Erratum: Simple way to incorporate loss when modeling multimode-entangled-state generation [Phys. Rev. A 105, 063707 (2022)]

Colin Vendromin and Marc M. Dignam

(Received 31 August 2023; published 8 February 2024)

DOI: 10.1103/PhysRevA.109.029902

There was an oversight in our original derivation of the proof that the general solution is a squeezed thermal state. We have since realized that the coupled Eqs. (71)–(73) are not valid for a *general* nonlinear Hamiltonian but only for a restricted set of Hamiltonians, which we will describe below.

We first note that Eq. (B1) in Appendix B is incorrect and should be replaced with

$$D_{ml} = -\tfrac{1}{2}(E_{ml} + E_{lm}). \tag{1}$$

This follows from setting the sum of the coefficients in front of the $b_m^\dagger b_l^\dagger$ operators in Eqs. (57) and (63) equal to zero and noting that $D_{ml} = D_{lm}$. Using the matrix elements D_{ml} defined in Eq. (58), we obtain the equation

$$\sum_\mu U_{m\mu}U_{l\mu}\left(\dot{r}_\mu + \frac{i}{2}\dot{\phi}_\mu \sinh(2r_\mu)\right)e^{i\phi_\mu} = -\frac{\sqrt{x_m x_l}}{x_m x_l - 1}(E_{ml} + E_{lm}), \tag{2}$$

which should replace Eq. (B2) in the original text. In the paper, we derived Eq. (B3) by multiplying Eq. (B2) by $U_{\mu m}^* U_{\mu l}^*$ and summing over m and l. Using Eq. (2) above, this leads to

$$\dot{r}_\mu + \frac{i}{2}\dot{\phi}_\mu \sinh(2r_\mu) = -e^{-i\phi_\mu}\sum_{k,p}\frac{\sqrt{x_k x_p}}{x_k x_p - 1}(E_{kp} + E_{pk})U_{k\mu}^* U_{p\mu}^*. \tag{3}$$

We now recognize that this equation is a necessary but not sufficient condition. This can be seen by putting Eq. (3) back into Eq. (2), to obtain

$$\sum_{k,p}\frac{\sqrt{x_k x_p}}{x_k x_p - 1}(E_{kp} + E_{pk})\sum_\mu U_{m\mu}U_{l\mu}U_{k\mu}^* U_{p\mu}^* = \frac{\sqrt{x_m x_l}}{x_m x_l - 1}(E_{ml} + E_{lm}). \tag{4}$$

One can easily show that this will not be satisfied by a general unitary matrix U. Therefore, Eq. (3) is correct only for particular unitary matrices U that satisfy Eq. (4). Consequently, Eqs. (71)–(73) are not valid for all nonlinear Hamiltonians of the form given in Eq. (16) in the original paper.

We now show that Eqs. (71)–(73) are valid for the important subset of systems where the nonlinear coupling matrix G only couples each mode to only one other mode, and no mode is coupled to itself. An important example of such a system is a side-coupled ring resonator with either third-order or second-order nonlinearity [1,2]. In this case, the nonlinear parameter can be written as a block-diagonal matrix,

$$G = \begin{bmatrix} 0 & G_{12} & 0 & 0 & \cdots \\ G_{12} & 0 & 0 & 0 & \\ 0 & 0 & 0 & G_{34} & \\ 0 & 0 & G_{34} & 0 & \\ \vdots & & & & \ddots \end{bmatrix}, \tag{5}$$

where G_{ij} are numbers that quantify the coupling between the two modes i and j. The factorization of this nonlinear parameter is given by

$$G = U\begin{bmatrix} G_{12} & 0 & 0 & 0 & \cdots \\ 0 & G_{12} & 0 & 0 & \\ 0 & 0 & G_{34} & 0 & \\ 0 & 0 & 0 & G_{34} & \\ \vdots & & & & \ddots \end{bmatrix}U^{\mathrm{T}}, \tag{6}$$

where U is a block-diagonal unitary matrix given by

$$U = \frac{1}{2}\begin{bmatrix} 1-i & 1+i & 0 & 0 & \cdots \\ 1+i & 1-i & 0 & 0 & \\ 0 & 0 & 1-i & 1+i & \\ 0 & 0 & 1+i & 1-i & \\ \vdots & & & & \ddots \end{bmatrix}. \tag{7}$$

Putting this U into Eq. (38) in the text, the squeezing parameter matrix is given by

$$z = \begin{bmatrix} 0 & z_{12} & 0 & 0 & \cdots \\ z_{12} & 0 & 0 & 0 & \\ 0 & 0 & 0 & z_{34} & \\ 0 & 0 & z_{34} & 0 & \\ \vdots & & & & \ddots \end{bmatrix} = U \begin{bmatrix} r_{12}e^{i\phi_{12}} & 0 & 0 & 0 & \cdots \\ 0 & r_{12}e^{i\phi_{12}} & 0 & 0 & \\ 0 & 0 & r_{34}e^{i\phi_{34}} & 0 & \\ 0 & 0 & 0 & r_{34}e^{i\phi_{34}} & \\ \vdots & & & & \ddots \end{bmatrix} U^{\mathrm{T}}, \tag{8}$$

where $z_{12} = r_{12}e^{i\phi_{12}}$ and $z_{34} = r_{34}e^{i\phi_{34}}$. Because z is block-diagonal, the density operator solution for this case will be a separable product of two-mode squeezed thermal states, where each two-mode squeezing operator has a single squeezing parameter and squeezing phase. For example, the squeezing parameter and squeezing phase for the first pair of modes is $r_1 \equiv r_{12}$ and $\phi_1 \equiv \phi_{12}$, and the second pair of modes is $r_2 \equiv r_{34}$ and $\phi_2 \equiv \phi_{34}$, etc. The differential equations for the squeezing amplitudes, squeezing phases, and thermal photon numbers are given by Eqs. (71)–(73) in the text. But in order to use those equations, we have to show that Eq. (4) is satisfied for the U given by Eq. (7). It is sufficient to show that Eq. (4) is satisfied for each 2×2 block of U. Putting the 2×2 matrix elements into E_{ml} in Eq. (65) of the text, it is easy to show that $E_{11} = E_{22} = 0$ for each block. Using this in Eq. (4), we obtain for $m = 1,\ l = 2$ or $m = 2,\ l = 1$,

$$2(E_{12} + E_{21})(U_{11}U_{21}U_{11}^{*}U_{21}^{*} + U_{12}U_{22}U_{12}^{*}U_{22}^{*}) = (E_{12} + E_{21}). \tag{9}$$

Evaluating the left-hand side of this equation using the U matrices given in Eq. (7), one can easily show that this equals the right-hand side, which shows that Eq. (4) is satisfied for the U given in Eq. (7).

Therefore, for a nonlinear interaction matrix that takes the form given in Eq. (5), all of the equations and results given in our paper remain correct, except for the numerical results presented in Sec. VII, which are not valid because the G matrix given in Eq. (113) is not of the form given in Eq. (5). As mentioned above, our Eqs. (71)–(73) are valid for a ring resonator system [1].

[1] Z. Yang, M. Jahanbozorgi, D. Jeong, S. Sun, O. Pfister, H. Lee, and X. Yi, A squeezed quantum microcomb on a chip, Nat. Commun. **12**, 4781 (2021).

[2] Y. K. Chembo, Quantum dynamics of Kerr optical frequency combs below and above threshold: Spontaneous four-wave mixing, entanglement, and squeezed states of light, Phys. Rev. A **93**, 033820 (2016).