Anyonic quantum multipartite maskers in the Kitaev model

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The structure of quantum mechanics forbids a bipartite scenario for masking quantum information, however, it allows multipartite maskers. The Latin squares are found to be closely related to a series of tripartite maskers. This adds another item, significantly different from the original no-cloning theorem, to the no-go theorems. On the other hand, anyonic excitations in two dimensions exhibit exotic collective behaviors of quantum physics, and open the avenue of fault-tolerant topological quantum computing. Here, we give the Latin-square construction of Abelian and Ising anyons in the Kitaev model and study the maskable space configuration in anyonic space. The circling and braiding of Kitaev anyons are masking operations on extended hyperdisks in anyonic space. We also realize quantum information masking in a teleportation way in the Kitaev Ising anyon model.

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I. INTRODUCTION

Several no-go theorems have been founded in the study of quantum information, such as no-cloning [1,2], no-deleting [3], no-broadcasting [4,5], and no-hiding theorems [6], which describe the difference between the quantum and classical worlds. Recently, Modi *et al.* [7] introduced another item, the no-masking theorem, to the family of no-go theorems. That is, it is impossible to mask an arbitrary state into bipartite quantum systems. However, the task can be achieved in multipartite scenarios, which reveals a significant difference between the no-masking theorem and the original no-cloning theorem. Li *et al.* [8] presented a unified construction of tripartite scenarios based on the Latin squares. Subsequently, probabilistic and approximate masking protocols are also studied [9].

Quantum information masking is crucial to many quantum communication topics, such as quantum secret sharing [10,11] and quantum bit commitments [12,13]. Optical experimental demonstrations of masking schemes have been reported very recently [14,15]. However, quantum masking in condensed-matter (many-body) systems is still absent.

This paper is devoted to the investigation of quantum information masking in anyons, which are quasiparticles living in two-dimensional condensed-matter systems. Anyons do not fit into the usual statistics of fermions and bosons, but obey a new form of fractional statistics, closely related with their famous braiding and fusion rules [16–25]. The nontrivial topological properties of anyons in the Kitaev spin-lattice model opened the avenue of topological quantum computing [26–31].

The quantum information masking experiment is a technology that uses quantum mechanics to mask and protect information. Although the quantum information masking experiment has many advantages in theory, it still faces many difficulties and challenges in practical applications. The Kitaev anyon system provides different ideas and methods for solving these problems in quantum information masking, making it a strong contender for experimental implementation.

(1) The fragility of quantum states: Quantum states are very fragile and are easily affected by noise and interference in the environment, leading to information loss and errors. Therefore, how to effectively protect and manipulate quantum states in quantum information masking experiments is a huge challenge. The Kitaev anyon system is a topologically protected system that can resist various forms of noise such as phase noise. There is an energy gap between the ground-state energy and excited-state energy of anyon systems, which can make the system immune to local errors. This makes anyon systems robust for quantum computation and quantum information [26–29].

(2) Operation and control of qubits: In practical applications, the operation and control of qubits require very precise and meticulous attention, otherwise it will lead to information loss and errors. Therefore, how to achieve precise and reliable control of quantum bits is also an important challenge faced by quantum information masking experiments. The toric code (or Kitaev surface code) provides a feasible and relatively simple manipulation method [27,30].

(3) Generation and control of quantum entanglement: Quantum entanglement is one of the key elements for realizing quantum information masking. However, in practical experiments, generating and controlling high-quality quantum entanglement remains a technical challenge. The nonlocal topological properties of particles in the Kitaev model provide a potential solution to this problem. The topological properties describe the overall properties of the system rather than the

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properties of individual particles. In addition, the ground state of a system of host non-Abelian anyons is usually highly entangled [26–29].

(4) The masking scheme should be universal for quantum circuits, and a typical example, Ising anyons, are not intuitively universal. In the schemes of Li *et al.* [8], Latin squares are used to realize masking in normal Hilbert spaces. But the anyonic space is composed of the direct product of each Hilbert subsystem space and the fusion Hilbert spaces of all charges,

$$H_{AB}^{c} = \bigoplus_{ab} H_A^a \otimes H_B^b \otimes V_{ab}^{c}, \tag{1}$$

where H_A^a is the space of charge a in the Hilbert subsystem space A, and the space V_{ab}^c is the fusion space containing the fusion rules (details are shown in the following). In the process of fusion (in the space V_{ab}^c), the braiding of charges induces the topological properties and causes entanglement. The braiding process of Ising anyons not only generates additional phases, but also accompanies the exchange of particles (the creation and the annihilation of new particles—see Figs. 3 and 4). It is intriguing whether they will interfere with the masking process. Our work demonstrates that masking can still be achieved, even in the case of Ising anyons.

In the present work, we construct the Latin-square masking scenarios of Abelian anyons and Ising anyons (the simplest non-Abelian anyons) in the Kitaev model. The braiding operations on anyons manipulate the states in the maximal maskable space, which are extended hyperdisks. This maskable space configuration of anyons allows a series of masking schemes in many-body systems. We also demonstrate the process of masking based on teleportation in the Kitaev Ising anyon model.

This paper is organized as following: In the next section, we present the masking protocol in a system of Abelian anyons using the Latin-square construction. Section III shows the process of masking in non-Abelian Ising anyons, which has an explanation based on quantum teleportation. Finally, the last section presents a summary.

II. THE KITAEV MODEL AND THE ABELIAN 1/2-ANYON

The Kitaev model is a honeycomb spin-lattice model in which each 1/2 spin is located on the vertex of the hexagon. Fermions and Z_2 vortices are the excitation states of this exactly solvable model. The excitations are divided into two cases, Abelian anyons and non-Abelian anyons, and furthermore 16 types of statistics with Chern number c mod 16. Bosons and gapped fermions are Abelian anyons which have an even Chern number c. Vortices and gapless fermions are non-Abelian anyons whose Chern numbers are odd [26,27].

A. Review of the Abelian 1/2-anyon and braiding rules

Wilczek [32,33] first introduced the Abelian anyons which are represented by the braiding group. He pointed out that the braiding of different quasiparticles for one circle caused a Aharonov-Bohm phase $\exp(i2\pi k\alpha)$ (α is the statistical parameter and k is the winding number) [32–34]. The Abelian anyons have four superselection sectors: 1 (the vacuum), vortices e and m, and fermion ε (we ignore another case in which

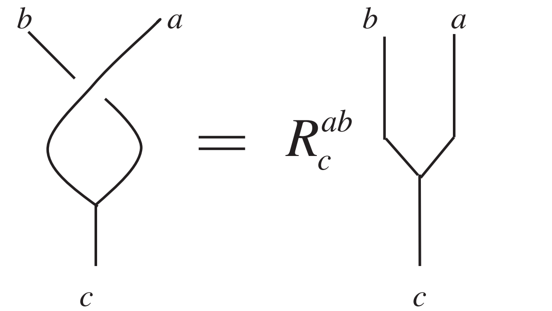

FIG. 1. The braiding operation of a and b. The fusion of a and b is c.

the vortices are two mutual antiparticles). There are two cases of Abelian anyons: One is the Chern number $c = 0, 8$, and the other is $c = \pm 4$. Both of them are mod 16 with different topological spins and Frobenius-Schur indicators. Here, we consider the simplest Abelian anyons (called Abelian anyons for short) with a Chern number $c = 0$, topological spin $\theta = 1$, and Frobenius-Schur indicator $\varkappa = 1$. The fusion rules of the Abelian anyons give

$$e \times e = m \times m = 1, \quad \varepsilon \times \varepsilon = 1, \tag{2}$$

$$\varepsilon \times e = m, \quad \varepsilon \times m = e, \quad e \times m = \varepsilon. \tag{3}$$

The Abelian anyons are mod 2, which means that two of the same quasiparticles annihilate to the vacuum. The braiding operation can be represented by R_z^{xy} as in Fig. 1.

In this case, as shown in Fig. 2, all associativity relations are trivial and the braiding rules are

$$\begin{aligned} &R_\varepsilon^{em} = 1, \quad R_\varepsilon^{me} = -1, \quad R_m^{e\varepsilon} = 1, \quad R_m^{\varepsilon e} = -1,\\ &R_e^{\varepsilon m} = 1, \quad R_e^{m\varepsilon} = -1, \quad R_1^{ee} = R_1^{mm} = 1, \quad R_1^{\varepsilon\varepsilon} = -1. \end{aligned} \tag{4}$$

The vortices e and m are bosons. The exchange of e and m is different from that of m and e ($R_\varepsilon^{em} = 1$, $R_\varepsilon^{me} = -1$). With the help of the representation of Gentile statistics, we gave a physical image for the exchange of different Abelian anyons [35]. Braiding different Abelian anyons depends on the topology of the particles. Braiding two different quasiparticles for one circle gives an additional phase -1 (e.g., $R_\varepsilon^{em} R_\varepsilon^{me} = -1$) to the state, so these kinds of anyons are called Abelian 1/2-anyons ($\exp[i2\pi \cdot (1/2) \cdot 1] = -1$).

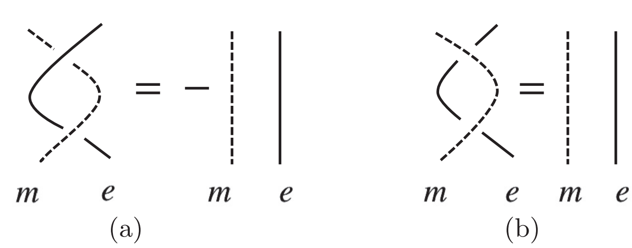

FIG. 2. The braiding (circling) of e and m. (a) An e particle circling an m particle counterclockwise corresponds to $R_\varepsilon^{me} \cdot R_\varepsilon^{em} = -1$. (b) There is no braiding in this case, $R_\varepsilon^{me} \cdot (R_\varepsilon^{me})^{-1} = 1$, where the two curves are separated.

B. Masking protocol of the Abelian 1/2-anyon

According to the orthogonality of the matrix elements, Latin squares are useful tools in quantum information masking. The matrix elements in each row and column of the Latin squares are all different, which leads to the disappearance of the cross terms during a trace. Two mutually orthogonal Latin squares are those matrices whose products of the matrix elements at the same locations are all different. Based on Theorem 2 in Ref. [8], for a d-dimensional space, the quantum states can be masked in $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$ systems. In the masking protocol of the Abelian 1/2-anyons, the space is four dimensional (four superselection sectors $\{|1\rangle, |e\rangle, |m\rangle, |\varepsilon\rangle\}$). We define three matrices A, B, C in $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$,

$$A \equiv \begin{pmatrix}1\\1\\1\\1\end{pmatrix} \otimes (1 \quad e \quad m \quad \varepsilon) \text{ or } \begin{pmatrix}1\\e\\m\\\varepsilon\end{pmatrix} \otimes (1 \quad 1 \quad 1 \quad 1), \tag{5}$$

where B and C are two mutually orthogonal Latin squares in $\mathbb{C}^4$. The more ideal case is where A, B, C are three mutually orthogonal Latin squares in $\mathbb{C}^4$. The matrix A can extend the restrictive condition to Eq. (5). The masking protocol process maps $\{|j\rangle || 1\rangle, |e\rangle, |m\rangle, |\varepsilon\rangle\}$ to

$$|\psi_j\rangle = \frac{1}{2}\sum_k |A_{jk}B_{jk}C_{jk}\rangle. \tag{6}$$

So we have the encoding process as

$$|\Psi\rangle = \sum_j \alpha_j|\psi_j\rangle = \frac{1}{2}\sum_{jk}\alpha_j\big|A_{jk}B_{jk}C_{jk}\big\rangle. \tag{7}$$

In this case,

$$\mathrm{Tr}_{AB}(|\Psi\rangle\langle\Psi|) = \mathrm{Tr}_{AC}(|\Psi\rangle\langle\Psi|) = \mathrm{Tr}_{BC}(|\Psi\rangle\langle\Psi|) = \frac{I}{4},$$

where I is the identity matrix. The quantum information is stored in the correlation of the tripartite system.

What calls for special attention is that the braiding of anyons (exchange) and the braiding for circles (circling), which are equal to operations on extended hyperdisks (which differ from hyperdisks on normal Hilbert space), do not affect the masking process, because the state vectors and their conjugations cancel out the phase factors attributed to the braidings. The cases in point are the exchange of B and C or braiding C around B for one circle. In other words, quantum information masking is invariant under the braiding operations of Abelian anyons. The details are given below.

Proof. We give one example and define

$$A = \begin{pmatrix}1 & e & m & \varepsilon\\1 & e & m & \varepsilon\\1 & e & m & \varepsilon\\1 & e & m & \varepsilon\end{pmatrix}, \quad B = \begin{pmatrix}1 & e & m & \varepsilon\\e & 1 & \varepsilon & m\\m & \varepsilon & 1 & e\\\varepsilon & m & e & 1\end{pmatrix},$$
$$C = \begin{pmatrix}1 & e & m & \varepsilon\\\varepsilon & m & e & 1\\e & 1 & \varepsilon & m\\m & \varepsilon & 1 & e\end{pmatrix}. \tag{8}$$

B and C are mutually orthogonal Latin squares. An arbitrary state $\alpha|1\rangle + \beta|e\rangle + \gamma|m\rangle + \delta|\varepsilon\rangle$ is mapped to a tripartite system $|\Psi\rangle$,

$$\begin{aligned}\to \tfrac{1}{2}[&\alpha(|111\rangle + |eee\rangle + |mmm\rangle + |\varepsilon\varepsilon\varepsilon\rangle)\\&+ \beta(|1e\varepsilon\rangle + |e1m\rangle + |m\varepsilon e\rangle + |\varepsilon m1\rangle)\\&+ \gamma(|1me\rangle + |e\varepsilon 1\rangle + |m1\varepsilon\rangle + |\varepsilon em\rangle)\\&+ \delta(|1\varepsilon m\rangle + |em\varepsilon\rangle + |me1\rangle + |\varepsilon 1e\rangle)],\end{aligned} \tag{9}$$

$$\mathrm{Tr}_{AB}(|\Psi\rangle\langle\Psi|) = \frac{1}{2}\sum_{jk}|\alpha_j|^2|C_{jk}\rangle\langle C_{jk}| = \frac{I}{4}. \tag{10}$$

With a similar operation, the other two partial traces are $\mathrm{Tr}_{AC}(|\Psi\rangle\langle\Psi|) = \mathrm{Tr}_{BC}(|\Psi\rangle\langle\Psi|) = I/4$. The mapping of Eq. (7) is indeed a masking protocol.

Now we show the braiding (exchange) of B and C. According to Eq. (4), braiding B and C gives

$$\begin{aligned}\to \tfrac{1}{2}[&\alpha(|111\rangle + |eee\rangle + |mmm\rangle + |\varepsilon\varepsilon\varepsilon\rangle)\\&+ \beta(R^{e\varepsilon}|1e\varepsilon\rangle + |e1m\rangle + R^{\varepsilon e}|m\varepsilon e\rangle + |\varepsilon m1\rangle)\\&+ \gamma(R^{me}|1me\rangle + |e\varepsilon 1\rangle + |m1\varepsilon\rangle + R^{em}|\varepsilon em\rangle)\\&+ \delta(R^{\varepsilon m}|1\varepsilon m\rangle + R^{m\varepsilon}|em\varepsilon\rangle + |me1\rangle + |\varepsilon 1e\rangle)].\end{aligned} \tag{11}$$

It can be proved that we still have

$$\mathrm{Tr}_{AB}(|\Psi\rangle\langle\Psi|) = \mathrm{Tr}_{AC}(|\Psi\rangle\langle\Psi|) = \mathrm{Tr}_{BC}(|\Psi\rangle\langle\Psi|) = \frac{I}{4}.$$

Exchanging A and C, and A and B is similar.

Next, when we braid C around B for one circle, so we have

$$\begin{aligned}\to \tfrac{1}{2}[&(\alpha|111\rangle + |eee\rangle + |mmm\rangle + |\varepsilon\varepsilon\varepsilon\rangle)\\&+ \beta(R^{e\varepsilon}R^{\varepsilon e}|1e\varepsilon\rangle + |e1m\rangle + R^{\varepsilon e}R^{e\varepsilon}|m\varepsilon e\rangle + |\varepsilon m1\rangle)\\&+ \gamma(R^{me}R^{em}|1me\rangle + |e\varepsilon 1\rangle + |m1\varepsilon\rangle + R^{em}R^{me}|\varepsilon em\rangle)\\&+ \delta(R^{\varepsilon m}R^{m\varepsilon}|1\varepsilon m\rangle + R^{m\varepsilon}R^{\varepsilon m}|em\varepsilon\rangle + |me1\rangle + |\varepsilon 1e\rangle)].\end{aligned} \tag{12}$$

The formula $\mathrm{Tr}_{AB}(|\Psi\rangle\langle\Psi|) = \mathrm{Tr}_{AC}(|\Psi\rangle\langle\Psi|) = \mathrm{Tr}_{BC}(|\Psi\rangle\langle\Psi|) = I/4$ is still tenable, and so does braiding any two of A, B, and C for circles. ■

As stated above, it can be said with certainty that all conclusions mentioned above are tenable when A, B, and C are three mutually orthogonal Latin squares.

III. THE ISING ANYON

The algebraic and topological structures in conformal theory lead to the exotic topological properties of non-Abelian anyons. They are represented in the framework of topological quantum field theory whose core is a unitary modular category. The properties of non-Abelian anyons cannot be described as simply as in the Abelian case. Non-Abelian anyons are those vortices and gapless fermions whose Chern numbers are odd.

A. Review of the Ising anyon and braiding rules

The simplest non-Abelian anyon is called an Ising anyon ($c = 1$ is the Chern number). There are three superselection sectors: the vacuum 1, the fermion ε, and the vortex σ. For

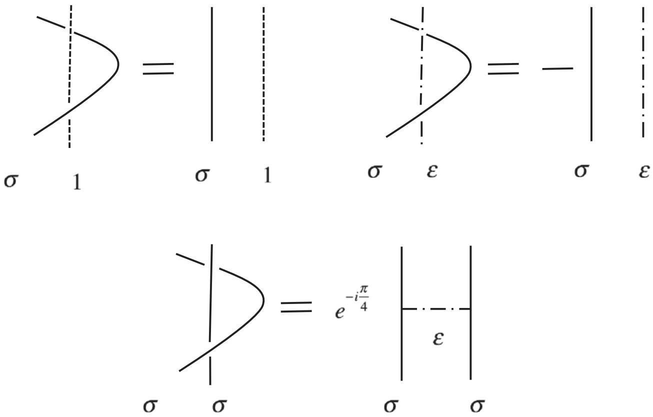

FIG. 3. The braiding operation of an Ising anyon. A σ circling an ε gives a phase factor -1. Two σ's circling induces $\exp(-i\pi/4)$ and is accompanied with an exchange of an ε.

non-Abelian anyons, the topological spin and the Frobenius-Schur indicators of vortices are divided into eight pieces, $\theta_\sigma = \exp(i\pi c/8)$ and $\varkappa_\sigma = (-1)^{(c^2-1)/8}$, and in addition $\theta_1 = 1$, $\theta_\varepsilon = -1$, $\varkappa_1 = \varkappa_\varepsilon = 1$. In the Ising anyon case, $\theta_\sigma = \exp(i\pi/8)$ and $\varkappa_\sigma = \varkappa = 1$. The fusion rules of non-Abelian anyons are

$$\varepsilon \times \varepsilon = 1, \quad \varepsilon \times \sigma = \sigma, \quad \sigma \times \sigma = 1 + \varepsilon. \tag{13}$$

According to the definition of braiding (Fig. 1), the braiding rules of Ising anyons give (Fig. 3)

$$\begin{aligned} &R_1^{\varepsilon\varepsilon} = -1, \quad R_1^{\sigma\sigma} = \varkappa e^{-\frac{i\pi c}{8}} = e^{-\frac{i\pi}{8}}, \\ &R_\sigma^{\varepsilon\sigma} = R_\sigma^{\sigma\varepsilon} = -i^c = -i, \quad R_\varepsilon^{\sigma\sigma} = \varkappa e^{\frac{i3\pi c}{8}} = e^{\frac{i3\pi}{8}}. \end{aligned} \tag{14}$$

Besides an additional phase factor $\exp(-i\pi/4)$, two σ's braiding exchange a fermion ε.

B. Masking protocol of the Ising anyon

In the Ising anyon case, the space is three dimensional (three superselection sectors $\{|1\rangle, |\varepsilon\rangle, |\sigma\rangle\}$). The quantum information can be masked in a $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ Latin-square construction. We also define the A matrix as

$$A \equiv \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \otimes (1 \quad \varepsilon \quad \sigma) \quad \text{or} \quad \begin{pmatrix} 1 \\ \varepsilon \\ \sigma \end{pmatrix} \otimes (1 \quad 1 \quad 1). \tag{15}$$

We show the first form of A as an example. The Latin squares in odd dimensions can be represented by cyclic permutation operators. Here, we define forward and backward cyclic permutation operators P_{cf} and P_{cb} as

$$\begin{aligned} P_{cf}|1, 2, 3, \ldots, d\rangle &= |d, 1, 2, \ldots, d-1\rangle, \\ P_{cf}^2|1, 2, 3, \ldots, d\rangle &= |d-1, d, 1, \ldots, d-2\rangle, \\ P_{cb}|1, 2, 3, \ldots, d\rangle &= |2, 3, 4, \ldots, d, 1\rangle, \\ P_{cb}^2|1, 2, 3, \ldots, d\rangle &= |3, 4, \ldots, d, 1, 2\rangle. \end{aligned} \tag{16}$$

So B and C are

$$B = \begin{pmatrix} 1 \\ P_{cf} \\ P_{cf}^2 \end{pmatrix} \otimes (1 \quad \varepsilon \quad \sigma) = \begin{pmatrix} 1 & \varepsilon & \sigma \\ \sigma & 1 & \varepsilon \\ \varepsilon & \sigma & 1 \end{pmatrix}, \tag{17}$$

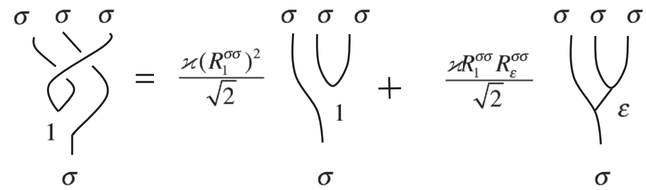

FIG. 4. The tripartite braiding operation of an Ising anyon.

$$C = \begin{pmatrix} 1 \\ P_{cb} \\ P_{cb}^2 \end{pmatrix} \otimes (1 \quad \varepsilon \quad \sigma) = \begin{pmatrix} 1 & \varepsilon & \sigma \\ \varepsilon & \sigma & 1 \\ \sigma & 1 & \varepsilon \end{pmatrix}. \tag{18}$$

Again, through the construction of Eqs. (6) and (7), an arbitrary state $\alpha|1\rangle + \beta|\varepsilon\rangle + \gamma|\sigma\rangle$ is mapped into

$$\begin{aligned} |\Psi\rangle = \frac{1}{\sqrt{3}}[&\alpha(|111\rangle + |\varepsilon\varepsilon\varepsilon\rangle + |\sigma\sigma\sigma\rangle) \\ &+ \beta(|1\sigma\varepsilon\rangle + |\varepsilon 1\sigma\rangle + |\sigma\varepsilon 1\rangle) \\ &+ \gamma(|1\varepsilon\sigma\rangle + |\varepsilon\sigma 1\rangle + |\sigma 1\varepsilon\rangle)]. \end{aligned} \tag{19}$$

In the same way, we can get $\mathrm{Tr}_{AB}(|\Psi\rangle\langle\Psi|) = \mathrm{Tr}_{AC}(|\Psi\rangle\langle\Psi|) = \mathrm{Tr}_{BC}(|\Psi\rangle\langle\Psi|) = I/3$, and the masking process is accomplished.

The braiding of non-Abelian anyons is a little complicated. Based on Eq. (14) and Fig. 3, the circling of ε and σ gives a phase factor -1, and the circling of two σ's induces a phase factor $\exp(-i\pi/4)$ that is accompanied by an exchange of one ε. When circling two of the three particles, circling B around A, for example, we have

$$\begin{aligned} |\Psi\rangle = \frac{1}{\sqrt{3}}[&\alpha(|111\rangle - |\varepsilon\varepsilon\varepsilon\rangle + e^{-i\frac{\pi}{4}}|\sigma\sigma\sigma\rangle) \\ &+ \beta(|1\sigma\varepsilon\rangle + |\varepsilon 1\sigma\rangle - |\sigma\varepsilon 1\rangle) \\ &+ \gamma(|1\varepsilon\sigma\rangle - |\varepsilon\sigma 1\rangle + |\sigma 1\varepsilon\rangle)]. \end{aligned}$$

In this case, the phases cased by circling cancel out each other in $|\Psi\rangle\langle\Psi|$ and $\mathrm{Tr}_{AB}(|\Psi\rangle\langle\Psi|) = \mathrm{Tr}_{AC}(|\Psi\rangle\langle\Psi|) = \mathrm{Tr}_{BC}(|\Psi\rangle\langle\Psi|) = I/3$. We can mask the information too, as does circling of any two particles in other cases. Braiding (exchange) two adjacent particles, such as A and B or B and C, is similar. For instance, braiding A and B shows

$$\begin{aligned} |\Psi\rangle = \frac{1}{\sqrt{3}}[&\alpha(|111\rangle + R^{\varepsilon\varepsilon}|\varepsilon\varepsilon\varepsilon\rangle + R^{\sigma\sigma}|\sigma\sigma\sigma\rangle) \\ &+ \beta(|1\sigma\varepsilon\rangle + |\varepsilon 1\sigma\rangle + R^{\sigma\varepsilon}|\sigma\varepsilon 1\rangle) \\ &+ \gamma(|1\varepsilon\sigma\rangle + R^{\varepsilon\sigma}|\varepsilon\sigma 1\rangle + |\sigma 1\varepsilon\rangle)]. \end{aligned}$$

Here, $R^{\sigma\sigma}$ has two forms $R_1^{\sigma\sigma}$ or $R_\varepsilon^{\sigma\sigma}$ according to the fusion rules. Regardless of the situation, $\mathrm{Tr}_{AB}(|\Psi\rangle\langle\Psi|) = \mathrm{Tr}_{AC}(|\Psi\rangle\langle\Psi|) = \mathrm{Tr}_{BC}(|\Psi\rangle\langle\Psi|) = I/3$. The masking process can be achieved. The exchange of A and C proceeds in two stages: One is a tripartite braiding (Fig. 4), and the other is the exchange of B and C. The tripartite braiding is divided into two parts with different probabilities (Fig. 4). So the mapping goes to

$$|\Psi\rangle=\frac{1}{\sqrt{3}}\bigg[\alpha\bigg(|111\rangle+(R^{\varepsilon\varepsilon})^3|\varepsilon\varepsilon\varepsilon\rangle+\frac{\varkappa\big(R_1^{\sigma\sigma}\big)^2}{\sqrt{2}}|\sigma\sigma\sigma\rangle_1+\frac{\varkappa R_1^{\sigma\sigma}R_\varepsilon^{\sigma\sigma}}{\sqrt{2}}|\sigma\sigma\sigma\rangle_\varepsilon\bigg)+\beta(R^{\sigma\varepsilon}|1\sigma\varepsilon\rangle+R^{\varepsilon\sigma}|\varepsilon1\sigma\rangle+R^{\sigma\varepsilon}|\sigma\varepsilon1\rangle)+\gamma(R^{\varepsilon\sigma}|1\varepsilon\sigma\rangle+R^{\varepsilon\sigma}|\varepsilon\sigma1\rangle+R^{\sigma\varepsilon}|\sigma1\varepsilon\rangle)\bigg]. \quad (20)$$

It is worth noting that the states $|\sigma\sigma\sigma\rangle_1$ and $|\sigma\sigma\sigma\rangle_\varepsilon$ are mutually orthogonal. Then, as mentioned above, the masking process is invariant under the exchange of two adjacent particles. Substituting the values of those braiding operations, we also conclude the partial traces $\mathrm{Tr}_{AB}(|\Psi\rangle\langle\Psi|)=\mathrm{Tr}_{AC}(|\Psi\rangle\langle\Psi|)=\mathrm{Tr}_{BC}(|\Psi\rangle\langle\Psi|)=I/3$. This also confirms that the braiding and circling of Ising anyons are actually on extended hyperdisks in the anyonic space of the Kitaev model, and thus gives an extended support of Ref. [36]. Similar to the above Abelian part, for Ising anyons, the ideal situation is that matrices A, B, and C are three mutually orthogonal Latin squares. In this ideal situation, all of the above discussed are still established.

C. Masking based on teleportation

It has been proved that there exist certain correlations between quantum information masking and teleportation. In Ref. [37], Shang *et al.* point out that teleportation is a process which masks the information first during the transference. Reference [22] gives a quantum teleportation scheme using Ising anyons. Here, we demonstrate quantum information masking in a teleportation way in the Kitaev Ising anyon model. Suppose the state to be teleported is $|\chi\rangle=\alpha|1\rangle+\beta|\varepsilon\rangle+\gamma|\sigma\rangle$ $(|\alpha|^2+|\beta|^2+|\gamma|^2=1)$. Alice and Bob share an entangled quantum channel

$$|\varphi\rangle_{23}=\frac{1}{\sqrt{3}}(|11\rangle+|\varepsilon\varepsilon\rangle+|\sigma\sigma\rangle). \quad (21)$$

Alice has the first and the second particles, and Bob holds the third one. The state of the whole system is

$$|\Psi\rangle_{123}=\frac{1}{\sqrt{3}}(\alpha|1\rangle+\beta|\varepsilon\rangle+\gamma|\sigma\rangle)_1(|11\rangle+|\varepsilon\varepsilon\rangle+|\sigma\sigma\rangle)_{23}. \quad (22)$$

With the help of Eq. (16), the above equation is mapped to

$$|\Psi'\rangle_{123}=\frac{1}{\sqrt{3}}(\alpha|1\rangle+\beta|\varepsilon\rangle+\gamma|\sigma\rangle)_1(|11\rangle+P_{cf2}P_{cb3}|\varepsilon\varepsilon\rangle+P_{cf2}^2P_{cb3}^2|\sigma\sigma\rangle)_{23}, \quad (23)$$

where P_{cf2} means the forward cyclic permutation operator which acts on the second particle. The matrices of the second and the third particles are mutually orthogonal Latin squares. The quantum information can be masked in the partial systems. After the operation,

$$\begin{aligned}|\Psi'\rangle_{123}=\tfrac{1}{3}[&(|\chi_1\rangle)_{12}(\alpha|1\rangle+\beta|\varepsilon\rangle+\gamma|\sigma\rangle)_3\\&+(|\chi_2\rangle)_{12}(\alpha|1\rangle+\beta\omega^2|\varepsilon\rangle+\gamma\omega|\sigma\rangle)_3\\&+(|\chi_3\rangle)_{12}(\alpha|1\rangle+\beta\omega|\varepsilon\rangle+\gamma\omega^2|\sigma\rangle)_3],\\(|\chi_1\rangle)_{12}=&|11\rangle+|1\varepsilon\rangle+|1\sigma\rangle+|\varepsilon1\rangle+|\varepsilon\varepsilon\rangle\\&+|\varepsilon\sigma\rangle+|\sigma1\rangle+|\sigma\varepsilon\rangle+|\sigma\sigma\rangle,\\(|\chi_2\rangle)_{12}=&|11\rangle+\omega|1\varepsilon\rangle+\omega^2|1\sigma\rangle+\omega^2|\varepsilon1\rangle+|\varepsilon\varepsilon\rangle\\&+\omega|\varepsilon\sigma\rangle+\omega|\sigma1\rangle+\omega^2|\sigma\varepsilon\rangle+|\sigma\sigma\rangle,\\(|\chi_3\rangle)_{12}=&|11\rangle+\omega^2|1\varepsilon\rangle+\omega|1\sigma\rangle+\omega|\varepsilon1\rangle+|\varepsilon\varepsilon\rangle\\&+\omega^2|\varepsilon\sigma\rangle+\omega^2|\sigma1\rangle+\omega|\sigma\varepsilon\rangle+|\sigma\sigma\rangle,\end{aligned} \quad (24)$$

where $\omega=e^{\frac{i2\pi}{3}}$. The information is transferred from Alice to Bob. This protocol demonstrates the teleportation process using a masking mapping.

IV. SUMMARY

In the task of quantum masking, the information information encoded in a single system is distributed to the correlations among a composite system. Modi *et al.* [7] pointed out that it is an impossible task in a bipartite system, while it can be achieved when more participants are allowed in the masking process. This phenomenon originated from the linearity, unitarity, and entanglement in quantum mechanics.

Here, we adopt exotic anyons in two-dimensional condensed-matter systems as the platform to realize multipartite masking scenarios. The anyonic space is quite different from the normal Hilbert space, and the framework of anyonic algebra is in a unitary braided fusion category. So how to mask information in the anyonic space and the maskable construction is worth researching and is conceivably complicated. Based on Theorem 2 in Ref. [8], we present the Latin-square masking protocols both in Abelian and non-Abelian anyons. We may safely draw the conclusion that the maskable constructions in anyonic space are extended hyperdisks, and the anyonic quantum entanglement which originates from the braiding operations is the source of the quantum information masking. We also realize quantum information masking in a teleportation way in the Kitaev Ising model, to reveal the relation between masking and teleportation in non-Abelian anyons.

More protocols of other Abelian and non-Abelian anyons cases and more realizations on the unitary evolution in quantum field systems will be discussed in our future works. Storing the information in the correlation is a material difference between a quantum computer and a classical computer. It goes without saying that quantum information masking is of great significance to the development of quantum computers and its related evidence collection in the future.

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