Thomas-Wigner rotation as a holonomy for spin-1/2 particles

Veiko Palge [*]
Laboratory of Theoretical Physics, Institute of Physics, University of Tartu, W. Ostwaldi 1, 50411 Tartu, Estonia

Christian Pfeifer [†]
Center of Applied Space Technology and Microgravity (ZARM), University of Bremen, Am Fallturm 2, 28359 Bremen, Germany

(Received 19 October 2023; accepted 12 February 2024; published 11 March 2024)

The Thomas-Wigner rotation (TWR) results from the fact that a combination of boosts leads to a nontrivial rotation of a physical system. Its origin lies in the structure of the Lorentz group. In this article we discuss the idea that the TWR can be understood in the geometric manner, being caused by the nontrivially curved relativistic momentum space, i.e., the mass shell, seen as a Riemannian manifold. We show explicitly how the TWR for a massive spin-1/2 particle can be calculated as a holonomy of the mass shell. To reach this conclusion we recall how to construct the spin bundle over the mass shell manifold. Interpreting TWR as a holonomy means it belongs to the same family of phenomena as Berry's phase.

DOI: 10.1103/PhysRevA.109.032206

I. INTRODUCTION

The Thomas-Wigner rotation (TWR) is a fascinating effect of special relativity, which originates in the fact that a combination of boosts results in a nontrivial rotation of a physical system [1–5]. If a non-relativistic system moving with velocity $\mathbf{v}_1$ is boosted by velocity $\mathbf{v}_2$, the resulting velocity is given by the familiar law of addition of velocities $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$. In special relativity this is not the case. Two successive noncollinear boosts lead to a boost *and rotation*. This phenomenon is called TWR and it originates in the structure of the Lorentz group, which encodes the fundamental symmetries of special relativity and Minkowski spacetime.

An alternative approach to understand the TWR is a geometric one. The TWR can be thought of as being caused by the nontrivially *curved* relativistic momentum space of massive particles, the mass shell, seen as a Riemannian manifold. The goal of this paper is to describe TWR in the context of relativistic quantum theory using the geometric approach. We focus on the free massive spin-1/2 particle. Whereas the standard approach in quantum theory is to use group theory and the Hilbert space formalism, the advantage of the geometric approach lies in its highly intuitive conceptualization of TWR. It explains the nonintuitive character of TWR in terms of the fact that the momentum space is a curved Riemannian manifold. This is in contrast to the nonrelativistic momentum space where the familiar law for addition of velocities means that it is a flat Euclidean space $\mathbb{R}^3$.

The geometric approach to TWR dates back to almost the birth of special relativity; see for instance [6] where the idea of a hyperbolic velocity space is tracked to the articles published between 1910 and 1919. In more recent literature, the hyperbolic velocity space is described and derived from somewhat different starting points [6–8], where the last reference in particular has inspired the present paper.

We build on prior work in the geometric approach. We start with the common underlying idea that the relativistic momentum or, equivalently, velocity space is a curved Riemannian manifold and use the language of differential geometry to develop the notion that the TWR is nothing but a holonomy of the relativistic momentum space. Holonomy is the idea that when a vector (or spinor) is parallel transported along a closed curve, then the initial and the final vector (or spinor) need not necessarily coincide because the manifold is curved. The transformation between the initial and the final, parallel transported vector is described by the holonomy matrix. This means that TWR belongs to the larger family of classical and quantum physical phenomena which can be described as holonomies, such as Berry's phase, the Aharonov-Bohm effect, or even classical effects like the Foucault pendulum [9,10].

The article is divided into two parts. In the first part (starting with Sec. II) we review the TWR in classical and quantum physics. Those familiar with this background may want to skip ahead to the second part (starting with Sec. III) where we present the main results of the article. We start by providing an intuitive picture which explains TWR in geometric terms. This is followed by the gradual formalization of the intuitive picture. We then describe the intrinsic geometry of the mass hyperbola using the standard differential geometric language of fiber bundles, connection, and curvature. After that we add the quantum spin field to the picture and show how the connection and curvature can be induced for the spinor bundle. Finally we explicitly calculate the holonomy matrix for the Thomas precession of the spin-1/2 particle. As a result we also reproduce the holonomy angle that coincides with the result obtained in [6]. Besides our application to the TWR, the vector bundle understanding of spinors has recently

[*]veiko.palge@ut.ee
[†]christian.pfeifer@zarm.uni-bremen.de

also gained attention in the context of relativistic quantum information [11,12].

We will use the following notational conventions. Throughout we denote four-vectors by normal font with greek indices μ, ν running from 0 to 3, where x^0 is the time component. Latin indices $\{1, 2, 3\}$ which run over spatial coordinates and spatial vectors $\mathbf{x}$ or 1-forms $\mathbf{p}$ are boldfaced. The Minkowski space with Minkowski metric will be denoted by $\mathbb{E}^{1,3}$ and described in canonical Cartesian coordinates in which the Minkowski metric η has the form $\eta = \mathrm{diag}(1, -1, -1, -1)$. The units are natural, $\hbar = c = 1$. The invariant four-momentum of a particle is denoted by the 1-form P, which in local Cartesian coordinates can be expanded as

$$P = p_\mu dx^\mu \tag{1}$$

so the four-momentum of a particle with mass m is given by $p_\mu = (p_0, \mathbf{p})$ with norm $\eta^{\mu\nu} p_\mu p_\nu = p^\mu p_\mu = (p_0)^2 - \mathbf{p}^2 = m^2$, where $p_0 = \sqrt{m^2 + \mathbf{p}^2} =: E(\mathbf{p})$.

II. THOMAS-WIGNER ROTATION IN CLASSICAL AND QUANTUM PHYSICS

In this section we will give a quick overview of the TWR in classical special relativistic physics and in relativistic quantum mechanics. We start by discussing how successive boosts act on classical point particle momenta. After this we examine their action on quantum mechanical spins.

There is no geometry involved at this point. Readers familiar with the standard account of TWR can skip ahead to Sec. III where we discuss the geometric approach to TWR in terms of the curved relativistic momentum space. Our approach complements and builds upon earlier treatments of TWR using projective geometry [6].

A. Boosts of relativistic momenta and TWR

Suppose a body with mass m is boosted from rest by velocity $\mathbf{v}_1$. We assume the boost is rotation free, i.e., pure, then momentum undergoes the following transformation:

$$L(\mathbf{v}_1) p_A = p_B, \tag{2}$$

where $p_A = (m, 0, 0, 0)$ is the four-momentum of the system at rest and $L(\mathbf{v}_1)$ is a pure boost that maps the rest momentum to p_B. A second pure boost with velocity $\mathbf{v}_2$ with respect to the frame with velocity $\mathbf{v}_1$ maps p_B to p_C:

$$L(\mathbf{v}_2) L(\mathbf{v}_1) p_A = L(\mathbf{v}_2) p_B = p_C. \tag{3}$$

If the velocity $\mathbf{v}_2$ is along the *same* direction, then the resulting velocity v_{12} of the final frame C with respect to the first frame A is given by the familiar formula for addition of relativistic velocities,

$$v_{12} = \frac{v_1 + v_2}{1 + v_1 v_2}, \tag{4}$$

where $v_i = |\mathbf{v}_i|$. However, if the second boost is *not* in the same direction but along a different direction that makes an angle θ relative to the first boost, then the velocity addition is more involved:

$$\mathbf{v}_{12} = \frac{1}{1 + \mathbf{v}_1 \cdot \mathbf{v}_2} \left[\left(1 + \frac{\gamma(\mathbf{v}_1)}{1 + \gamma(\mathbf{v}_1)} \mathbf{v}_1 \cdot \mathbf{v}_2 \right) \mathbf{v}_1 + \frac{1}{\gamma(\mathbf{v}_1)} \mathbf{v}_2 \right], \tag{5}$$

where $\gamma(\mathbf{v})^{-1} = \sqrt{1 - \mathbf{v}^2}$. Importantly, the resulting momentum p_C is not the momentum that one would get with a single boost from rest by composite velocity $\mathbf{v}_{12}$. The final momentum generally is also additionally subject to a *rotation* $R(\mathbf{v}_1, \mathbf{v}_2)$ called the *Thomas-Wigner rotation*. Formally, this means

$$L(\mathbf{v}_2) L(\mathbf{v}_1) = L(\mathbf{v}_{12}) R(\mathbf{v}_1, \mathbf{v}_2), \tag{6}$$

where $R(\mathbf{v}_1, \mathbf{v}_2)$ is a rotation by angle α that depends on velocities $\mathbf{v}_1$ and $\mathbf{v}_2$. Using Eq. (6) we can express the rotation as a sequence of boosts,

$$R(\mathbf{v}_1, \mathbf{v}_2) = L(\mathbf{v}_{12})^{-1} L(\mathbf{v}_2) L(\mathbf{v}_1). \tag{7}$$

This is the standard expression of the TWR as a result of a sequence of three boosts: first boosting the system from rest along arbitrary directions $\mathbf{v}_1$ and $\mathbf{v}_2$, and then bringing it back to rest by the third boost as shown in Fig. 1. For nonrelativistic velocities, the angle α by which the TWR rotates the system is negligible. In ultrarelativistic scenarios it can approach 180°, depending on the geometry of the boost situation, i.e., the angle between the velocities $\mathbf{v}_1$ and $\mathbf{v}_2$, and the magnitudes of the velocities.

In summary, the TWR always occurs when boosts are *noncollinear*. This can be generalized to an arbitrary number of boosts where at least two are noncollinear. Group theoretically, the reason is that the subset of boosts in the Lorentz group does not form a subgroup.[1] Instead, a combination of two boosts results in a boost *and* a rotation as expressed in Eq. (6).

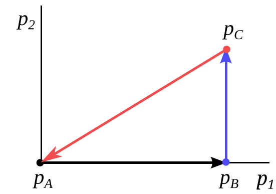


FIG. 1. Sequence of three boosts in the two-dimensional (2D) plane of the four-momentum space. The energy axis p_0 is perpendicular to the plane and the component $p_3 = 0$ of the four-momentum is suppressed. The plane consists of two spatial components $\mathbf{p} = (p_1, p_2)$, where each point corresponds to a system with momentum $\mathbf{p}$. The first boost (black arrow) with velocity $\mathbf{v}_1$ along the p_1 axis takes a system from rest momentum $\mathbf{p}_A = (0, 0)$ to $\mathbf{p}_B = (p_B, 0)$. The second boost (blue arrow) with velocity $\mathbf{v}_2$ is along the p_2 axis to $\mathbf{p}_C$, and the third (red arrow) with velocity $\mathbf{v}_{12}$ brings the system back to rest. As a result, the 2D plane undergoes TWR by $R(\mathbf{v}_1, \mathbf{v}_2)$ about the p_0 axis.

[1]This is in contrast to the nonrelativistic situation: boosts in the Galilei group do form a subgroup.

TWR gives rise to a plethora of effects in different branches of physics. It can be measured for satellites moving around the Earth, it manifests as a correction term for the nonrelativistic Hamiltonian of the hydrogen atom and it must be taken into account when calculating scattering cross sections in quantum field theory. It is also the reason why the behavior of quantum entanglement in relativity is significantly different from its nonrelativistic counterpart [13–19]. Because it is ultimately a feature of the structure of relativistic spacetime,[2] the TWR is independent of the dynamics that caused the boost [6].

Finally, one might relax the assumption we made above that only pure Lorentz transformations or pure boosts are used, thus considering a more generic treatment of boosts. The latter can be realized by multiplying both sides of Eq. (2) on the right by a momentum dependent rotation such that the rest momentum is unchanged, leading to different kinds of boosts. Each of these boosts is generated by a different momentum dependent rotation. In this more general scheme, our choice of pure boosts corresponds to the case where rotation is identity. The other two most often used options are helicity boosts and light front boosts [20]. In this paper, our goal is carry out the first stage of the analysis which involves only pure boosts. Once this analysis is carried out, one can consider more general boosts which involve momentum dependent rotations.[3]

B. Quantum mechanical spins and TWR

In this section we will summarize the standard treatment of massive spin-1/2 particles in the Hilbert space formalism in relativistic quantum mechanics.

In the Hilbert space theory of quantum systems, particles with spin are described as representation spaces of the relevant symmetry group. Free massive spin-1/2 particles can be described by two different but equivalent theories, the first using the unitary irreducible representations of the Poincaré-group and the second the finite-dimensional representations of the Lorentz group.[4] We will work in the first approach which relies on the Wigner representation (also called the Wigner-Bargmann or the spin basis), and is presented in Refs. [24–26]. Here the single particle states are given by the unitary representations of the Poincarégroup which are labeled by mass $m > 0$ and the intrinsic spin s, where the latter takes both integer and half-integer values. The representations are realized in the space $\mathscr{H}^+_{m,s} = \bigoplus^{2s+1} L^2(\mathscr{V}^+_m)$ of square integrable functions on the forward mass hyperboloid $\mathscr{V}^+_m = \{p_\mu \in \mathbb{E}^{1,3} \mid \eta^{\mu\nu} p_\mu p_\nu = m^2,\, p_0 > 0\}$, where the scalar product is defined as

$$\langle\phi|\psi\rangle = \sum_{\sigma=1}^{2s+1} \int d\mu(\mathbf{p})\, \phi^*_\sigma(\mathbf{p})\psi_\sigma(\mathbf{p}) \tag{8}$$

with $d\mu(\mathbf{p}) = [2E(\mathbf{p})]^{-1} d^3\mathbf{p}$ being the Lorentz invariant integration measure and $\phi_\sigma(\mathbf{p})$, $\psi_\sigma(\mathbf{p})$ elements of $L^2(\mathscr{V}^+_m)$. The state space $\mathscr{H}^+_{m,1/2}$ of single spin-1/2 particle with mass m is given by $L^2(\mathscr{V}^+_m) \otimes \mathbb{C}^2$. Using basis states which are labeled by the three-momentum $\mathbf{p}$ and spin σ, a generic state can be written as

$$|\psi\rangle = \sum_\sigma \int d\mu(\mathbf{p})\, \psi_\sigma(\mathbf{p})\, |\mathbf{p}, \sigma\rangle\,. \tag{9}$$

The general Lorentz transformation Λ acts on the basis element as follows:

$$U(\Lambda)\,|\mathbf{p}, \sigma\rangle = \sum_\lambda |\Lambda\mathbf{p}, \lambda\rangle\, D_{\lambda\sigma}[W(\Lambda, \mathbf{p})], \tag{10}$$

where we write $\Lambda\mathbf{p}$ for the spatial part of the vector Λp, with $p = (E(\mathbf{p}), \mathbf{p})$, and $W(\Lambda, \mathbf{p})$ is the Wigner rotation,

$$W(\Lambda, \mathbf{p}) = L^{-1}(\Lambda\mathbf{p})\Lambda L(\mathbf{p}) \tag{11}$$

which leaves p_0 invariant. Note that $W(\Lambda, \mathbf{p})$ exhibits the same form as $R(\mathbf{v}_1, \mathbf{v}_2)$ in (7) since $L(\mathbf{p})$ is the boost that maps the rest momentum to $\mathbf{p}$, Λ performs an arbitrary boost and $L^{-1}(\Lambda\mathbf{p})$ maps the system back to rest. For massive particles, W is an SO(3) rotation and $D[W(\Lambda, \mathbf{p})]$ the corresponding representation. The latter is an element of SU(2) for spin-1/2 particles and it can be generally written as

$$D(\alpha) = \exp(-\mathrm{i}\,\alpha\,\hat{\boldsymbol{n}} \cdot \boldsymbol{\sigma}/2), \tag{12}$$

where α is the Wigner rotation angle and the three unit vector $\hat{\boldsymbol{n}}$ defines the rotation axis. This rotation matrix can be parametrized in terms of momenta and rapidities [27].

We can now relate the action of boost operator $U(\Lambda)$ in Eq. (10) to the two-boost scenario described above in Fig. 1. The label $\mathbf{p}$ in the quantum state $|\mathbf{p}, \sigma\rangle$ of Eq. (10) refers to a system moving with velocity $\mathbf{v}_1 = \mathbf{p}/E(\mathbf{p})$ after the first boost. This corresponds to $\mathbf{p}_A$ in Fig. 1. Boost Λ in corresponds to the second boost by velocity $\mathbf{v}_2$ in Fig. 1. The TWR $W(\Lambda, \mathbf{p})$ is the rotation that the quantum system undergoes as a result of boost Λ when its state of motion changes from $|\mathbf{p}, \sigma\rangle$ at point $\mathbf{p}_A$ to state $U(\Lambda)\,|\mathbf{p}, \sigma\rangle$ at point $\mathbf{p}_B$ in Fig. 1.

But why do two noncollinear boosts lead to a rotation? In the next section we will see that interpreting the boost sequence in Fig. 1 in the geometric context provides a natural explanation of the phenomenon.

III. THE GEOMETRY OF CURVED MOMENTUM SPACE AND TWR

In this section we embark on the geometric study of TWR. We begin by discussing the intuitive picture of boosts from the perspective of the curved momentum space. Section III B describes the geometry of the mass hyperboloid in terms of standard differential geometry. Thereafter we discuss

[2] This is sometimes expressed by saying that the TWR is a *kinematic* effect.

[3] We note that from the methodological point of view it makes sense to first focus on pure boosts alone since they are the transformations that give rise to the unexpected phenomenon of the Wigner rotation that surprised many researches, including Einstein himself (see [21] cited in [22]). Choosing a more general boost convention which involves a rotation might obfuscate the situation by possibly masking or amplifying the Wigner rotation which arises from pure boosts alone. We would like to thank an anonymous referee for raising this point.

[4] See [23] for a good overview, including a detailed explanation of how the two approaches are related.

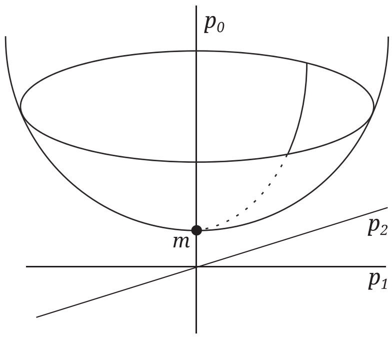


FIG. 2. The 2D relativistic mass hyperboloid. Component $p_3 = 0$ is not shown. The black dot indicates rest momentum at $p_A = (m, 0, 0, 0)$ where the hyperboloid intersects the p_0 axis.

quantum spins in terms of spinor bundles over the mass hyperboloid. We give an explicit derivation of the bundle connection in Secs. III C and III C 3. Finally in Sec. III D we conceptualize TWR as the holonomy of the curved momentum space.

A. Prelude: The intuitive picture

The account of how a sequence of noncollinear Lorentz boosts results in a rotation, and why they behave differently from the more familiar Galilei boosts discussed in Sec. II, relies on *group theoretic* properties of Lorentz boosts. However, an alternative perspective on TWR gives a *geometric* explanation for its appearance. In terms of projective geometry this is discussed in [6], and in [25] the authors consider the curved velocity space.

Our aim is to discuss the TWR as being caused by curved momentum space. This picture is particularly suitable since quantum mechanical spinors in momentum space representation can be understood precisely in terms of a spinor bundle over the relativistic curved momentum space. The latter is also key to understanding the reason why Lorentz boosts behave differently from Galilei boosts. While the nonrelativistic momentum is given by $\mathbb{R}^3$, which a flat space, the relativistic momentum space is given by a curved manifold: the mass hyperboloid, shown in the 2D case in Fig. 2. Points on the hyperboloid correspond to physically viable momenta, satisfying the relativistic dispersion relation $\eta^{\mu\nu} p_\mu p_\nu = m^2$. For instance, we can consider a physical system (an inertial frame or a spin of a particle or an observer) at point $p_A = (m, 0, 0, 0)$ on the hyperboloid, which corresponds to the system being at rest. There is a natural way to think of boosts of the physical system in the curved setting. A *pure boost* corresponds to parallel transporting the system along a geodesic from the initial to the final state of motion. For example, boosting a system from momentum p_A to p_B means the system is parallel transported along a geodesic from point p_A to p_B on the hyperboloid.

Using the hyperboloid, let us visualize the action of the boost sequence previously shown in Fig. 1. For the visualization, we consider boosting a 2D frame $F(p)$; see Fig. 3. Without loss of generality, let us orient this 2D frame initially, when it is at rest at $p_A = (m, 0, 0, 0)$, along the p_1 and p_2 axes. The first boost $L(\mathbf{v}_1)$ transports the frame along a geodesic of

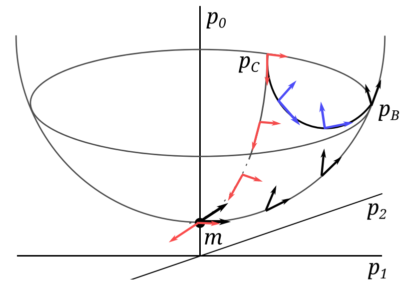


FIG. 3. Sequence of three boosts on the 2D mass hyperboloid. The black dot indicates rest momentum at $p_A = (m, 0, 0, 0)$ where the hyperboloid intersects the p_0 axis. The first (black) boost is along the geodesic from rest to point p_B, the second (blue) along the geodesic from point p_B to p_C. The third (red) brings the system back to rest. The resulting final frame (red) at the origin is Thomas-Wigner rotated relative to the initial (black) frame.

the hyperboloid (shown black) to p_B. There is no rotation as a result of the boost, the frame remains oriented in the original direction. Now, applying the second boost $L(\mathbf{v}_2)$ along the p_2 axis means the frame (shown blue) travels along a geodesic which does not intersect with the origin. Parallel transporting the frame along that geodesic to point p_C means the frame is *rotated* relative to the frame at the origin. We can see this clearly when we boost the frame (shown red) back to the origin p_A along a geodesic. This geodesic *does* intersect the origin, hence there is *no rotation* involved when the frame is parallel transported along that geodesic. The final (red) frame shown in Fig. 3 is TWR rotated relative to the initial (black) frame.

This example demonstrates the power of the geometric picture. Since boosting is conceptualized as parallel transport between points on the hyperboloid, one can easily see why the final frame is rotated relative the initial one. Other scenarios can be analyzed in a similar manner. In general, one recognizes that whether or not a frame will be rotated as a result of a series of boosts depends on the path that the frame follows on the hyperboloid. For instance, all geodesics that intersect the origin do not rotate the frame. Rotation occurs only when the system is boosted along a geodesic that does not intersect the origin [6]. Hence parallel transport along a generic path on the hyperboloid will give rise to rotation.

With this intuitive picture at hand, we will next give a formal, differential geometric description of the mass hyperboloid.

B. The geometry of the mass hyperboloid

In the previous section we described the momentum space as a hyperboloid and argued heuristically that boosting a system means parallel transporting it along a particular trajectory. We now turn to the formal, differential geometric description of the relativistic curved momentum space. The state space of a spin-1/2 particle as a field over this manifold will be introduced below in Sec. III C and the TWR as holonomy in Sec. III D.

We start with Minkowski spacetime $\mathbb{E}^{1,3}$ and note that, as a manifold, this space comes equipped with a tangent $T_x\mathbb{E}^{1,3}$ and cotangent space $T_x^*\mathbb{E}^{1,3}$ at each point $x \in \mathbb{E}^{1,3}$. The mass hyperboloid $\mathscr{V}_m^+$ is defined as a subset of the cotangent space,

$$\mathscr{V}_{m,x}^+ = \{P \in T_x^*\mathbb{E}^{1,3} \mid \eta(P,P) = \eta^{\mu\nu} p_\mu p_\nu = m^2, p_0 > 0\}. \tag{13}$$

Since the Minkowski space is flat, we can identify the mass hyperboloids at different points of spacetime and omit the subscript x that labels the base point, $\mathscr{V}_{m,x}^+ = \mathscr{V}_m^+$. As a result, we speak of a single momentum hyperboloid in which the particle moves.[5]

To describe the intrinsic geometry of the hyperboloid, we consider $\mathscr{V}_m^+$ as the image of a mapping of spherical polar coordinates $z = (\rho, \theta, \phi)$, where $\rho \in [0, \infty)$, $\theta \in [0, \pi)$, $\phi \in [0, 2\pi)$, into the cotangent space $T_x^*\mathbb{E}^{1,3}$,

$$f(\rho,\theta,\phi) \mapsto p(\rho,\theta,\phi),$$
$$p(\rho,\theta,\phi) = \left(\sqrt{m^2+\rho^2}, \rho\sin\theta\cos\phi, \rho\sin\theta\sin\phi, \rho\cos\theta\right), \tag{14}$$

where

$$\rho = |\mathbf{p}| = \sqrt{\mathbf{p}\cdot\mathbf{p}}, \quad \tan\theta = \frac{\sqrt{p_1^2+p_2^2}}{p_3}, \quad \tan\phi = \frac{p_2}{p_1}. \tag{15}$$

For $\theta = \pi/2$ this parametrization of the mass hyperboloid encodes motion of a particle in the spatial 1-2 plane. The energy $E := E(\rho)$ of the particle is identified as

$$E(\rho) = \sqrt{m^2+\rho^2}, \tag{16}$$

and the relativistic $\gamma(v) = 1/\sqrt{1-v^2}$ factor for a particle with velocity v in this language is given by

$$\gamma(v) = \frac{E}{m} = \frac{\sqrt{m^2+\rho^2}}{m} \;\Leftrightarrow\; \rho = \frac{mv}{\sqrt{1-v^2}}. \tag{17}$$

We next induce a metric tensor g on the hyperboloid via pullback of the Minkowski inner product from $T_x^*\mathbb{E}^{1,3}$ to $\mathscr{V}_m^+$:

$$g = -\left(\tfrac{m^2}{E^2} d\rho\otimes d\rho + \rho^2 d\theta\otimes d\theta + \rho^2\sin^2\theta d\phi\otimes d\phi\right). \tag{18}$$

This metric is negative definite since we started from the Minkowski metric of signature $(+,-,-,-)$. It defines the intrinsic geometry of $\mathscr{V}_m^+$ via the Levi-Civita connection which determines the parallel transport of vectors, and whose nonvanishing Christoffel symbols $\Gamma^i{}_{jk} = \frac{1}{2}g^{ip}(\partial_j g_{pk} + \partial_k g_{pj} - \partial_p g_{jk})$ are given as follows:

$$\begin{aligned}
&\Gamma^\rho{}_{\rho\rho} = -\tfrac{\rho}{m^2+\rho^2}, \\
&\Gamma^\rho{}_{\theta\theta}\sin^2\theta = \Gamma^\rho{}_{\phi\phi} = -\tfrac{\rho}{m^2}(m^2+\rho^2)\sin^2\theta, \\
&\Gamma^\theta{}_{\rho\theta} = \Gamma^\theta{}_{\theta\rho} = \tfrac{1}{\rho}, \\
&\Gamma^\theta{}_{\phi\phi} = -\cos\theta\sin\theta, \\
&\Gamma^\phi{}_{\rho\phi} = \Gamma^\phi{}_{\phi\rho} = \tfrac{1}{\rho}, \\
&\Gamma^\phi{}_{\theta\theta} = \cot\theta.
\end{aligned} \tag{19}$$

[5]In general, this cannot be done in curved spacetimes since there is no canonical way to identify hyperboloids at neighboring points.

The Riemann curvature tensor of the hyperboloid can be also easily evaluated, $R^p{}_{qij} = \partial_i\Gamma^p{}_{qj} - \partial_j\Gamma^p{}_{qi} + \Gamma^p{}_{si}\Gamma^s{}_{qj} - \Gamma^p{}_{sj}\Gamma^s{}_{qi}$, and its Ricci scalar is constant $R = g^{qj}R^p{}_{qpj} = 6/m^2$. The latter is not surprising since the hyperboloid is a maximally symmetric space with constant curvature.

In order to describe the parallel transport of spinors later in Secs. III C–III D, we need another geometric ingredient: the spin connection coefficients induced by the Levi-Civita connection. Loosely speaking, we need to express the information encoded in the Levi-Civita connection as a collection of 1-forms $\omega^A{}_B$ on the curved momentum manifold $\mathscr{V}_m^+$,

$$\omega^A{}_B = \sum_{i\in\{\rho,\phi,\theta\}} \omega^A{}_{Bi}\, dz^i, \tag{20}$$

whose components can be computed using the relation

$$\omega^A{}_{Bi} = e^A{}_k e_B{}^j \Gamma^k{}_{ij} + e^A{}_k \partial_i e_B{}^k, \tag{21}$$

where $e^A{}_i$ are components of an orthonormal coframe $\Theta^A = e^A{}_i\, dz^i$ of the momentum space metric g. In other words, we write the Christoffel symbols in an orthonormal frame basis θ^A, while reserving the index i for the coordinate basis. The momentum space 1-forms $\omega^A{}_B$ are the components of a SO(3) connection 1-form ω on $\mathscr{V}_m^+$, which we will later on map to the SU(2) spinor connection over $\mathscr{V}_m^+$.

Using the following frame and coframe of the metric (18),

$$\begin{aligned}
&e_1 = \frac{E}{m}\partial_\rho, \quad e_2 = \frac{1}{\rho}\partial_\theta, \quad e_3 = \frac{1}{\rho\sin\theta}\partial_\phi, \\
&\Theta^1 = \frac{m}{E}d\rho, \quad \Theta^2 = \rho\, d\theta, \quad \Theta^3 = \rho\sin\theta\, d\phi,
\end{aligned} \tag{22}$$

we can display the coefficients $\omega^A{}_B$ as the matrix

$$\omega = \begin{bmatrix} 0 & -\frac{E}{m}d\theta & -\sin\theta\,\frac{E}{m}d\phi \\ \frac{E}{m}d\theta & 0 & -\cos\theta\, d\phi \\ \sin\theta\,\frac{E}{m}d\phi & \cos\theta\, d\phi & 0 \end{bmatrix}. \tag{23}$$

Similarly, we can express the curvature of the Levi-Civita connection as a collection of 2-forms $\Omega^A{}_B$:

$$\begin{aligned}
\Omega^A{}_B &= \tfrac{1}{2}R^A{}_{Bij}dz^i\wedge dz^j = \tfrac{1}{4}R^A{}_{Bij}(dz^i\otimes dz^j - dz^j\otimes dz^i) \\
&= \tfrac{1}{2}R^A{}_{Bij}dz^i\otimes dz^j,
\end{aligned} \tag{24}$$

where $R^A{}_{Bij} = R^a{}_{bij}e^A{}_a e^b{}_B$ are the components of the Riemann curvature tensor of the metric g partially expressed in the orthonormal frame and coframe e and Θ. The momentum-space 2-forms $\Omega^A{}_B$ are the components of the SO(3) curvature 2-form of the over $\mathscr{V}_m^+$. For the frame (22) we obtain, again in matrix notation,

$$\Omega = \begin{bmatrix} 0 & -\frac{\sqrt{E^2-m^2}}{Em}\, d\rho \wedge d\theta & -\frac{\sqrt{E^2-m^2}}{Em} \sin\theta\, d\rho \wedge d\phi \\ \frac{\sqrt{E^2-m^2}}{Em}\, d\rho \wedge d\theta & 0 & -\frac{(E^2-m^2)}{m^2} \sin\theta\, d\theta \wedge d\phi \\ \frac{\sqrt{E^2-m^2}}{Em} \sin\theta\, d\rho \wedge d\phi & \frac{(E^2-m^2)}{m^2} \sin\theta\, d\theta \wedge d\phi & 0 \end{bmatrix}. \tag{25}$$

Constructing the connection 1-form ω and the curvature 2-form Ω of the mass hyperboloid $\mathscr{V}_m^+$ concludes the necessary geometric ingredients we need to describe spinors on the mass hyperboloid. However, note that at this point the picture is incomplete since it does not contain the quantum state space; all we have is a bare basis manifold with the tangent structure. In the next section we will finish the construction: we will add the state space of a free spin-1/2 particle over the basis manifold.

C. Spinors on the mass hyperboloid

In this section we turn to the geometric approach of the same Hilbert space that describes a free massive spin-1/2 particle discussed in Sec. II B. We stress that we are not constructing a new state space. Rather, we focus on the geometric structure that is inherent in the same space while using the language of differential geometry to describe how the state space of a relativistic particle arises in the geometric context. This is common practice when working with the gauge theoretic structure of quantum theory. We begin with an intuitive picture and then move on to a more formal description.

Intuitively, when we think about the particle in flat spacetime, its wave function can be represented in the position representation $\psi(x)$ or the momentum representation $\psi(p)$. For relativistic particles, momentum space is the mass hyperboloid $\mathscr{V}_m^+$, which, as we have seen above, is a Riemannian space with constant curvature. Then, ψ is a map from $\mathscr{V}_m^+$ to the Hilbert space $\mathscr{H}_p$ at $p \in \mathscr{V}_m^+$. Technically, we can interpret this in the language of fiber bundles, where $\mathscr{V}_m^+$ is the base manifold, $\mathscr{H}_p$ is the typical fiber, and ψ is a section of the fiber bundle given by the union of spaces $\mathscr{H}_p$ over all $p \in \mathscr{V}_m^+$. Common notation in the literature for fiber bundles is $(\pi : E \to M, F)$, where E is the total space of the bundle, M the base manifold, F the fiber, and π the projection from the total space to the base manifold. Hence in the case under study we write $(\pi : E \to \mathscr{V}_m^+, \mathscr{H}_p)$. For single particles, i.e., for spinors, the state space of the particle arises as the space of square integrable sections of the vector bundle with a suitable representation of the group SU(2). This state space is the space $\mathscr{H}_{m,s}^+$ we referred to in Sec. II B.

From the differential geometric perspective it is clear that one cannot directly compare wave functions $\psi(p)$ at different points p since they belong to different spaces $\mathscr{H}_p$. In order to compare them we need to map the wave functions into the same space, which is usually done by a nontrivial path dependent parallel transport. The latter in general leads to nontrivial state change as shown in Fig. 3.

For clarity, let us next recall how to construct the bundle for spin-1/2 particles, which is the spinor bundle over the three-dimensional curved manifold $\mathscr{V}_m^+$ equipped with the metric g. In this case the typical fiber $\mathscr{H}_p = \mathbb{C}^2$ is endowed with an action of SU(2) and spinors are sections $\psi : \mathscr{V}_m^+ \to \mathbb{C}^2$.

1. Spin group and spinors: Algebra

In order to construct spinors, one starts with a Clifford algebra and then identifies the spin group as a particular subset of the algebra. A Clifford algebra is defined as a pair $(\mathcal{A}, \gamma)$ for a quadratic space (V, g), where V is a vector space over $\mathbb{R}$, g a scalar product on V, and $\gamma : V \to \mathcal{A}$ a linear map. The elements of the algebra satisfy the Clifford multiplication rule

$$\gamma(\mathbf{v})\gamma(\mathbf{u}) + \gamma(\mathbf{u})\gamma(\mathbf{v}) = 2g(\mathbf{v}, \mathbf{u})1_{\mathcal{A}} \tag{26}$$

for all $\mathbf{u}, \mathbf{v} \in V$. In the case where g has signature $p + q = n$, the corresponding Clifford algebra is denoted $\mathrm{Cl}(p, q) := \mathrm{Cl}(\mathbb{R}^{p,q})$. A generic element of the algebra need not have an inverse, hence $\mathrm{Cl}(p, q)$ is not a group. However, the subset of elements that do have inverses are normalized and consist only of products of even number of elements, forms the spin group $\mathrm{Spin}(p, q)$. Specifically for our purposes $p = 3$ and $q = 0$, then the algebra is Cl(3, 0) and the corresponding spin group is Spin(3). It can be shown that the latter is isomorphic to SU(2).

The isomorphism provides the matrix representation of Spin(3). This way we have arrived at *spinors*: they are the elements of spaces on which the spin group acts. In other words, spinors are real or complex column vectors which come with the rule that specifies how they are transformed by the elements of the Clifford algebra [28].

Thus, Clifford algebras lead to the identification of the spin group, which in turn gives rise to spinors.

2. Spin and spinor bundles

Having recalled the spin group and spinors we can construct the corresponding bundles. How does one accomplish this technically? Historically, it was not clear how to construct spinors on Riemannian manifolds. It was only after the development of the formalism of principal fiber bundles at the end of 1940s that spinors could be transferred from flat spaces to Riemannian manifolds [29]. Following treatments in [28,30], we will construct the spinor bundle as an associated bundle of a spin bundle. Spinor fields are defined as cross sections of the spinor bundle.

To construct a *spin* bundle $(P_{\mathrm{Spin}(3)} \xrightarrow{\tilde{\pi}_s} \mathscr{V}_m^+, \mathrm{Spin}(3))$, we first focus on the tangent bundle of the manifold under consideration; here the tangent bundle of the mass shell, $T\mathscr{V}_m^+ = E$, which is a six-dimensional vector bundle over $(\mathscr{V}_m^+, g)$. The bundle of orthonormal frames $P_{\mathrm{SO}(3)}(E)$ over the mass shell is a principal bundle $(P_{\mathrm{SO}(3)}(E) \xrightarrow{\tilde{\pi}} \mathscr{V}_m^+, \mathrm{SO}(3))$, with group SO(3) as typical fiber.

The spin bundle is now the principle bundle in which the orthogonal group, here SO(3), is replaced by its double covering, here Spin(3) ≅ SU(2), such that two sheeted covering map $\xi : P_{\mathrm{Spin}(3)}(E) \longrightarrow P_{\mathrm{SO}(3)}(E)$ exists and the following relation holds:

$$\begin{array}{ccc} P_{\mathrm{Spin}(3)}(E) & \xrightarrow{\quad \xi \quad} & P_{\mathrm{SO}(3)}(E) \\ & {\scriptstyle \widetilde{\pi}_s} \searrow \quad \swarrow {\scriptstyle \widetilde{\pi}} & \\ & \mathscr{V}_m^+ & \end{array} \tag{27}$$

where $\widetilde{\pi}$ and $\widetilde{\pi}_s$ are, respectively, projections that define the $P_{\mathrm{SO}(3)}(E)$ and $P_{\mathrm{Spin}(3)}(E)$ bundles.

A *spinor* bundle $S(E)$ of E is defined as an associated vector bundle to the spin bundle $P_{\mathrm{Spin}}(E)$,

$$S(E) = P_{\mathrm{Spin}}(E) \times_\rho V, \tag{28}$$

where V is a vector space which carries the representation ρ and action of the group Spin. For spin-1/2 particles over the base manifold $(\mathscr{V}_m^+, g)$, we have Spin = Spin(3), the vector space is $V = \mathbb{C}^2$, and ρ is the fundamental 2×2 matrix representation of Spin(3) ≅ SU(2).[6]

With this the construction is complete and we can regard spinor fields as sections ψ of the spinor bundle $S(E)$.

3. *The spinor connection and curvature*

Let us take stock of where we are and what needs to be accomplished next. We have constructed the spinor bundle and defined spin states as sections of the bundle. We have collected almost all the components needed to realize the geometric idea of a boost as parallel transport except for one crucial component: a connection that tells us how to transport spinors. Recall, however, that above in (23) we computed the connection for the tangent bundle; this describes parallel transport of tangent vectors. It turns out that that computation was not in vain. We can use the connection on the tangent bundle to *generate* a connection on the spinor bundle. In this section, we will explicitly compute the spinor connection, following the discussion in Refs. [28,30].

The key idea is that the two-sheeted covering ξ in (27) gives rise to a Lie algebra isomorphism between the su(2) and so(3) algebras, which enables one to lift the SO(3) connection to the spin connection, which in turn induces a connection on the associated spinor bundle. Let us elaborate on this.

We start by noting that we regarded the 1-form ω as an SO(3) connection over momentum space $\mathscr{V}_m^+$. This means it maps a given vector field X on $\mathscr{V}_m^+$ to $\omega(X)$, which is an element of the Lie algebra so(3), and thus ω can be written in the basis of the algebra as

$$\omega = \omega^1{}_2 E_1{}^2 + \omega^1{}_3 E_1{}^3 + \omega^2{}_3 E_2{}^3, \tag{29}$$

where $E_i{}^j$ are the antisymmetric matrices with -1 at the i, j entry, 1 at the j, i entry, and 0 elsewhere:

$$\begin{bmatrix} & (i) & & (j) & \\ & \vdots & & \vdots & \\ (i) \cdots & 0 & \cdots & -1 & \cdots \\ (j) \cdots & 1 & \cdots & 0 & \cdots \\ & \vdots & & \vdots & \end{bmatrix}, \tag{30}$$

which form a basis of the Lie algebra so(3). The components are given by the 1-form components we computed in (23),

$$\omega^1{}_2 = \frac{E}{m}\, d\theta, \quad \omega^1{}_3 = \sin\theta\, \frac{E}{m}\, d\phi, \quad \omega^2{}_3 = \cos\theta\, d\phi. \tag{31}$$

The connection 1-form (29) can be elevated to a *global* connection $\widetilde{\omega}$ over the frame bundle $P_{\mathrm{SO}(3)}(E)$; see an overview of structures used in the paper in Appendix A. The resulting connection contains the same information as the collection of local connections defined on patches $\mathcal{O} \subset \mathscr{V}_m^+$. The two connections are related by a pullback $\omega = \widetilde{\sigma}^* \widetilde{\omega}$ with a section $\widetilde{\sigma} : \mathscr{V}_m^+ \rightarrow P_{\mathrm{SO}(3)}(E)$ of the principal bundle. Formally, both connections are 1-forms taking values in the Lie algebra of the structure group of their respective bundles. In fact, although formally ω and $\widetilde{\omega}$ live in different spaces, the coordinate expressions of both connections turn out to be identical, so the matrix of $\widetilde{\omega}$ is the same as ω.

The central idea in constructing the spinor connection is that once a connection on the frame bundle $P_{\mathrm{SO}(3)}(E)$ is fixed, a connection on the spin bundle $P_{\mathrm{Spin}(3)}(E)$ is uniquely determined. This is because the double covering ξ induces an isomorphism ϕ that determines the spin connection $\widetilde{\omega}_s$ from the frame bundle connection $\widetilde{\omega}$, where the precise relationship is given by [28]

$$\xi^* \widetilde{\omega} = \phi(\widetilde{\omega}_s). \tag{32}$$

To obtain the spinor connection ω_s we need to pull down $\widetilde{\omega}_s$ to the base manifold $\mathscr{V}_m^+$ using a section $\widetilde{\sigma}_s$ of $P_{\mathrm{Spin}(3)}$,

$$\omega_s = \widetilde{\sigma}_s^* \widetilde{\omega}_s. \tag{33}$$

Using also the fact that a section $\widetilde{\sigma}_s$ on the spin and a section $\widetilde{\sigma}$ of the frame bundle are related by $\widetilde{\sigma} = \xi \circ \widetilde{\sigma}_s$, we obtain a simple relationship between the connections on the base manifold $\mathscr{V}_m^+$,

$$\omega_s = \phi(\omega), \tag{34}$$

which tells us that as a practical computation for obtaining the spinor connection ω_s from the local Levi-Civita connection ω, one needs to express the connection ω in the su(2) basis (see details in Appendix B). The latter is given in terms of Pauli matrices by

$$\begin{aligned} J_1 &= -\frac{1}{2} i\sigma_1 = -\frac{1}{2} i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ J_2 &= -\frac{1}{2} i\sigma_2 = -\frac{1}{2} i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \\ J_3 &= -\frac{1}{2} i\sigma_3 = -\frac{1}{2} i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \tag{35}$$

[6]Treating spinors as sections of a vector bundle with $V = \mathbb{C}^2$ over the mass shell has also been discussed in the context of relativistic quantum information theory in [11,12].

This basis satisfies

$$[J_i, J_k] = \sum_j \epsilon_{ikj} J^j. \tag{36}$$

The isomorphism ϕ between the so(3) and su(2) bases is given by

$$E_i{}^j \leftrightarrow \epsilon_i{}^{jk} J_k, \tag{37}$$

where ϵ is the totally antisymmetric Levi-Civita tensor of g in the frame basis. Explicitly we have

$$\begin{aligned} &E_1{}^2 \leftrightarrow -\tfrac{1}{2} i\sigma_3 = J_3, \quad E_1{}^3 \leftrightarrow \tfrac{1}{2} i\sigma_2 = -J_2, \\ &E_2{}^3 \leftrightarrow -\tfrac{1}{2} i\sigma_1 = J_1. \end{aligned} \tag{38}$$

Using (29), the spinor connection ω_s takes the form

$$\begin{aligned} \omega_s &= -\frac{i}{2}\left(\omega^1{}_2\sigma^3 - \omega^1{}_3\sigma^2 + \omega^2{}_3\sigma^1\right) \\ &= -\frac{i}{2}\left(\frac{E}{m} d\theta\, \sigma^3 - \frac{E}{m}\sin\theta\, d\phi\, \sigma^2 + \cos\theta\, d\phi\, \sigma^1\right). \end{aligned} \tag{39}$$

Similar reasoning applies to the computation of spinor curvature. Starting with (25), which is expressed in the so(3) basis, we rewrite curvature in the su(2) basis and obtain

$$\begin{aligned} \Omega_s = \frac{i}{2}\Bigg(&-\frac{\sqrt{E^2 - m^2}}{Em}\, d\rho \wedge d\theta\, \sigma_3 \\ &+ \frac{\sqrt{E^2 - m^2}}{Em} \sin\theta\, d\rho \wedge d\phi\, \sigma_2 \\ &- \frac{E^2 - m^2}{m^2} \sin\theta\, d\theta \wedge d\phi\, \sigma_1\Bigg). \end{aligned} \tag{40}$$

These two quantities—spinor connection and curvature—represent an important milestone. They encode the necessary geometric information for realizing the goal we have been working for: to describe how the spin of a quantum particle changes when it follows a path in the curved momentum space. In the next section, we will look at a concrete example and calculate the holonomy matrix that characterizes the state change of the spin.

D. TWR as holonomy

Up until now we have been claiming that in the geometric framework boosting the particle can be understood as parallel transporting the vector which represents the state of the particle from the initial to the final momentum. We saw above that the simplest form of TWR occurs when the particle is boosted along a triangular, closed path in the momentum space where the initial and the final momenta correspond to the rest momenta of the system. This leads to the notion of *holonomy*. Holonomy is the idea that we can associate with every closed curve C a transformation matrix which maps the initial state ψ_i of the system to its final state ψ_f when the particle has been parallel transported along the closed curve. The set of all such transformation matrices forms a group which is called the *holonomy group*. To determine the transformation matrices belonging to a specific loop C, one needs to solve the parallel transport equation along this loop. Formally, one can express the transformation with help of the path ordered exponential [31] as

$$\mathrm{Hol}(\omega_s, C) = \mathcal{P}\left[\exp\left(-\int_C \omega_s\right)\right]. \tag{41}$$

This means TWR can be understood as the holonomy transformation which arises due to the curvature of the relativistic momentum space.

By way of an example, let us calculate the holonomy matrix for a particular boost scenario. Consider a spin-1/2 particle that follows a circular path C in the momentum space given by

$$\begin{aligned} C : [0, 2\pi] &\to \mathscr{V}_m^+ \\ \tau &\mapsto C(\tau) = (\rho(\tau), \theta(\tau), \phi(\tau)) = (\rho_0, \pi/2, \tau). \end{aligned} \tag{42}$$

One can think of the particle that is moving with momentum of constant norm $\rho_0 = mV/\sqrt{1 - V^2}$, or with speed V [see (17)], as undergoing infinitesimal parallel transports when it travels around the circular trajectory. Each small boost gives rise to TWR, all of which accumulate when the particle has completed one revolution. This is the famous case of Thomas precession. The holonomy matrix Hol is a function of connection ω and path C along which the vector is parallel transported. The holonomy of C can be expressed in terms of the curvature of the connection as

$$\mathrm{Hol}(\omega_s, C) = \exp\left(-\int_D \Omega_s\right), \tag{43}$$

where D is the disk with boundary C. Using (40) with $d\theta = 0$, the holonomy integral becomes

$$\begin{aligned} -\int_D \Omega_s &= -\int_0^{\rho_0}\int_0^{2\pi} d\rho\, d\phi\, \frac{i}{2}\frac{\rho}{\sqrt{m^2 + \rho^2}m}\sigma_2 \\ &= -i\sigma_2\pi\frac{\left(\sqrt{m^2 + \rho_0^2} - m\right)}{m} = -i\sigma_2\pi\left(\frac{E(\rho_0)}{m} - 1\right) \\ &= -i\sigma_2\pi(\gamma(V) - 1). \end{aligned} \tag{44}$$

Introducing $\alpha = 2\pi(\gamma(V) - 1)$ we get the following holonomy matrix:

$$\mathrm{Hol}(\omega_s, C) = \exp\left(-i\frac{\alpha}{2}\sigma_2\right) = \begin{pmatrix} \cos(\alpha/2) & -\sin(\alpha/2) \\ \sin(\alpha/2) & \cos(\alpha/2) \end{pmatrix}. \tag{45}$$

This SU(2) matrix acting on spin-1/2 particles corresponds to an SO(3) rotation R_{e_2} by angle α around the $e_2 = \frac{1}{\rho}\partial_\theta$ axis; see Appendix C.

With this result we have demonstrated how to derive the TWR and in particular the Thomas precession of spinors, using the formalism of the differential geometry of curved relativistic momentum space, the mass hyperboloid. Our results coincide with the results obtained by the authors in [6] for the same scenario in terms of projective geometry. The fact that the two approaches converge on the same result demonstrates that both reproduce the essential characteristics of the phenomenon albeit with somewhat different means.

Let us briefly discuss the advantages offered by the two approaches. They belong to the same family since both offer geometric conceptualization of TWR. The approach in [6] provides valuable insight into TWR by making the phenomenon easy to grasp visually as well as conceptually. It also serves as an inspiration for the current paper.[7] On the other hand, the approach adopted here builds on this using the language of modern differential geometry. The advantage of the modern theory lies in that it provides a toolbox for calculating interesting quantities in a coherent framework. For instance, it allows one to compute the holonomy matrix based on the quantities that describe the intrinsic geometry of the mass shell manifold: the connection and curvature. In other words, it allows one to conceptualize the TWR as holonomy, associate it with structure, and relate it to the characteristics of the manifold: holonomy forms a group which is nontrivial if the manifold has curvature.

IV. CONCLUSIONS

We have discussed how the TWR of a spin-1/2 particle can be understood in a geometric manner as the holonomy of the curved relativistic momentum space, the mass hyperboloid. We explicitly demonstrated how to construct the spin bundle over the mass hyperboloid and how to derive the holonomy matrix.

Since holonomies describe numerous physical effects, for example Berry's phase, the Aharonov-Bohm effect, the Foucault pendulum, or gravitational effects on matter wave interferometers, one sometimes groups them as being geometric or topological, and classical or quantum holonomies [9]. For instance, the famous case of Berry's phase is a geometric and a quantum holonomy because it originates in the curvature of the quantum bundle. TWR, on the other hand, is a classical and a geometric holonomy since it arises from the curvature of the relativistic momentum space.

Having understood the TWR geometrically on Minkowski spacetime, our work paves the way to extending this approach to curved spacetimes. To do so, we aim to develop a geometric framework which allows one to describe spinors on a curved spacetime as sections of the spinor bundle over the nonequivalent curved momentum spaces—the mass hyperboloids—at each point of spacetime. This necessitates the construction of differential geometric structures that incorporate both curved spacetimes and curved momentum spaces. A promising candidate here is Hamilton geometry or its generalizations [32,33]. Previous approaches to describe TWR on curved spacetimes rely on the Dirac approach to the spin-1/2 particle, and do not refer to curved momentum spaces [34]. Moreover, the geometric approach to TWR allows us to study how deformed mass shells change the predictions for the TWR. Deformed mass shells appear in deformed or doubly special relativity (DSR) theories employed in quantum gravity phenomenology [35], the most studied one being the κ-Poincaré framework, or in Lorentz invariance violating theories like the standard model extension.

Since it has also been demonstrated that viewing spinors as vector bundles over the curved momentum space, or mass shell, is well suited for analyzing the ambiguities in relativistic quantum information (RQI) theory [11,12], extending our approach to curved spacetimes will also lead to a better understanding of RQI on curved spacetimes.

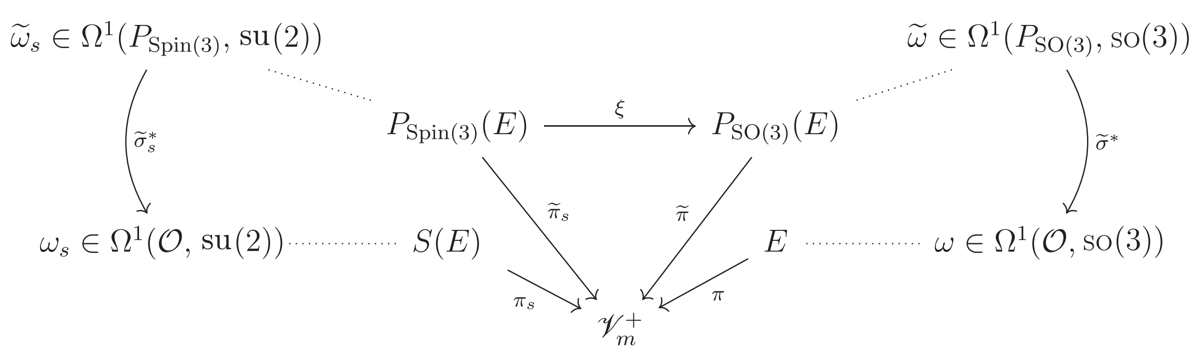
Thus the treatment of TWR presented here in terms of the geometry of spinor bundles over curved momentum space explicitly exploits the connection between special relativistic effects and the geometry of momentum space. It lays the foundation for future investigations in the context of momentum spaces whose geometry differs from the special relativistic one, and also for the addition of spacetime curvature to the picture.

ACKNOWLEDGMENTS

C.P. is funded by the excellence cluster QuantumFrontiers of the German Research Foundation (Deutsche Forschungsgemeinschaft, DFG) under Germany's Excellence Strategy, EXC-2123 QuantumFrontiers, 390837967. V.P. acknowledges partial support by the Estonian Research Council (Eesti Teadusagentuur, ETAG) through Grant No. PSG489.

APPENDIX A: SUMMARY OF STRUCTURES USED IN THE PAPER

The following diagram summarizes the structure and objects used in constructing the spinor connection:



(A1)

[7] Many facts about the behavior of boosts can be readily grasped using projective geometry. Reference [6] provides an excellent and accessible treatment of TWR in the geometric setting.

In the diagram, $E = T\mathscr{V}_m^+$ is the vector bundle: the tangent bundle of the forward mass hyperboloid $\mathscr{V}_m^+$ for particle with mass m. We denote $\mathcal{O} \subset \mathscr{V}_m^+$ and, to avoid making the figure even more crowded, we omit the spin section $\widetilde{\sigma}_s : \mathscr{V}_m^+ \to P_{\mathrm{Spin}(3)}(E)$ and the frame bundle section $\widetilde{\sigma} : \mathscr{V}_m^+ \to P_{\mathrm{SO}(3)}(E)$, but show how their pullbacks map connections over principal bundles to connections for the respective associated bundles, for instance $\omega_s = \widetilde{\sigma}_s^* \widetilde{\omega}_s$. We also occasionally abbreviate $P_{\mathrm{Spin}(3)}(E) \equiv P_{\mathrm{Spin}(3)}$ and $P_{\mathrm{SO}(3)}(E) \equiv P_{\mathrm{SO}(3)}$. The dotted lines show which space a particular connection operates on. For instance, ω belongs to the space Ω^1 of so(3) algebra valued 1-forms defined on $\mathcal{O}$, and it defines a map on E. In the same vein, $\widetilde{\omega}$ is a connection on the SO(3) principal bundle, $\widetilde{\omega}_s$ on the Spin(3) principal bundle, and ω_s on the spinor bundle $S(E)$.

APPENDIX B: THE SPINOR CONNECTION

Given the (local) Levi-Civita connection 1-form ω on $\mathscr{V}_m^+$, we can compute the (local) spinor connection ω_s on $\mathscr{V}_m^+$. We follow the discussions in [28,30]. Using the notation of (A1), we denote connections on the corresponding principal bundles with $\widetilde{\ }$, i.e. the global connection $\widetilde{\omega}$ on $P_{\mathrm{SO}(3)}(E)$ and the spin connection $\widetilde{\omega}_s$ on $P_{\mathrm{Spin}}(E)$. The relationship between the two connections is given by

$$\xi^* \widetilde{\omega} = \phi(\widetilde{\omega}_s). \tag{B1}$$

In order to obtain the spinor connection ω_s we need to pull down $\widetilde{\omega}_s$ to the base manifold $\mathscr{V}_m^*$ using a section $\widetilde{\sigma}_s$ of the spin bundle $P_{\mathrm{Spin}(3)}(E)$,

$$\omega_s = \widetilde{\sigma}_s^* \widetilde{\omega}_s. \tag{B2}$$

We can now check that the pulled down sections $\widetilde{\sigma}^* \widetilde{\omega}$ and $\widetilde{\sigma}_s^* \widetilde{\omega}_s$ are related by the isomorphism ϕ. Using the fact that a section $\widetilde{\sigma}_s$ on the spin and a section $\widetilde{\sigma}$ of the frame bundle are related by $\widetilde{\sigma} = \xi \circ \widetilde{\sigma}_s$, we compute

$$\widetilde{\sigma}^* \widetilde{\omega} = (\xi \circ \widetilde{\sigma}_s)^* \widetilde{\omega} \tag{B3}$$
$$= \widetilde{\sigma}_s^* (\xi^* \widetilde{\omega}) \tag{B4}$$
$$= \widetilde{\sigma}_s^* (\phi(\widetilde{\omega}_s)) \tag{B5}$$
$$= \widetilde{\sigma}_s^* \big(\widetilde{\omega}_s^A \, \phi(E_A)\big) \tag{B6}$$
$$= \big(\widetilde{\sigma}_s^* \widetilde{\omega}_s^A\big) \phi(E_A) \tag{B7}$$
$$= \phi\big(\big(\widetilde{\sigma}_s^* \widetilde{\omega}_s^A\big) E_A\big) \tag{B8}$$
$$= \phi(\widetilde{\sigma}_s^* \widetilde{\omega}_s). \tag{B9}$$

Since the pulldown of the Levi-Civita connection $\omega = \widetilde{\sigma}^* \widetilde{\omega}$, we obtain the isomorphism between connections on the base manifold $\mathscr{V}_m^+$,

$$\omega = \phi(\omega_s). \tag{B10}$$

APPENDIX C: IDENTIFYING SU(2) AND SO(3)

Below equation (45) we claimed that the obtained SU(2) holonomy transformation corresponds to an SO(3) rotation around the e_2 axis. Here we quickly recall the relation between SU(2) elements and SO(3) rotations for completeness and for a self-contained discussion.

Consider (M, h) being a three-dimensional Riemannian manifold. Let $Z = Z^a e_a \in T_x M \sim \mathbb{R}^3$, where e_a is a orthonormal basis of $T_x M$, i.e., $h(e_a, e_b) = \delta_{ab}$. Let $\hat{Z} = \frac{Z}{h(Z,Z)}$, be the normalization of Z and define the su(2) representation $\mathbf{Z}$ of Z as $\mathbf{Z} = Z^a J_a$ [the J_a are defined in (35)]. Then

$$\begin{aligned} U(\mathbf{Z}) &= \exp(\mathbf{Z}) \\ &= \mathbf{1} \cos\left(\frac{\sqrt{h(Z,Z)}}{2}\right) - 2\left(\frac{\mathbf{Z}}{\sqrt{h(Z,Z)}}\right) \\ &\quad \times \sin\left(\frac{\sqrt{h(Z,Z)}}{2}\right), \end{aligned} \tag{C1}$$

is a representation of an element $U(\mathbf{Z})$ of SU(2), generated by the su(2) element $\mathbf{Z}$, on $\mathbb{C}^2$, parametrized by the components Z^a of the vector Z. It is easy to check that $\det(U(\mathbf{Z})) = 1$. Call $\sqrt{h(Z,Z)} = \varphi$; then the SU(2) elements become

$$\begin{aligned} U(\mathbf{Z}) = \exp(\mathbf{Z}) &= \mathbf{1} \cos\left(\frac{\varphi}{2}\right) - 2\hat{\mathbf{Z}} \sin\left(\frac{\varphi}{2}\right) \\ &= \mathbf{1} \cos\left(\frac{\varphi}{2}\right) + i\hat{Z}^a \sigma_a \sin\left(\frac{\varphi}{2}\right). \end{aligned} \tag{C2}$$

They corresponds to a SO(3) element R_Z which represents a rotation around the Z axis of an angle φ in the following way.

Let $X^a e_a$ and $Y = Y^a e_a$ be vectors in $T_x M$ expanded in a orthonormal basis of $T_x M$. Then, they can be identified with the elements $\mathbf{X} = X^a J_a$ and $\mathbf{Y} = Y^a J_a$ in su(2). The scalar product is encoded as

$$h(X, Y) = -2\mathrm{Tr}(\mathbf{X}\,\mathbf{Y}). \tag{C3}$$

Define

$$R_Z(X) = R_{U(\mathbf{Z})}(\mathbf{X}) = U(\mathbf{Z})\,\mathbf{X}\,U(\mathbf{Z})^{-1}, \tag{C4}$$

which is a map from $T_x M$ to $T_x M$. It preserves the scalar product

$$\begin{aligned} h(X, Y) &= -2\,\mathrm{Tr}\,(\mathbf{X}\,\mathbf{Y}) = -2\,\mathrm{Tr}\,(R_{U(\mathbf{Z})}(\mathbf{X}) R_{U(\mathbf{Z})}(\mathbf{Y})) \\ &= h(R_Z(X), R_Z(Y)). \end{aligned} \tag{C5}$$

Thus $R_Z(X)$ are the rotations of X around the Z axis by an angle $\varphi = h(Z, Z)$ expressed in terms of SU(2) elements $U(\mathbf{Z})$. It is clear that $U(Z)$ and $-U(Z)$ generate the same rotations, which makes visible the fact that SU(2) is the double covering of SO(3).

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