Simple formula for the Jones product and the Pancharatnam connection in optics

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An inner (Hermitian) product of two polarization states describes the interference of polarized beams, detection and production of polarized radiation, entanglement of polarized photons, etc. Here we pose the question of expressing the inner product $\gamma \equiv \mathbf{u}_1{}^\dagger \mathbf{u}_2$ of two Jones vectors (polarization states $\mathbf{u}_1$ and $\mathbf{u}_2$) explicitly in terms of ellipticities (ϵ_1, ϵ_2) and tilts (τ_1, τ_2) of the associated polarization ellipses. To that end, we derive a remarkably simple equation, $\gamma = \cos(\tau_2 - \tau_1)\cos(\epsilon_2 - \epsilon_1) + i\sin(\tau_2 - \tau_1)\sin(\epsilon_2 + \epsilon_1)$. The Poincaré sphere interpretation in terms of distance between polarization states is given, and the Pancharatnam phase is set to $\arg(\gamma)$ to show its invariance under the parity operation.

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I. INTRODUCTION

Applications of polarimetry in physics are broad, e.g., trapping molecules [1], signal compression and multiplexing [2], anisotropy sensing of random media via circularly polarized light [3,4], novel metamaterial-based Poincaré sphere polarizers [5], and experimental manipulation of polarized light [6], to name a few. Naturally, there is a vast interdisciplinary literature on the subject (e.g., [6,7]). In this paper we are concerned with the interference term $\Gamma \equiv \mathbf{E_1}^\dagger \mathbf{E}_2$ (E denotes the electric field) that arises whenever the interference of coherent or partially coherent polarized waves occurs. This expression also arises when a polarized wave $\mathbf{E}_1$ is received (projected) by a detector in polarization state $\mathbf{E}_2$, and the applications are broad, including lidar and radar polarimetric detection problems [8,9]. In addition, various manifestations of partially polarized light lead to a suitably chosen correlation coefficient, i.e., degree of polarization, and insofar as the latter has all the properties of an inner product in a space of random functions, Γ plays a central role in the formalism [6,10]. In this paper, however, we shall focus on the implications for the geometric phase.

In polarization (classical or quantum) optics, normalized polarization states are typically represented by a two-dimensional complex Jones vector $\mathbf{u} = [\cos(\alpha),\ \sin(\alpha)e^{i\phi}]^T$ (u for unitary). The Jones (inner) product $\mathbf{u}_1^\dagger \mathbf{u}_2$ then describes the *scaled* interference term, including the emerging field of interferometry with geometric phases [11]. The classic work of Pancharatnam on the *generalized* interference of polarized beams in anisotropic crystal optics [12,13] was also centered on $\gamma \equiv \mathbf{u_1}^\dagger \mathbf{u}_2 \sim \mathbf{E}_1{}^\dagger \mathbf{E}_2$ on the Poincaré sphere (see Fig. 1), and subsequently, it was linked to geometric phases by Berry [13–15]; for a historical account see [16,17].

As the Hermitian (scaled interference) product γ is of central interest, we wish to derive here a closed-form expression for it in terms of the Poincaré sphere variables, ellipticity ϵ and tilt τ, as illustrated in the left panel of Fig. 1. This is important for several reasons because these parameters are the most intuitive, provide a natural distance between polarization states on the Poincaré sphere, and lead to a new and concrete expression for geometric phases. However, to the best of our knowledge, the question has never been asked. A surprisingly simple answer is found in this paper and then used to explore the phase symmetries.

II. DERIVATION OF AN EXPLICIT EXPRESSION FOR THE INNER PRODUCT IN POLARIZATION SPACE

To fix notation, let $\mathbf{E}$ denote the electric field of a plane wave, including the explicit time dependence with phase shifts in the x and y components, e.g., as in [18]:

$$\mathbf{E} = E_{0x}\cos(kz - \omega t + \phi_x)\hat{x} + E_{0y}\cos(kz - \omega t + \phi_y)\hat{y}, \quad (1)$$

with E_0 denoting the amplitudes.

The plane wave is then given by

$$\mathbf{E} = \mathrm{Re}\left(\begin{bmatrix} E_{0,x}e^{i\phi_x} \\ E_{0,y}e^{i\phi_y} \end{bmatrix} e^{i(kz-\omega t)}\right). \quad (2)$$

The Jones vector $\mathbf{u}$ is the complex unit magnitude (hence u for unitary) two-component vector formed by electric-field amplitudes and phases:

$$\mathbf{u} \equiv \frac{1}{\sqrt{\left(E_{0x}^2 + E_{0y}^2\right)}} \begin{bmatrix} E_{0x}e^{i\phi_x} \\ E_{0y}e^{i\phi_y} \end{bmatrix}. \quad (3)$$

The Jones vector can be expressed in any orthogonal basis $\mathbf{e}_1$ and $\mathbf{e}_2$ with the coefficients $\mathbf{E} = (\mathbf{e}_1^\dagger \mathbf{E})\mathbf{e}_1 + (\mathbf{e}_2^\dagger \mathbf{E})\mathbf{e}_2$ for linear, circular, or elliptical polarizations. The traditional representation used throughout the literature is $\mathbf{u} = [\cos(\alpha),\ \sin(\alpha)e^{i\phi}]^T$.

The term containing the physics of the generalized interference is the inner product γ of the two interfering arbitrary polarization states (Jones vectors) $\mathbf{u}_1$ and $\mathbf{u}_2$ († denotes the complex-conjugate transpose), which is given by

$$\gamma = \mathbf{u}_1^\dagger \mathbf{u}_2, \quad (4)$$

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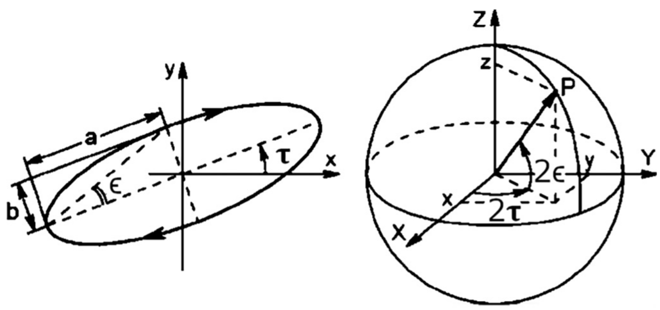

FIG. 1. Polarization ellipse and Poincaré sphere. Ellipticity angle $\epsilon = \pm\tan^{-1}(\frac{b}{a})$, $b \leqslant a$, positive for right-handed polarizations (RCPs), defined as clockwise rotation looking into the beam (e.g., [19]), and negative for left-handed ones (LCPs), ranging from $-\pi/4$ to $\pi/4$ from LCP to RCP. For linear polarization $\epsilon = 0$. The angle of the major axis with respect to the $+x$ axis is the tilt angle τ, ranging from $-\pi/2$ to $\pi/2$, consistent with the range of the inverse tangent [20]. The Poincaré sphere angles at the associated point P are 2ϵ and 2τ, ranging from $-\pi/2$ to $\pi/2$ and $-\pi$ to π, respectively.

and our task is to derive a closed-form expression of $\gamma(\epsilon_1, \epsilon_2, \tau_1, \tau_2)$ so that one can interpret it on the Poincaré sphere, obtain another route to the Pancharatnam-Berry geometric phase, and explore symmetry invariants in polarization space.

To that end, the key is to realize that the Jones vector expressed in terms of ellipticity, tilt, and phase must be a rotation $R(\tau)$ away from its expression in principal ellipticity axes $\mathbf{u} = [\cos(\epsilon), \sin(\epsilon)]$, where

$$R(\tau) = \begin{bmatrix} \cos(\tau) & -\sin(\tau) \\ \sin(\tau) & \cos(\tau) \end{bmatrix} \tag{5}$$

is the rotation matrix. The general Jones vector is then $\mathbf{u}(\epsilon, \tau, \phi)$, where the arguments denote ellipticity, tilt, and absolute phase, respectively [21]:

$$\mathbf{u}(\epsilon, 0, \phi) = \begin{bmatrix} \cos(\epsilon) \\ -i\sin(\epsilon) \end{bmatrix} e^{i\phi}. \tag{6}$$

Then,

$$\mathbf{u}(\epsilon, \tau, \phi) = R(\tau)\mathbf{u}(\epsilon, 0, \phi). \tag{7}$$

The inner product γ is then given by

$$\gamma \equiv \mathbf{u}(\epsilon_1, \tau_1, \phi_1)^\dagger \mathbf{u}(\epsilon_2, \tau_2, \phi_2). \tag{8}$$

Substituting (7) into (8),

$$\gamma = \mathbf{u}(\epsilon_1, 0, \phi_1)^\dagger R(\tau_1)^\dagger R(\tau_2)\mathbf{u}(\epsilon_2, 0, \phi_2). \tag{9}$$

Because R is a rotation matrix, it is unitary, and $R(\tau_1)^\dagger = R^{-1}(\tau_1)$. The inverse is $R^{-1}(\tau_1) = R(-\tau_1)$. Hence,

$$R(\tau_1)^\dagger R(\tau_2) = R^{-1}(\tau_1)R(\tau_2) = R(\tau_2 - \tau_1). \tag{10}$$

Substituting (10) into (9) yields

$$\gamma = \mathbf{u}(\epsilon_1, 0, \phi_1)^\dagger R(\tau_2 - \tau_1)\mathbf{u}(\epsilon_2, 0, \phi_2). \tag{11}$$

In matrix form, first treating the practically important case of zero overall phase (but trivially generalized below), γ equals

$$\begin{bmatrix} \cos(\epsilon_1) \\ -i\sin(\epsilon_1) \end{bmatrix}^\dagger \begin{bmatrix} \cos(\tau_2 - \tau_1) & -\sin(\tau_2 - \tau_1) \\ \sin(\tau_2 - \tau_1) & \cos(\tau_2 - \tau_1) \end{bmatrix} \begin{bmatrix} \cos(\epsilon_2) \\ -i\sin(\epsilon_2) \end{bmatrix}. \tag{12}$$

Matrix multiplication of the above, with elementary trigonometry, leads to an elegant expression:

$$\gamma = \cos(\tau_2 - \tau_1)\cos(\epsilon_2 - \epsilon_1) + i\sin(\tau_2 - \tau_1)\sin(\epsilon_2 + \epsilon_1). \tag{13}$$

This is the sought-for expression for the Jones inner product in terms of the ellipticities and tilts of the constituent Jones vectors. Note the interesting lack of symmetry in the arguments: three differences and one sum. Let us check a few special cases: (1) For orthogonal polarization states, $\epsilon_2 = -\epsilon_1$ and $\tau_2 = \tau_1 \pm \frac{\pi}{2}$, with the plus or minus sign chosen to keep the tilt in the range $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, the inner product vanishes, $\gamma = 0$. (2) For identical polarizations, the inner product is unity, $\gamma = 1$. (3) When both polarizations are linear, the inner product is the dot product of the two lines, $\gamma = \cos(\tau_2 - \tau_1)$. (4) For linearly polarized at tilt angle β and right-circular polarization, $\gamma = \frac{1}{\sqrt{2}}e^{i\beta}$. We also performed a direct computational check on a fine mesh in the ϵ-τ polarization space, leaving no doubt about the validity of Eq. (13).

With an eye on applications such as producing "magic" ellipticity with high accuracy in order to trap molecules [1], monitoring the orthogonality of polarimeters [6], and optimizing radar or lidar detection to "match" a given target as closely as possible [20,22], we develop simple approximations for the near-orthogonal and near-equal cases. In the former case, tilts differ by nearly $\frac{\pi}{2}$, $\tau_2 = \tau_1 \pm \frac{\pi}{2} + \delta_\tau$, and ellipticities are near negatives of each other: $\epsilon_2 = -\epsilon_1 + \delta_\epsilon$, with δ_ϵ and δ_τ defined as the deviations from the orthogonality of the polarization states, with the latter being antipodal on the Poincaré sphere. Then from Eq. (13) and these near-orthogonal conditions on ϵ_1, τ_1 and ϵ_2, τ_2,

$$\begin{aligned} \gamma &= \mp\sin(\delta_\tau)\cos(2\epsilon_1 - \delta_\epsilon) \pm i\cos(\delta_\tau)\sin(\delta_\epsilon) \\ &= \mp\left(\delta_\tau - \frac{1}{3!}\delta_\tau^3 + \cdots\right) \\ &\quad \times \left(\cos(2\epsilon_1) + \delta_\epsilon\sin(2\epsilon_1) - \delta_\epsilon^2\frac{1}{2!}\cos(2\epsilon_1) + \cdots\right) \\ &\quad \pm i\left(1 - \frac{1}{2!}\delta_\tau^2 + \cdots\right)\left(\delta_\epsilon - \frac{1}{3!}\delta_\epsilon^3 + \cdots\right) \\ &= \mathrm{Re}(\gamma) + \mathrm{Im}(\gamma) \\ &= \mp\left(\delta_\tau\cos(2\epsilon_1) + \delta_\tau\delta_\epsilon\sin(2\epsilon_1) + \cdots\right) \\ &\quad \pm i\left(\delta_\epsilon - \delta_\tau^2\delta_\epsilon\frac{1}{2!} - \delta_\epsilon^3\frac{1}{3!} + \cdots\right). \end{aligned} \tag{14}$$

Interestingly, to first order, ellipticity and tilt contributions decouple, yielding

$$\gamma \approx \mp\cos(2\epsilon_1)\delta_\tau \pm i\delta_\epsilon, \tag{15}$$

with the ellipticity "error" alone being responsible for the imaginary part.

For the near-parallel case with small tilt difference $\delta_\tau = \tau_2 - \tau_1$ and small ellipticity difference $\delta_\epsilon = \epsilon_2 - \epsilon_1$,

$$\gamma = \cos(\delta_\tau)\cos(\delta_\epsilon) + i\sin(\delta_\tau)\sin(2\epsilon_1 + \delta_\epsilon). \tag{16}$$

Expanding Eq. (16) in Taylor series about the parallel arrangement yields

$$\gamma = \left(1 - \frac{1}{2}\delta_\tau^2 + \dots\right)\left(1 - \frac{1}{2}\delta_\epsilon^2 + \dots\right) + i\left(\delta_\tau + \frac{1}{3!}\delta_\tau^3 \dots\right)(\sin(2\epsilon_1) + 2\cos(2\epsilon_1)\delta_\epsilon + \dots) \tag{17}$$

The near-equal inner product to second order is

$$\gamma \approx 1 + i\delta_\tau \sin(2\epsilon_1) - \tfrac{1}{2}\delta_\tau^2 - \tfrac{1}{2}\delta_\epsilon^2 + i\delta_\tau\delta_\epsilon 2\cos(2\epsilon_1), \tag{18}$$

and to first order, it reduces to $\gamma \approx 1 + i\delta_\tau \sin(2\epsilon_1)$, so only the tilt deviation contributes. This simple result is good news for laboratory and field experiments as geometric alignment is typically easier to monitor.

III. THE PANCHARATNAM CONNECTION

To place our results within the broader context of geometric phases in physics, we now prove that the Pancharatnam phase (one maximizing interference) is given by the argument of expression (13). To that end, we return to the traditional representation $\mathbf{u} = [\cos(\alpha),\ \sin(\alpha)e^{i\phi}]^T$ for calculating the inner product γ because convenient expressions for the Poincaré coordinates in terms of α and ϕ are available in, e.g., [7] (their Fig. 3.2.2, in particular). Thus,

$$\mathbf{u}_1^\dagger\mathbf{u}_2 = [\cos(\alpha_1)\ \sin(\alpha_1)e^{-i\phi_1}]\begin{bmatrix}\cos(\alpha_2)\\ \sin(\alpha_2)e^{i\phi_2}\end{bmatrix} = \cos(\alpha_1)\cos(\alpha_2) + \sin(\alpha_1)\sin(\alpha_2)e^{i(\phi_2-\phi_1)}. \tag{19}$$

A generic polarization state $\mathbf{u} = [\cos(\alpha),\ \sin(\alpha)e^{i\phi}]^T$ corresponds to a point P on the Poincaré sphere with a great-circle arc (geodesic) length of 2α and phase ϕ, measured from the equatorial plane as in Fig. 2 (see also [7]). To obtain explicit expressions for the Pancharatnam connection (an expression coined by Berry [23] for the rule of comparing phases), we next work towards the polar form of the inner product in Eq. (19). The goal is to confirm that $\arg(\gamma)$ is the Pancharatnam phase between the two polarization states $\mathbf{u}_1$ and $\mathbf{u}_2$ on the Poincaré sphere. The details are supplied in Appendix A, where spherical trigonometry and other identities are used to identify the relevant phases, guided by the expectation that c, the geodesic (great circle) arc length between $\mathbf{u}_1$ and $\mathbf{u}_2$ on the sphere (the symbol used in [12,16]), will appear in the final result as it did in Pancharatnam's work. Indeed, from

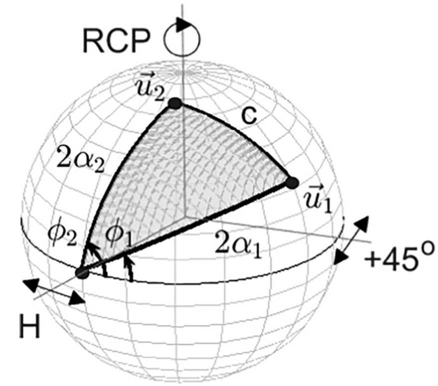

FIG. 2. A sketch of Jones vectors on the Poincaré sphere. Two polarization states (points) $\mathbf{u}_1$ and $\mathbf{u}_2$ form the shaded triangle with the great-circle arcs $2\alpha_1$ and $2\alpha_2$, emanating from the third (reference) point: linear horizontal polarization. These arcs subtend angles $2\alpha_1$ and $2\alpha_2$ with respect to the associated circular sectors, centered at the sphere origin. The corresponding phases ϕ_1 and ϕ_2 are measured from the equatorial plane as indicated by arrows. Although this plot deals with two polarization states, the triangle is needed to invoke the spherical law of cosines. Then, as detailed in Appendix A, the arc length c between $\mathbf{u}_1$ and $\mathbf{u}_2$ is given by $\cos(c) = \cos(2\alpha_1)\cos(2\alpha_2) + \sin(2\alpha_1)\sin(2\alpha_2)\cos(\phi_2 - \phi_1)$. Once derived, the result $|\mathbf{u}_1^\dagger\mathbf{u}_2| = \cos(c/2)$ is seen to be independent of the coordinate system and valid for any pair of points on the sphere.

Appendix A,

$$|\mathbf{u}_1^\dagger\mathbf{u}_2| = \cos(c/2). \tag{20}$$

This result is frame independent, and its plausibility can be seen at once by choosing $\mathbf{u}_1$ and $\mathbf{u}_2$ to lie on the equator (linear polarizations), for example, and employing spherical symmetry (see also [24]). To avoid confusion we note that the discussion here and in Fig. 2 is about the Pancharatnam *connection* and only two polarization states are involved. The triangle in Fig. 2 is needed to take advantage of the spherical law of cosines detailed in Appendix A. It is the application of the spherical law of cosines that instantly brings the geodesic arc length c connecting $\mathbf{u}_1$ and $\mathbf{u}_2$ into (20), as shown in Fig. 2 [25]. To see the emergence of the Pancharatnam phase within the Jones formalism, observe that if one appends, in addition to polarization-defining phases, a dynamic phase θ as $\mathbf{u}_1 \to e^{i\theta}\mathbf{u}_1$, the inner product transforms as $\gamma = \mathbf{u}_1^\dagger\mathbf{u}_2 \to \gamma e^{-i\theta}$ (rotating phasor). We then ask, Given two polarization states $\mathbf{u}_1$ and $\mathbf{u}_2$, what is the value of θ that maximizes the total intensity of their superposition? The answer is $\theta = \arg(\mathbf{u}_1^\dagger\mathbf{u}_2) \equiv \delta$, which is the classical Pancharatnam phase (connection), as demonstrated below. To show this, we calculate the phase of the inner product from its real and imaginary parts in Eq. (19) as follows:

$$\arg(\mathbf{u}_1^\dagger\mathbf{u}_2) = \tan^{-1}\left(\frac{\mathrm{Im}(\mathbf{u}_1^\dagger\mathbf{u}_2)}{\mathrm{Re}(\mathbf{u}_1^\dagger\mathbf{u}_2)}\right) = \tan^{-1}\left(\frac{\sin(\alpha_1)\sin(\alpha_2)\sin(\phi_2-\phi_1)}{\cos(\alpha_1)\cos(\alpha_2) + \sin(\alpha_1)\sin(\alpha_2)\cos(\phi_2-\phi_1)}\right) \equiv \frac{\Omega}{2}, \tag{21}$$

and the thus-defined phase Ω should be compared with the Pancharatnam phase δ occurring in the generalized-interference term $2\cos(\frac{c}{2})\cos(\delta)$ of [12,23]. The capital-letter notation is because of the solid-angle interpretation of Ω on the Poincaré sphere as sketched in Fig. 2: it is the solid angle subtended by a spherical triangle with side lengths $2\alpha_1$ and $2\alpha_2$ and angle $\phi_2 - \phi_1$ in between the two sides. This can be surmised by direct comparison of Eq. (21) with the expression for the spherical excess of a triangle, $E \equiv A + B + C - \pi$, with A, B, and C denoting the angles of a spherical triangle, measured in radians. Recall that E equals the triangle area on a unit sphere and the relevant formula is given in Appendix A.

Thus, the argument of the Jones scalar product is the solid angle subtended by the triangle on the Poincaré sphere, with the vertices being the two Jones vectors and the (reference) horizontal linear polarization, all connected by geodesics (great-circle arcs). This Pancharatnam phase is still related to the connection between two polarization states and is distinct from the broader Berry geometric phase, with the latter applying to any closed circuit and any number of states on the sphere. To facilitate comparison with similar results in the literature (e.g., [12,13,16,17]) we write γ as

$$\mathbf{u}_1^{\dagger}\mathbf{u}_2 = \cos\left(\frac{c}{2}\right)e^{i\frac{\Omega}{2}}. \tag{22}$$

When two unit intensity polarized beams interfere, their total intensity is given by $(\mathbf{u}_1 + \mathbf{u}_2)^{\dagger}(\mathbf{u}_1 + \mathbf{u}_2) = \mathbf{u}_1^{\dagger}\mathbf{u}_1 + \mathbf{u}_2^{\dagger}\mathbf{u}_2 + \mathbf{u}_1^{\dagger}\mathbf{u}_2 + \mathbf{u}_2^{\dagger}\mathbf{u}_1$, with the interference term $\mathbf{u}_1^{\dagger}\mathbf{u}_2 + \mathbf{u}_2^{\dagger}\mathbf{u}_1 = 2\mathrm{Re}(\mathbf{u}_1^{\dagger}\mathbf{u}_2)$. By Eq. (22), for two Jones vectors with the dynamical phase $e^{i\theta}$ on $\mathbf{u}_1$ in Eq. (19), we obtain $2\mathrm{Re}(\mathbf{u}_1^{\dagger}\mathbf{u}_2) = 2\mathrm{Re}[\cos(\frac{c}{2})e^{i\frac{\Omega}{2}}e^{-i\theta}] = 2\cos(\frac{c}{2})\cos(\frac{\Omega}{2} - \theta)$. On the other hand, Pancharatnam [12] obtained the following interference term expression: $2\cos(\frac{c}{2})\cos(\delta)$. Thus, direct comparison of the two at $\theta = 0$ yields $\frac{\Omega}{2} = \delta$, with δ being the classic Pancharatnam's phase [12] between two polarization states. Interference is maximized when the dynamical phase difference $\theta = \frac{\Omega}{2} = \delta$. In the expression $\gamma e^{-i\theta}$ this compensates for the $\delta = \frac{\Omega}{2}$ arising solely from the difference in the two polarization states. Thus, the maximum interference occurs at $\theta = \arg(\mathbf{u}_1^{\dagger}\mathbf{u}_2) = \delta$ as claimed.

As was discovered by Pancharatnam, this phase δ is geometric insofar as it is nontransitive (path dependent): for $\mathbf{u}_A$ in phase with $\mathbf{u}_B$, and $\mathbf{u}_B$ in phase with $\mathbf{u}_C$, $\mathbf{u}_A$ and $\mathbf{u}_C$ are, in general, not in phase [12,14]. To recap, the Jones-product magnitude equals the cosine of half the great-arc length (geodesic) c between the Jones vectors, and its argument matches Pancharatnam's geometric phase δ.

Let us now use Eq. (13) and briefly examine the phase δ evolution. The case of nearby polarization states leads to $\gamma \approx 1 + i\delta_\tau \sin(2\epsilon)$ to first order. Therefore, $\delta = \arg(\gamma) = \tan^{-1}[\sin(2\epsilon)\delta_\tau] \approx \sin(2\epsilon)d\tau$, with d denoting the differential. Hence, the Pancharatnam phase does not accrue along the meridians ($\tau = \mathrm{const}$) or locally along the equator where $\sin(2\epsilon) = 0$. If one takes the entire equator, the solid-angle formula for the general Pancharatnam-Berry phase (solid angle subtended by the upper hemisphere) gives π rather than zero for the total accumulated phase. The discrepancy is resolved by noting that there is a singularity at the antipodal point where the π discontinuity occurs [11,26–28]. This is where the horizontal component of a linearly polarized wave flips as the vertical polarization state is crossed. A similar argument leads to Pancharatnam phases not accumulating locally along meridians.

IV. CONCLUDING REMARKS

To conclude, let us return to our equation $\gamma = e^{-\theta}[\cos(\tau_2 - \tau_1)\cos(\epsilon_2 - \epsilon_1) + i\sin(\tau_2 - \tau_1)\sin(\epsilon_2 + \epsilon_1)]$. Because of its explicit dependence on the ellipsometric variables, Eq. (13) is likely to bring intuition and insights not easily found otherwise. For example, geometric phases were obtained in [15,23] by requiring that γ be real valued. Let us use Eq. (13) to inquire under what conditions $\mathrm{Im}(\gamma)$ vanishes, and they are clear at once: either $\epsilon_1 = \epsilon_2 = 0$ or $\tau_1 = \tau_2$. Note the lack of symmetry in the two variables: unlike the case of tilts, $\epsilon_1 = \epsilon_2$ does not suffice. Ellipticities must vanish separately. While the former condition corresponds to linearly polarized states (equator on the Poincaré sphere), the latter is less obvious: two aligned but otherwise arbitrary polarization ellipses (Poincaré meridians).

This last case of aligned ellipses (meridian arcs) provides a simple illustration of how Eq. (13) satisfies $|\mathbf{u}_1^{\dagger}\mathbf{u}_2| = \cos(c/2)$. The left side is $|\mathbf{u}_1^{\dagger}\mathbf{u}_2| = \cos(\epsilon_2 - \epsilon_1)$, and $2\epsilon_2 - 2\epsilon_1$ is the arc length between $\mathbf{u}_1$ and $\mathbf{u}_2$ lying on the same meridian of the Poincaré sphere so that $\cos(\frac{c}{2}) = \cos(\frac{2\epsilon_2 - 2\epsilon_1}{2}) = \cos(\epsilon_2 - \epsilon_1)$.

More broadly, Eq. (13) allows one to bypass tedious algebra and spherical trigonometry and work directly with the intuitive ellipticities and tilts, e.g., to compute the required phase directly via $\theta = \arg(\mathbf{u}_1^{\dagger}\mathbf{u}_2)$. This requires no loops, no triplet of states, and no solid angles to characterize the Pancharatnam phase and supplies a concrete expression for the Pancharatnam connection.

As a specific application, we use Eq. (13) to explore symmetries of geometric phase by employing recent results on symmetries in polarization space reported in Ref. [18], which did not address the topic of geometric phase. Beginning with Eq. (13), we observe that reversing the sign of both ϵ_1 and ϵ_2 transforms γ to γ^*. This corresponds to flipping the geometric phase. Flipping of the geometric phase also occurs when the signs of τ_1 and τ_2 are reversed, and γ transforms to γ^*. When both ellipticities and tilts change sign, γ and the geometric phase remain invariant.

Next, these symmetry properties of the geometric phase can be related to space-time reflection symmetries within the Jones formalism, which was explored in [18]. Specifically, results derived in [18] show that the parity operator P flips the handedness and tilt of the polarization ellipse (up to a π rotation). These results can be shown to imply that (see Appendix B) $P\mathbf{u}(\epsilon, \tau) = -\mathbf{u}(-\epsilon, -\tau)$. In conjunction with our Eq. (13) this, in turn, implies that the geometric phase is an invariant of a parity operator. Similar reasoning can be used for time reversal. The imaginary part of the Jones product flips, and so does the geometric phase. Such results based on Eq. (13) are likely to stimulate further developments in geometric-phase interferometry by allowing one to design phases with desired properties.

Sometimes desired (magic) ellipticities are known from elsewhere, e.g., in traps [1], and the small-angle approximations as given by (18) can be used to evaluate experimental precision. Another example is matching a receiver polarization to detect a given incident polarization state [22]. Also, when the applications at hand include randomness so that γ acquires stochastic character, Eq. (13) allows direct calculations of γ distributions from those prescribed for the ellipticities and tilts. Then, the cross-correlation coefficient, having all the properties of an inner product in the abstract space of random functions, can also be related to the Jones product.

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APPENDIX A: ON THE POINCARÉ SPHERE INTERPRETATION OF THE JONES-PRODUCT MAGNITUDE AND PHASE

Our task here is to interpret the magnitude of the Jones product $\mathbf{u}_1^\dagger\mathbf{u}_2$ on the Poincaré sphere for a completely general pair of states. Taking the absolute magnitude of Eq. (19) yields

$$|\mathbf{u}_1^\dagger\mathbf{u}_2| = |\cos(\alpha_1)\cos(\alpha_2) + \sin(\alpha_1)\sin(\alpha_2)e^{i(\phi_2-\phi_1)}|. \quad \text{(A1)}$$

Using trigonometric identities for double angles, $\sin^2(x) = \frac{1}{2}[1-\cos(2x)]$ and so on, reduces (A1) to

$$|\mathbf{u}_1^\dagger\mathbf{u}_2| = \left[\tfrac{1}{2} + \tfrac{1}{2}\cos(2\alpha_1)\cos(2\alpha_2) + \tfrac{1}{2}\sin(2\alpha_1)\sin(2\alpha_2)\cos(\phi_2-\phi_1)\right]^{\frac{1}{2}}. \quad \text{(A2)}$$

To interpret Eq. (A2) geometrically, we now employ the spherical law of cosines: $\cos(c) = \cos(a)\cos(b) + \sin(a)\sin(b)\cos(C)$, with c denoting the side opposite of angle C. The right-hand side of (A2) is of the form $[\frac{1}{2} + \frac{1}{2}\cos(a)\cos(b) + \frac{1}{2}\sin(a)\sin(b)\cos(C)]^{\frac{1}{2}}$, with $a \equiv 2\alpha_1$, $\phi_2 - \phi_1 \equiv C$, etc. This brings in the sought-after half arc of a great circle c connecting $\mathbf{u_1}$ and $\mathbf{u_2}$ on the Poincaré sphere as shown in Fig. 2. Thus,

$$|\mathbf{u}_1^\dagger\mathbf{u}_2| = \left(\tfrac{1}{2} + \tfrac{1}{2}\cos(c)\right)^{\frac{1}{2}} = \cos(c/2). \quad \text{(A3)}$$

To the best of our knowledge, this is the simplest yet general derivation of $|\mathbf{u}_1^\dagger\mathbf{u}_2| = \cos(c/2)$.

Next, to interpret the phase in Eq. (21), we appeal to the formula for spherical excess E (see, e.g., [29], Article 103.2). To compare this formula with Eq. (21) at a glance, we take $\tan^{-1}$ of both sides of Article 103.2 of [29] to obtain

$$\tan^{-1}\left(\frac{\sin(a/2)\sin(b/2)\sin(C)}{\cos(a/2)\cos(b/2) + \sin(a/2)\sin(b/2)\cos(C)}\right) = \frac{E}{2}. \quad \text{(A4)}$$

As E also equals the solid angle subtended by the shaded triangle in Fig. 2 for a unit sphere, the argument of the Jones scalar product is then the solid angle subtended by the triangle on the Poincaré sphere.

APPENDIX B: PARITY OPERATOR IN THE ELLIPTICITY AND TILT SPACE

According to [18], the parity operator in the horizontal-vertical polarization basis is given by the Jones matrix:

$$P = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{(B1)}$$

and we seek an algebraic expression for the action of the parity operator P on a general Jones vector (spinor).

The Jones vector as a function of ellipticity and tilt is given by

$$\mathbf{u}(\epsilon,\tau) = R(\tau)\mathbf{u}(\epsilon,0), \quad \text{(B2)}$$

where the rotation matrix is

$$R(\tau) = \begin{pmatrix} \cos(\tau) & -\sin(\tau) \\ \sin(\tau) & \cos(\tau) \end{pmatrix}. \quad \text{(B3)}$$

The Jones vector in principal axes is

$$\mathbf{u}(\epsilon,0) = \begin{pmatrix} \cos(\epsilon) \\ -i\sin(\epsilon) \end{pmatrix}. \quad \text{(B4)}$$

The effect of the parity operator P on the rotation matrix is found simply as

$$\begin{aligned} PR(\tau) &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos(\tau) & -\sin(\tau) \\ \sin(\tau) & \cos(\tau) \end{pmatrix} \\ &= \begin{pmatrix} -\cos(\tau) & \sin(\tau) \\ \sin(\tau) & \cos(\tau) \end{pmatrix} \\ &= \begin{pmatrix} \cos(\tau) & \sin(\tau) \\ -\sin(\tau) & \cos(\tau) \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= R(-\tau)P. \end{aligned} \quad \text{(B5)}$$

Moreover, P acting on $\mathbf{u}(\epsilon,\tau)$ gives

$$P\mathbf{u}(\epsilon,\tau) = R(-\tau)P\mathbf{u}(\epsilon,0), \quad \text{(B6)}$$

and P acting on $\mathbf{u}(\epsilon,0)$ is

$$\begin{aligned} P\mathbf{u}(\epsilon,0) &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos(\epsilon) \\ -i\sin(\epsilon) \end{pmatrix} \\ &= \begin{pmatrix} -\cos(\epsilon) \\ -i\sin(\epsilon) \end{pmatrix} \\ &= -\begin{pmatrix} \cos(\epsilon) \\ -i\sin(-\epsilon) \end{pmatrix} = -\mathbf{u}(-\epsilon,0). \end{aligned} \quad \text{(B7)}$$

Therefore,

$$\begin{aligned} P\mathbf{u}(\epsilon,\tau) &= R(-\tau)P\mathbf{u}(\epsilon,0) \\ &= -R(-\tau)\mathbf{u}(-\epsilon,0) \\ &= -\mathbf{u}(-\epsilon,-\tau). \end{aligned} \quad \text{(B8)}$$

While the Jones vector is generally changed by the parity operation, Eq. (13) shows that the two-state quantities, $\gamma = \mathbf{u}_1{}^\dagger\mathbf{u}_2$ and its argument (geometric phase), are unchanged by the parity operation.

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[21] To supply the reader with all the necessary details to trace this derivation, we note that Ref. [7] uses ψ for tilt and χ for ellipticity, where $\chi \equiv \epsilon$ but $\psi \neq \tau$. The tilt of Ref. [7] ranges over quadrants 1 and 2, instead of our quadrants 1 and 4. This is merely a formal difference, as either range suffices to describe ellipse tilt because of the symmetry. Relating the two definitions requires an addition of π if τ is in the fourth quadrant, $\psi = \tau + \pi$ if $-\frac{\pi}{2} < \tau \leqslant 0$ and $\psi = \tau$ if $0 \leqslant \tau \leqslant \frac{\pi}{2}$. Note that Ref. [7] has a typo at the bottom of p. 82 in the unnumbered equation following (3.1.45). The author has tilt ψ, rather than ellipticity χ, describing an ellipse in its principal-axes system. Also, while (3.1.46a) of Ref. [7] and Eq. (1.75) of Ref. [6] agree, care is required because of a convention difference: while Ref. [7] defines positive ellipticity for counterclockwise rotation looking into the beam [Eq. (3.1.42) and following], Ref. [6] does the opposite (p. 8). Reference [7] takes the phase winding $e^{-i\omega t}$ (p. 78), and Ref. [6] again has the opposite (p. 11). The two differences cancel, resulting in formal agreement.

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[25] To test construction of Fig. 2 by way of a simple example, take $\alpha_1 = 0$ so that $u_1 = (1, 0)^T$ and $\alpha_2 = \pi/4$ so that $u_2 = 1/\sqrt{2}(1, i)$. Then, the triangle collapses onto the $2\alpha_2$ arc, and $|\gamma| = 1/\sqrt{2}$, so the corresponding arc length $c = 2\alpha_2 = \pi/2$, in agreement with $|\gamma| = \cos(c/2)$ from Eq. (20).

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