Superfluid fraction and Leggett bound in a density-modulated strongly interacting Fermi gas at zero temperature

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(Received 28 November 2023; accepted 5 January 2024; published 1 February 2024)

We calculate the superfluid fraction of an interacting Fermi gas, in the presence of a one-dimensional periodic potential of strength V_0 and wave vector q. Special focus is given to the unitary Fermi gas, characterized by the divergent behavior of the s-wave scattering length. Comparison with the Leggett's upper bound $(\langle n_{1D}\rangle < 1/n_{1D} >)^{-1}$, with n_{1D} the one-dimensional column density, explicitly shows that differently from the case of a dilute interacting Bose gas, the bound significantly overestimates the value of the superfluid fraction, except in the phonon regime of small q. Sum rule arguments show that the combined knowledge of the Leggett bound and of the actual value of the superfluid fraction allows for the determination of curvature effects, providing the deviation of the dispersion of the Anderson-Bogoliubov mode from the linear phonon dependence. The comparison with the predictions of the weakly interacting BCS Fermi gas points out the crucial role of two-body interactions. The implications of our predictions on the anisotropic behavior of the sound velocity are also discussed.

DOI: 10.1103/PhysRevA.109.023301

I. INTRODUCTION

In 1970, Leggett [1] derived an upper bound to the superfluid fraction $f_S = n_S/\bar{n}$, given by the ratio between the superfluid density relative to the motion of the fluid along the x direction and the mean density, in terms of the spatial average of the inverse of the one-dimensional (1D) column density $n_{1D}(x) = \int n(\mathbf{r})dydz$ [2],

$$f_S \leqslant \left(\langle n_{1D}(x)\rangle\left\langle\frac{1}{n_{1D}(x)}\right\rangle\right)^{-1} \equiv f_S^L. \quad (1)$$

Result (1) is particularly relevant at zero temperature, where the superfluid fraction is affected by the density modulations induced by the breaking of translational invariance, and holds independent of whether the breaking occurs spontaneously or is induced by an external force. At finite temperature, the superfluid fraction is instead affected by thermal effects even for uniform systems, with important consequences on the propagation of second sound [3–5]. Under the assumption that the many-body wave function is described by a single phase equal for all the particles, i.e., an assumption applicable to a dilute Bose gas at zero temperature described by the Gross-Pitaevskii equation, Leggett derived a further lower bound [6] to the superfluid fraction. The Leggett upper and lower bounds coincide if the 3D density depends only on the x variable, so that in this case the inequality (1) reduces to an equality. In a recent experiment [7], the Leggett's integral entering Eq. (1) was actually measured in an ultracold Bose gas confined in a 2D box, in the presence of a periodically modulated external 1D potential, causing important dishomogeneities in the density profile, and the bound (1) was shown to agree, with good accuracy, with the result for the superfluid fraction measured through the velocity of sound, in agreement with the predictions of Gross-Pitaevskii theory.

The question of the accuracy of the Leggett bound in other interacting many-body systems is relatively unexplored and the aim of our work is to address the question in the case of an interacting Fermi gas at zero temperature, with special focus on the so-called unitary limit, where the s-wave scattering length is infinite and the system behaves as a strongly interacting superfluid. We will consider a gas confined in a box in the presence of a periodic potential of the form

$$V_{\text{ext}} = V_0\cos(qx), \quad (2)$$

which induces density modulation of period $d = 2\pi/q$ in the ground-state density profile. We will attack the problem following two different lines. In Sec. II, we develop a sum-rule approach to explore the behavior of the superfluid fraction in the presence of a weak perturbation (small V_0) and calculate the Leggett's integral as well as the exact value of the superfluid fraction. The sum-rule results provide the proof that only if the excitation spectrum of the unperturbed uniform superfluid consists of a single excitation exhausting the relevant sum rules of the dynamic structure factor, will the inequality (1) reduce to an identity. This is the case of the dilute BEC gas for all values of the wave vector q, but remains an inequality in the most general case. The results of the sum-rule approach are also used to infer on the curvature of the dispersion of the

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Anderson-Bogoliubov collective mode of a Fermi superfluid at long wavelengths. In Sec. III, we first provide a quantitative implementation of the sum-rule approach developed in Sec. II by applying the dynamical Bardeen-Cooper-Schrieffer (BCS) theory [8]. We then present our numerical results for the superfluid fraction of the unitary Fermi gas by solving the Bogoliubov–de Gennes (BdG) equations in the presence of the external periodic potential (2) in order to provide a quantitative comparison between the Leggett bound and the actual value of the superfluid fraction, beyond the weak perturbation limit of small V_0, in a wider range of parameters of possible relevance for future experiments. In Sec. IV, we discuss how our predictions for the superfluid density could be tested by measuring the effects of anisotropy in the velocity of sound.

II. SUM RULE APPROACH

At zero temperature, the dynamic structure factor $S(\omega, q)$ of a many-body system is related to the imaginary part of the density response function $\chi(\omega, q)$ by

$$S(\omega, \mathbf{q}) = -\frac{1}{\pi}\mathrm{Im}\chi(\omega + i0^+, \mathbf{q}), \tag{3}$$

so that its spectral representation is

$$S(\omega, \mathbf{q}) = \frac{1}{V}\sum_n |\langle 0|\rho_{\mathbf{q}}|n\rangle|^2\delta(\omega + E_0 - E_n), \tag{4}$$

where V is the volume of the box, $\rho_{\mathbf{q}} = \sum_k \exp(i\mathbf{q}\cdot\mathbf{x}_k)$ is the Fourier transform of the density operator, $|n\rangle$ is a complete set of eigenstates of the many-body Hamiltonian, and $(E_n - E_0)$ are the corresponding excitation energies. Here, and in the following, we use the convention $\hbar = 1$ and consider configurations where the dynamic structure factor $S(\omega, \mathbf{q})$ depends on the wave vector only through its modulus $q = |\mathbf{q}|$. From the dynamic structure factor, one can extract the energy weighted moments [9],

$$m_j(q) = \frac{1}{\bar{n}}\int_0^\infty d\omega\omega^j S(\omega, q), \tag{5}$$

where j is an integer and $\bar{n} = N/V$ is the average fermion density. The energy weighted moment $m_1(q)$ is model independent and equal to $q^2/(2m)$, as follows from the number conservation law (f-sum rule), with m the particle mass. All the other moments are generally model dependent and do carry crucial information on the many-body properties of the system, as we shall see below.

A. Superfluid fraction f_S vs Leggett bound f_S^L

If the periodic potential (2) is weak (small V_0), the density changes induced by the perturbation can be calculated via linear response theory and take the form

$$\delta n(x) = V_0\chi(0, q)\cos(qx), \tag{6}$$

where $\chi(\omega = 0, q)$ is the static response function, related to the inverse energy weighted moment of the dynamic structure factor $S(\omega, q)$ by [10]

$$-\frac{1}{\bar{n}}\chi(0, q) = 2m_{-1}(q). \tag{7}$$

By expanding the Leggett's integral up to terms quadratic in V_0, one finds the following result for the Leggett's upper bound:

$$f_S^L = 1 - 2V_0^2[m_{-1}(q)]^2, \tag{8}$$

in terms of V_0 and $m_{-1}(q)$. Since Eq. (8) is based on linear response theory, the moment $m_{-1}(q)$ in Eq. (8) has to be evaluated using the unperturbed ($V_0 = 0$) uniform system.

In the following, we will consider many-body systems, characterized by the absence of transverse gapless excitations. This includes, among others, single-component Bose and Fermi superfluids in the presence of an external periodic perturbation of the form (2). In order to calculate the superfluid fraction along the x direction, it is convenient to calculate the expectation value $\langle P_x\rangle$ of the longitudinal momentum $P_x = \sum_{k=1}^N p_{k,x}$ acquired by the system in the presence of the longitudinal perturbation $-vP_x$ and use the definition

$$f_S = 1 - \lim_{v\to 0}\frac{\langle P_x\rangle}{Nmv}. \tag{9}$$

Notice that Eq. (9) holds true for both Bose and Fermi superfluids, with m being the single-particle mass and N the total number of individual particles in the system. By applying perturbation theory, the superfluid fraction can then be written in the form

$$f_S = 1 - \frac{2}{Nm}\sum_{n\neq 0}\frac{|\langle 0|P_x|n\rangle|^2}{E_n - E_0}. \tag{10}$$

If the Hamiltonian is translationally invariant, i.e., if $[H, P_x] = 0$, then $f_S = 1$. If instead the Hamiltonian contains the additional term (2) breaking translational invariance, then the system will acquire a normal component. Using the commutation relation

$$[H, P_x] = -\frac{q}{2}V_0(\rho_q - \rho_{-q}), \tag{11}$$

one can rewrite Eq. (10) in the form

$$\begin{aligned} f_S &= 1 - \frac{1}{Nm}V_0^2q^2\sum_{n\neq 0}\frac{|\langle 0|\rho_q|n\rangle|^2}{(E_n - E_0)^3} \\ &= 1 - 2V_0^2 m_1(q)m_{-3}(q), \end{aligned} \tag{12}$$

showing that the superfluid fraction is controlled by the inverse cubic energy weighted moment $m_{-3}(q)$. While Eq. (8) is valid only for a weak periodic potential, Eq. (12) holds for arbitrary values of V_0. Of course, for small V_0, the moment m_{-3} can be safely evaluated using the unperturbed uniform system.

Equations (8) and (12) are the main results of this section. They show that in the considered limit of weak perturbations (small V_0), the Leggett bound coincides with the exact value of the superfluid fraction only if the general inequality [9]

$$\frac{m_1}{m_{-1}} \geqslant \frac{m_{-1}}{m_{-3}}, \tag{13}$$

which follows from the positiveness of the domain of the dynamic structure factor, reduces to an equality. The equality holds only in a few notable cases: (i) for a dilute BEC gas, where the Bogoliubov excitation spectrum is characterized by a single mode for all values of the wave vector q, with

TABLE I. Low momentum behavior of the leading contributions to a given physical quantity (left column), coming from gapless collective modes (central column) and from gapped single-particle excitations (right column). Notice that the sums of the q^4 contributions of the gapless and gapped excitations to the energy weighted moment m_1 cancel out in order to ensure the fulfillment of the f-sum rule, $m_1(q) = q^2/(2m)$.

	Gapless	Gapped
$\lvert\langle 0\lvert\rho_q\rvert n\rangle\rvert^2$	$q + q^3$	q^4
$\omega(q)$	$q + q^3$	const
$m_1(q)$	$q^2 + q^4$	q^4
$m_{-1}(q)$	const $+ q^2$	q^4
$m_{-3}(q)$	$1/q^2 +$ const	q^4

frequency $\omega = \sqrt{q^2\mu_b/m + q^4/(2m)^2}$, where μ_b is the mean-field chemical potential of the BEC gas. (ii) In general, for superfluids in the phonon limit, $q \to 0$, as a consequence of the fact that in this limit, the phonon mode exhausts the relevant sum rules entering the inequality (13). This includes, in particular, the case of the interacting superfluid Fermi gas considered in this paper, but also the case of superfluid ^{4}He. In general, the inequality (13) implies that $f_S^L \geqslant f_S$, in agreement with Leggett's prescription.

It is useful to stress that the existence of a finite m_{-3} for a Fermi gas requires the particle-hole excitations spectrum to be gapped (as is indeed the case in the superfluid state), otherwise m_{-3} diverges due to the presence of gapless particle-hole excitations at the Fermi surface [10].

B. Curvature effect in the excitation spectrum

Interestingly, the moments m_{-1} and m_{-3}, controlling the behavior of the Leggett bound f_S^L and of the actual value f_S of the superfluid fraction in the presence of the periodic potential (2), can also be used to characterize the dispersion relation of the Anderson-Bogoliubov mode of a uniform system at small q,

$$\omega^2 = c^2 q^2 (1 + \gamma q^2), \tag{14}$$

where c is the sound velocity, related to the inverse compressibility $\kappa^{-1} = n(\partial\mu/\partial n)$, through the thermodynamic relation $c^2 = \kappa^{-1}/m$, and γ measures the curvature effect beyond the linear dispersion. The coefficient γ is positive in the case of a dilute BEC gas [$\gamma = 1/(4\mu_b)$], while it is expected to become negative in the BCS region of the BEC-BCS crossover. Its change of sign takes place around unitarity and has been the object of several theoretical calculations [11–14] and of a recent measurement in a superfluid atomic Fermi gas using Bragg spectroscopy [15].

To understand the connection between the results for the superfluid fraction in the presence of the perturbation $V_0\cos(qx)$ and the curvature effect, it is useful to explore the behavior of the contributions to the various sum rules originating from the gapless branch and from the gapped excitations in the small-q regime (see Table I). The q^4 dependence of the strength contribution of the gapped states follows from general arguments based on Galilean invariance (see, for example, [10]) and implies that the m_{-3} and m_{-1} moments of the dynamic structure factor are exhausted by the gapless mode, not only in the lowest-q phonon regime, but also including the first corrections in q^2. A nontrivial consequence is that at low q, the average square excitation energy

$$\omega^2 = m_{-1}(q)/m_{-3}(q) \tag{15}$$

exactly coincides with the expansion (14) of the square frequency, including the curvature correction in γ. Vice versa, the most popular ratio $m_1(q)/m_{-1}(q)$ correctly reproduces only the leading phonon term c^2q^2. Using the small-q expansions $m_{-1}(q) = (\kappa/2)(1 - \beta_{-1}q^2)$ and $q^2 m_{-3}(q) = (m\kappa^2/2)(1 - \beta_{-3}q^2)$ in Eq. (15) and comparing the obtained expression with Eq. (14), one finds

$$\gamma = \beta_{-3} - \beta_{-1}. \tag{16}$$

By substituting the same expansions in Eqs. (8) and (12), we note that the inequality $f_S \leqslant f_S^L$ yields $\beta_{-3} \leqslant 2\beta_{-1}$. When γ vanishes, corresponding to $\beta_{-1} = \beta_{-3}$, the q^2 corrections to the Leggett bound and to the exact superfluid fraction differ exactly by a factor of 2. In the case of a superfluid Fermi gas, this happens very close to unitarity [15]. The situation is different for a dilute BEC gas, where $\beta_{-3} = 2\beta_{-1}$, since $f_S^L = f_S$ for all values of q.

It is also interesting to compare the above results with the effective field theory for the unitary Fermi gas developed in [11], which includes the leading-order corrections in an expansion in small gradients of the quantum hydrodynamic Lagrangian density (see, also, the related discussion in Ref. [12]). The corresponding predictions for the static response function and for the dispersion of the collective mode at small q are given by [11]

$$\chi(0,q) = -n\kappa\left[1 + A\left(c_1 - \frac{9}{2}c_2\right)\frac{q^2}{k_F^2}\right], \tag{17}$$

$$\omega^2 = c^2q^2\left[1 - A\left(c_1 + \frac{3}{2}c_2\right)\frac{q^2}{k_F^2}\right], \tag{18}$$

where c_1 and c_2 are dimensionless parameters entering the Lagrangian density, $k_F = (3\pi^2 n)^{1/3}$ is the Fermi wave vector, $A = 2\pi^2\sqrt{2(1+\beta)}$, and β is the dimensionless Bertsch parameter entering the equation of state of the homogeneous Fermi gas at unitarity: $\mu = (1+\beta)k_F^2/(2m)$. The BdG equations yield $\beta = -0.41$ (see Sec. III), while more accurate Monte Carlo calculations predict $\beta = -0.58$ [16,17], which is quite close to the value $\beta = -0.63$ reported in the experiment [18].

By comparing Eqs. (17) and (18) with Eqs. (7) and (15), respectively, we find that the parameters of the effective field theory can be expressed as

$$c_1 = (2\beta_{-1} - 3\beta_{-3})\frac{k_F^2}{4A},$$
$$c_2 = (2\beta_{-1} - \beta_{-3})\frac{k_F^2}{6A}. \tag{19}$$

In Ref. [12], it was demonstrated that the coefficient c_2 must be positive. From Eq. (19), we see that this is equivalent to the condition $f_S < f_S^L$.

III. NUMERICAL RESULTS

A. Dynamical BCS theory and sum rules

In this section, we evaluate the sum-rule moments of a unitary Fermi gas with the help of the dynamical BCS theory developed in Ref. [8] (see, also, an earlier work by Minguzzi and collaborators [19]). In particular, we will make use of the general expression for the density response function,

$$\chi(\omega, q) = -\frac{1}{2\pi^2}\left[I''(\omega, q) - \frac{F(\omega, q)}{I_{11}(\omega, q)I_{22}(\omega, q) - \omega^2 I_{12}^2(\omega, q)}\right], \tag{20}$$

where I'', F, and I_{ij} are functions whose definition in terms of integrals over momenta can be found in Ref. [8]. Notice that with respect to the result quoted in Ref. [8] [see their Eq. (25)], the right-hand side of Eq. (20) contains an extra factor of 2 because, in our definition of the density response function, we account for the spin degeneracy [see Eq. (6)]. The first term in Eq. (20), proportional to I'', represents the bare contribution from the linear response formalism applied to the BCS state, while the second term accounts for the fluctuations of the order parameter over its mean-field value, which are included via the random phase approximation (RPA). In particular, the vanishing of the denominator of the second term, $I_{11}(\omega, q)I_{22}(\omega, q) - \omega^2 I_{12}^2(\omega, q) = 0$, yields the dispersion relation of the Anderson-Bogoliubov mode, $\omega = \omega_{AB}(q)$, which has a phononlike behavior at low momentum.

Although not explicitly written, all functions in Eq. (20) also depend on the two mean-field parameters describing the BCS-BEC crossover in Fermi gases, namely, the energy gap Δ and the chemical potential μ. Their values can be obtained by inverting the BCS gap equation and the number equation describing the BCS-BEC crossover for uniform systems; see, for instance, [8]. In particular, for a unitary Fermi gas, corresponding to $1/(k_F a) = 0$, one finds $\mu = 0.591 E_F$ and $\Delta = 0.686 E_F$. The dynamic structure factor can be calculated numerically from Eqs. (3) and (20). For a given momentum q, there is a threshold frequency $\omega_{tr}(q)$ where the collective mode merges with the continuum of single-particle excitations [8],

$$\omega_{tr}(q) = \begin{cases} 2\Delta & \text{for } \mu > 0 \text{ and } q < 2\sqrt{2m\mu}, \\ 2\sqrt{(q^2/(8m) - \mu)^2 + \Delta^2} & \text{otherwise.} \end{cases} \tag{21}$$

For $\omega < \omega_{tr}(q)$, all the functions I'', F, and I_{ij} in Eq. (20) are real due to the energy gap in the single-particle energy spectrum. In this case, the only contribution to the dynamic structure factor originates from the collective mode, as the vanishing of the denominator in the second term in Eq. (20) causes the appearance of an imaginary part, via the formula $1/(x + i0^+) = P1/x - i\pi\delta(x)$, where P stands for the principal part. In contrast, for $\omega > \omega_{tr}(q)$, the only contribution to $S(q, \omega)$ comes from the continuum of single-particle excitations, with all the functions I'', F, and I_{ij} in Eq. (20) acquiring an imaginary part by themselves.

From the dynamic structure factor, we extract the moments (5) and use them to estimate the superfluid density and the associated Leggett bound in the presence of a weak periodic potential, corresponding to $V_0 \ll E_F$, with $E_F = k_F^2/(2m)$ being the Fermi energy of the unperturbed system. Since the potential induces a quadratic in V_0 corrections to both quantities, we write $f_S = 1 - \alpha_S(V_0/E_F)^2$ and $f_S = 1 - \alpha_S^L(V_0/E_F)^2$, where α_S and α_S^L are dimensionless functions. From Eqs. (8) and (12), we find, respectively, $\alpha_S^L(q) = 2[m_{-1}(q)]^2 E_F^2$ and $\alpha_S(q) = 2m_1(q)m_{-3}(q)E_F^2$. In Fig. 1, we plot such quantities as a function of the ratio q/k_F for a unitary Fermi gas. We see that in general $\alpha_S^L(q) \neq \alpha_S(q)$, showing that the Leggett bound is not exhausted in the presence of the external periodic potential. A notable exception occurs for vanishing q, where the two quantities coincide and reduce to $\kappa^2 E_F^2/2 = 9E_F^2/(8\mu^2) = 3.221$. Moreover, we see from Fig. 1 that the corrections to the superfluid density become less and less important as q increases, and vanish for infinite q. In this extreme limit, sum rules are dominated by the single-particle kinetic energy $\epsilon(q) = q^2/(2m)$, implying that $m_j(q) = \epsilon(q)^j$ and therefore $\alpha_S(q) = \alpha_S^L(q) = 2k_F^4/q^4$.

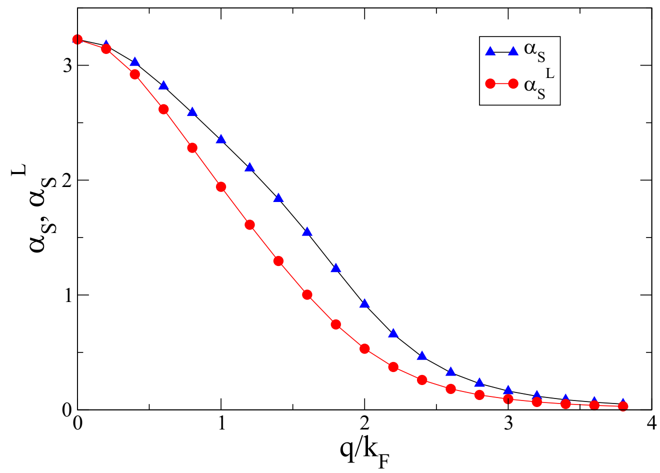

FIG. 1. Lattice-induced perturbative corrections to the superfluid density, $\alpha_S = (1 - f_S)E_F^2/V_0^2$, and to the associated Leggett bound, $\alpha_S^L = (1 - f_S^L)E_F^2/V_0^2$, of a unitary Fermi gas, calculated from the sum-rule approach [see Eqs. (8) and (12)].

In Fig. 2, we plot the low-momentum estimates $\omega(q) = \sqrt{m_1(q)/m_{-1}(q)}$ (green circles) and $\omega(q) = \sqrt{m_{-1}(q)/m_{-3}(q)}$ (red diamonds) for the energy dispersion of the Anderson-Bogoliubov collective mode of a unitary Fermi gas based on the sum-rule approach. The black solid line represents the full numerical result, $\omega = \omega_{AB}(q)$, showing a positive, albeit small curvature [13]. At higher momentum, however, the correction to the phononic dispersion becomes negative, as the frequency of the collective mode is bounded above by the threshold frequency ω_{tr}. We see that the sum-rule estimate $\omega(q) = \sqrt{m_1(q)/m_{-1}(q)}$ reproduces only the sound velocity of the superfluid in the phononic regime, while missing the curvature correction. In contrast, the sum-rule estimate $\omega(q) = \sqrt{m_{-1}(q)/m_{-3}(q)}$ is more accurate as it includes the first nonlinear correction.

B. BdG equations for systems in a periodic potential

So far, we have considered a weak external periodic potential, so that its effect on the transport properties of the unitary

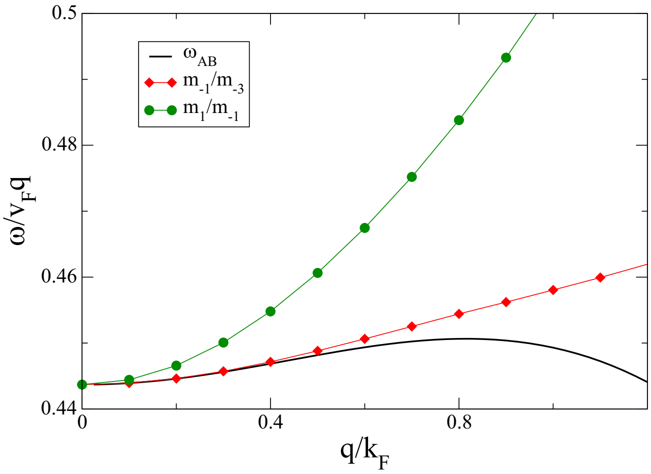

FIG. 2. Dispersion relation of the Anderson-Bogoliubov collective mode of the unitary Fermi gas. The sum-rule estimates (symbols) in the long-wavelength limit are compared with the full numerical result $\omega = \omega_{AB}(q)$ (black solid line), first discussed in Ref. [13]. Here, k_F is the Fermi momentum and $v_F = k_F/m$ is the Fermi velocity. The sound velocity of the system is $c = 0.444 v_F$.

Fermi gas can be calculated within perturbation theory. Below we compute the superfluid fraction and the Leggett bound for a periodic potential of arbitrary strength by numerically solving the BdG equations. Since the external potential (2) is periodic, the superfluid fraction defined in Eq. (9) coincides with the effective mass ratio m/m^* of the Fermi gas, calculated from the curvature of the lowest-energy band according to

$$f_S = \frac{m}{m^*} \equiv \lim_{Q\to 0} \frac{1}{\bar{n}} \frac{\partial^2 e(\bar{n}, Q)}{\partial Q^2}, \tag{22}$$

where $e = E/V$ is the energy density and Q is the wave vector of the supercurrent. To see this, we note that $-\langle P_x \rangle$ in Eq. (9) represents the total momentum of the system along the x axis measured with respect to the moving frame, so that the supercurrent density j in the laboratory frame is given by

$$j = \bar{n} v - \frac{\langle P_x \rangle}{mV} = f_S \bar{n} v + O(v^2). \tag{23}$$

Since the supercurrent density is related to the wave vector by $j = \partial e/\partial Q$, from Eq. (23) we find $v = Q/m$ and

$$e(\bar{n}, Q) = e(\bar{n}, 0) + f_S \bar{n} \frac{Q^2}{2m} + O(Q^3), \tag{24}$$

which is consistent with Eq. (22). In the absence of the lattice ($V_0 = 0$), one has $e(\bar{n}, Q) = e(\bar{n}, 0) + \bar{n} Q^2/(2m)$, and hence Eq. (22) yields $m^* = m$ and $f_S = 1$.

In order to calculate the superfluid fraction via Eq. (22), we proceed as in Refs. [20,21] by imposing an order parameter of the Bloch form $\Delta(\mathbf{r}) = e^{i2Qx} \tilde{\Delta}(x)$ in the BdG equations, where $\tilde{\Delta}(x)$ is a periodic function with period d. This implies that the corresponding eigenfunctions take the form $u(\mathbf{r}) = \tilde{u}_{\mathbf{k}}(x) e^{iQx} e^{i\mathbf{k}\cdot\mathbf{r}}/\sqrt{V}$ and $v(\mathbf{r}) = \tilde{v}_{\mathbf{k}}(x) e^{-iQx} e^{i\mathbf{k}\cdot\mathbf{r}}/\sqrt{V}$, where k_x lies in the first Brillouin zone and the amplitudes $\tilde{u}_{\mathbf{k}}(x)$ and $\tilde{v}_{\mathbf{k}}(x)$ are periodic functions with period d obeying

$$\begin{pmatrix} H_Q(x) & \tilde{\Delta}(x) \\ \tilde{\Delta}^*(x) & -H_{-Q}(x) \end{pmatrix} \begin{pmatrix} \tilde{u}_{\mathbf{k}}(x) \\ \tilde{v}_{\mathbf{k}}(x) \end{pmatrix} = \epsilon_{\mathbf{k}} \begin{pmatrix} \tilde{u}_{\mathbf{k}}(x) \\ \tilde{v}_{\mathbf{k}}(x) \end{pmatrix}, \tag{25}$$

where

$$H_Q(x) \equiv \frac{1}{2m}\left[\mathbf{k}_\perp^2 + (-i\partial_x + Q + k_x)^2\right] + V_{\text{ext}}(x) - \mu. \tag{26}$$

For a given value of the momentum $\mathbf{k}$, the BdG equations (25) represent an eigenvalue problem with the corresponding eigenfunctions being normalized according to $\int_0^d [\tilde{u}^*_{\mathbf{k}\alpha}(x)\tilde{u}_{\mathbf{k}\beta}(x) + \tilde{v}^*_{\mathbf{k}\alpha}(x)\tilde{v}_{\mathbf{k}\beta}(x)]dx = d\delta_{\alpha,\beta}$.

The local pairing field $\tilde{\Delta}(x)$ and the chemical potential μ, appearing in Eqs. (25) and (26), are variational parameters to be determined self-consistently. The first constraint is provided by the gap equation

$$\tilde{\Delta}(x) = -g \frac{1}{V} \sum_{\mathbf{k}\alpha} \tilde{u}_{\mathbf{k}\alpha}(x) \tilde{v}^*_{\mathbf{k}\alpha}(x), \tag{27}$$

where g is the bare coupling constant of the contact interaction potential and the sum is restricted over the solutions with $\epsilon_{\mathbf{k}\alpha} \geqslant 0$. Notice that Eq. (27) is well defined, as by construction both the left-hand side and the right-hand side are periodic functions with period d. It is well known that the right-hand side of Eq. (27) is actually plagued by a ultraviolet divergence and must be regularized. Following the procedure suggested in Ref. [22], we introduce an explicit high-energy cutoff E_c in the sum over eigenenergies, and replace g with a position-dependent effective coupling constant. To solve the BdG equations, we write all periodic functions as truncated Fourier series, for instance, $\tilde{u}_{\mathbf{k}}(x) = \sum_{j=-N_c}^{N_c} u_j e^{iqjx}$, and set $E_c = (2N_c + 1)^2 \epsilon(q)/4 \approx N_c^2 \epsilon(q)$. The value of N_c must be chosen such that E_c is much larger than all energy scales in the problem, $E_c \gg \max[E_F, V_0, \epsilon(q), \mu, \Delta]$. The second constraint to be satisfied is that the spatial average of the density profile must yield the mean density $\bar{n}$, that is, $\bar{n} = \int_0^d n(x)dx/d$, where

$$n(x) = \frac{1}{V} \sum_{\mathbf{k}\alpha} 2|\tilde{v}_{\mathbf{k}\alpha}(x)|^2. \tag{28}$$

Once the mean-field parameters are known, we compute the energy density $e = E/V$ of the system according to

$$e = \frac{1}{V} \sum_{\mathbf{k}\alpha} \int_0^d \frac{dx}{d} [2(\mu - \epsilon_{\mathbf{k}\alpha})|\tilde{v}_{\mathbf{k}\alpha}(x)|^2 + \tilde{\Delta}^*(x)\tilde{u}_{\mathbf{k}\alpha}(x)\tilde{v}^*_{\mathbf{k}\alpha}(x)]. \tag{29}$$

and use Eq. (22) to extract the superfluid fraction from the obtained result. Notice that the two contributions in the right-hand side of Eq. (29) are separately divergent, but their sum is finite, implying that this equation is well defined.

For the Leggett bound, we instead solve the BdG equations (25) at equilibrium, i.e., with $Q = 0$, and then extract f_S^L from Eqs. (1) and (28). In Fig. 3, we display f_S and f_S^L as a function of the ratio V_0/ϵ_F for the unitary Fermi gas, with the two panels corresponding to two different values of the dimensionless parameter $k_F d = 4$ [Fig. 3(a)] and $k_F d = 10$ [Fig. 3(b)], which correspond to typical values available

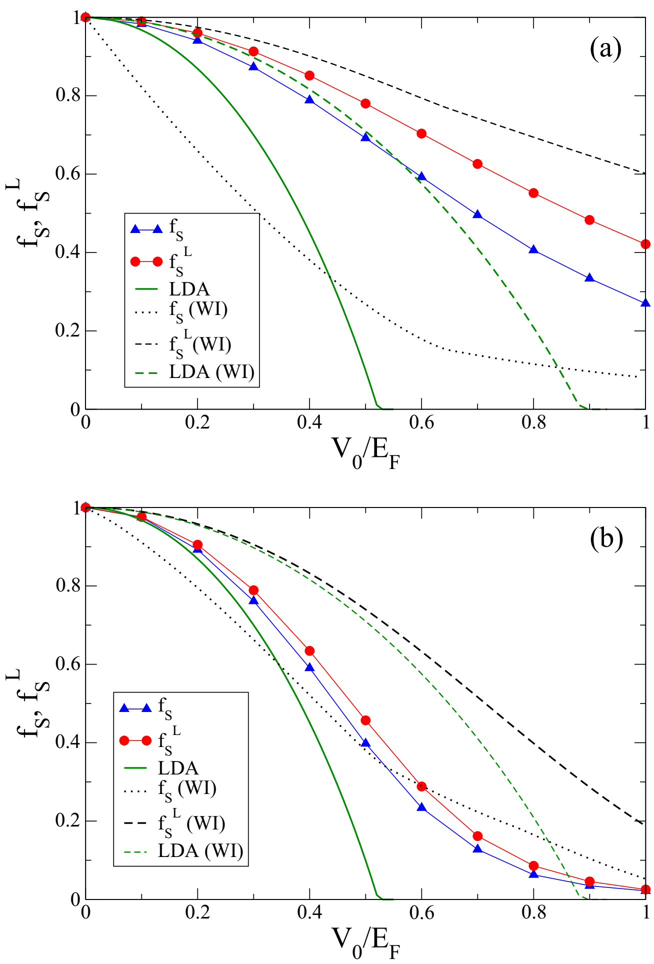

FIG. 3. Superfluid fraction f_S (blue triangles) and its Leggett bound f_S^L (red circles) for the unitary Fermi gas as a function of the ratio V_0/E_F for (a) $k_F d = 4$ and (b) $k_F d = 10$. The green solid line refers to the local density approximation (LDA) at unitarity. In each plot, we also include the corresponding predictions for the weakly interacting BCS regime for f_S (black dotted line), f_S^L (black dashed line), and the LDA (green dashed line).

in current experiments with atomic superfluid Fermi gases. For $k_F d = 4$, which corresponds to $q/k_F = 1.57$, the Leggett bound remains well above the superfluid fraction, which is consistent with the sizable difference observed in the perturbative corrections for the two quantities displayed in Fig. 1. For $k_F d = 10$, the Leggett bound is instead quite close to the superfluid fraction. In particular in the limit $k_F d \gg 1$, the two values of f_S and f_S^L are expected to approach the prediction of the local density approximation (LDA), where the density profile of the Fermi gas can be calculated starting from the expression $\mu_0 = \mu_{\text{hom}}[n(x)] + V_0 \cos(qx)$ for the chemical potential, with $\mu_{\text{hom}}(n) = (1+\beta)E_F = (1+\beta)(3\pi^2 n)^{2/3}/(2m)$ the equation of state of the uniform Fermi gas at unitarity [9]. The density profile in the LDA then takes the form

$$n_{\text{LDA}}(x) = \frac{1}{3\pi^2}\left(\frac{2m}{1+\beta}\right)^{3/2} [\mu_0 - V_0 \cos(2\pi x/d)]^{3/2}, \quad (30)$$

yielding the prediction

$$\frac{1}{f_S^{\text{LDA}}} = \langle n \rangle \left\langle \frac{1}{n} \right\rangle = h\left(\frac{V_0}{\mu_0}\right) \int_0^1 d\tilde{x} \left[1 - \frac{V_0}{\mu_0}\cos(2\pi\tilde{x})\right]^{-3/2}, \quad (31)$$

for the inverse superfluid fraction, where $\tilde{x} = x/d$ and the function h is defined as $h(V_0/\mu_0) = \int_0^1 d\tilde{x}[1 - (V_0/\mu)\cos(2\pi\tilde{x})]^{3/2}$. The chemical potential μ_0 is related to the Fermi energy E_F and the potential strength V_0 via the normalization condition $\bar{n} = \int_0^d n_{\text{LDA}}(x)dx/d$, yielding

$$\mu_0 = \frac{(1+\beta)E_F}{h(V_0/\mu_0)^{2/3}}. \quad (32)$$

Equations (31) and (30) show that in the LDA, the superfluid density is independent of the lattice period d and only depends on the ratio V_0/E_F (see green solid line in Fig. 3). In particular, the superfluid fraction vanishes for $V_0/E_F = (1+\beta)/h(1)^{2/3} = 0.522$, corresponding to $V_0/\mu_0 = 1$, where the integral in Eq. (31) diverges. Differently from the LDA prediction, the actual value of the Leggett bound (and, consequently, of the superfluid fraction) differs from zero for all values of V_0, as shown in Fig. 3. It vanishes asymptotically as $V_0 \to \infty$, the spatial average $\langle 1/n_{1D}(x) \rangle$ entering Eq. (1) becoming larger and larger.

For small values of V_0/E_F, the LDA expression for the superfluid density exhibits the quadratic dependence (see, also, [20] and Eq. (12) in the $q \to 0$ limit)

$$f_S^{\text{LDA}} = 1 - \frac{9}{8(1+\beta)^2}\frac{V_0^2}{E_F^2}. \quad (33)$$

It is interesting to compare the above expansion with the corresponding expansion for the compressibility, in the same LDA regime ($q \to 0$ limit):

$$\kappa^{\text{LDA}} = \frac{2}{3}(1+\beta)E_F\left[1 + \frac{1}{8(1+\beta)^2}\frac{V_0^2}{E_F^2}\right]. \quad (34)$$

Equation (34) is easily obtained by applying perturbation theory for the calculation of the energy in the presence of the perturbation (2) and plays a crucial role in the propagation of sound (see Sec. IV below). Notice, in particular, that the effect of the perturbation is smaller by a factor of 9, with respect to the analogous expansion (33) for the superfluid fraction. It is also interesting to compare the above results with the expansions holding in a dilute BEC gas [7] where, in the LDA, the V_0^2 correction to the compressibility is exactly vanishing, as a consequence of the linear density dependence exhibited by the equation of state.

In Fig. 3, we also show the superfluid fraction (dotted line) and its Leggett bound (dashed line) for a Fermi gas in the weakly interacting (WI) BCS regime, corresponding to $\Delta \ll E_F$. In this limit, the density profile and the energy of the system can be calculated by assuming that the Fermi gas is noninteracting. Of course, the noninteracting Fermi gas is not superfluid because of the occurrence of transverse gapless excitations. However, in the superfluid phase, the superfluid fraction f_S coincides with the effective mass [see (Eq. (22)]. The evaluation of the effective mass in the noninteracting Fermi gas is expected to provide the actual value of f_S^{WI}, if the condition $qk_F/m \gg \Delta$, corresponding to wavelengths of the perturbation that are much smaller than the size of Cooper pairs, is satisfied. One can then write [23]

$$f_S^{WI} \simeq \frac{1}{\bar{n}V}\sum_{\mathbf{k}j}\frac{\partial^2\epsilon_j(k_x)}{\partial k_x^2}2\Theta\left[\mu-\epsilon_j(k_x)-\frac{\mathbf{k}_\perp^2}{2m}\right], \tag{35}$$

where $\epsilon_j(k_x)$ are the single-particle dispersion relations induced by the 1D external potential, with j being the band index, and the integration over k_x is restricted to the first Brillouin zone.

The Leggett bound in the WI regime can be obtained from Eq. (1) by using the density profile of the noninteracting Fermi gas,

$$n(x) = \frac{1}{\bar{n}V}\sum_{\mathbf{k}j}|\psi_{k_xj}(x)|^2 2\Theta\left[\mu-\epsilon_j(k_x)-\frac{\mathbf{k}_\perp^2}{2m}\right], \tag{36}$$

where $\psi_{k_xj}(x)$ are the amplitudes of the Bloch states.

A first useful comparison between the superfluid fractions calculated at unitarity and in the weekly interacting BCS gas concerns the behavior of the Leggett bound in the local density approximation [see Eq. (31)]. Since the chemical potential of the noninteracting Fermi gas and of the gas at unitarity have the same density dependence, except for the factor $(1+\beta)$ [see Eq. (32)], one finds that in the weakly interacting case, $f_S^{\rm LDA}$ always stays significantly *above* the corresponding value at unitarity. In particular, the LDA superfluid fraction vanishes for a larger value of the ratio V_0/E_F, equal to $0.522/(1+\beta) = 0.883$. The situation can, however, be very different for the actual value f_S of the superfluid fraction. For example, for small values of V_0, the superfluid fraction of the weakly interacting BCS gas always stays *below* the value at unitarity. For large wave vectors [see Fig. 3(a)], the same also happens for larger values of V_0, suggesting that interactions play a crucial role to facilitate superfluid transport, as previously reported by Watanabe and Pethick with important implications on the behavior of neutrons stars [21].

Another interesting feature emerging from Figs. 3(a) and 3(b) is that in the WI regime, f_S exhibits a linear dependence for small V_0. We have checked that this behavior occurs only for $k_Fd \geqslant \pi$, while for $k_Fd < \pi$, we find that $1 - f_S \propto V_0^2$, as previously observed for the interacting system. This effect can be simply understood by noting that for a noninteracting uniform Fermi gas, the moment $m_{-3}(q)$ in Eq. (12) diverges for $q < 2k_F$ because the dynamic structure factor $S(q,\omega)$ is linear in ω at low frequency, reflecting the presence of gapless single-particle excitations at the Fermi surface. In the superfluid Fermi gas, these excitations are instead gapped due to Cooper pairing. In the Appendix, we provide an explicit derivation of the behavior of f_S for small V_0 in the WI regime based on degenerate perturbation theory.

IV. BEHAVIOR OF THE SOUND VELOCITIES

The results discussed in the previous section on the superfluid density of the interacting Fermi gas have important implications for the value of the sound velocity and, in particular, its anisotropy in the presence of the periodic perturbation (2). According to the hydrodynamic theory of superfluids, the square sound velocities along the longitudinal and transverse directions are given by

$$c_x^2 = f_S\kappa^{-1},$$
$$c_y^2 = c_z^2 = \kappa^{-1}, \tag{37}$$

emphasizing that their anisotropy, caused by the presence of the 1D periodic potential, follows from the anisotropy of the superfluid fraction. From an experimental point of view, a natural strategy to determine the value of the superfluid fraction, associated with the motion of the superfluid along the x direction, then consists of measuring the ratio

$$\frac{c_x^2}{c_y^2} = f_S, \tag{38}$$

which is independent of the value of the compressibility. This procedure has been recently successfully employed in the case of a dilute Bose gas confined in a 2D box trap [7].

V. CONCLUSIONS

In this work, we have investigated the superfluid fraction of a Fermi gas at unitarity, in the presence of a one-dimensional periodic potential. We have shown that in sharp contrast with the dilute BEC gas case, the Leggett's upper bound for the superfluid fraction can significantly overestimate its actual value at zero temperature, unless the period $2\pi/q$ of the applied lattice becomes much larger than the size of Cooper pairs, i.e., if $q \ll \Delta/(mk_F)$, where Δ is the pairing gap energy. We have pointed out the occurrence of important deviations of Leggett's bound from the actual value f_S of the superfluid fraction at unitarity and shown that these deviations become more and more important as one approaches the weakly interacting BCS regime of small and negative values of the s-wave scattering length. Using sum-rule arguments, we have also shown that the deviations of the Leggett bound from the superfluid fraction explain the change of sign of the curvature of the Anderson-Bogoliubov mode dispersion taking place around unitarity.

ACKNOWLEDGMENTS

We thank Y. Castin, F. Dalfovo, S. Giorgini, A. Leggett, and W. Zwerger for fruitful discussions. This work was granted access to the HPC resources of TGCC under the allocations AD010513635 and AD010513635R1 supplied by GENCI (Grand Equipement National de Calcul Intensif). S.S. acknowledges continuous financial support from PAT (Provincia Autonoma di Trento).

APPENDIX: ANALYTICAL RESULTS IN THE WEAKLY INTERACTING BCS REGIME

For completeness, in this Appendix, we provide an analytical calculation of f_S for a weak periodic potential in the weakly interacting BCS regime, under the assumption that $q \gg \Delta m/k_F$, which is equivalent to $k_Fd \ll E_F/\Delta$. As discussed in the main text, under this condition, one is allowed to identify the inverse effective mass, calculated using the ideal Fermi gas, with the superfluid fraction of the weakling interacting BCS superfluid. Below we will separately consider two cases, $k_Fd < \pi$ and $k_Fd \geqslant \pi$.

1. Case $k_F d < \pi$

We calculate f_S^{WI} from Eq. (12) since $m_{-3}(q)$ is expected to be finite for a uniform system. For $q > 2k_F$, the dynamic structure factor of the noninteracting Fermi gas is given by

$$S(\omega, q) = \frac{m}{4\pi^2 q}\left[k_F^2 - \frac{m^2}{q^2}\left(\omega - \frac{q^2}{2m}\right)^2\right], \qquad (A1)$$

if the right-hand side of Eq. (A1) is positive, that is, for $|\omega - q^2/(2m)| < qk_F/m$, and zero otherwise,

$$m_{-3}(q) = \frac{m}{4\pi^2 q\bar{n}}\int_{q^2/(2m)-qk_F/m}^{q^2/(2m)+qk_F/m}\frac{d\omega}{\omega^3}\times\left[k_F^2 - \frac{m^2}{q^2}\left(\omega - \frac{q^2}{2m}\right)^2\right]. \qquad (A2)$$

To perform the integral, we introduce dimensionless variable $y = [\omega - q^2/(2m)]/E_F$ and $p = q/k_F$, so that the integral becomes

$$m_{-3}(q) = \frac{m^3}{\pi^2 k_F^3 \bar{n} p}\int_{-2p}^{2p}\left(1 - \frac{y^2}{4p^2}\right)\frac{dy}{(y+p^2)^3}. \qquad (A3)$$

By performing the integration in Eq. (A3) and recalling that $\bar{n} = k_F^3/(3\pi^2)$, we obtain $m_{-3}(q) = F(q/k_F)/E_F^3$, where

$$F(p) = \frac{3}{8p^3}\left[\ln\left(\frac{p-2}{p+2}\right) + \frac{4p}{p^2-4}\right]. \qquad (A4)$$

Since $m_1(q) = q^2/(2m)$, from Eq. (12) we find

$$1 - f_S^{WI} \simeq 2\left(\frac{q}{k_F}\right)^2 F\left(\frac{q}{k_F}\right)\left(\frac{V_0}{E_F}\right)^2. \qquad (A5)$$

Notice that the function $F(p)$ in Eq. (A4) diverges at $p = 2$ as $F(p) \approx \frac{3}{128(p-2)}$, implying that for $q = 2k_F$, the correction to the superfluid fraction induced by the periodic potential is no longer quadratic but linear in V_0, as we shall see below.

2. Case $k_F d \geqslant \pi$

We use degenerate perturbation theory applied to the external periodic potential (2) to show that $1 - f_S \propto V_0$. For simplicity, we will prove the claim for the special case $k_F d = \pi$, so that only the lowest-energy band is populated. The proof can be easily adapted to the general case where higher bands are also populated.

We consider two states with wave vector k_x and $k_x - G$, where G is a reciprocal lattice point. The two energy bands are obtained by the condition

$$\begin{vmatrix} \frac{k_x^2}{2m} - E & \frac{V_0}{2} \\ \frac{V_0}{2} & \frac{(k_x-G)^2}{2m} - E \end{vmatrix} = 0, \qquad (A6)$$

whose solutions are

$$E_\pm(k_x) = \frac{1}{2}\left(\frac{k_x^2}{2m} + \frac{(k_x-G)^2}{2m}\right) \pm\left[\frac{1}{4}\left(\frac{k_x^2}{2m} - \frac{(k_x-G)^2}{2m}\right)^2 + \frac{V_0^2}{4}\right]^{1/2}. \qquad (A7)$$

We consider $0 \leqslant k_x \leqslant \pi$ and choose $G = 2\pi/d$ in Eq. (A7) so that $E_\pm(\pi/d) = E_R \pm V_0/2$, where $E_R = \pi^2/(2md^2)$. For $k_F = \pi/d$, the Fermi energy $E_F = E_R$ sits between the two bands, so the upper band is completely empty and, according to Eq. (35), it does not contribute to the superfluid fraction. Let us introduce the dimensionless variables $\tilde{E}_- = E_-/E_R$ and $x = k_x\pi/d$. Then,

$$\tilde{E}_-(x) = x^2 + 2 - 2x - \left[4(1-x)^2 + V_r^2\right]^{1/2}, \qquad (A8)$$

where $V_r = V_0/(2E_R)$ is the rescaled potential strength. From Eq. (A8), we find

$$\frac{\partial^2\tilde{E}_-(x)}{\partial x^2} = 2 - \frac{4V_r^2}{\left[4(1-x)^2 + V_r^2\right]^{3/2}}, \qquad (A9)$$

showing that the curvature of the bands diverges as $2 - 4/V_r$ for vanishing V_r. Substituting Eqs. (A8) and (A9) into Eq. (35), we find

$$f_S^{WI} \simeq 1 - \frac{4V_r^2\int_0^1\frac{\left[-(x-1)^2+\sqrt{4(1-x)^2+V_r^2}\right]}{\left[4(1-x)^2+V_r^2\right]^{3/2}}dx}{2\int_0^1\left[-(x-1)^2 + \sqrt{4(1-x)^2+V_r^2}\right]dx}. \qquad (A10)$$

Let us discuss the behavior of the two integrals in Eq. (A10), starting from the upper one. Since the major contribution to the integral comes from the region around $x = 1$, the term proportional to $-(x-1)^2$ in the numerator can be neglected as it vanishes for $x = 1$ (its contribution to the integral is of the order of $V_r^2\ln V_r$). For the lower integral, we can safely set $V_r = 0$ as no singularity is present, so that Eq. (A10) reduces to

$$f_S^{WI} \simeq 1 - \frac{4V_r^2\int_0^1\left[4(1-x)^2 + V_r^2\right]^{-1}dx}{2\int_0^1[-(x-1)^2 + 2(1-x)]dx}. \qquad (A11)$$

Since V_r is small, in the upper integral we can safely replace 0 with $-\infty$, so that the integration becomes elementary, yielding

$$f_S^{WI} \simeq 1 - \frac{\pi V_r}{4/3} = 1 - \frac{3\pi}{8}\frac{V_0}{E_F}, \qquad (A12)$$

showing that the correction to the superfluid fraction is linear in V_0, which agrees with our numerics.

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