Sharing Bell nonlocality of bipartite high-dimensional pure states using only projective measurements

Tinggui Zhang,[1,*] Hong Yang,[2] and Shao-Ming Fei[3]

[1]*School of Mathematics and Statistics, Hainan Normal University, Haikou 571158, China*
[2]*College of Physics and Electronic Engineering, Hainan Normal University, Haikou 571158, China*
[3]*School of Mathematical Sciences, Capital Normal University, Beijing 100048, China*

(Received 6 September 2023; revised 8 December 2023; accepted 24 January 2024; published 13 February 2024)

Bell nonlocality is the key quantum resource in some device-independent quantum information processing. It is of great importance to study the efficient sharing of this resource. Unsharp measurements are widely used in sharing the nonlocality of an entangled state shared among several sequential observers. Recently, the authors in [Phys. Rev. Lett. **129**, 230402 (2022)] showed that the Bell nonlocality of two-qubit pure states can be shared even when one only uses projective measurements and local randomness. We demonstrate that projective measurements are also sufficient for sharing the Bell nonlocality of arbitrary high-dimensional pure bipartite states. Our results promote further understanding of the nonlocality sharing of high-dimensional quantum states under projective measurements.

DOI: 10.1103/PhysRevA.109.022419

I. INTRODUCTION

Bell nonlocality, revealed by violating the Bell inequalities of quantum entangled states, is one of the most startling predictions of quantum mechanics [1]. It plays an important role in device-independent quantum information processing, such as quantum key distribution [2,3], quantum secure direct communication [4,5], and communication complexity reduction [6,7].

In recent years an interesting question about the shareability of Bell nonlocality has been extensively studied [8–35]. The question is whether the postmeasurement state in a Bell experiment can be reused for showcasing nonlocality between several observers who perform sequential quantum measurements; see Fig. 1(a) for the schematic diagram. In 2015 Silva *et al.* showed that the Bell nonlocality from an entangled pair can be utilized for multiple parties with sequential unsharp measurements of intermediate strength [8]. Since then most of the studies on nonlocality sharing adopt weak measurements or unsharp measurements. In 2020 Brown and Colbeck used average probability positive operator-valued measure (POVM) [19] and showed that arbitrarily many independent Bobs can share the nonlocality of the maximally entangled pure two-qubit state $|\phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ with the single Alice.

Various measurements have been used in demonstrating the Bell nonlocality. Among them the projective measurement is the simplest one. Nevertheless, the projective measurement is also the most destructive one to the quantum states. Generally, an entangled state would become separable after such measurements. Recently, in [30] the authors showed that if the Bobs apply standard projective measurements [a random combination of three projective measurement strategies with different probabilities, see Fig. 1(b)], then two and three Bobs can share the nonlocality of the two-qubit state $|\phi\rangle$ with the single Alice.

The high-dimensional quantum systems can carry more information and are more resistant to noises. High-dimensional quantum systems are important in improving the performance of quantum networks, quantum key distribution, quantum teleportation, and quantum internet [36–39]. Therefore in this article we study the nonlocal correlation sharing scenario for arbitrary high-dimensional bipartite entangled pure states along the line of [30] (Fig. 1). We show that projective measurement is also a sufficient condition for two observers to share the Bell nonlocality of any arbitrary dimensional bipartite entangled pure states.

II. NONLOCAL SHARING OF BIPARTITE HIGH-DIMENSIONAL PURE STATES

Let H_A and H_B be Hilbert spaces with dimensions $\dim(H_A) = s$ and $\dim(H_B) = t$, respectively (without loss of generality, we assume $s \leqslant t$). A bipartite pure state $|\psi\rangle \in H_A \otimes H_B$ has the Schmidt decomposition form $|\psi\rangle = \sum_{i=1}^{s} c_i |i_A\rangle |i_B\rangle$, where $c_i \in [0, 1]$, $\sum_{i=1}^{s} c_i^2 = 1$, and $\{i_A\}_1^s$ and $\{i_B\}_1^t$ are the orthonormal bases of H_A and H_B, respectively. $|\psi\rangle$ is entangled if and only if at least two c_is are nonzero. Without loss of generality, below we assume that c_i are arranged in descending order.

We focus on the sequential scenario shown in Fig. 1. To begin with, Alice and Bob_1 share an arbitrary entangled bipartite pure state $\rho_{AB}^{(1)} = |\psi\rangle\langle\psi|$. Bob_k ($k = 1, 2, \ldots, n$) are restricted to perform two different projective measurement settings: PM(1) where both choose a projection measurement ($\lambda = 1$), and PM(2) where one chooses a projection measurement and the other chooses the identity operator

[*]Corresponding author: tinggui333@163.com

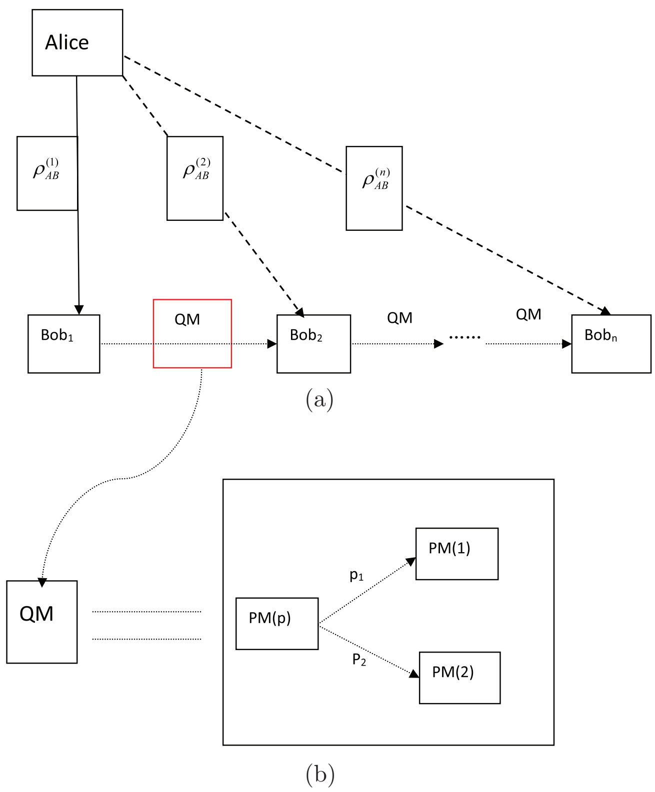

FIG. 1. (a) A quantum state $\rho_{AB}^{(1)}$ is initially shared by Alice and Bob$_{(1)}$. Bob$_{(1)}$ performs some kind of quantum measurements on his part and then passes it to Bob$_{(2)}$. The postmeasurement state is $\rho_{AB}^{(2)}$. Bob$_{(2)}$ measures $\rho_{AB}^{(2)}$ on his part and passes it to Bob$_{(3)}$ and so on. (b) Quantum measurement (QM) given by a random combination of several projective measurements (PMs) with different probabilities p. Before the experiment begins, all the parties agree to share correlated strings of classical data $p = \{p_i\}$ satisfying $\sum_i p_i = 1$.

$(\lambda = 2)$. Denote the binary input and output of Alice (Bob$_k$) by X $(Y^k) \in \{0, 1\}$ and A $(B^{(k)}) \in \{0, 1\}$, respectively. Before the experiment begins, all the parties agree to share the correlated strings of classical data λ subjected to probability distribution $\{p_\lambda\}_{\lambda=1,2}$. Suppose Bob$_1$ performs the measurement according to $Y^1 = y$ with outcome $B^1 = b$. Averaged over the inputs and outputs of Bob$_1$, the un-normalized state shared between Alice and Bob$_2$ is given by

$$\rho_{AB}^{(2,\lambda)} = \frac{1}{2}\Sigma_{b,y}\big(I_s \otimes \sqrt{B_{b|y}^{(1,\lambda)}}\big)\rho_{AB}^{(1)}\big(I_s \otimes \sqrt{B_{b|y}^{(1,\lambda)}}\big), \quad (1)$$

where $B_{b|y}^{(1,\lambda)}$ $(\lambda = 1, 2)$ is the projective measurement $[(B_{b|y}^{(1,\lambda)})^2 = B_{b|y}^{(1,\lambda)}]$, corresponding to outcome b of Bob$^{(1,\lambda)}$'s measurement for input y, and I_s is the $s \times s$ identity matrix. Repeating this process, one gets the state $\rho_{AB}^{(k,\lambda)}$ shared between Alice and Bob$^{(k)}$, $k = 2, 3, \dots, n$.

The Bell nonlocality is verified by the violation of the Clauser-Horne-Shimony-Holt (CHSH) inequality [40]. Each pair Alice-Bob$_k$ tests the CHSH inequality,

$$S_k \equiv \sum_{\lambda=1}^{2} p_\lambda S_k^\lambda \leqslant 2, \quad (2)$$

where

$$S_k^\lambda \equiv \sum_{x,y}(-1)^{xy}\mathrm{Tr}\big(A_x \otimes B_y^{(k,\lambda)}\big)\rho_{AB}^{(k,\lambda)}. \quad (3)$$

Here, $\{A_x, B_y^{(k,\lambda)}\}_{k=1,2,\dots}$ denotes the observables of respective parties conditioned on λ. Only when $k = 1$, $\rho_{AB}^{(1,\lambda)} = \rho_{AB}^{(1)}$.

Let us consider the simplest scenario, namely, $n = 2$ and s, t are even. We set Alice's quantum measurements to be given by the observables

$$A_0 = I_{\frac{s}{2}} \otimes (\cos\theta\sigma_3 + \sin\theta\sigma_1) \quad (4)$$

and

$$A_1 = I_{\frac{s}{2}} \otimes (\cos\theta\sigma_3 - \sin\theta\sigma_1) \quad (5)$$

for some $\theta \in [0, \frac{\pi}{2}]$.

Case (i): $(\lambda = 1)$. We set Bob$_1$'s projective measurements to be given by the observables

$$B_{0|0}^{(1,1)} = \tfrac{1}{2}[I_t + (I_{\frac{t}{2}} \otimes \sigma_3)] \quad (6)$$

and

$$B_{0|1}^{(1,1)} = \tfrac{1}{2}[I_t + (I_{\frac{t}{2}} \otimes \sigma_1)]. \quad (7)$$

Denote $B_{1|y}^{(1,1)} = I_t - B_{0|y}^{(1,1)}$ and $B_y^{(1,1)} = B_{0|y}^{(1,1)} - B_{1|y}^{(1,1)}$ for $y = 0, 1$. Similar to the calculations in Ref. [20], it is not difficult to obtain that $S_1^1 \geqslant 2(\cos\theta + K\sin\theta) := \widehat{S_1^1}$, where $K = 2(c_1c_2 + c_3c_4 + \cdots + c_{s-1}c_s)$, $0 < K \leqslant 1$.

Using Eq. (1) we obtain

$$\begin{aligned}\rho_{AB}^{(2,1)} = {} & \tfrac{1}{2}\rho_{AB}^{(1)} + \tfrac{1}{4}[I_s \otimes (I_{\frac{t}{2}} \otimes \sigma_3)]\rho_{AB}^{(1)}[I_s \otimes (I_{\frac{t}{2}} \otimes \sigma_3)] \\ & + \tfrac{1}{4}[I_s \otimes (I_{\frac{t}{2}} \otimes \sigma_1)]\rho_{AB}^{(1)}[I_s \otimes (I_{\frac{t}{2}} \otimes \sigma_1)]. \end{aligned} \quad (8)$$

Then taking $B_y^{(2,1)} = B_y^{(1,1)}$ for $y = 0, 1$, we get $S_2^1 \geqslant (\cos\theta + K\sin\theta) := \widehat{S_1^1}$. The tradeoff relationship between $\widehat{S_1^1}$ and $\widehat{S_2^1}$ is given by

$$\widehat{S_2^1} = \tfrac{1}{2}\widehat{S_1^1}. \quad (9)$$

When $\theta = \arctan K$, $\widehat{S_1^1}$ attains the maximum value $2\sqrt{1+K^2}$. At this moment $\widehat{S_2^1} = \sqrt{1+K^2}$. Moreover, when $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, that is, $K = 1$, we obtain the same maximum value of $\widehat{S_1^1}$ as in Ref. [30].

Case (ii): $(\lambda = 2)$. We take Bob$_1$'s projective measurements to be

$$B_{0|0}^{(1,2)} = I_t \quad (10)$$

and

$$B_{0|1}^{(1,2)} = \tfrac{1}{2}[I_t + (I_{\frac{t}{2}} \otimes \sigma_1)]. \quad (11)$$

Denote $B_{1|y}^{(1,2)} = I_t - B_{0|y}^{(1,2)}$ and $B_y^{(1,2)} = B_{0|y}^{(1,2)} - B_{1|y}^{(1,2)}$ for $y = 0, 1$. Similarly, we can obtain $S_1^2 \geqslant 2K\sin\theta := \widehat{S_1^2}$. By calculation, we have

$$\rho_{AB}^{(2,2)} = \tfrac{3}{4}\rho_{AB}^{(1)} + \tfrac{1}{4}[I_s \otimes (I_{\frac{t}{2}} \otimes \sigma_1)]\rho_{AB}^{(1)}[I_s \otimes (I_{\frac{t}{2}} \otimes \sigma_1)]. \quad (12)$$

Then take $B_0^{(2,2)} = I_{\frac{t}{2}} \otimes \sigma_3$, $B_1^{(2,2)} = I_{\frac{t}{2}} \otimes \sigma_1$, and we obtain $S_2^2 \geqslant 2K\sin\theta + \cos\theta := \widehat{S_2^2}$. The tradeoff relationship

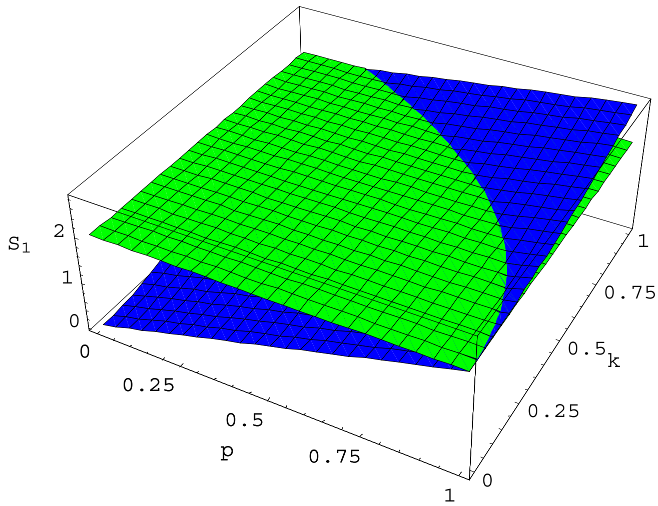

FIG. 2. p and K as parameters of S_1 (blue), contour surface $S_1 = 2$ (green).

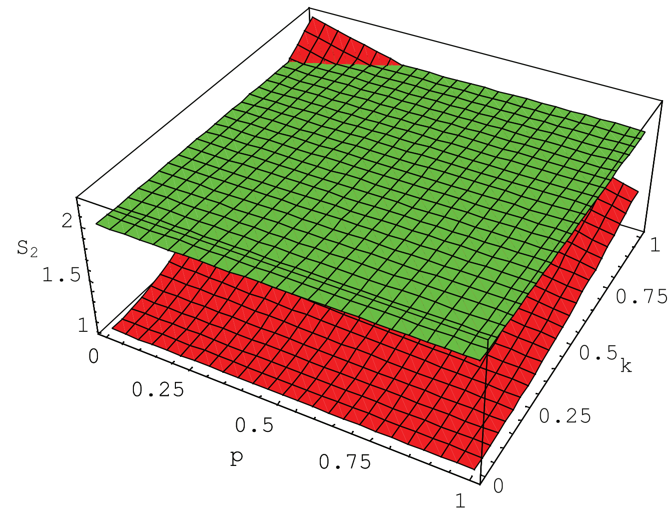

FIG. 3. p and K as parameters of S_2 (red), contour surface $S_2 = 2$ (green).

between $\widehat{S_1^2}$ and $\widehat{S_2^2}$ becomes

$$\widehat{S_2^2} = \widehat{S_1^2} + \frac{1}{2K}\sqrt{4K^2 - \left(\widehat{S_1^2}\right)^2}. \tag{13}$$

When $\theta = \arctan 2K$, $\widehat{S_2^2}$ achieves the maximum value $\sqrt{4K^2+1}$. Meanwhile, $\widehat{S_1^2} = \frac{4K^2}{\sqrt{4K^2+1}}$. In particular, for $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ $(K = 1)$, our the tradeoff relation Eq. (13) gives rise to Eq. (6) of Ref. [30].

Concerning the Bell nonlocality, we assume that the probability of choosing the first (second) measurement is p $(1-p)$. According to the definition (2), it can be seen that $S_1 = pS_1^1 + (1-p)S_1^2 \geqslant p\widehat{S_1^1} + (1-p)\widehat{S_1^2} = 2p\sqrt{1+K^2} + (1-p)\frac{4K^2}{\sqrt{4K^2+1}}$ and $S_2 = pS_2^1 + (1-p)S_2^2 \geqslant p\widehat{S_2^1} + (1-p)\widehat{S_2^2} = p\sqrt{1+K^2} + (1-p)\sqrt{4K^2+1}$. The nonlocality sharing problem is then transformed to find parameters p and K such that S_1 and S_2 are both greater than 2, as long as the right-hand formulas of the two inequalities above are both greater than 2 is sufficient. From Figs. 2 and 3 we see that they can both be greater than 2 for some p and K. For example, when $K = 1$, $p \in [\frac{2\sqrt{5}-4}{2\sqrt{10}-4}, \frac{\sqrt{5}-2}{\sqrt{5}-\sqrt{2}}] \approx [0.203, 0.286]$, where S_1 and S_2 are simultaneously greater than 2. Because S_1 and S_2 are both continuous functions of p and K, there is still a finite domain in which S_1 and S_2 are both greater than 2. This also fully demonstrates that projective measurements are sufficient for sharing Bell nonlocality for bipartite high-dimension pure states.

When s and t are odd numbers, we only need to take the following measurement operators and follow the calculation method in Ref. [20] to obtain the same conclusion as when s is even, except that the expression of K is changed to be $2(c_1c_2 + c_3c_4 + \cdots + c_{s-2}c_{s-1})$. The measurement operators can be selected as

$$A_0 = \begin{pmatrix} I_{[\frac{s}{2}]} \otimes (\cos\theta\sigma_3 + \sin\theta\sigma_1) & 0 \\ 0 & 1 \end{pmatrix} \tag{14}$$

and

$$A_1 = \begin{pmatrix} I_{[\frac{s}{2}]} \otimes (\cos\theta\sigma_3 - \sin\theta\sigma_1) & 0 \\ 0 & 1 \end{pmatrix} \tag{15}$$

for some $\theta \in [0, \frac{\pi}{2}]$, where $[m]$ represents the integer less than or equal to m.

Similarly, in case (i) the corresponding projective measurements of Bob_1 are taken as

$$B_{0|0}^{(1,1)} = \frac{1}{2}\left[I_t + \begin{pmatrix} (I_{[\frac{t}{2}]} \otimes \sigma_3) & 0 \\ 0 & 1 \end{pmatrix}\right];$$

$$B_{0|1}^{(1,1)} = \frac{1}{2}\left[I_t + \begin{pmatrix} (I_{[\frac{t}{2}]} \otimes \sigma_1) & 0 \\ 0 & 1 \end{pmatrix}\right].$$

In case (ii) the corresponding projective measurements of Bob_1 are taken as

$$B_{0|0}^{(1,2)} = I_t;$$

$$B_{0|1}^{(1,2)} = \frac{1}{2}\left[I_t + \begin{pmatrix} (I_{[\frac{t}{2}]} \otimes \sigma_1) & 0 \\ 0 & 1 \end{pmatrix}\right].$$

The corresponding projective measurements of Bob_2 are taken as

$$B_0^{(2,2)} = \begin{pmatrix} (I_{[\frac{t}{2}]} \otimes \sigma_3) & 0 \\ 0 & 1 \end{pmatrix};$$

$$B_1^{(2,2)} = \begin{pmatrix} (I_{[\frac{t}{2}]} \otimes \sigma_1) & 0 \\ 0 & 1 \end{pmatrix}.$$

One derives again that the Bell nonlocality of bipartite high-dimension pure states can be shared under projective measurements.

III. CONCLUSIONS AND OUTLOOK

We have shown that projective measurements are sufficient for sharing the Bell nonlocality of high-dimensional entangled pure states. Namely, two independent Bobs may share states with a single Alice such that all the shared states violate the CHSH inequality. Our work greatly expands the range of quantum states given in Ref. [30]. These quantum states share the Bell nonlocality through projection measurements and the distribution of shared classical randomness. These results are not only theoretically interesting but also are of significance for experimental implementation, since projective measurements enable one to demonstrate sequential nonlocality sharing in much simpler setups than previous nonprojective measurements [10,11,17,18,21]. In fact, unsharp quantum measurements have been proven to be useful for device-independent self-testing and recycling quantum communication [41–43]. Our results show that with only projection measurements it might be feasible for some sequential quantum information protocols related to quantum coherence [44], entanglement witnessing [45,46], quantum steering [47,48], and quantum contextuality [49].

We have proven that based on POVMs, a high-dimension bipartite entangled pure state may produce n sequential violations of the CHSH inequality [20]. For projective measurements how many sequence violations can occur for high-dimensional entangled pure states is still unknown. It is also a meaningful problem to design the optimal projection measurement scheme. Instead of pure states, one may ask if any special mixed states can also be used for nonlocality sharing under projection measurements and shared randomness. Moreover, conclusions about simultaneous bilateral measurements and shared randomness would also be of importance.

ACKNOWLEDGMENTS

We thank the anonymous reviewers for their helpful suggestions. This work is supported by the National Natural Science Foundation of China (NSFC) under Grants No. 12204137, No. 12075159, and No. 12171044; the Hainan Provincial Natural Science Foundation of China under Grant No. 121RC539; the specific research fund of the Innovation Platform for Academicians of Hainan Province under Grant No. YSPTZX202215; and the Beijing Natural Science Foundation (Grant No. Z190005).

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