Qubit-environment entanglement in time-dependent pure dephasing

Małgorzata Strzałka,[1,2] Radim Filip,[3] and Katarzyna Roszak[1]
[1]*FZU, Institute of Physics of the Czech Academy of Sciences, Na Slovance 2, 182 00 Prague, Czech Republic*
[2]*Faculty of Mathematics and Physics, Charles University, Ke Karlovu 5, 121 16 Prague, Czech Republic*
[3]*Department of Optics, Palacký University, 17 Listopadu 12, 771 46 Olomouc, Czech Republic*

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We show that the methods for quantification of system-environment entanglement that were recently developed for interactions that lead to pure decoherence of the system can be straightforwardly generalized to time-dependent Hamiltonians of the same type. This includes the if-and-only-if criteria of separability, as well as the entanglement measure applicable to qubit systems, and methods of detection of entanglement by operations and measurements performed solely on the system without accessing the environment. We use these methods to study the nature of the decoherence of a qubit-oscillator system. Qubit-oscillator entanglement is essential for developing bosonic quantum technology with quantum non-Gaussian states and its applications in quantum sensing and computing. The dominating bosonic platforms—trapped ions, electromechanics, and superconducting circuits—are based on the time-dependent gates that use such entanglement to achieve new quantum sensors and quantum error correction. The steplike time dependence of the Hamiltonian that is taken into account allows us to capture the complex interplay between the buildups of classical and quantum correlations, which could not be replicated in time-independent scenarios.

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I. INTRODUCTION

The study of the generation of entanglement between a system and its environment is typically hard, because of the size and limited accessibility of the environment. The environment can be complex, but a single oscillator accessible only through a single qubit behaves as an environment, too. Although methods to calculate entanglement directly from a density matrix are available for two qubits [1–4], when either of the systems becomes large proper quantification of entanglement requires many-parameter optimization [5,6]. The only available direct tool is Negativity [7,8], but being based on the positive-partial-transpose (PPT) criterion [9,10] it does not capture bound entangled states [11–16] and is still numerically demanding for larger systems.

Recently, large progress has been made which allows efficient study of entanglement generation for a class of system-environment (SE) Hamiltonians that lead to pure decoherence (PD) of the system, as long as the initial state is of product form and the system of interest is in a pure state (the state of the environment can be arbitrary). The pure dephasing types of interactions are essential in current quantum technology with superconducting circuits [17], trapped ions [18], electromechanical oscillators [19], and, for a long time, cavity QED [20].

First, the system-environment state at a given time can now be qualified as separable or entangled with relative ease [21,22] and it has been shown that this type of interactions can lead to two distinct types of entanglement for larger systems [22]. Furthermore, understanding of the nature of the correlations that can be formed during the evolution allowed the design of schemes that detect this type of entanglement that are operated solely on the system of interest with no need to access the environment [23–26]. The schemes work, because the buildup of entanglement leaves a distinct trace on the state of the environment (which is related to the equivalence of this entanglement with quantum discord from the point of view of the environment [27]), which can in turn affect the system evolution. The ease with which such entanglement can be detected suggests that any quantum algorithm operated in a noisy setting will react differently to decoherence of quantum and of classical origins. This has already been shown on the simplest algorithms, such as teleportation [28,29] and spin echo [26].

Beyond being a resource for quantum technology, this type of Hamiltonian describes the most fundamental type of decoherence [30] that is not accompanied by energy exchange between the system and the environment. It has been widely used in fundamental studies of the nature of decoherence, and forms the basis for quantum Darwinism studies and investigation of the nature of the quantum-to-classical transition today [31–41]. Furthermore, this type of decoherence tends to dominate in realistic solid-state systems where the energy of environment quanta is much smaller than the energy-level separation in the system of interest, such as excitonic and electronic states confined in quantum dots [42–48] and various types of spin qubits [49–54].

In this paper we show that the methods previously devised for pure decoherence can be generalized to time-dependent Hamiltonians of the same type, because time dependence does not change the nature of the correlations that can be generated in any fundamental way. This includes the if-and-only-if criteria of separability for a single qubit system [21], and for a system of any size [22], as well as the single qubit entanglement (QE) measure [55]. Also all of the schemes for entanglement detection [23–26], which are a direct consequence of the form of the separability criteria, can be used in case of a time-dependent interaction.

As described above, it is of special interest for the study of hybrid solid-state-optical systems, such as superconducting transmon qubits [56–60] and trapped ions [61–68]. The biggest difference with respect to standard solid-state qubits here is the possibility of engineering and control of the interaction with the optical environment. This means that for such systems, the interaction can be specially tailored to control the level of entanglement buildup between the system and its environment in order to be used as a resource [69,70], e.g., for decoherence control [29,71,72], and hence a deeper understanding of such entanglement is critical.

We use the time-dependent methods for the study of qubit-environment entanglement (QEE) generated via a tunable Hamiltonian which describes the interaction with a single bosonic mode that is used for the description of both a transmon qubit interacting with microwave cavity photons [56,57] as well as trapped ions interacting with mechanical oscillator modes [66], or electrically controlled mechanical modes [19]. For these systems, the interaction leads to pure decoherence and it is experimentally controllable to a high extent. We change the interaction in a steplike manner between systems that do not lead to entanglement generation of the qubit with an initial mixture of Fock states and systems that do. Both the nonentangling and entangling interactions lead to decoherence of the qubit, but regardless of the similarities in the qubit evolution for the two interactions studied separately, their nature is very different. This is visible when the interactions act consecutively on the qubit, which leads to the buildup of quantum correlations affecting entanglement-driven decoherence, and vice versa. We observe nontrivial effects such as the simultaneous growth of coherence and entanglement at certain time periods, as well as the nonentangling Hamiltonian driving entanglement when it is preceded by a time when the interaction is entangling. These effects could not be observed using the time-independent methods, even though the evolution of the density matrix is obtained in a series of time-independent steps, because any correlations (quantum or classical) formed between the system and the environment preclude their application.

The paper is organized as follows. In Sec. II we introduce the time-dependent pure dephasing Hamiltonian and describe the formalism to obtain the resulting time evolution that we use in the rest of the paper. In Sec. III we show how the time-independent separability criteria for system-environment entanglement (SEE) are generalized to the time-dependent scenario. This holds true also for the qubit-environment entanglement measure and we use it in Sec. IV to study the evolution of entanglement between a transmon or trapped-ion qubit and its environment. Section V concludes the paper.

II. THE HAMILTONIAN AND THE EVOLUTION

We are interested in time-dependent Hamiltonians that describe SE evolution which leads to PD of the system when the degrees of freedom of the environment are traced out. To this end we must first specify the general form of such Hamiltonians. The general conditions for PD evolutions hold regardless of time dependence, namely that the free system Hamiltonian must commute with the interaction term, but now time dependence imposes that this condition must be fulfilled at all times t and t':

$$[\hat{H}_S(t), \hat{H}_{\rm int}(t')] = 0. \tag{1}$$

Here we assumed that the full SE Hamiltonian is of the form $\hat{H}_{\rm PD}(t) = \hat{H}_S(t) + \hat{H}_E(t) + \hat{H}_{\rm int}(t)$, where the first two terms on the right-hand side describe the free Hamiltonians of the system and the environment, respectively, while the third term describes their interaction.

The commutation relation (1) translates into limitations on the possible forms of the system and interaction Hamiltonian. Most importantly its fulfillment requires that there exists a well-defined and time-independent pointer basis of the system, which we will denote as $\{|i\rangle\}$, with $i = 0, \dots, N-1$, where N is the dimension of the system. Hence the time dependence of the system Hamiltonian has to be limited to the eigenvalues, while the time dependence of the interaction is fully described by environmental operators. We can now explicitly write the most general form of a time-dependent PD Hamiltonian:

$$\hat{H}_{\rm PD}(t) = \sum_i |i\rangle\langle i| \otimes \hat{V}_i(t), \tag{2}$$

where only the environmental operators $\hat{V}_i(t)$ are time dependent. They describe contributions to the Hamiltonian from all three terms and can be written as

$$\hat{V}_i(t) = \varepsilon_i(t) + \hat{H}_E(t) + \tilde{V}_i(t), \tag{3}$$

where $\varepsilon_i(t)$ is the eigenvalue of $\hat{H}_S(t)$ corresponding to pointer state $|i\rangle$ and $\tilde{V}_i(t)$ are environmental operators which describe the effect of a given system pointer state on the environment, obtained by writing the interaction Hamiltonian in the form

$$\hat{H}_{\rm int}(t) = \sum_i |i\rangle\langle i| \otimes \tilde{V}_i(t). \tag{4}$$

Once the general form of PD Hamiltonians is specified, one can easily find the form of the evolution operator, which is analogous to the time-independent PD evolution operator [21,22]:

$$\hat{U}_{\rm PD}(t) = \sum_i |i\rangle\langle i| \otimes \hat{w}_i(t). \tag{5}$$

The critical difference here lies in the form of the conditional evolution operators of the environment $\hat{w}_i(t)$ which are given by

$$\hat{w}_i(t) = \mathrm{Texp}\left[-\frac{i}{\hbar}\int_0^t dt' \hat{V}_i(t')\right], \tag{6}$$

where Texp[...] denotes the time-ordered exponential function. It is important to note here that although the operators (6) can have a much more complicated structure than their time-independent counterparts, they are still unitary operators.

Having found the evolution operator, one can write the SE density matrix at time t for any initial conditions. For a product initial SE state with a pure system state given by $|\psi(0)\rangle = \sum_i c_i|i\rangle$ and an arbitrary initial state of the

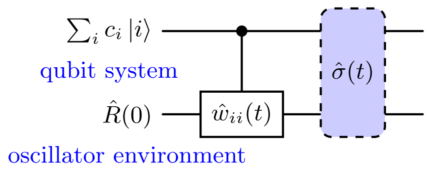

FIG. 1. Circuit representing entanglement generation during pure decoherence. The system state is initially in a superposition of pointer states ($\sum_i c_i|i\rangle$) and the environment is in an arbitrary, possibly mixed state $\hat{R}(0)$. The interaction acts as a gate $\hat{w}_{ii}(t)$ on the environment which is conditional on the pointer state of the system, $|i\rangle$, yielding the system-environment state at time t, $\hat{\sigma}(t)$, given by Eq. (7).

environment described by the density matrix $\hat{R}(0)$, this is given by

$$\hat{\sigma}(t) = \sum_{ij} c_i c_j^* |i\rangle\langle j| \otimes \hat{R}_{ij}(t), \tag{7}$$

with

$$\hat{R}_{ij}(t) = \hat{w}_i(t)\hat{R}(0)\hat{w}_j^\dagger(t). \tag{8}$$

The effect of the interaction, which can be interpreted as a conditional gate, where the evolution of the environment is conditional on the pointer state of the qubit, is illustrated in Fig. 1.

III. SYSTEM-ENVIRONMENT ENTANGLEMENT

The SE density matrix (7) has exactly the same structure as the one studied in Ref. [22] in order to qualify SE states obtained during time-independent PD evolution as entangled or separable. This structure, together with the fact that the conditional evolution operators of the environment (6) are unitary, allows us to directly transcribe the complete set of separability conditions from time-independent PD Hamiltonians to the time-dependent case. The proofs from Ref. [21] for QEE, and the generalized proofs for SEE of Ref. [22], hold for time-dependent PD described by the Hamiltonian (2) as long as the initial SE state is of product form with a pure initial state of the system [which is required to obtain Eq. (7)].

If the system under study is a qubit, there exists a unique separability criterion for pure dephasing [21], namely a qubit is separable from its environment if and only if

$$\hat{R}_{00}(t) = \hat{R}_{11}(t), \tag{9}$$

where $\hat{R}_{00}(t)$ and $\hat{R}_{11}(t)$ are given by Eq. (8). Otherwise there is QEE in the system. This makes checking for QEE particularly straightforward and allows for the existence of an entanglement measure which can be calculated directly from a density matrix obtained during PD evolution [55].

This measure is also valid for time-dependent PD Hamiltonians, since it retains all of the properties that have been proven in Ref. [55] for the time-independent case (and generalizing the proofs to time-dependent PD Hamiltonians is straightforward because the structure of the SE density matrix is the same in both cases). The measure is zero if and only if the SE state is separable, has a well-defined maximum value (equal to 1) that is reached for the same set of states as by entanglement measures using the convex-roof construction, is invariant under local unitary operations, and is monotonic under local operations that preserve the structure of the SE density matrix (7) (otherwise it does not exist). It is given by

$$E(t) = 4|c_0|^2|c_1|^2[1 - F(\hat{R}_{00}(t), \hat{R}_{11}(t))], \tag{10}$$

where $F(\hat{\rho}_1, \hat{\rho}_2) = [\mathrm{Tr}\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}}]^2$ is the fidelity. Hence the amount of entanglement that forms between a qubit and its environment during PD depends on the initial qubit coherence, $|\rho_{01}(0)|^2 = |c_0|^2|c_1|^2$, and evolves proportionally to how different the state of the environment becomes for the two qubit pointer states.

For larger systems, there are two types of separability criteria and a system of size N; there exist $N-1$ independent criteria of the first type and $(N-1)(N-2)/2$ independent criteria of the second type [22]. If any one of the following criteria is broken at time t this means that there is entanglement between the system and the environment at this time.

Separability criteria of the first type state that for all $i \neq j$ we must have

$$\hat{R}_{ii}(t) = \hat{R}_{jj}(t), \tag{11}$$

meaning that at a given time the state of the environment under the condition that the qubit is in pointer state $|i\rangle$ is the same as its state when the qubit is in state $|j\rangle$. The QEE criterion is a separability criterion of this type.

Criteria of the second type are more abstract in interpretation and relate to commutation between pairs of conditional evolution operators. Namely, they state that for all i, j, k, l we must have

$$[\hat{w}_i(t)\hat{w}_j^\dagger(t), \hat{w}_k(t)\hat{w}_l^\dagger(t)] = 0 \tag{12}$$

for separability. These criteria are related to internal SE coherences and criteria of this type do not exist if the system under study is a qubit.

Incidentally, conditions of the first type (11) cannot be broken when the initial density matrix of the environment is a fully mixed state, but conditions of the second type (12) can. This means that if a system is a qubit (so there are no conditions of the second type), PD interactions cannot lead to entanglement with a maximally mixed environment, but for larger systems it is possible. This was shown for a qutrit system in an example in Ref. [22].

Entanglement which is accompanied by the violation of any criterion of the first type (11) can be detected experimentally, because it manifests itself directly in the state of the environment. One could measure observables on the environment to witness SEE, but such measurements are hard in general (with the actual experimental feasibility depending strongly on the physical system under study). Yet, for time-independent PD Hamiltonians, it has been shown that the operation of simple algorithms on the system without the need to access the environment is sufficient for the detection of QEE in many situations for qubits [23,24] and for larger systems [25]. Since there is no qualitative change in the SE density matrix (7) which is obtained as a result of a time-dependent Hamiltonian, the schemes introduced in Refs. [23–25] can be operated as entanglement witnesses also in time-dependent scenarios.

IV. TRANSMON QUBIT AND MICROWAVE CAVITY OR TRAPPED ION AND MECHANICAL OSCILLATOR MODE

As an example we will study the evolution of entanglement using the measure (10) for an interaction Hamiltonian which can describe the effective coupling of a superconducting transmon qubit to the microwave cavity modes, as well as the interaction between a qubit defined on a trapped ion and the environment of a mechanical oscillator mode. The Hamiltonian is given by [18,56,57]

$$\hat{H}(t) = \hat{\sigma}_z \otimes \{[\alpha(t)\hat{a}^\dagger + \alpha^*(t)\hat{a}] + \beta\hat{a}^\dagger\hat{a} + \gamma(t)\}. \tag{13}$$

Here, $\hat{\sigma}_z$ is the appropriate Pauli operator acting on the qubit subspace, while operators $\hat{a}^\dagger$ and $\hat{a}$ are creation and annihilation operators in the subspace of the environment. Time dependence is explicitly marked when applicable and $\gamma(t)$ is responsible for free evolution of the qubit.

The Hamiltonian (13) can be easily rewritten into the PD form given by Eq. (2), with $i = 0, 1$ and

$$\hat{V}_{0/1}(t) = \pm\{[\alpha(t)\hat{a}^\dagger + \alpha^*(t)\hat{a}] + \beta\hat{a}^\dagger\hat{a} + \gamma(t)\}. \tag{14}$$

Since the environmental operators $\hat{V}_{0/1}(t)$ not only commute, but differ only by the sign, it is easy to show that the functions $\hat{w}_{0/1}(t)$ commute at any given time:

$$\forall_t[\hat{w}_0(t), \hat{w}_1(t)] = 0, \tag{15}$$

since $\hat{w}_0(t) = \hat{w}_1^\dagger(t)$. This does not translate however into them commuting at different times as it would in time-independent cases.

In the following, we will be considering the simplest case in terms of the time dependence of the Hamiltonian (13), namely such that the parameter $\alpha(t)$ is a step function. We assume that initially $\alpha(t) = 0$ until time t_1, then it is constant $\alpha(t) = \alpha$ for duration t_2, and again $\alpha(t) = 0$ for a time duration t_3:

$$\alpha(t) = \begin{cases} 0 & \text{for } t \in [0, t_1) \\ \alpha & \text{for } t \in [t_1, t_1 + t_2) \\ 0 & \text{for } t \in [t_1 + t_2, t_1 + t_2 + t_3]. \end{cases} \tag{16}$$

The actual value of $\gamma(t)$ is irrelevant, since it does not influence the generated entanglement, or the evolution of the degree of qubit coherence (absolute value of the off-diagonal element of the density matrix). We choose the step-function time dependence of the Hamiltonian, because it allows us to observe behaviors of the time evolution of entanglement which are not possible for time-independent Hamiltonians, while its simplicity allows for a straightforward interpretation of the observed results in terms of the generation of different types of correlations between the qubit and the environment.

For this scenario the conditional evolution operators of the environment $\hat{w}_k(t)$, with $k = 0, 1$, consist of three parts:

$$\hat{w}_k(t) = \hat{w}_k^3(t_3)\hat{w}_k^2(t_2)\hat{w}_k^1(t_1), \tag{17}$$

where the operators $\hat{w}_k^i(t_i)$, over time durations t_i defined by Eq. (16), are given by

$$\hat{w}_{0/1}^{1/3}(t) = e^{\mp\frac{i}{\hbar}\beta\hat{a}^\dagger\hat{a}t}, \tag{18a}$$

$$\hat{w}_{0/1}^{2}(t) = e^{Y_{0/1}(t)}e^{i\Phi_{0/1}(t)}e^{\mp\frac{i}{\hbar}\beta\hat{a}^\dagger\hat{a}t}, \tag{18b}$$

with

$$\Phi_{0/1}(t) = \mp\frac{|\alpha|^2}{\beta^2}\sin\frac{\beta t}{\hbar},$$

$$Y_{0/1}(t) = \frac{\alpha}{\beta}\left(e^{\mp\frac{i}{\hbar}\beta t} - 1\right)\hat{a}^\dagger - \frac{\alpha^*}{\beta}\left(e^{\pm\frac{i}{\hbar}\beta t} - 1\right)\hat{a}.$$

We will consider two types of initial states for the environment, while the qubit will always initially be in an equal superposition state. First, the environment will be initially at a thermal equilibrium of Fock states, meaning the Gibbs state corresponding to the Hamiltonian $\hat{H}_0 = \Gamma\hat{a}^\dagger\hat{a}$, and later we will show plots for coherent states for comparison.

In Fig. 2 QEE measured by Eq. (10) is plotted by the solid lines as a function of time. Complementarily, the absolute value of the qubit coherence normalized by its initial value, $\frac{\hat{\rho}_{01}(t)}{\hat{\rho}_{01}(0)}$, is plotted using dashed lines. Figures 2(a)–2(d) contain the evolution for the three-step time dependence of Hamiltonian (13) with the parameter α changing as given by Eq. (16) at $\beta t/\hbar = \beta t_1/\hbar = 2$ and $\beta t/\hbar = \beta(t_1 + t_2)/\hbar = 4$ which are marked by gray vertical lines on the plots. Figure 2(a) contains zero-temperature results, while progressively higher temperatures are taken into account in Figs. 2(b)–2(d): $k_BT/\Gamma = 0.5$, 1, and 2, respectively.

Let us first note that there are no qualitative changes in the evolution of entanglement when the temperature is increased, but there is a stark difference in the decoherence before time t_1 is reached at zero temperature (which is the only situation when the first part of the evolution does not display decoherence). This is because for $t \leqslant t_1$ decoherence is not an outcome of the generation of entanglement between the qubit and its environment, but rather the establishment of classical correlations between them. For pure states, classical correlations cannot be generated through a unitary evolution and thus decoherence is not possible at zero temperature.

Entanglement starts being generated after $\beta t/\hbar = 2$ because terms of the Hamiltonian (13) with $\alpha \neq 0$ which are responsible for the conditional evolution of the environment do not commute with $\hat{R}_{00}(t_1) = \hat{R}_{11}(t_1)$. After time $\beta t/\hbar = 4$ when α is again set to zero, the qubit-environment interaction is nevertheless capable of driving the evolution of entanglement, because of the QE correlations that have been established in the previous phase of the evolution. Note that the third part of evolution is different both for entanglement and coherence, which manifests itself most visibly in the sharp change observed at $\beta t/\hbar = 4$.

For comparison, we have additionally plotted the evolution of entanglement and coherence for the same Hamiltonian, but without time dependence, with $\alpha = 0$ in Fig. 2(e) and with $\alpha \neq 0$ in Fig. 2(f) (here the evolution starts at $\beta t/\hbar = 2$ in order to ease the comparison between these curves and analogous evolution that has been preceded by an interaction with $\alpha = 0$). The plots correspond to $k_BT/\Gamma = 2$ as in Fig. 2(d). The oscillatory behavior observed in Figs. 2(e) and 2(f) would also be present in the time-dependent evolution [Figs. 2(a)–2(d)] if the transition times between different values of α were chosen longer, as this is a trivial consequence of only one bosonic mode being taken into account. Comparison of Figs. 2(e) and 2(f) shows that there is no qualitative change in the evolution of coherence, even though the nature

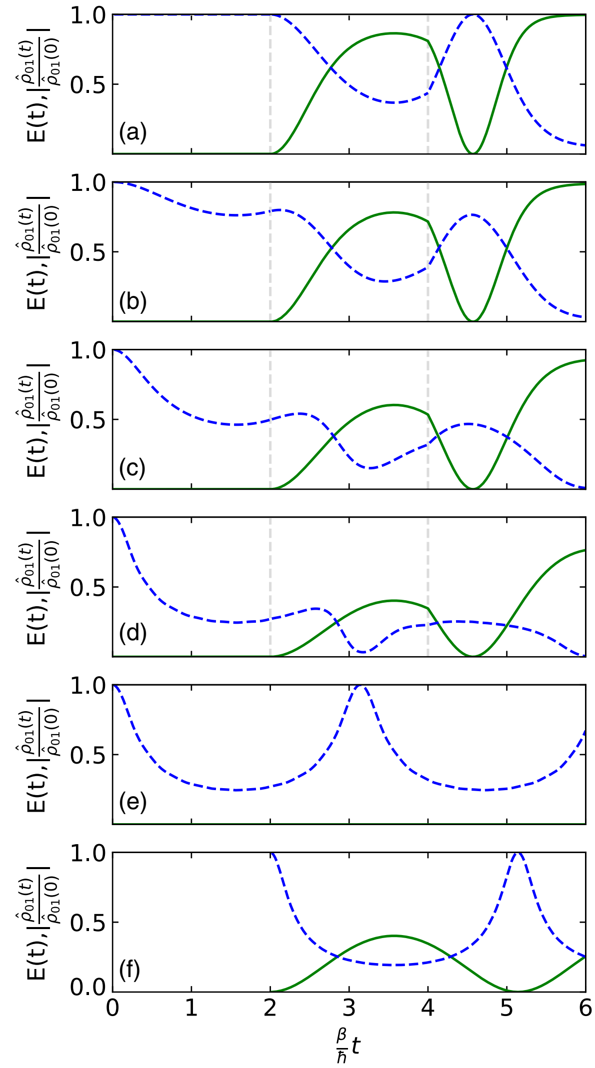

FIG. 2. Evolution of QEE (solid lines) and normalized qubit coherence (dashed lines) for the initial Gibbs state of the environment at different temperatures: (a) $k_B T/\Gamma = 0$, (b) $k_B T/\Gamma = 0.5$, (c) $k_B T/\Gamma = 1$, and (d) $k_B T/\Gamma = 2$. Vertical dashed lines denote the times when parameter α is changed from $\alpha = 0$ to $\alpha \neq 0$ at $\beta t/\hbar = 2$ and back from $\alpha \neq 0$ to $\alpha = 0$ at $\beta t/\hbar = 4$. The Hamiltonian parameters are set to $\alpha/\beta = (1+i)/2$ for $\alpha \neq 0$. (e) Evolution with constant Hamiltonian and $\alpha = 0$ for $k_B T/\Gamma = 2$. (f) Evolution with constant Hamiltonian with $\alpha/\beta = (1+i)/2$ for $k_B T/\Gamma = 2$. The plot is shifted for easier comparison and the evolution starts at $\beta t/\hbar = 2$.

of the decoherence is fundamentally different, as one is the result of classical SE correlations being established, while the other is driven by entanglement generation. Quite surprisingly, when the switch between the two types of interactions is made in Fig. 2(d), we observe a stark qualitative change in decoherence, due to the interplay of classical and quantum correlations, even though the actual generations of entanglement in Figs. 2(d) and 2(f) resemble each other closely.

Because of the time dependence of the Hamiltonian (13) we are able to observe specific features of entanglement evolution which are otherwise rare. First there is a transition between decoherence classical in nature and decoherence which is induced by QEE, as described above, but in several

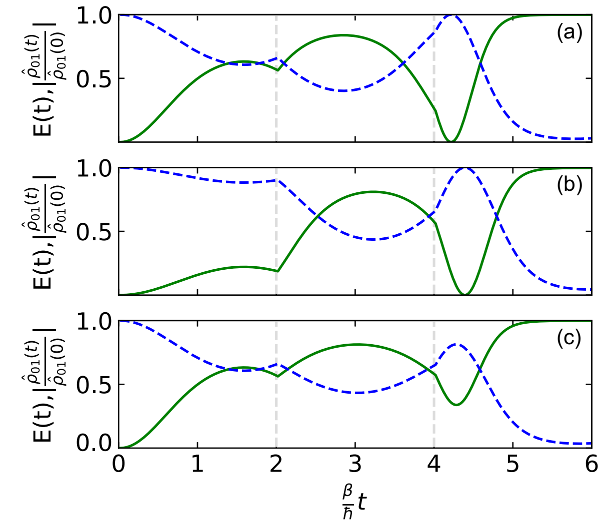

FIG. 3. Evolution of QEE (solid lines) and normalized qubit coherence (dashed lines) for the initial coherent state of the environment with (a) $\zeta = 0.5e^{i\pi/4}$, (b) $\zeta = 0.25e^{i\pi/4}$, and (c) $\zeta = 0.5$.

time instances we see that the qubit coherence can grow at the same time as entanglement does. This is only possible at finite temperatures and is the outcome of the competition between quantum and classical decoherence mechanisms. This is most distinct just after $\beta t/\hbar = 2$ when the classical dephasing process leads to the enhancement of qubit coherence due to the unitary nature of the QE evolution and the single bosonic mode taken into account, while $\alpha(t) = \text{const}$ ensures the establishment of quantum correlations which start to lead to qubit decoherence.

It is important to note here that although the way that the time dependence of the Hamiltonian is taken into account allows us to obtain the QE evolution by superposing time-independent evolution operators, time-independent methods for the quantification or qualification of QEE would not be sufficient here. This is because the SE states at $\beta t/\hbar = 2$ and 4 contain SE correlations (classical for $\beta t/\hbar = 2$ and both quantum and classical for $\beta t/\hbar = 4$) and do not fulfill the requirements for initial SE states in the time-independent methods.

For completeness in Fig. 3 we plot the evolution of entanglement and coherence analogous to the plots in Fig. 2 for the situation when the initial state of the environment is a coherent state:

$$|\zeta\rangle = e^{-\frac{1}{2}|\zeta|^2} e^{\zeta \hat{a}^\dagger} e^{-\zeta^* \hat{a}} |0\rangle \quad (19)$$

where ζ is a complex number. The panels correspond to different values of ζ, and it varies only in amplitude between Figs. 3(a) and 3(b) and only in phase between Figs. 3(a) and 3(c). This yields to a stark difference between the observed curves in the first phase when $\alpha = 0$ between Figs. 3(a) and 3(b) which diminishes in the later phases, while between Figs. 3(a) and 3(c) the biggest difference in the evolution is in the third phase, when the parameter α is again set to zero.

Nevertheless, the most important difference manifests itself in the comparison between Figs. 2 and 3, since these differences are most distinctly qualitative. For coherent states,

$\alpha = 0$ does not preclude the generation of SEE from the initial product state, so entanglement is generated throughout the evolution. This exemplifies that in the generation of entanglement, not only the SE interaction is important, but its interplay with the initial state of the environment plays a critical part.

As a last remark in this section, it is relevant to note that although the type of time dependence that has been included in the example under study is as simple as possible, it is sufficient to demonstrate nontrivial properties of the evolution of entanglement. These results could not be obtained outside of the time-dependent formalism. Furthermore modeling time dependence as a number of small, consecutive steps is a fairly standard procedure [19,73], so in principle the same method could be used to model any time dependence in the Hamiltonian.

V. CONCLUSION

We have shown that the whole array of methods developed for the qualification and quantification of entanglement that can be generated during the joint evolution of a system and its environment which leads to pure decoherence of the system can be generalized to time-dependent pure-decoherence Hamiltonians. This is because the SE density matrix obtained during such an evolution is qualitatively the same as in the time-independent case and the nature of the correlations that can be formed does not change. Hence, the same criteria for the qualification of SE states as separable or mixed can be used.

Nevertheless, time dependence in the Hamiltonian allows for a much more complex evolution of entanglement, e.g., in the extreme case interactions that only lead to buildup of classical correlations during decoherence can be interchanged with interactions that rely on entanglement generation. We demonstrate this studying an interaction Hamiltonian that is used to describe a transmon qubit interacting with a microwave cavity as well as a trapped ion interacting with mechanical modes. Such systems are good examples of systems that effectively undergo pure decoherence while the environment is engineered and the parameters of the interaction can be experimentally manipulated.

We use a steplike time dependence of the Hamiltonian, changing from an interaction which is nonentangling for an initial thermal-equilibrium state of the environment, to an entangling one, and back. This allows us to show the stark change in both the nature and time dependence of the decoherence and of entanglement. In the first part of the evolution, decoherence is not an effect of entanglement generation, but of the formation of classical correlations between the system and the environment. Once the interaction is switched to entangling, there is an interplay between classical and quantum correlations which is reflected in the decoherence, that can now display counterintuitive behaviors, such as the reversal of decoherence while entanglement grows. In the third part, when the entangling part of the evolution is switched off, we still observe entanglement evolution, and we show that entanglement can grow in this phase to higher levels than the maximum obtained while the entangling interaction is turned on. This is again due to the interplay of quantum and classical correlations that are present in the system at the moment when the nature of the interaction is changed.

It is important to stress here that the nontrivial features of the presented results could not be replicated without the use of time-dependent methods. The evolution of coherence for the entangling Hamiltonian is very different if it is not preceded by a period of classical-correlation-driven decoherence. Similarly, the fact that entanglement evolves and can grow in the third part of the evolution would not be possible if it were not preceded by a period of the evolution when SEE is generated.

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[1] W. K. Wootters, Entanglement of formation of an arbitrary state of two qubits, Phys. Rev. Lett. **80**, 2245 (1998).

[2] T. Seidelmann, C. Schimpf, T. K. Bracht, M. Cosacchi, A. Vagov, A. Rastelli, D. E. Reiter, and V. M. Axt, Two-photon excitation sets limit to entangled photon pair generation from quantum emitters, Phys. Rev. Lett. **129**, 193604 (2022).

[3] F. Schmolke and E. Lutz, Noise-induced quantum synchronization, Phys. Rev. Lett. **129**, 250601 (2022).

[4] T. Vovk and H. Pichler, Entanglement-optimal trajectories of many-body quantum Markov processes, Phys. Rev. Lett. **128**, 243601 (2022).

[5] V. Vedral, M. B. Plenio, M. A. Rippin, and P. L. Knight, Quantifying entanglement, Phys. Rev. Lett. **78**, 2275 (1997).

[6] E. M. Rains, Rigorous treatment of distillable entanglement, Phys. Rev. A **60**, 173 (1999).

[7] G. Vidal and R. F. Werner, Computable measure of entanglement, Phys. Rev. A **65**, 032314 (2002).

[8] M. B. Plenio, Logarithmic negativity: A full entanglement monotone that is not convex, Phys. Rev. Lett. **95**, 090503 (2005).

[9] A. Peres, Separability criterion for density matrices, Phys. Rev. Lett. **77**, 1413 (1996).

[10] M. Horodecki, P. Horodecki, and R. Horodecki, Separability of mixed states: necessary and sufficient conditions, Phys. Lett. A **223**, 1 (1996).

[11] M. Horodecki, P. Horodecki, and R. Horodecki, Mixed-state entanglement and distillation: Is there a "bound" entanglement in nature? Phys. Rev. Lett. **80**, 5239 (1998).

[12] J. A. Smolin, Four-party unlockable bound entangled state, Phys. Rev. A **63**, 032306 (2001).

[13] J. DiGuglielmo, A. Samblowski, B. Hage, C. Pineda, J. Eisert, and R. Schnabel, Experimental unconditional preparation and detection of a continuous bound entangled state of light, Phys. Rev. Lett. **107**, 240503 (2011).
[14] B. C. Hiesmayr and W. Löffler, Complementarity reveals bound entanglement of two twisted photons, New J. Phys. **15**, 083036 (2013).
[15] G. Sentís, J. N. Greiner, J. Shang, J. Siewert, and M. Kleinmann, Bound entangled states fit for robust experimental verification, Quantum **2**, 113 (2018).
[16] A. Gabdulin and A. Mandilara, Investigating bound entangled two-qutrit states via the best separable approximation, Phys. Rev. A **100**, 062322 (2019).
[17] A. Eickbusch, V. Sivak, A. Z. Ding, S. S. Elder, S. R. Jha, J. Venkatraman, B. Royer, S. M. Girvin, R. J. Schoelkopf, and M. H. Devoret, Fast universal control of an oscillator with weak dispersive coupling to a qubit, Nat. Phys. **18**, 1464 (2022).
[18] O. Katz and C. Monroe, Programmable quantum simulations of bosonic systems with trapped ions, Phys. Rev. Lett. **131**, 033604 (2023).
[19] W.-L. Ma, S. Puri, R. J. Schoelkopf, M. H. Devoret, S. M. Girvin, and L. Jiang, Quantum control of bosonic modes with superconducting circuits, Sci. Bull. **66**, 1789 (2021).
[20] S. Haroche, M. Brune, and J. M. Raimond, From cavity to circuit quantum electrodynamics, Nat. Phys. **16**, 243 (2020).
[21] K. Roszak and L. Cywiński, Characterization and measurement of qubit-environment-entanglement generation during pure dephasing, Phys. Rev. A **92**, 032310 (2015).
[22] K. Roszak, Criteria for system-environment entanglement generation for systems of any size in pure-dephasing evolutions, Phys. Rev. A **98**, 052344 (2018).
[23] K. Roszak, D. Kwiatkowski, and L. Cywiński, How to detect qubit-environment entanglement generated during qubit dephasing, Phys. Rev. A **100**, 022318 (2019).
[24] B. Rzepkowski and K. Roszak, A scheme for direct detection of qubit-environment entanglement generated during qubit pure dephasing, Quantum Inf. Process **20**, 1 (2021).
[25] M. Strzałka and K. Roszak, Detection of entanglement during pure dephasing evolutions for systems and environments of any size, Phys. Rev. A **104**, 042411 (2021).
[26] K. Roszak and L. Cywiński, Qubit-environment-entanglement generation and the spin echo, Phys. Rev. A **103**, 032208 (2021).
[27] K. Roszak and L. Cywiński, Equivalence of qubit-environment entanglement and discord generation via pure dephasing interactions and the resulting consequences, Phys. Rev. A **97**, 012306 (2018).
[28] T. Harlender and K. Roszak, Transfer and teleportation of system-environment entanglement, Phys. Rev. A **105**, 012407 (2022).
[29] K. Roszak and J. K. Korbicz, Purifying teleportation, Quantum **7**, 923 (2023).
[30] W. H. Zurek, Decoherence, einselection, and the quantum origins of the classical, Rev. Mod. Phys. **75**, 715 (2003).
[31] H. Ollivier, D. Poulin, and W. H. Zurek, Objective properties from subjective quantum states: Environment as a witness, Phys. Rev. Lett. **93**, 220401 (2004).
[32] H. Ollivier, D. Poulin, and W. H. Zurek, Environment as a witness: Selective proliferation of information and emergence of objectivity in a quantum universe, Phys. Rev. A **72**, 042113 (2005).
[33] W. H. Zurek, Quantum Darwinism, Nat. Phys. **5**, 181 (2009).
[34] J. K. Korbicz, P. Horodecki, and R. Horodecki, Objectivity in a noisy photonic environment through quantum state information broadcasting, Phys. Rev. Lett. **112**, 120402 (2014).
[35] R. Horodecki, J. K. Korbicz, and P. Horodecki, Quantum origins of objectivity, Phys. Rev. A **91**, 032122 (2015).
[36] P. Mironowicz, J. K. Korbicz, and P. Horodecki, Monitoring of the process of system information broadcasting in time, Phys. Rev. Lett. **118**, 150501 (2017).
[37] A. Lampo, J. Tuziemski, M. Lewenstein, and J. K. Korbicz, Objectivity in the non-Markovian spin-boson model, Phys. Rev. A **96**, 012120 (2017).
[38] K. Roszak and J. K. Korbicz, Entanglement and objectivity in pure dephasing models, Phys. Rev. A **100**, 062127 (2019).
[39] S. Lorenzo, M. Paternostro, and G. M. Palma, Anti-Zeno-based dynamical control of the unfolding of quantum Darwinism, Phys. Rev. Res. **2**, 013164 (2020).
[40] K. Roszak and J. K. Korbicz, Glimpse of objectivity in bipartite systems for nonentangling pure dephasing evolutions, Phys. Rev. A **101**, 052120 (2020).
[41] R. D. Baldijão, R. Wagner, C. Duarte, B. Amaral, and M. T. Cunha, Emergence of noncontextuality under quantum Darwinism, PRX Quantum **2**, 030351 (2021).
[42] P. Borri, W. Langbein, S. Schneider, U. Woggon, R. L. Sellin, D. Ouyang, and D. Bimberg, Ultralong dephasing time in InGaAs quantum dots, Phys. Rev. Lett. **87**, 157401 (2001).
[43] A. Vagov, V. M. Axt, and T. Kuhn, Impact of pure dephasing on the nonlinear optical response of single quantum dots and dot ensembles, Phys. Rev. B **67**, 115338 (2003).
[44] A. Vagov, V. M. Axt, T. Kuhn, W. Langbein, P. Borri, and U. Woggon, Nonmonotonous temperature dependence of the initial decoherence in quantum dots, Phys. Rev. B **70**, 201305(R) (2004).
[45] M. Glässl, A. M. Barth, and V. M. Axt, Proposed robust and high-fidelity preparation of excitons and biexcitons in semiconductor quantum dots making active use of phonons, Phys. Rev. Lett. **110**, 147401 (2013).
[46] H. Tahara, Y. Ogawa, F. Minami, K. Akahane, and M. Sasaki, Long-time correlation in non-Markovian dephasing of an exciton-phonon system in InAs quantum dots, Phys. Rev. Lett. **112**, 147404 (2014).
[47] T. Salamon and K. Roszak, Entanglement generation between a charge qubit and its bosonic environment during pure dephasing: Dependence on the environment size, Phys. Rev. A **96**, 032333 (2017).
[48] T. Seidelmann, F. Ungar, A. M. Barth, A. Vagov, V. M. Axt, M. Cygorek, and T. Kuhn, Phonon-induced enhancement of photon entanglement in quantum dot-cavity systems, Phys. Rev. Lett. **123**, 137401 (2019).
[49] N. Zhao, S.-W. Ho, and R.-B. Liu, Decoherence and dynamical decoupling control of nitrogen vacancy center electron spins in nuclear spin baths, Phys. Rev. B **85**, 115303 (2012).
[50] D. Kwiatkowski and Ł. Cywiński, Decoherence of two entangled spin qubits coupled to an interacting sparse nuclear spin bath: Application to nitrogen vacancy centers, Phys. Rev. B **98**, 155202 (2018).
[51] H. P. Bartling, M. H. Abobeih, B. Pingault, M. J. Degen, S. J. H. Loenen, C. E. Bradley, J. Randall, M. Markham, D. J. Twitchen, and T. H. Taminiau, Entanglement of spin-pair qubits with

intrinsic dephasing times exceeding a minute, Phys. Rev. X **12**, 011048 (2022).
[52] S. L. Bayliss, P. Deb, D. W. Laorenza, M. Onizhuk, G. Galli, D. E. Freedman, and D. D. Awschalom, Enhancing spin coherence in optically addressable molecular qubits through host-matrix control, Phys. Rev. X **12**, 031028 (2022).
[53] N. Wang, C.-F. Liu, J.-W. Fan, X. Feng, W.-H. Leong, A. Finkler, A. Denisenko, J. Wrachtrup, Q. Li, and R.-B. Liu, Zero-field magnetometry using hyperfine-biased nitrogen-vacancy centers near diamond surfaces, Phys. Rev. Res. **4**, 013098 (2022).
[54] M. Onizhuk and G. Galli, Bath-limited dynamics of nuclear spins in solid-state spin platforms, Phys. Rev. B **108**, 075306 (2023).
[55] K. Roszak, Measure of qubit-environment entanglement for pure dephasing evolutions, Phys. Rev. Res. **2**, 043062 (2020).
[56] S. Touzard, A. Kou, N. E. Frattini, V. V. Sivak, S. Puri, A. Grimm, L. Frunzio, S. Shankar, and M. H. Devoret, Gated conditional displacement readout of superconducting qubits, Phys. Rev. Lett. **122**, 080502 (2019).
[57] P. Campagne-Ibarcq, A. Eickbusch, S. Touzard, E. Zalys-Geller, N. E. Frattini, V. V. Sivak, P. Reinhold, S. Puri, S. Shankar, R. J. Schoelkopf *et al.*, Quantum error correction of a qubit encoded in grid states of an oscillator, Nature (London) **584**, 368 (2020).
[58] Y. Y. Gao, M. A. Rol, S. Touzard, and C. Wang, Practical guide for building superconducting quantum devices, PRX Quantum **2**, 040202 (2021).
[59] R. D. Delaney, M. D. Urmey, S. Mittal, B. M. Brubaker, J. M. Kindem, P. S. Burns, C. A. Regal, and K. W. Lehnert, Superconducting-qubit readout via low-backaction electro-optic transduction, Nature (London) **606**, 489 (2022).
[60] F. Hassani, M. Peruzzo, L. N. Kapoor, A. Trioni, M. Zemlicka, and J. M. Fink, Inductively shunted transmons exhibit noise insensitive plasmon states and a fluxon decay exceeding 3 hours, Nat. Commun. **14**, 3968 (2023).
[61] D. Leibfried, R. Blatt, C. Monroe, and D. Wineland, Quantum dynamics of single trapped ions, Rev. Mod. Phys. **75**, 281 (2003).
[62] D. Kienzler, C. Flühmann, V. Negnevitsky, H.-Y. Lo, M. Marinelli, D. Nadlinger, and J. P. Home, Observation of quantum interference between separated mechanical oscillator wave packets, Phys. Rev. Lett. **116**, 140402 (2016).
[63] D. Lv, S. An, Z. Liu, J.-N. Zhang, J. S. Pedernales, L. Lamata, E. Solano, and K. Kim, Quantum simulation of the quantum Rabi model in a trapped ion, Phys. Rev. X **8**, 021027 (2018).
[64] M. Bock, P. Eich, S. Kucera, M. Kreis, A. Lenhard, C. Becher, and J. Eschner, High-fidelity entanglement between a trapped ion and a telecom photon via quantum frequency conversion, Nat. Commun. **9**, 1998 (2018).
[65] K. A. Landsman, C. Figgatt, T. Schuster, N. M. Linke, B. Yoshida, N. Y. Yao, and C. Monroe, Verified quantum information scrambling, Nature (London) **567**, 61 (2019).
[66] C. Monroe, W. C. Campbell, L.-M. Duan, Z.-X. Gong, A. V. Gorshkov, P. W. Hess, R. Islam, K. Kim, N. M. Linke, G. Pagano, P. Richerme, C. Senko, and N. Y. Yao, Programmable quantum simulations of spin systems with trapped ions, Rev. Mod. Phys. **93**, 025001 (2021).
[67] C. Kokail, R. van Bijnen, A. Elben, B. Vermersch, and P. Zoller, Entanglement Hamiltonian tomography in quantum simulation, Nat. Phys. **17**, 936 (2021).
[68] V. G. Matsos, C. H. Valahu, T. Navickas, A. D. Rao, M. J. Millican, M. J. Biercuk, and T. R. Tan, Robust and deterministic preparation of bosonic logical states in a trapped ion, arXiv:2310.15546.
[69] R. Chaves and L. Davidovich, Robustness of entanglement as a resource, Phys. Rev. A **82**, 052308 (2010).
[70] E. Chitambar and G. Gour, Quantum resource theories, Rev. Mod. Phys. **91**, 025001 (2019).
[71] K. Roszak and Ł. Cywiński, The relation between the quantum discord and quantum teleportation: The physical interpretation of the transition point between different quantum discord decay regimes, Europhys. Lett. **112**, 10002 (2015).
[72] B. Rzepkowski and K. Roszak, Signature of quantumness in pure decoherence control, Phys. Rev. A **108**, 012412 (2023).
[73] C. Flühmann, Encoding a qubit in the motion of a trapped ion using superpositions of displaced squeezed states, Ph.D. thesis, ETH Zurich, 2019.