)$ are finitely completable.

Now we show that for any positive real number R, the criterion $(\mathbb{W}_R, D_R, I_R)$ is finitely incompletable. If not, some $R > 0$ and a set $\mathcal{W} := \{W_i\}_{i=1}^n \subseteq \mathbb{W}_R^c$ such that $\mathbb{D} \setminus \Delta = \cup_{i=1}^n \mathbb{C}_R[W_i]$ exist. That is, for any coherent states ρ, some $W \in \mathcal{W}$ such that $\mathrm{Tr}[W\rho] < 0$ or $\mathrm{Tr}[W\rho] > R$ always exists. On the other hand, we set $\rho_\epsilon = \frac{\mathbb{I}_d}{d} + \epsilon H$, where $H = |1\rangle\langle 2| + |2\rangle\langle 1|$ and $\epsilon \in (0, \frac{1}{d})$, and $M = \max(|\mathrm{Tr}\{[W - \Delta(W)]H\}| : W \in \mathcal{W}) + 1$. Note that

$$\mathrm{Tr}[W\rho_\epsilon] = \frac{R}{d} + \epsilon \mathrm{Tr}\{[W - \Delta(W)]H\}.$$

If we set $\epsilon = \min\{\frac{R}{2dM}, \frac{1}{2d}\}$, then we have $0 < \frac{1}{2d}R \leqslant \mathrm{Tr}[W\rho_\epsilon] \leqslant \frac{3}{2d}R < R$ for every $W \in \mathcal{W}$. However, ρ_ϵ is a coherent state which cannot be detected by all the witnesses in $\mathcal{W}$.

Now we show that the criterion $(\mathbb{W}_>, D_>, I_>)$ is also finitely incompletable. For any finite set $\mathcal{W}_> := \{W_i\}_{i=1}^n \subseteq \mathbb{W}_>^c$, we define $M_1 := \max(|\mathrm{Tr}\{[W - \Delta(W)]H\}| : W \in \mathcal{W}) + 1$, and $m_1 := \min\{\mathrm{Tr}[W] : W \in \mathcal{W}_>\} > 0$. If we set $\epsilon_1 = \min\{\frac{m_1}{2dM_1}, \frac{1}{2d}\}$, then $\mathrm{Tr}[W\rho_{\epsilon_1}] \geqslant \frac{m_1}{2d} > 0$ for every $W \in \mathcal{W}_>$. That is, ρ_{ϵ_1} is a coherent state whose coherence cannot be detected by the witnesses in $\mathcal{W}_>$.

From Theorem 2, only those criteria that include all observables with traces being zero are finitely completable. In this sense, types A and B_1 are quite different from types B_2 and C. Although the criteria and the plots (presented in Fig. 3) of types B_1 and B_2 seem the same, they are, in fact, inequivalent.

In the following we discuss the conditions when the finite intersections of $\mathbb{C}_R[W]$ may be empty, for which the case with R being $\geqslant$ was discussed in Ref. [54].

Theorem 3. Property of finite intersection. Let $R \in \mathcal{R}$ and $\mathcal{W} := \{W_i\}_{i=1}^n \subseteq \mathbb{W}_R^c$. Then the following statements hold.

(1) For the case $R \in (0, \infty)$ and any subset $\mathcal{W}'$ with cardinality $|\mathcal{W}'| \geqslant \lceil \frac{n}{2} \rceil$, if $t_W > 0$ for each $W \in \mathcal{W}'$ and $\sum_{W \in \mathcal{W}'} t_W = 1$ always exist such that $\sum_{W \in \mathcal{W}'} t_W W \geqslant \mathbf{0}$, then $\bigcap_{i=1}^n \mathbb{C}_R[W_i] = \emptyset$.

(2) If R is $>$ or $\geqslant$, then $\bigcap_{i=1}^n \mathbb{C}_R[W_i] = \emptyset$ if and only if there is some non-negative $t_i \geqslant 0$ and $\sum_{i=1}^n t_i = 1$ such that $\sum_{i=1}^n t_i W_i$ is positive semidefinite.

(3) If $R = 0$, a common coherent state exists that can be detected by all W_i via the criterion of type C. That is, we always have

$$C_0[\mathcal{W}] := \bigcap_{i=1}^n \mathbb{C}_0[W_i] \neq \emptyset.$$

Moreover, the cardinality of $C_0[\mathcal{W}]$ is infinity. That is, given any set of finitely many coherent witnesses, they share infinitely many common coherent states.

Proof. (1) If $\rho \in \bigcap_{i=1}^n \mathbb{C}_R[W_i]$, then $\mathrm{Tr}[W_i\rho] < 0$ or $\mathrm{Tr}[W_i\rho] > R$. Let $\mathcal{I}_0 := \{i \mid \mathrm{Tr}[W_i\rho] < 0, \ 1 \leqslant i \leqslant n\}$ and $\mathcal{I}_R := \{i \mid \mathrm{Tr}[W_i\rho] > R, \ 1 \leqslant i \leqslant n\}$. Without loss of generality, we assume the set $\mathcal{I}_R$ is larger than $\mathcal{I}_0$. Then, correspondingly, the cardinality must be greater than or equal to $\lceil \frac{n}{2} \rceil$. Define $\mathcal{W}' := \{W_i | i \in \mathcal{I}_R\}$. By assumption, there are $t_i > 0$ for $i \in \mathcal{I}_R$ such that $\sum_{i \in \mathcal{I}_R} t_i = 1$ and $W := \sum_{i \in \mathcal{I}_R} t_i W_i \geqslant \mathbf{0}$. Therefore, all eigenvalues of W are nonnegative. Moreover, as $\mathrm{Tr}[W] = R$, all the eigenvalues of W are less than or equal to R. Hence, $W \leqslant R\mathbb{I}_d$, which implies $\mathrm{Tr}[W\rho] \leqslant R$. On the other hand, for each $i \in \mathcal{I}_R$, we have $\mathrm{Tr}[W_i\rho] > R$, which leads to

$$\mathrm{Tr}[W\rho] = \sum_{i \in \mathcal{I}_R} t_i \mathrm{Tr}[W_i\rho] > R \sum_{i \in \mathcal{I}_R} t_i = R$$

and thus a contradiction. Hence, $\bigcap_{i=1}^n \mathbb{C}_R[W_i] = \emptyset$.

(2) The argument for the case that R is $\geqslant$ can be directly referred to Ref. [54]. When R is $>$, as $D_>$ and $I_>$ are the same as $D_\geqslant$ and $I_\geqslant$, the proof goes similarly to the one given in [54].

(3) We prove the statement by induction on n. First, we consider the case with $n = 2$. As both $\mathbb{C}_0[W_1]$ and $\mathbb{C}_0[W_2]$ are nonempty, we assume $\rho \in \mathbb{C}_0[W_1]$ and $\sigma \in \mathbb{C}_0[W_2]$. We may assume that $\rho, \sigma \notin \mathbb{C}_0[W_1] \cap \mathbb{C}_0[W_2]$, i.e., $\rho \notin \mathbb{C}_0[W_2]$ and $\sigma \notin \mathbb{C}_0[W_1]$. Equivalently, we have $\mathrm{Tr}[W_2\rho] = \mathrm{Tr}[W_1\sigma] = 0$. Then for any $\lambda \in (0, 1)$, the matrix $\pi_\lambda := \lambda\rho + (1-\lambda)\sigma \in \mathbb{D}$. Moreover, $\mathrm{Tr}[W_1\pi_\lambda] = \lambda \mathrm{Tr}[W_1\rho] \neq 0$ and $\mathrm{Tr}[W_2\pi_\lambda] = (1-\lambda)\mathrm{Tr}[W_2\sigma] \neq 0$. That is, for each $\lambda \in (0, 1)$, the state π_λ belongs to $\mathbb{C}_0[W_1] \cap \mathbb{C}_0[W_2]$. Now suppose that the statement holds when the cardinality of the set of coherence witnesses is equal to $n-1$. Let $\mathcal{W} = \{W_1, W_2, \ldots, W_n\}$. We define $\mathcal{W}_1 = \{W_1, \ldots, W_{n-1}\}$ and $\mathcal{W}_2 = \{W_2, \ldots, W_n\}$. By induction, the two sets

$$C_0[\mathcal{W}_1] := \bigcap_{W \in \mathcal{W}_1} \mathbb{C}_0[W], \quad C_0[\mathcal{W}_2] := \bigcap_{W \in \mathcal{W}_2} \mathbb{C}_0[W]$$

are nonempty. Suppose $\rho \in C_0[\mathcal{W}_1]$ and $\sigma \in C_0[\mathcal{W}_2]$, i.e., $\mathrm{Tr}[W_i \ \rho] \neq 0$ and $\mathrm{Tr}[W_{i+1}\sigma] \neq 0$ for all $i = 1, 2, \ldots, (n-1)$. If $\mathrm{Tr}[W_n\rho] \neq 0$ ($\mathrm{Tr}[W_1\sigma] \neq 0$), we have $\rho \in C_0[\mathcal{W}]$ ($\sigma \in C_0[\mathcal{W}]$), which proves the nonemptiness of $C_0[\mathcal{W}]$. Therefore, we might assume $\mathrm{Tr}[W_n\rho] = 0$ and $\mathrm{Tr}[W_1\sigma] = 0$. Similarly, for each $\lambda \in (0, 1)$, we define $\pi_\lambda := \lambda\rho + (1-\lambda)\sigma \in \mathbb{D}$. Then we have $\mathrm{Tr}[W_1\pi_\lambda] = \lambda \mathrm{Tr}[W_1\rho] \neq 0$ and $\mathrm{Tr}[W_n\pi_\lambda] = (1-\lambda)\mathrm{Tr}[W_n\sigma] \neq 0$. Moreover, for each integer $i \in [2, n-1]$, we have $a_i := \mathrm{Tr}[W_i\rho] \neq 0$ and $b_i := \mathrm{Tr}[W_i\sigma] \neq 0$. Define $f_i(\lambda) := \mathrm{Tr}[W_i\pi_\lambda] = \lambda a_i + (1-\lambda)b_i = (a_i - b_i)\lambda + b_i$ and $f(\lambda) := \prod_{i=2}^{n-1} f_i(\lambda)$. If $a_i = b_i$ for all $i \in [2, n-1]$, we have $\mathrm{Tr}[W_i\pi_\lambda] = b_i \neq 0$. Then $\pi_\lambda \in C_0[\mathcal{W}]$. If not, $f(\lambda)$ is a nontrivial (not a constant) polynomial of λ. Therefore, there are only finite λ's, say, λ_j $(j = 1, 2, \ldots, N)$ such that $f(\lambda_j) = 0$. Therefore, for each $\lambda \in (0, 1) \setminus \{\lambda_j\}_{j=1}^N$, we always have $f(\lambda) \neq 0$. Hence, $\mathrm{Tr}[W_i\pi_\lambda] = f_i(\lambda) \neq 0$ for all integer $i \in [2, n-1]$, which also holds for $i = 1, n$. Therefore, they all belong to the set $C_0[\mathcal{W}]$.

For each coherent state ρ, a witness W_ρ exists which cannot detect the coherence of ρ, i.e., $\rho \notin \mathbb{C}_0[W_\rho]$. If $C_0[\mathcal{W}]$ is finite, suppose $C_0[\mathcal{W}] = \{\rho_j\}_{j=1}^m$. Then we have

$$\left(\cap_{i=1}^n \mathbb{C}_0[W_i]\right) \bigcap \left(\cap_{j=1}^m \mathbb{C}_0[W_{\rho_j}]\right) = \emptyset,$$

which leads to a contradiction. ■

From Theorem 3, we conclude that the coherence criterion of type C is different from the other types. Moreover, we can also conclude that the coherence criterion of type A is also different from type B (that is, types B_1 and B_2) from the following examples.

First, there do exist W_i's $\in \mathbb{W}_R^c$ which satisfy all the conditions of statement 1 in Theorem 3. Take $d = n = 3$, for example,

$$W_1 = \begin{bmatrix} \frac{R}{2} & 0 & \frac{R}{6} \\ 0 & \frac{R}{2} & \frac{R}{6} \\ \frac{R}{6} & \frac{R}{6} & 0 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 0 & \frac{R}{6} & \frac{R}{6} \\ \frac{R}{6} & \frac{R}{2} & 0 \\ \frac{R}{6} & 0 & \frac{R}{2} \end{bmatrix},$$

$$W_3 = \begin{bmatrix} \frac{R}{2} & \frac{R}{6} & 0 \\ \frac{R}{6} & 0 & \frac{R}{6} \\ 0 & \frac{R}{6} & \frac{R}{2} \end{bmatrix}.$$

We easily check that $\frac{1}{2}W_1 + \frac{1}{2}W_2$, $\frac{1}{2}W_2 + \frac{1}{2}W_3$, $\frac{1}{2}W_1 + \frac{1}{2}W_3$, and $\frac{1}{3}W_1 + \frac{1}{3}W_2 + \frac{1}{3}W_3$ satisfy the assumed conditions. Hence, $\mathbb{C}_R[W_1] \cap \mathbb{C}_R[W_2] \cap \mathbb{C}_R[W_3] = \emptyset$. Moreover, for the case with $d = n = 2$, if we set

$$W_1 = \begin{bmatrix} R & -\frac{R}{2} \\ -\frac{R}{2} & 0 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 0 & \frac{R}{2} \\ \frac{R}{2} & R \end{bmatrix}, \quad \rho = \begin{bmatrix} \frac{1}{4} & \frac{1}{3} \\ \frac{1}{3} & \frac{3}{4} \end{bmatrix},$$

then $\frac{1}{2}W_1 + \frac{1}{2}W_2 = R\mathbb{I}_2$. Hence, $\mathbb{C}_>[W_1] \cap \mathbb{C}_>[W_2] = \mathbb{C}_\geqslant[W_1] \cap \mathbb{C}_\geqslant[W_2] = \emptyset$. However, $\mathrm{Tr}[W_1\rho] = -\frac{R}{12} < 0$, and $\mathrm{Tr}[W_2\rho] = \frac{13}{12}R > R$. Therefore, $\rho \in \mathbb{C}_R[W_1] \cap \mathbb{C}_R[W_2]$.

Now we study more about the inclusion and identity relations among the family of sets $\mathbb{C}_R[W]$, where $W \in \mathbb{W}_R$.

Theorem 4. Property of inclusion or identity. Let $R \in \mathcal{R}$ and $W_1, W_2 \in \mathbb{W}_R^c$. The following statements hold.

(1) If $R \in (0, \infty)$, then $\mathbb{C}_R[W_1] = \mathbb{C}_R[W_2]$ if and only if $W_1 = W_2$ or $W_1 + W_2 = R\mathbb{I}_d$ (the statement is true only if $d = 2$).

(2) If R is $>$ or $\geqslant$, $\mathbb{C}_R[W_1] = \mathbb{C}_R[W_2]$ if and only if $r \in \mathbb{R}_+ := \{r \in \mathbb{R} \mid r > 0\}$ such that $W_2 = rW_1$ exists. Moreover, $\mathbb{C}_R[W_2] \subseteq \mathbb{C}_R[W_1]$ if and only if $a \in \mathbb{R}_+$ and a positive-semidefinite operator P such that $W_2 = aW_1 + P$ exist.

(3) If $R = 0$, then $\mathbb{C}_0[W_1] = \mathbb{C}_0[W_2]$ if and only if some $r \in \mathbb{R} \setminus \{0\}$ such that $W_2 = rW_1$ exist (we call two such witnesses $\mathbb{R}$ equivalent). If W_1 and W_2 are not $\mathbb{R}$ equivalent, then $\mathbb{C}_0[W_1] \not\subseteq \mathbb{C}_0[W_2]$, and $\mathbb{C}_0[W_2] \not\subseteq \mathbb{C}_0[W_1]$.

Proof. We first prove the statements 2 and 3.

For statement 2, for any Hermitian matrix $W \in \mathbb{H}$, define $S_W := \{\rho \in \mathbb{D} | \ \mathrm{Tr}[W\rho] = 0\}$ and $M_W := \{X \in \mathrm{Mat}_d(\mathbb{C}) | \ \mathrm{Tr}[WX] = 0\}$. Suppose that $\mathbb{C}_R[W_1] = \mathbb{C}_R[W_2]$. We claim that $S_{W_1} = S_{W_2}$. Otherwise, without loss of generality, we assume that $\rho \in S_{W_1}$ and $\rho \notin S_{W_2}$, i.e., $\mathrm{Tr}[W_1\rho] = 0$ and $\mathrm{Tr}[W_2\rho] \neq 0$. If $\mathrm{Tr}[W_2\rho] < 0$, then $\rho \in \mathbb{C}_R[W_2]$ and $\rho \notin \mathbb{C}_R[W_1]$, which contradicts the assumption. Therefore, $\mathrm{Tr}[W_2\rho] > 0$. Now choosing any $\sigma \in \mathbb{C}_R[W_1]$, we have $\mathrm{Tr}[W_1\sigma] < 0$ and $\mathrm{Tr}[W_2\sigma] < 0$. For each $\epsilon \in (0, 1)$, define $\rho_\epsilon = \epsilon\sigma + (1 - \epsilon)\rho \in \mathbb{D}$. Then for small enough ϵ, we have $\mathrm{Tr}[W_1\rho_\epsilon] < 0$ but $\mathrm{Tr}[W_2\rho_\epsilon] > 0$, which implies that $\rho_\epsilon \in \mathbb{C}_R[W_1]$ but $\rho_\epsilon \notin \mathbb{C}_R[W_2]$. Hence, we obtain again a contradiction. Therefore, $S_{W_1} = S_{W_2}$. By Lemma 2 in the Appendix, $M_{W_1} = M_{W_2}$ implies $W_2 = cW_1$ for some nonzero complex number. With this at hand, it is easy to show that c is, in fact, a positive real number. The other direction of the first statement in statement 2 is obvious.

Now we prove the second statement. Suppose that there are $a \in \mathbb{R}_+$ and a positive-semidefinite operator P such that $W_2 = aW_1 + P$. For any $\rho \in \mathbb{C}_R[W_2]$, we have $0 > \mathrm{Tr}[W_2\rho] = a\mathrm{Tr}[W_1\rho] + \mathrm{Tr}[P\rho]$. This implies that $\mathrm{Tr}[W_1\rho] < 0$ because we always have $\mathrm{Tr}[P\rho] \geqslant 0$ for positive-semidefinite P. Therefore, $\rho \in \mathbb{C}_R[W_1]$ and $\mathbb{C}_R[W_2] \subseteq \mathbb{C}_R[W_1]$.

Suppose that $\mathbb{C}_R[W_2] \subseteq \mathbb{C}_R[W_1]$. We prove that $a \in \mathbb{R}_+$ and a positive-semidefinite operator P such that $W_2 = aW_1 + P$ exist. Set $t_1 = \mathrm{Tr}[W_1]$ and $t_2 = \mathrm{Tr}[W_2]$. Clearly, $t_1, t_2 \geqslant 0$ as $W_1, W_2 \in \mathbb{H}_\geqslant$. We prove the conclusion according to the following four cases.

(a) Both t_1 and t_2 are positive. Note that $\mathbb{C}_R[W_1] = \mathbb{C}_R[W_1/t_1]$ and $\mathbb{C}_R[W_2] = \mathbb{C}_R[W_2/t_2]$. Following the proof of Lemma 1 and Corollary 1 in Ref. [55] (which is correct under the assumption that the traces of the observables are 1), we have $W_2/t_2 = a_1W_1/t_1 + Q$ for some positive a_1 and positive-semidefinite Q. Therefore, $W_2 = aW_1 + P$, where $a = a_1t_2/t_1$ and $P = t_2Q$.

(b) $t_1 = 0$, but $t_2 > 0$. It is easily verified that $\mathbb{C}_R[W_2] \subseteq \mathbb{C}_R[W_1 + W_2] \subseteq \mathbb{C}_R[W_1]$. Similarly, we have

$$W_2/t_2 = a_1(W_1 + W_2)/t_2 + Q \tag{5}$$

for some positive a_1 and positive-semidefinite Q. From the trace of Eq. (5), we obtain $1 = a_1 + \mathrm{Tr}[Q]$. Therefore, $0 < a_1 \leqslant 1$. Clearly, $a_1 \neq 1$; otherwise, $\mathrm{Tr}[Q] = 0$ implies that $Q = \mathbf{0}$ and $W_1 = -t_2/a_1Q = \mathbf{0}$ is not a nontrivial coherent witness. Hence, $0 < a_1 < 1$. Equation (5) can be reexpressed as

$$W_2 = \frac{a_1}{1 - a_1}W_1 + \frac{t_2}{1 - a_1}Q.$$

(c) $t_1 > 0$, but $t_2 = 0$. We easily show that $\mathbb{C}_R[W_2] \subseteq \mathbb{C}_R[W_1 + W_2] \subseteq \mathbb{C}_R[W_1]$. Similarly, we have

$$(W_1 + W_2)/t_1 = a_1W_1/t_1 + Q \tag{6}$$

for some positive a_1 and positive-semidefinite Q. The trace of Eq. (6) gives rise to $1 = a_1 + \mathrm{Tr}[Q]$. Therefore, $0 < a_1 \leqslant 1$. Clearly, $a_1 \neq 1$; otherwise, $W_2 = t_1Q = \mathbf{0}$ cannot be a nontrivial coherent witness. We can rewrite Eq. (6) as $(1 - a_1)W_1 + W_2 = t_1Q$ and obtain $\mathbb{C}_R[W_2] \subseteq \mathbb{C}_R[(1 - a_1)W_1 + W_2] \subseteq \mathbb{C}_R[W_1]$. On the other hand, $\mathbb{C}_R[t_1Q] = \emptyset$. Therefore, this case cannot happen as $\mathbb{C}_R[W_2] \neq \emptyset$.

(d) $t_1 = t_2 = 0$. For this case, we show that this can happen only when $W_2 = aW_1$. In fact, for small enough $\epsilon > 0$, $\mathbb{C}_R[W_2 + \epsilon\mathbb{I}_d] \subseteq \mathbb{C}_R[W_2] \subseteq \mathbb{C}_R[W_1]$. By the argument of case b, we have $a_\epsilon > 0$ and a positive-semidefinite operator P_ϵ such that $W_2 + \epsilon\mathbb{I}_d = a_\epsilon W_1 + P_\epsilon$, which implies that the diagonals of P_ϵ are all ϵ. Moreover, as P_ϵ is positive semidefinite, the module of each off-diagonal element must be smaller than ϵ. The matrix P_ϵ converges to the zero matrix (up to the entries) when $\epsilon \to 0$. Taking the limit $\epsilon \to 0$, we have $W_2 = aW_1$.

For statement 3, if $W_2 = rW_1$ for $r \neq 0$, we always have $\mathrm{Tr}[W_2\rho] = r\mathrm{Tr}[W_1\rho]$ for all $\rho \in \mathbb{D}$. In this case, $\mathrm{Tr}[W_2\rho] = 0$ if and only if $\mathrm{Tr}[W_1\rho] = 0$. Hence, $\mathbb{C}_0[W_2] = \mathbb{C}_0[W_1]$. Conversely, if $\mathbb{C}_0[W_1] = \mathbb{C}_0[W_2]$, we have, equivalently, $S_{W_1} = S_{W_2}$. By Lemma 2 in the Appendix, we get $M_{W_1} = M_{W_2}$. By definition, M_{W_1} is just the orthogonal complement space to the vector W_1. Hence, the space has dimension $d^2 - 1$. So the vectors that are orthogonal to each element in M_{W_1} are just $\mathbb{C}W_1$. Therefore, there is a nonzero $c \in \mathbb{C}$ such that $W_2 = cW_1$. Since W_2 is Hermitian, we have $c \in \mathbb{R}$.

We prove the second statement in statement 3 by contradiction. Without loss of generality, we may assume that $\mathbb{C}_0[W_1] \subseteq \mathbb{C}_0[W_2]$. Then we have $M_{W_1} \subseteq M_{W_2}$. As both are a linear space of the same dimension $d^2 - 1$, they must be equal, i.e., $M_{W_1} = M_{W_2}$. Similar to the above derivations, we deduce that W_2 and W_1 are $\mathbb{R}$ equivalent, which contradicts the assumption.

For case 1, if R is positive real and $W \in \mathbb{W}_R^c$, we have $\mathbb{C}_R[W] = \mathbb{C}_\geqslant[W] \cup \mathbb{C}_\geqslant[R\mathbb{I}_d - W]$. If $R\mathbb{I}_d - W$ is not positive semidefinite, then $\mathbb{C}_\geqslant[R\mathbb{I}_d - W] \neq \emptyset$. In this setting, $\mathbb{C}_R[W]$ is a disjoint union of two convex sets.

Suppose that $\mathbb{C}_R[W_1] = \mathbb{C}_R[W_2]$. If $\mathbb{C}_\geqslant[R\mathbb{I}_d - W_1] = \emptyset$, we must have $\mathbb{C}_\geqslant[R\mathbb{I}_d - W_2] = \emptyset$, which yields $\mathbb{C}_\geqslant[W_1] = \mathbb{C}_\geqslant[W_2]$. By case 2, $W_2 = rW_1$ for some positive r. Taking into account the trace, we must have $r = 1$. If $\mathbb{C}_\geqslant[R\mathbb{I}_d - W_1] \neq \emptyset$, we have $\mathbb{C}_\geqslant[R\mathbb{I}_d - W_2] \neq \emptyset$. Moreover, either $\mathbb{C}_\geqslant[W_1] = \mathbb{C}_\geqslant[W_2]$ or $\mathbb{C}_\geqslant[W_1] = \mathbb{C}_\geqslant[R\mathbb{I}_d - W_2]$. For the former case, we have shown that $W_1 = W_2$. For the latter case, we also have $\mathbb{C}_\geqslant[W_2] = \mathbb{C}_\geqslant[R\mathbb{I}_d - W_1]$, from which we can deduce $(d - 1)W_1 = R\mathbb{I}_d - W_2$ and $(d - 1)W_2 = R\mathbb{I}_d - W_1$. If $d > 2$, we have $W_1, W_2 \propto \mathbb{I}_d$, which contradicts $W_1, W_2 \in \mathbb{W}_R^c$. However, for the case with $d = 2$, the two equations merge into $W_1 + W_2 = R\mathbb{I}_d$.

Moreover, for $d = 2$ we have $\mathrm{Tr}[W_1\rho] + \mathrm{Tr}[W_2\rho] = R$ for any $\rho \in \mathbb{D}$. Hence, $\mathrm{Tr}[W_1\rho] < 0$ if and only if $\mathrm{Tr}[W_2\rho] >$

R, and $\mathrm{Tr}[W_1\rho] > R$ if and only if $\mathrm{Tr}[W_2\rho] < 0$. Therefore, $\mathbb{C}_R[W_1] = \mathbb{C}_R[W_2]$. ■

With the above results, we are ready to give a full classification of the variant coherence criteria.

Theorem 5. There are exactly four inequivalent classes among the variant coherence criteria $\{(\mathbb{W}_R, D_R, I_R) \mid R \in \mathcal{R}\}$.

Proof. First, we have shown that the criteria of type A are all equivalent.

Second, the criterion of type C is not equivalent to other types. In fact, in the settings of A, B_1, and B_2, there are always two witnesses, W_1 and W_2, such that $\mathbb{C}_R[W_1] \cap \mathbb{C}_R[W_2] = \emptyset$. However, that is not the case when $R = 0$.

Third, the criterion of type B_2, i.e., $(\mathbb{W}_\geqslant, D_\geqslant, I_\geqslant)$, is not equivalent to types A and B_1. In fact, the criterion of type B_2 is finitely completable, but the other two are not.

Moreover, $\mathbb{C}_>[W]$ are always convex for each $W \in \mathbb{W}_>^c$. But this fact may not be true for $\mathbb{C}_R[W]$ when $R > 0$. Therefore, criteria of types A and B_1 are inequivalent. To conclude we have exactly four inequivalent classes of coherence criteria. ■

IV. CONCLUSION AND DISCUSSIONS

Based on prior knowledge of observables, we have presented a series coherence criteria which detect coherence better than the usual ones without prior knowledge of observables. Moreover, through a systematic and rigorous study of the properties of criteria such as completeness, finite completeness, finite intersection, and inclusion, we have singled out four classes of inequivalent coherence criteria. These results help to deepen our understanding of coherence-detection methodologies and thereby highlight advancements in quantum technologies.

There are also some interesting problems left, such as the condition of the inclusion of $\mathbb{C}_R[W]$ when $R > 0$ for statement 1 of Theorem 4. From a practical perspective, delving into the advantages of coherence witnesses when prior knowledge is retained, particularly in scenarios involving noise in the measurement of the operator value, is a compelling and worthwhile research avenue. Further exploration of coherence detection enriched by prior knowledge may promise to unlock even greater potential. It would also be appealing to extend our scheme to deal with the case of entanglement witnesses. Our research may serve as a catalyst for future investigations of quantum coherence detection, as well as the detection of other resources like quantum correlations.

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APPENDIX: TWO LEMMAS AND A PROOF OF THE CLAIM $\mathbb{W}_R^c = \mathbb{W}_R \cap \Lambda_-$

In order to prove the claim $\mathbb{W}_R^c = \mathbb{W}_R \cap \Lambda_-$, we need the following lemma.

Lemma 1. If $W \in \mathbb{H}_\geqslant \setminus \{\mathbf{0}\}$ is not positive semidefinite, then some density matrices $\rho \in \mathbb{D}$ exist such that $\mathrm{Tr}[W\rho] < 0$. Moreover, a positive density matrix $\sigma \in \mathbb{D}$ also exists such that $\mathrm{Tr}[W\sigma] = 0$.

Proof. From the given assumption, the matrix W has both positive and negative eigenvalues. Suppose the spectral decomposition of W is

$$W = \sum_{\lambda_i>0} \lambda_i |e_i\rangle\langle e_i| + \sum_{\lambda_j<0} \lambda_j |e_j\rangle\langle e_j| + \sum_{\lambda_k=0} \lambda_k |e_k\rangle\langle e_k|,$$

where $\{|e_i\rangle\}_{\lambda_i>0} \cup \{|e_j\rangle\}_{\lambda_j<0} \cup \{|e_k\rangle\}_{\lambda_k=0}$ is an orthonormal basis of the system $\mathcal{H}$. For any fixed j such that $\lambda_j < 0$, the state $\rho = |e_j\rangle\langle e_j|$ satisfies $\mathrm{Tr}[W\rho] = \lambda_j < 0$. Set $\alpha := \sum_{\lambda_i>0} \lambda_i$ and $\beta := \sum_{\lambda_j<0} |\lambda_j|$. Clearly, $\alpha, \beta > 0$. We define

$$Q = \sum_{\lambda_i>0} \beta |e_i\rangle\langle e_i| + \sum_{\lambda_j<0} \alpha |e_j\rangle\langle e_j| + \sum_{\lambda_k=0} |e_k\rangle\langle e_k|.$$

Then Q is a positive matrix. Moreover, we have

$$\mathrm{Tr}[WQ] = \beta \sum_{\lambda_i>0} \lambda_i + \alpha \sum_{\lambda_j<0} \lambda_j = \beta\alpha + \alpha \times (-\beta) = 0.$$

Then $\sigma = Q/\mathrm{Tr}[Q]$ is the density matrix we wanted. That is, σ is a positive matrix and satisfies $\mathrm{Tr}[W\sigma] = 0$.

Proof of the claim $\mathbb{W}_R^c = \mathbb{W}_R \cap \Lambda_-$. Clearly, both sets are contained in $\mathbb{W}_R \setminus \{\mathbf{0}\}$. For any $W \in \mathbb{W}_R \cap \Lambda_-$, W has some negative eigenvalue. From Lemma 1, a ρ exists such that $\mathrm{Tr}[W\rho] < 0$. Hence, $\mathbb{C}_R[W] \neq \emptyset$ for such W. Therefore, $\mathbb{W}_R \cap \Lambda_- \subseteq \mathbb{W}_R^c$.

Now suppose that $W \in \mathbb{W}_R \setminus \{\mathbf{0}\}$ but $W \notin \mathbb{W}_R \cap \Lambda_-$. Therefore, W must be positive semidefinite (which is impossible when $R = 0$), which implies that $\mathrm{Tr}[W\rho] \geqslant 0$ for all $\rho \in \mathbb{D}$. Therefore, when R is $\geqslant$ or $>$, $\mathbb{C}_R[W] = \emptyset$. If R is a positive real number, all the eigenvalues of W must be less than or equal to R. Hence, we have $\mathbf{0} \leqslant W \leqslant R\mathbb{I}_d$. Therefore, $0 \leqslant \mathrm{Tr}[W\rho] \leqslant R$ for all $\rho \in \mathbb{D}$ and $\mathbb{C}_R[W] = \emptyset$ for such W. No matter what R is, the set $\mathbb{C}_R[W] = \emptyset$. This implies that such a W is not an effective coherence witness, i.e., $W \notin \mathbb{W}_R^c$. Hence, we must have $\mathbb{W}_R \cap \Lambda_- = \mathbb{W}_R^c$. ■

In the following, we present another lemma that is reused multiple times throughout this paper.

Lemma 2. Let $R \in \mathcal{R}$ and $W \in \mathbb{W}_R^c$ be a coherent witness. Define $S_W := \{\rho \in \mathbb{D} \mid \mathrm{Tr}[W\rho] = 0\}$, $H_W := \{H \in \mathbb{H} \mid \mathrm{Tr}[WH] = 0\}$, and $M_W := \{X \in \mathrm{Mat}_d(\mathbb{C}) \mid \mathrm{Tr}[WX] = 0\}$. Then the following sets are equal: $\mathrm{span}_\mathbb{C}(S_W) = \mathrm{span}_\mathbb{C}(H_W) = M_W$.

Proof. Notice that each $W \in \mathbb{W}_R^c$ is not positive semidefinite. From Lemma 1, there is a positive state $\rho \in \mathbb{D}$ such that $\mathrm{Tr}[W\rho] = 0$. So $\rho \in S_W$ by definition.

Clearly, $S_W \subseteq H_W \subseteq M_W$ by definition. Therefore, $\text{span}_{\mathbb{C}}(S_W) \subseteq \text{span}_{\mathbb{C}}(H_W) \subseteq M_W$. First, we show that $M_W \subseteq \text{span}_{\mathbb{C}}(H_W)$. In fact, for any $X \in M_W$, we have $\text{Tr}[WX] = 0$. Therefore, $\text{Tr}[WX^\dagger] = \text{Tr}[X^\dagger W] = \text{Tr}[X^\dagger W^\dagger] = \text{Tr}[(WX)^\dagger] = \text{Tr}[WX]^\dagger = 0$. Then we have $\text{Tr}[W(X + X^\dagger)] = 0$ and $\text{Tr}[W(\text{i}X - \text{i}X^\dagger)] = 0$. However, since $H_1 = (X + X^\dagger)$ and $H_2 = \text{i}(X - X^\dagger) \in \mathbb{H}$, they both belong to H_W. Notice that $X = (H_1 - \text{i}H_2)/2$. Hence, it belongs to $\text{span}_{\mathbb{C}}(H_W)$.

Now we prove that $\text{span}_{\mathbb{C}}(H_W) \subseteq \text{span}_{\mathbb{C}}(S_W)$. It suffices to show that $H_W \subseteq \text{span}_{\mathbb{C}}(S_W)$. For any $H \in H_W$, we have $H \in \mathbb{H}$ and $\text{Tr}[WH] = 0$. We can find small enough $\epsilon \in (0, 1)$ such that $P_\epsilon := \epsilon H + (1 - \epsilon)\rho$ is positive. Set $N = \text{Tr}(P_\epsilon)$. Then $\rho_\epsilon := P_\epsilon/N = \frac{\epsilon}{N}H + \frac{1-\epsilon}{N}\rho \in \mathbb{D}$. Moreover, we have the equality

$$\text{Tr}[W\rho_\epsilon] = \frac{\epsilon}{N}\text{Tr}[WH] + \frac{1-\epsilon}{N}\text{Tr}[W\rho] = 0.$$

Therefore, $\rho_\epsilon \in S_W$. Note that H can be written as a linear combination of ρ_ϵ and ρ. Hence, $H_W \subseteq \text{span}_{\mathbb{C}}(S_W)$, which completes the proof. ■

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