Steering inequality for pairs of states restricted by a particle-number superselection rule

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We consider violations of a Clauser-Horne-Shimony-Holt-type steering inequality for quantum states of systems of indistinguishable particles restricted by a particle-number superselection rule. We check for violations in noninteracting Bose-Einstein condensate states, NOON states, and relative phase eigenstates, by using two copies of the states for bypassing the superselection rule. The superselection rule prevents the states from maximally violating the steering inequality, but the steering inequality violations are higher than Bell inequality violations for the same states. This in particular implies, in certain cases, that visibilities of the steering inequality violations are higher than the same for Bell inequality violations, for admixtures with white noise. We also found that an increase in the number of particles in the noninteracting condensate states results in a decrease of the violation amount of the steering inequality.

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I. INTRODUCTION

Entanglement is a fundamental resource of quantum information processing [1]. Entanglement is typically considered between separated systems or different degrees of freedom of the same physical system, for which the tensor-product structure is well defined. For the case of indistinguishable particles, it is therefore natural to consider entanglement between different modes.

Violation of Bell inequalities forms an important method for detecting entanglement present in quantum states of shared systems [2]. The Bell Clauser-Horne-Shimony-Holt (CHSH) inequality [3] for two distinguishable systems, each having the option to choose between two dichotomic measurement settings, A_1, A_2 and B_1, B_2, respectively for the two parties, is given by $|B| \leqslant 2$, where

$$B = \langle A_1 \otimes B_1 \rangle + \langle A_1 \otimes B_2 \rangle + \langle A_2 \otimes B_1 \rangle - \langle A_2 \otimes B_2 \rangle. \quad (1)$$

The maximum violation of this inequality that can be reached by any quantum state is $2\sqrt{2}$ [4].

Quantum steering, introduced by Schrödinger [5], is a concept that lies in between those of entanglement and Bell inequality violation [6]. If two observers share an entangled state, then unsteerability implies that there will exist a "local hidden state" model of the state. Steerability can be tested through the violation of steering inequalities (see, e.g., Refs. [7–13]). The violation of steering inequalities, like that of Bell inequalities, indicates the presence of entanglement in the shared state involved.

Recently, Cavalcanti, Foster, Fuwa, and Wiseman [14] have proposed a CHSH-type inequality to check for steerability in the two-party, two-setting (per party), and two-outcome (per measurement setting) scenario. Using the same notations as for the Bell inequality, the steering inequality is given by $S \leqslant 2$, where

$$S = \sqrt{\langle (A_1 + A_2) \otimes B_1 \rangle^2 + \langle (A_1 + A_2) \otimes B_2 \rangle^2} + \sqrt{\langle (A_1 - A_2) \otimes B_1 \rangle^2 + \langle (A_1 - A_2) \otimes B_2 \rangle^2}. \quad (2)$$

In quantum mechanics, the optimal violation of this steering inequality is again $2\sqrt{2}$.

Superselection rules are a set of axioms in quantum mechanics which restricts effective superposition rules like quantum superposition of states of elementary particles with different electric charge [15]. In this paper, we are concerned with the superselection rule which forbids quantum superposition of states with different numbers of the total number of particles [16,17]. A straightforward testing of Bell's inequality may not be possible for systems of indistinguishable particles since rotations away from fixed particle-number bases may not be allowed due to the restriction imposed by a superselection rule on indistinguishable systems [18–24]. Interestingly, Heaney, Lee, and Jaksch [25] derived a method for testing Bell inequalities for states of indistinguishable particles even within the superselection-rule restrictions, using which the above obstacles can be removed. In order to overcome the superselection rule, they use two copies of the system for performing rotated measurements. Another method that has been proposed to overcome the difficulty posed by the superselection rule uses interaction with a local reservoir and a subsequent local postselection [26–28]. As mentioned in Ref. [25], this group of experiments and proposals require postselection, and thereby, while very useful for entanglement detection, are not so for the more rigid requirements of Bell and steering inequalities. See also Refs. [29–32]. See Ref. [33] and references therein for a discussion on loopholes in steering inequality violations.

Our goal in this paper is to consider the violation of the steering inequality on quantum systems of indistinguishable particles that are restricted by particle-number superselection rules. We check for violations in noninteracting Bose-Einstein condensate states, "NOON" states, and relative phase eigenstates. We find that compared to the Bell-CHSH inequality violation, the steering inequality violation reaches a higher visibility for admixture with white noise, in certain cases. Just like for the Bell inequality, a maximum violation for the steering inequality is however still less than what is possible without the superselection rule.

The rest of the paper is organized as follows. In Sec. II, we discuss the motivation for using two copies of the system state. In Sec. III, we set the notations corresponding to the steering inequality, and propose to use it for two copies of bimodal states. In Sec. IV, we test this inequality for noninteracting Bose-Einstein-condensate states and compare their violations with Bell inequality violations for the same states. In Secs. V and VI, we consider the same test for NOON states and relative phase eigenstates respectively. In Secs. VII and VIII, we briefly mention the implications of the findings for white noise in the environment, and the generalization of the results to the case of massless bosons like photons. Finally, in Sec. IX we conclude our findings.

II. MOTIVATION FOR USING TWO COPIES OF THE SYSTEM STATE

In Ref. [25], it has been demonstrated that if a single copy of a massive particle is used, measurement in particle-number basis may not distinguish separable states from entangled ones, where entanglement and separability are considered among different modes. This can be seen by considering a single copy of the two-mode system in the state,

$$\rho_{AB} = p|\psi_1^+\rangle\langle\psi_1^+|_{AB} + (1-p)|\psi_1^-\rangle\langle\psi_1^-|_{AB}, \tag{3}$$

where $|\psi_1^\pm\rangle = \frac{1}{\sqrt{2}}(|10\rangle \pm |01\rangle)$. The probability p $(0 \leqslant p \leqslant 1)$ can be used to vary the entanglement content of ρ_{AB}. In particular, it can be seen that for $p = 0$ and 1, ρ_{AB} is maximally entangled, but for $p = 1/2$, it is a separable state. Particle-number measurement is the only valid local measurement due to the restriction imposed by the superselection rule. The two-party correlation function of the corresponding observables is then independent of p. However, this problem of discriminating entangled states from separable ones can be solved by using two copies of the system state: $\sigma = \rho_{AB} \otimes \rho_{CD}$. Here, modes A and C belong to the first party and B and D belong to the second. If each party makes two-mode local measurements such that $A_\phi = |\phi+\rangle\langle\phi+| - |\phi-\rangle\langle\phi-|$ with $|\phi+\rangle = \alpha|10\rangle + \beta e^{i\phi}|01\rangle$ and $|\phi-\rangle = \beta e^{i\phi}|10\rangle - \alpha|01\rangle$ is the measuring observable of the first party and $B_\theta = |\theta+\rangle\langle\theta+| - |\theta-\rangle\langle\theta-|$ with $|\theta+\rangle = \alpha|10\rangle + \beta e^{i\theta}|01\rangle$ and $|\phi-\rangle = \beta e^{i\theta}|10\rangle - \alpha|01\rangle$ is the measuring observable of the second party, then the two-party correlation function for the shared state $\sigma = \rho_{AB} \otimes \rho_{CD}$ and measuring observables A_ϕ and B_θ is obtained as

$$\langle A_i \otimes B_j\rangle = 8(p - 1/2)\alpha^2\beta^2\cos(\phi_i - \theta_j). \tag{4}$$

Since the joint correlation function in Eq. (4) is p dependent, we can say that if two copies of the system state are used, it is—in principle—possible to distinguish separable and entangled states. We will be analyzing a CHSH-type steering inequality, and therefore, the use of two copies of the system state will again be important as in the case of the CHSH-type Bell inequality.

III. STEERING INEQUALITY

Let us consider two systems, which are each split into two spatially nonoverlapping modes. Suppose that N_1 and N_2 are particle numbers of the first and second systems respectively, and let the composite system state be $\sigma^{N_1+N_2} = \rho_{ab}^{N_1} \otimes \rho_{AB}^{N_2}$. Here a and A are two modes of the two systems, controlled by say, Alice, and b and B are the two further modes of the two systems, supervised by say, Bob. Let us now assume that Alice performs a joint measurement on her two modes by using the operator,

$$A(\phi_j) = \sum_{N_A=0}^{N_1+N_2}\sum_{n_c=0}^{N_A}\epsilon(n_c, m_C)(|n_c, m_C\rangle\langle n_c, m_C|)_{cC}, \tag{5}$$

where $m_C = N_A - n_c$ and $\epsilon(n_c, m_C)$ is a weighting coefficient. Similarly as Alice, Bob makes a joint measurement on his two modes, b and B, which are denoted by $B(\theta_k)$. Here $j, k = 1, 2$ denotes the two measurement settings each of Alice and Bob. Measurements by Alice and Bob on their respective modes only allow one to perform local particle-number measurements. However, in order to perform general measurements, the spatial modes of both systems are separately allowed to pass through separate beam splitters. For Alice, the beam-splitter transformation is given by

$$\hat{c} = \alpha\hat{a} + \beta\exp(-i\phi_j)\hat{A}, \;\; \hat{C} = \beta\hat{a} - \alpha\exp(-i\phi_j)\hat{A}, \tag{6}$$

where $\hat{a}$ and $\hat{A}$ are annihilation operators corresponding to the two input modes of Alice's beam splitter, and $\hat{c}$ and $\hat{C}$ are annihilation operators for the two output modes of the same. A similar transformation is true on Bob's side:

$$\hat{d} = \alpha\hat{b} + \beta\exp(-i\theta_k)\hat{B}, \;\; \hat{D} = \beta\hat{b} - \alpha\exp(-i\theta_k)\hat{B}. \tag{7}$$

Here $\hat{b}$ and $\hat{B}$ are annihilation operators for the two input modes of Bob's beam splitter and $\hat{d}$ and $\hat{D}$ are annihilation operators for the two output modes of the same. It is noted that each party measures in a particle-number basis. The output of the measurement depends on the local angles, ϕ_j and θ_k, of Alice's and Bob's beam-splitter settings respectively. The measurement vector $|n_c, m_C\rangle$ associated with the observable $A(\phi_j)$ for the particle numbers n_c and m_C in the output modes c and C is given by

$$|n_c, m_C\rangle_{cC} = \frac{(\alpha\hat{a}^\dagger + \beta e^{-i\phi_j}\hat{A}^\dagger)^{n_c}}{\sqrt{n_c!}}\frac{(\beta\hat{a} - \alpha e^{-i\phi^j}\hat{A})^{m_C}}{\sqrt{m_C!}}|0, 0\rangle_{aA}, \tag{8}$$

where $|0, 0\rangle_{aA}$ is the vacuum state corresponding to the modes a and A. We now set the weighting coefficient $\epsilon(n_c, m_c)$ of $A(\phi_j)$ as [25]

$$\epsilon(n_c, m_C) = (-1)^{m_C + \frac{(m_C+n_c)(m_C+n_c+1)}{2}}. \tag{9}$$

Since in the composite system, there are $N_1 + N_2$ particles, the number of outcomes of the measurements is

$$O = \left[\tfrac{1}{2}(N_1 + N_2) + 1\right](N_1 + N_2 + 1). \tag{10}$$

The Bell-CHSH inequality proposed in Ref. [25] for pairs of states restricted with the particle-number superselection rule is given by

$$|B_p| \equiv |\langle A(\phi_1)\otimes B(\theta_1)\rangle + \langle A(\phi_1)\otimes B(\theta_2)\rangle + \langle A(\phi_2)\otimes B(\theta_1)\rangle - \langle A(\phi_2)\otimes B(\theta_2)\rangle| \leqslant 2, \quad (11)$$

where the correlation between the observables is defined as

$$\langle A(\phi_j)\otimes B(\theta_k)\rangle = \sum_{n_c+m_C+n_d+m_D=N_1+N_2} \epsilon(n_c, m_C)\times \times\, \epsilon(n_d, m_D) P(A(\phi_j)\otimes B(\theta_k)), \quad (12)$$

with $P(A(\phi_j)\otimes B(\theta_k))$ denoting the joint probability of outcomes of measurement of the local observables $A(\phi_j)$ and $B(\theta_k)$. Similar to how this Bell inequality was formed using local observables $A(\phi_j)$ and $B(\theta_k)$, we can write the CHSH-type steering inequality in Eq. (2) using the same local observables. Each observable on Alice's or Bob's side is bounded: $\| A(\phi_j) \| \leqslant 1$, $\| B(\theta_k) \| \leqslant 1$. Therefore, the steering inequality which bypasses the superselection rule can be formulated as

$$\begin{aligned} S_p \equiv &\{\langle [A(\phi_1)+A(\phi_2)]\otimes B(\theta_1)\rangle^2 \\ &+ \langle [A(\phi_1)+A(\phi_2)]\otimes B(\theta_2)\rangle^2\}^{1/2} \\ &+ (\langle [A(\phi_1)-A(\phi_2)]\otimes B(\theta_1)\rangle^2 \\ &+ \langle [A(\phi_1)-A(\phi_2)]\otimes B(\theta_2)\rangle^2\}^{1/2} \leqslant 2. \end{aligned} \quad (13)$$

In the three succeeding sections, we apply this steering inequality to the cases of noninteracting Bose-Einstein condensate states, the NOON states, and relative phase eigenstates, known to be useful in precision measurements. In order to compare the violation of the steering inequality with that of the Bell inequality for the states, we first recapitulate in each case the known results for the Bell inequality in (11).

IV. NONINTERACTING BOSE-EINSTEIN CONDENSATE STATES

In order to test the steering inequality for noninteracting Bose-Einstein condensate states, we have considered two different cases.

A. Case i: Both systems share the same number of particles, i.e., $N_1 = N_2 = N$

Consider the zero-temperature noninteracting Bose-Einstein condensate state with a fixed particle number, symmetrically distributed between two modes:

$$|\psi_N\rangle = \frac{1}{\sqrt{2}^N}\sum_{n=0}^{N}\frac{\sqrt{N!}}{\sqrt{n!(N-n)!}}|n, N-n\rangle. \quad (14)$$

If the number of particles of each system is unity ($N = 1$), then $|\psi_N\rangle$ reduces to $|\psi_1\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$, and the composite system is in the state

$$|\psi_1'\rangle = \tfrac{1}{2}(|10\rangle + |01\rangle)\otimes(|10\rangle + |01\rangle). \quad (15)$$

Since the composite system has a total of two particles ($N_1 + N_2 = 2N = 2$), the number of measurement outcomes is 6 [see Eq. (10)]. In this case, the beam splitter is balanced, i.e., $\alpha = \beta = 1/\sqrt{2}$, and the six measurement basis vectors of $A(\phi_1)$ acting on modes a and A with weighting coefficients $\epsilon(n_c, m_C)$ on Alice's side are given in Table I.

TABLE I. The information contained here was given in Ref. [25]. When we make a measurement on the modes c and C on Alice's side, in the particle-number basis, an effective measurement is carried out on Alice's side in the modes a and A. For the case when $N = 1$, this effective measurement basis is given in the middle column. The corresponding $\epsilon(n_c, m_C)$ is given in the last column.

$\vert nm\rangle_{c,C}$	Effective measurement basis for Alice's modes a and A	$\epsilon(n_c, m_C)$
$\vert 00\rangle$	$\vert 00\rangle$	1
$\vert 10\rangle$	$\frac{1}{\sqrt{2}}[\vert 10\rangle + \exp(-i\phi_1)\vert 01\rangle]$	−1
$\vert 01\rangle$	$\frac{1}{\sqrt{2}}[\vert 10\rangle - \exp(-i\phi_1)\vert 01\rangle]$	1
$\vert 11\rangle$	$\frac{1}{\sqrt{2}}[\vert 20\rangle - \exp(-2i\phi_1)\vert 02\rangle]$	1
$\vert 20\rangle$	$\frac{1}{2}[\vert 20\rangle + \sqrt{2}\exp(-i\phi_1)\vert 11\rangle + \exp(-2i\phi_1)\vert 02\rangle]$	−1
$\vert 02\rangle$	$\frac{1}{2}[\vert 20\rangle - \sqrt{2}\exp(-i\phi_1)\vert 11\rangle + \exp(-2i\phi_1)\vert 02\rangle]$	−1

The six measurement basis vectors on the modes b and B of Bob's side can be similarly obtained. Now, using these basis vectors of the measurement settings of Alice and Bob for the system state $|\psi_1'\rangle$, and the correlation function $\langle A(\phi_j)\otimes B(\theta_k)\rangle = \sin^2(\frac{\phi_j-\theta_k}{2})$, the steering inequality expression is given by

$$\begin{aligned} S_{p,1}^{\rm BEC} = \frac{1}{2}\Big(&\{[\cos(\theta_1-\phi_1)+\cos(\theta_1-\phi_2)-2]^2 \\ &+[\cos(\theta_2-\phi_1)+\cos(\theta_2-\phi_2)-2]^2\}^{1/2} \\ &+\sqrt{2}\Big[\sin^2\Big(\frac{\phi_1-\phi_2}{2}\Big)[2-\cos(2\theta_1-\phi_1-\phi_2) \\ &-\cos(2\theta_2-\phi_1-\phi_2)]\Big]^{1/2}\Big). \end{aligned} \quad (16)$$

The maximum value of $S_{p,1}^{\rm BEC}$ is ≈2.79, and is reached at $\phi_1 = 0$, $\phi_2 = \pi/2$, $\theta_1 \approx 3.93$, and $\theta_2 \approx 2.90$. The quantum-mechanical expression for Bell-CHSH expression from (11) for the same quantum state and the same operators is given by

$$\begin{aligned} B_{p,1}^{\rm BEC} = \tfrac{1}{2}[&-\cos(\theta_1-\phi_1)-\cos(\theta_1-\phi_2) \\ &-\cos(\theta_2-\phi_1)+\cos(\theta_2-\phi_2)+2]. \end{aligned} \quad (17)$$

The maximum value of $|B_{p,1}^{\rm BEC}|$ is ≈2.41, and is reached at $\phi_1 = 0$, $\phi_2 = \pi/2$, $\theta_1 \approx 3.93$, and $\theta_2 \approx 2.36$, as obtained in Ref. [25]. We therefore have reached the following two observations for the state under consideration.

(1) Constraints of the particle-number superselection rule on measurement space prevent the maximum violation from reaching $2\sqrt{2}$ for both Bell as well as steering inequality.

(2) Optimal violation of steering inequality is higher than that for the Bell inequality, and therefore the steering inequality violation will have greater visibility than the Bell inequality violation for admixture with white noise, at least in cases where the observables involved are traceless.

Admixing with white noise for a state $|\psi\rangle$ is defined as the creation of the state $p|\psi\rangle\langle\psi| + (1-p)\rho_w$, where $(1-p)\in[0,1]$ is the admixing probability (of the white noise) and ρ_w is the completely depolarized state of the relevant dimension. We refer the reader to Ref. [39] for a discussion

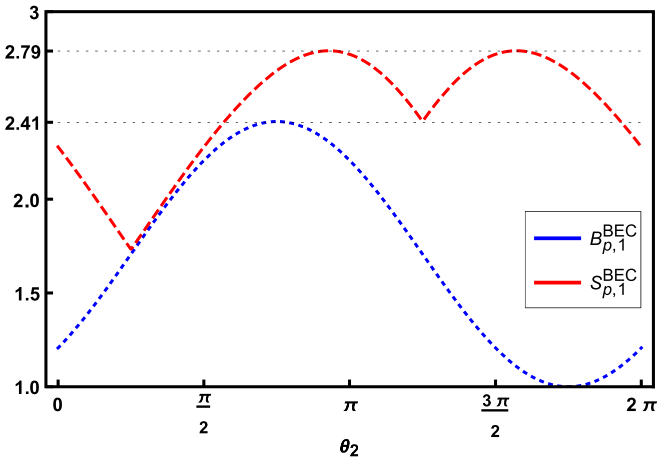

FIG. 1. Comparing steering inequality violation with Bell inequality violation. The quantities $S_{p,1}^{\rm BEC}$ and $|B_{p,1}^{\rm BEC}|$ are plotted with respect to the measurement parameter, θ_2, at $\phi_1 = 0, \phi_2 = \pi/2, \theta_1 = 3.93$. The state involved is given by two copies of the state given in Eq. (14), for $N = 1$. The horizontal axis is in radians, while the vertical one is dimensionless.

on this point. For usage of this noise model, one may refer, e.g., to Refs. [40–42]. The claim made in item 2 above about visibility holds if $A_i \otimes B_j$ for $i, j = 1, 2$ and $(A_1 \pm A_2) \otimes B_k$ for $k = 1, 2$ are traceless. This is true, e.g., for the case when either N_1 or N_2 is unity.

In Fig. 1, we have plotted $S_{p,1}^{\rm BEC}$ and $|B_{p,1}^{\rm BEC}|$ at $\phi_1 = 0, \phi_2 = \pi/2, \theta_1 = 3.93$ with respect to θ_2. From the figure, we can see that there are two maxima of steering inequality violation in the region $\theta_2 \in [0, 2\pi)$ compared to a single maximum for violation of Bell inequality. This can potentially be useful in an actual implementation to check for violations of the steering inequality.

Next, let us consider two particles in each system, i.e., $N_1 = N_2 = N = 2$. Then, $|\psi_N\rangle$ reduces to $|\psi_2\rangle = \frac{1}{2}(|02\rangle + \sqrt{2}|11\rangle + |20\rangle)$, and the composite state of the two systems is

$$|\psi_2'\rangle = \left(\tfrac{1}{2}(|02\rangle + \sqrt{2}|11\rangle + |20\rangle)\right)^{\otimes 2}. \tag{18}$$

The number of measurement outcomes is now 15 [compare with Eq. (10)]. The 15 measurement basis vectors correspond to 15 effective outcomes of the modes a and A of Alice with weighting coefficient ϵ, as given in Table II.

Similarly, the 15 effective measurement basis vectors on the modes b and B on Bob's side can be obtained. Now, using these measurement settings of Alice and Bob for the composite system state $|\psi_2'\rangle$, the quantum expression of the steering inequality is given by

$$\begin{aligned} S_{p,2}^{\rm BEC} = &\left\{\left[\sin^4\left(\frac{\phi_1-\theta_1}{2}\right) - \sin^4\left(\frac{\phi_2-\theta_1}{2}\right)\right]^2\right.\\ &\left.+\left[\sin^4\left(\frac{\phi_1-\theta_2}{2}\right) - \sin^4\left(\frac{\phi_2-\theta_2}{2}\right)\right]^2\right\}^{1/2}\\ &+\left\{\left[\sin^4\left(\frac{\phi_1-\theta_1}{2}\right) + \sin^4\left(\frac{\phi_2-\theta_1}{2}\right)\right]^2\right.\\ &\left.+\left[\sin^4\left(\frac{\phi_1-\theta_2}{2}\right) + \sin^4\left(\frac{\phi_2-\theta_2}{2}\right)\right]^2\right\}^{1/2}. \end{aligned} \tag{19}$$

The maximum value of $S_{p,2}^{\rm BEC}$ is $\approx$2.78 at $\phi_1 = 0, \phi_2 \approx 1.07, \theta_1 \approx 3.93$, and $\theta_2 \approx 3.00$. For the same state and the same measurement settings, the Bell operator [see (11)] has

TABLE II. The considerations are the same as in Table I, except that we have $N_1 = N_2 = 2$ here.

$\vert nm\rangle_{c,C}$	Effective measurement basis for Alice's modes a and A	$\epsilon(n_c, m_C)$
$\vert 00\rangle$	$\vert 00\rangle$	1
$\vert 10\rangle$	$\frac{1}{\sqrt{2}}[\vert 10\rangle + \exp(-i\phi_1)\vert 01\rangle]$	−1
$\vert 01\rangle$	$\frac{1}{\sqrt{2}}[\vert 10\rangle - \exp(-i\phi_1)\vert 01\rangle]$	1
$\vert 11\rangle$	$\frac{1}{\sqrt{2}}[\vert 20\rangle - \exp(-2i\phi_1)\vert 02\rangle]$	1
$\vert 20\rangle$	$\frac{1}{2}[\vert 20\rangle + \sqrt{2}\exp(-i\phi_1)\vert 11\rangle + \exp(-2i\phi_1)\vert 02\rangle]$	−1
$\vert 02\rangle$	$\frac{1}{2}[\vert 20\rangle - \sqrt{2}\exp(-i\phi_1)\vert 11\rangle + \exp(-2i\phi_1)\vert 02\rangle]$	−1
$\vert 12\rangle$	$\frac{1}{4}[\vert 30\rangle - \sqrt{2}\exp(-i\phi_1)\vert 21\rangle - \sqrt{2}\exp(-2i\phi_1)\vert 12\rangle + \exp(-3i\phi_1)\vert 03\rangle]$	1
$\vert 21\rangle$	$\frac{1}{4}[\vert 30\rangle + \sqrt{2}\exp(-i\phi_1)\vert 21\rangle - \sqrt{2}\exp(-2i\phi_1)\vert 12\rangle - \exp(-3i\phi_1)\vert 03\rangle]$	−1
$\vert 03\rangle$	$\frac{1}{4\sqrt{3}}[\vert 30\rangle - 3\exp(-i\phi_1)\vert 21\rangle + 3\exp(-2i\phi_1)\vert 12\rangle - \exp(-3i\phi_1)\vert 03\rangle]$	−1
$\vert 30\rangle$	$\frac{1}{4\sqrt{3}}[\vert 30\rangle + 3\exp(-i\phi_1)\vert 21\rangle + 3\exp(-2i\phi_1)\vert 12\rangle + \exp(-3i\phi_1)\vert 03\rangle]$	1
$\vert 22\rangle$	$\frac{1}{8}[\vert 40\rangle - 2\exp(-2i\phi_1)\vert 22\rangle + \exp(-4i\phi_1)\vert 04\rangle]$	1
$\vert 40\rangle$	$\frac{1}{8\sqrt{6}}[\vert 40\rangle + 4\exp(-i\phi_1)\vert 31\rangle + 6\exp(-2i\phi_1)\vert 22\rangle + 4\exp(-3i\phi_1)\vert 13\rangle + \exp(-4i\phi_1)\vert 04\rangle]$	1
$\vert 04\rangle$	$\frac{1}{8\sqrt{6}}[\vert 40\rangle - 4\exp(-i\phi_1)\vert 31\rangle + 6\exp(-2i\phi_1)\vert 22\rangle - 4\exp(-3i\phi_1)\vert 13\rangle + \exp(-4i\phi_1)\vert 04\rangle]$	1
$\vert 13\rangle$	$\frac{1}{4\sqrt{6}}[\vert 40\rangle - 2\exp(-i\phi_1)\vert 31\rangle + 2\exp(-3i\phi_1)\vert 13\rangle - \exp(-4i\phi_1)\vert 04\rangle]$	−1
$\vert 31\rangle$	$\frac{1}{4\sqrt{6}}[\vert 40\rangle + 2\exp(-i\phi_1)\vert 31\rangle - 2\exp(-3i\phi_1)\vert 13\rangle - \exp(-4i\phi_1)\vert 04\rangle]$	−1

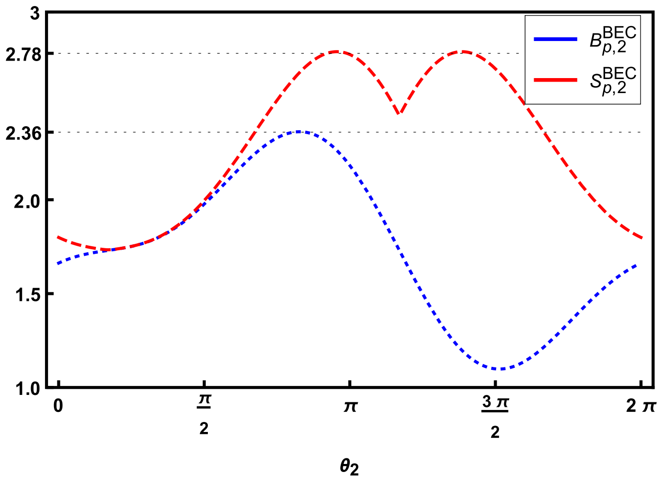

FIG. 2. The considerations are the same as in Fig. 1, except that $N_1 = N_2 = 2$ here. Also, the Bell inequality curve is plotted for $\theta_1 = 3.68$, while the steering inequality one is plotted for $\theta_1 = 3.93$. $\phi_1 = 0$, $\phi_2 = 1.07$ for both the curves.

the quantum mechanical average value of

$$B_{p,2}^{\rm BEC} = \sin^4\left(\frac{\phi_1-\theta_1}{2}\right) + \sin^4\left(\frac{\phi_2-\theta_1}{2}\right) + \sin^4\left(\frac{\phi_1-\theta_2}{2}\right) - \sin^4\left(\frac{\phi_2-\theta_2}{2}\right). \quad (20)$$

The maximum value of $|B_{p,2}^{\rm BEC}|$ is $\approx$2.36 and is reached at $\phi_1 = 0, \phi_2 \approx 1.07, \theta_1 \approx 3.68$, and $\theta_2 \approx 2.60$. Comparing with the $N_1 = N_2 = 1$ case, we find that the quantum violation has decreased with increasing particle number for both the steering as well as the Bell inequality. However, this decrease in violation for the steering inequality is lower than that for the Bell inequality. In Fig. 2 we have plotted $S_{p,2}^{\rm BEC}$ and $|B_{p,2}^{\rm BEC}|$ at $\phi_1 = 0, \phi_2 = 1.07$, with respect to θ_2. We have chosen $\theta_1 = 3.93$ for the former, while $\theta_1 = 3.68$ for the latter. We see in Fig. 2 that similar to the case when $N_1 = N_2 = 1$, there are two maxima of the steering inequality violation in the θ_2-parameter space, while there is a single maximum for the Bell inequality violation in the same space. From the above study, we can say that for the cases studied, as the number of particles in each system increases equally, the maximum violations of both Bell and steering inequalities of CHSH type decrease. However, the decrease in violation of CHSH-type steering inequality is less than that of the Bell-CHSH inequality. Hence, for $N > 2$, there can be a situation in which there is a violation of steering inequality but no violation of Bell inequality, for the types considered. As the number of particles in each system increases, the number of measurement basis elements of the observable also increases, which makes the correlation between their observables complicated, and we were able to study up to only two particles in each system.

B. Case ii: Both systems share different numbers of particles, i.e., $N_1 \neq N_2$

The steering inequality is now to be checked for the state $|\psi_{N_1}\rangle \otimes |\psi_{N_2}\rangle$. In the case when $N_1 = 1$ and $N_2 = 2$, the composite state is

$$|\psi_{1,2}\rangle = \frac{1}{2\sqrt{2}}(|10\rangle + |01\rangle) \otimes (|02\rangle + \sqrt{2}|11\rangle + |20\rangle). \quad (21)$$

The number of measurement outcomes is now 10 [compare with Eq. (10)]. These can be read off from the first ten rows in Table II. If the beam splitter is balanced, i.e., if $\alpha = \beta = 1/\sqrt{2}$, all correlation functions of the steering inequality vanish. Next, we found that for any unequal number of particles in each system ($N_1 \neq N_2$), and balanced beam splitters, the correlation functions of the steering inequality vanish. However, for unbalanced beam splitters, the correlation functions are not vanishing, but still no violation of the steering inequality is obtained for the cases considered.

V. NOON STATES

Consider the two-mode state,

$$|\{N, m\}\rangle = \frac{1}{\sqrt{2}}(|N-m, m\rangle + |m, N-m\rangle), \quad (22)$$

and for definiteness, suppose that $N = 2$ and $m = 0$. Suppose also that the composite state of Alice and Bob is now

$$|\psi_3'\rangle = |\{2,0\}\rangle^{\otimes 2} = \left(\frac{1}{\sqrt{2}}(|2,0\rangle + |0,2\rangle)\right)^{\otimes 2}. \quad (23)$$

For this state, the correlation function between the Alice and Bob observables is given by $\langle A(\phi_j) \otimes B(\theta_k)\rangle = \cos^2(\phi_j - \theta_k)$. It can be seen that by scaling $\phi_j \rightarrow \frac{\phi_j}{2}$ and $\theta_k \rightarrow \frac{\theta_k}{2} - \frac{\pi}{2}$ in $\cos^2(\phi_j - \theta_k)$, we obtain the same correlation function between observables of Alice and Bob as obtained for noninteracting Bose-Einstein condensate states with two particles in the composite system. Hence, in the case of the NOON state represented by $|\psi_3'\rangle$, the same amount of maximum quantum violation of both steering and Bell inequalities can be obtained as for the case of the noninteracting Bose-Einstein condensate state with a single particle in each system.

VI. RELATIVE PHASE EIGENSTATE

Let us now consider the relative phase eigenstate for a two-mode system [34,35], given by

$$\left|\frac{N}{2}, \theta_p\right\rangle = \frac{1}{\sqrt{N+1}} \sum_{l=-N/2}^{l=N/2} \exp(il\theta_p) \frac{(\hat{a}^\dagger)^{N/2-l}}{\sqrt{(N/2-l)!}} \times \frac{(\hat{b}^\dagger)^{N/2+l}}{\sqrt{(N/2+l)!}} |0,0\rangle.$$

It is a pure entangled state of N bosons, where $\theta_p = p[2\pi/(N+1)]$ with $p = -N/2, -N/2+1, \ldots, +N/2$ is the relative phase [36]. The relative phase eigenstate is an example of a spin-squeezed state [35], and is consistent with the superselection rule. When $N = 2$, the relative phase state given in Eq. (24) reduces to

$$|1,\theta_p\rangle = \frac{1}{\sqrt{3}}\sum_{l=-1}^{l=1}\exp(il\theta_p)\frac{(\hat{a}^\dagger)^{1-l}}{\sqrt{(1-l)!}}\frac{(\hat{b}^\dagger)^{1+l}}{\sqrt{(1+l)!}}|0,0\rangle$$
$$= \frac{1}{\sqrt{6}}[\exp(-i\theta_p)|0,2\rangle + \exp(i\theta_p)|2,0\rangle]. \quad (24)$$

Since the number of bosons in each system is 2, the number of measurement outcomes is 15 [see Eq. (10)]. Using the 15 measurement basis outcomes corresponding to the 15 outcomes given in Table II, for modes a and A of Alice, and a similar 15 measurement basis outcomes of the modes b and B of Bob, for two copies of the relative phase eigenstate, viz. for $|1,\theta_p\rangle^{\otimes 2}$, the correlation between the Alice and Bob observables is obtained as $\langle A(\phi_j)\otimes B(\theta_k)\rangle = \cos^2(\phi_j-\theta_k)$. Note that the correlation obtained between the Alice and Bob observables for the relative phase eigenstate is phase independent, as also obtained for the NOON state. Hence, when $N=2$, the quantum value of the CHSH type of steering and Bell inequality expressions for the relative phase eigenstate is equal to that for the NOON state.

Another phase-dependent state, introduced in Refs. [37,38], is given by

$$|\omega,\chi\rangle = \frac{1}{\sqrt{N!}}[\cos\omega\exp(-i\chi/2)\hat{a}^\dagger + \sin\omega\exp(i\chi/2)\hat{b}^\dagger]^N|0,0\rangle. \quad (25)$$

When $N=1$, $|\omega,\chi\rangle$ reduces to

$$|\omega,\chi\rangle' = \cos\omega\exp(-i\chi/2)|1,0\rangle + \sin\omega\exp(i\chi/2)|0,1\rangle. \quad (26)$$

At $\omega=\pi/4$ and $\chi=0$, the composite system state $|\omega,\chi\rangle'$ reduces to the noninteracting Bose-Einstein condensate state $|\psi_1'\rangle$ given in Eq. (15).

When $N=1$, the number of bosons in the composite system state $|\omega,\chi\rangle'^{\otimes 2}$ is 2 and the number of measurement outcomes is 6. Now, using measurement basis corresponding to the six outcomes of Alice given in Table I, and similarly of Bob, for $|\omega,\chi\rangle'^{\otimes 2}$ in Eq. (26), the quantum-mechanical expressions corresponding to the CHSH-type steering and Bell inequality expressions are given by

$$\begin{aligned}S^{ps}_{p,1} = 2\Big[&(\{\sin^4\omega+\cos^4\omega\\ &-\sin^2\omega\cos^2\omega[\cos(\theta_1-\phi_1)+\cos(\theta_1-\phi_2)]\}^2\\ &+\{\sin^4\omega+\cos^4\omega\\ &-\sin^2\omega\cos^2\omega[\cos(\theta_2-\phi_1)+\cos(\theta_2-\phi_2)]\}^2)^{1/2}\\ &+2\sqrt{2}\sin^2\omega\cos^2\omega\left|\sin\left(\frac{\phi_1-\phi_2}{2}\right)\right|\\ &\times[\cos(2\theta_1+\phi_1-\phi_2)-1+\cos(2\theta_2-\phi_1-\phi_2)]^{1/2}\Big]\end{aligned} \quad (27)$$

and

$$\begin{aligned}B^{ps}_{p,1} = 2\{&\sin^4\omega+\cos^4\omega-\sin^2\omega\cos^2\omega\\ &\times[\cos(\theta_1-\phi_1)+\cos(\theta_1-\phi_2)+\cos(\theta_2-\phi_1)\\ &-\cos(\theta_2-\phi_2)]\}\end{aligned} \quad (28)$$

respectively. Note that at $\omega=\pi/4$, the state in Eq. (26) is maximally entangled, and the maximum quantum value of $S^{ps}_{p,1}$ is approximately 2.78, obtained at $\omega=\pi/4$ for $\phi_1=0$, $\phi_2\approx 1.07$, $\theta_1\approx 3.93$, and $\theta_2\approx 3.00$. The Bell inequality is maximally violated at $\omega=\pi/4$. The maximum quantum value of $B^{ps}_{p,1}$ is $\approx$2.36 and is reached at $\omega=\pi/4$, $\phi_1=0$, $\phi_2\approx 1.07$, $\theta_1\approx 3.68$, and $\theta_2\approx 2.60$. For the values of ω other than $\pi/4$, entanglement of the phase state decreases. For example, at $\omega=\pi/3$, a maximum quantum value of $S^{ps}_{p,1}\approx 2.70$ is obtained for approximately $\phi_1=\pi/12$, $\phi_2=0$, $\theta_1=2.36$, and $\theta_2=3.24$, and of $B^{ps}_{p,1}$ is ≈ 2.2, obtained approximately at $\phi_1=0$, $\phi_2=\pi/2$, $\theta_1=\pi$, and $\theta_2=2.06$.

VII. WHITE NOISE VS VIOLATION OF BELL AND STEERING INEQUALITIES

The higher amounts of violations obtained for steering inequalities, in comparison to that for Bell inequalities, indicates that at least in the case of white noise (see, e.g., Refs. [40–42]) from environmental effects, the steering inequality violation will be easier to obtain experimentally. A state ϱ_{AB} is said to be affected by white noise if it is transformed as

$$\varrho_{AB}\longrightarrow\varrho^{\gamma}_{AB}\equiv\gamma\varrho_{AB}+(1-\gamma)I_{AB}/d, \quad (29)$$

where the mixing probability of white noise, $1-\gamma$, lies in [0,1], and I_{AB} is the identity operator on the Hilbert space on which ϱ_{AB} is defined, and d is the dimension of that joint Hilbert space. See, e.g., Refs. [39–42] for discussions on this noise model. Suppose that state ϱ_{AB} violates a Bell inequality $|B|\leqslant 2$ [see Eq. (1)] such that

$$|B_\varrho| = 2+\mathcal{B}, \quad (30)$$

with $\mathcal{B}\in(2,2\sqrt{2}-2]$. Then, corresponding to the white-noise-affected state ϱ^{γ}_{AB}, we have

$$|B_{\varrho^\gamma}| = \gamma(2+\mathcal{B}), \quad (31)$$

provided the observables used for the Bell experiment are traceless. Therefore, $|B_{\varrho^\gamma}|$ is still greater than 2, if

$$\gamma > 2/(2+\mathcal{B}). \quad (32)$$

The situation is similar for the steering inequality in Eq. (2), and in particular, we again have

$$S_{\varrho^\gamma} = \gamma S_\varrho. \quad (33)$$

Therefore, in either case, a higher violation in the noiseless case implies that the noisy state will still violate for a higher noise strength, with the noise strength being quantified by $1-\gamma$.

VIII. MASSLESS BOSONS

Our results can also be generalized to massless bosons like photons, regardless of the fact that they obey the local particle-number superselection rule.

The particle-number superselection rule in case of massive bosons applies at the global level. The state of a two-mode bosonic system of N bosons is represented by

$$|\Psi_N\rangle = \sum_{n=0}^{N} C(N,n)|n\rangle_A|N-n\rangle_B, \tag{34}$$

where n bosons are associated with mode A and $N-n$ with mode B, and the global particle-number superselection rule is active in that the same number of bosons (N) is present in each term of the superposition. However, the situation is different in the case of (massless) photons. For example, the Bell states, $|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A|1\rangle_B + |1\rangle_A|0\rangle_B)$ and $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A|1\rangle_B - |1\rangle_A|0\rangle_B)$ for $N=1$ [and $C(N,n) = \frac{1}{\sqrt{2}}$], can be states of a massive bosonic system obeying the global particle-number superselection rule, but the other two Bell states, $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B)$ and $|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B - |1\rangle_A|1\rangle_B)$, follow the local particle-number superselection rule instead of global. This local particle-number superselection rule is applicable for (massless) photons. Photons are associated with the local particle-number superselection rule in the sense that observers cannot prepare states which involve a superposition of different photon numbers [43,44].

Loophole-free tests of steering inequalities with mode-entangled states of photons have been demonstrated in Refs. [45,46]. In the case of the setup used in this paper, the entangled state shared between untrusted Alice and trusted Bob can be generated by sending two lots of N photons through a balanced beam splitter such that one output is in possession of Alice and the other of Bob. Similarly, as mentioned in Ref. [25], the output ports of Alice and Bob can again be passed through another beam splitter of suitable reflectivity. The photon number of each party belongs to the final output port of the second beam splitter. In order to get the joint correlations in steering inequality, Alice does the local measurement on her part, and Bob performs local measurements on his part without trusting the measurement performed by Alice.

IX. CONCLUSIONS

In summary, we have checked for violation of a steering inequality for quantum states of systems of indistinguishable particles that are restricted by a particle-number superselection rule. In order to bypass the superselection rule, we have formulated a steering inequality, where measurements on two copies of the system state are considered, a strategy that had previously been applied for Bell inequality violation. We found that the restriction of the superselection rule on the measurement space prevents the violation of the steering inequality from reaching its maximal quantum value. The violation was checked for noninteracting Bose-Einstein condensate states, for which we also found that as the particle number increases, the violation of the steering inequality decreases. We also checked for violations in NOON states and relative phase eigenstates. For all considered states, steering inequality violations are higher than Bell inequality violations, implying that the former will have a higher visibility against the admixture of white noise, in certain situations.

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[1] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009); O. Gühne and G. Tóth, Phys. Rep. **474**, 1 (2009); S. Das, T. Chanda, M. Lewenstein, A. Sanpera, A. Sen(De), and U. Sen, The separability versus entanglement problem, in *Quantum Information*, edited by D. Bruß and G. Leuchs (Wiley, New York, 2019), Chap. 8.

[2] J. S. Bell, Phys. Phys. Fiz. **1**, 195 (1964).

[3] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Phys. Rev. Lett. **23**, 880 (1969).

[4] B. S. Cirel'son, Lett. Math. Phys. **4**, 93 (1980).

[5] E. Schrödinger, Naturwissenschaften **23**, 807 (1935).

[6] M. T. Quintino, T. Vértesi, D. Cavalcanti, R. Augusiak, M. Demianowicz, A. Acín, and N. Brunner, Phys. Rev. A **92**, 032107 (2015).

[7] M. D. Reid, Phys. Rev. A **40**, 913 (1989).

[8] E. G. Cavalcanti, S. J. Jones, H. M. Wiseman, and M. D. Reid, Phys. Rev. A **80**, 032112 (2009).

[9] Z. Y. Ou, S. F. Pereira, H. J. Kimble, and K. C. Peng, Phys. Rev. Lett. **68**, 3663 (1992).

[10] S. P. Walborn, A. Salles, R. M. Gomes, F. Toscano, and P. H. Souto Ribeiro, Phys. Rev. Lett. **106**, 130402 (2011).

[11] J. Schneeloch, C. J. Broadbent, S. P. Walborn, E. G. Cavalcanti, and J. C. Howell, Phys. Rev. A **87**, 062103 (2013).

[12] P. Girdhar and E. G. Cavalcanti, Phys. Rev. A **94**, 032317 (2016).

[13] D. J. Saunders, S. J. Jones, H. M. Wiseman, and G. J. Pryde, Nat. Phys. **6**, 845 (2010).

[14] E. G. Cavalcanti, C. J. Foster, M. Fuwa, and H. M. Wiseman, J. Opt. Soc. Am. B **32**, A74 (2015).

[15] A. Galindo and P. Pascual, *Quantum Mechanics* (Springer-Verlag, Berlin, 1990), Vol. 1.

[16] E. A. Cornell and C. E. Wieman, Rev. Mod. Phys. **74**, 875 (2002); W. Ketterle, *ibid.* **74**, 1131 (2002).

[17] F. Verstraete and J. I. Cirac, Phys. Rev. Lett. **91**, 010404 (2003).

[18] G. C. Wick, A. S. Wightman, and E. P. Wigner, Phys. Rev. **88**, 101 (1952).

[19] H. M. Wiseman and J. A. Vaccaro, Phys. Rev. Lett. **91**, 097902 (2003).

[20] N. Schuch, F. Verstraete, and J. I. Cirac, Phys. Rev. Lett. **92**, 087904 (2004).

[21] S. D. Bartlett, T. Rudolph, and R. W. Spekkens, Rev. Mod. Phys. **79**, 555 (2007).

[22] J. A. Vaccaro, F. Anselmi, H. M. Wiseman, and K. Jacobs, Phys. Rev. A **77**, 032114 (2008).

[23] G. A. White, J. A. Vaccaro, and H. M. Wiseman, Phys. Rev. A **79**, 032109 (2009).

[24] K. Nakahira, Phys. Rev. A **101**, 022104 (2020).
[25] L. Heaney, S.-W. Lee, and D. Jaksch, Phys. Rev. A **82**, 042116 (2010).
[26] S. M. Tan, D. F. Walls, and M. J. Collett, Phys. Rev. Lett. **66**, 252 (1991).
[27] B. Hessmo, P. Usachev, H. Heydari, and G. Björk, Phys. Rev. Lett. **92**, 180401 (2004).
[28] L. Heaney and J. Anders, Phys. Rev. A **80**, 032104 (2009).
[29] M. S. Williamson, L. Heaney, and W. Son, Phys. Rev. A **82**, 032105 (2010).
[30] Y.-C. Ou and M. S. Byrd, Quantum Inf. Comput. **10**, 0223 (2010).
[31] S. Ashhab, K. Maruyama, and F. Nori, Phys. Rev. A **75**, 022108 (2007).
[32] S. Ashhab, K. Maruyama, C. Brukner, and F. Nori, Phys. Rev. A **80**, 062106 (2009).
[33] B. Wittmann, S. Ramelow, F. Steinlechner, N. K. Langford, N. Brunner, H. M. Wiseman, R. Ursin, and A. Zeilinger, New J. Phys. **14**, 053030 (2012).
[34] B. J. Dalton, L. Heaney, J. Goold, B. M. Garraway, and T. Busch, New J. Phys. **16**, 013026 (2014).
[35] B. J. Dalton, J. Goold, B. M. Garraway, and M. D. Reid, Phys. Scr. **92**, 023005 (2017).
[36] D. T. Pegg and S. M. Barnett, Phys. Rev. A **39**, 1665 (1989).
[37] Y. Li, P. Treutlein, J. Reichel, and A. Sinatra, Eur. Phys. J. B **68**, 365 (2009).
[38] B. J. Dalton, arXiv:1108.0756v2.
[39] J. Preskill, Lecture notes on quantum computation, http://theory.caltech.edu/~preskill/ph229/.
[40] A. Sen(De), U. Sen, M. Wieśniak, D. Kaszlikowski, and M. Żukowski, Phys. Rev. A **68**, 062306 (2003).
[41] A. Cabello, A. Feito, and A. Lamas-Linares, Phys. Rev. A **72**, 052112 (2005).
[42] A. Dutta, J. Ryu, W. Laskowski, and M. Żukowski, Phys. Lett. A **380**, 2191 (2016).
[43] S. D. Bartlett, A. C. Doherty, R. W. Spekkens, and H. M. Wiseman, Phys. Rev. A **73**, 022311 (2006).
[44] B. J. Dalton, J. Goold, B. M. Garraway, and M. D. Reid, Phys. Scr. **92**, 023004 (2017).
[45] T. Guerreiro, F. Monteiro, A. Martin, J. B. Brask, T. Vertesi, B. Korzh, M. Caloz, F. Bussieres, V. B. Verma, A. E. Lita, R. P. Mirin, S. W. Nam, F. Marsilli, M. D. Shaw, N. Gisin, N. Brunner, H. Zbinden, and R. T. Thew, Phys. Rev. Lett. **117**, 070404 (2016).
[46] M. Fuwa, S. Takeda, M. Zwierz, H. M. Wiseman, and A. Furusawa, Nat. Commun. **6**, 6665 (2015).