Quantum nonlocality determined by fine-grained uncertainty relations

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Quantum nonlocality and the uncertainty principle are two fundamental cornerstones of quantum theory, reflecting totally different aspects that distinguish it from classical theory. However, Oppenheim and Wehner introduced fine-grained uncertainty relations (FGURs) to account for the degree of quantum nonlocality. In the simplest Bell scenario, they showed that the quantum nonlocality measured by the maximal violation of some Bell inequality was determined by the FGUR. Along this line of research, we first derive explicit FGURs with general weights about probability distributions of two general projective measurements on local systems. Then we split those joint probabilities of Bell inequalities into a single-party conditional probability and the remaining joint probabilities so that the weighted FGURs can be applied straightforwardly to obtain the maximal violation in various scenarios ranging from bipartite, tripartite, to a complete set of multipartite Bell inequalities. Furthermore, in the bipartite scenario, the exact correlation boundary can be shown to be solely determined by our weighted local FGURs.

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I. INTRODUCTION

In the realm of quantum mechanics, the profound distinctions from classical mechanics are readily apparent, with two key concepts standing out as the most distinguishing: quantum nonlocality and the uncertainty principle. Quantum nonlocality, often synonymous with Bell nonlocality, emerges when examining correlations in multipartite scenarios, where local measurements on space-like separated quantum systems defy explanation within the confines of local realistic models [1–5]. On the other hand, the uncertainty principle, initially posited by Heisenberg [6–8], speaks to the fundamental limitation that arises when attempting to simultaneously measure incompatible physical quantities. It finds expression through a range of uncertainty relations, revealing the tradeoffs between precision in incompatible measurements. While these two distinctive features of quantum theory seemingly pertain to different facets of the quantum world, an intriguing connection emerges: the degree of quantum nonlocality is, in fact, determined by the uncertainty relations governing local subsystems.

Quantum nonlocality is also referred to Bell nonlocality and was first discovered by Bell in his research [3] on the Einstein—Podolsky—Rosen (EPR) paradox [1]. Bell showed that the classical correlations, which are characterized by locality and realism, satisfy the so-called Bell inequalities. In the simplest bipartite scenario, in which each of two observers A, B performs two local dichotomic measurements $A_{0,1}$ and $B_{0,1}$, the resulting correlations satisfy the well-known Clauser-Horne-Shimony-Holt (CHSH) inequality [9]

$$B_{\mathrm{CHSH}} := \langle A_0B_0 + A_0B_1 + A_1B_0 - A_1B_1\rangle \leqslant 2, \tag{1}$$

if the underlying theory is local and realistic. However, the quantum correlations produced by suitable local measurements on entangled states of space-like separated systems can violate Bell inequalities, signaling a profound difference between quantum and classical models. In fact, we have a maximal violation of the CHSH inequality as large as $2\sqrt{2}$ in quantum theory, which is known as the Tsirelson bound [10] defining the degree of quantum nonlocality in the given Bell scenario. Interestingly, quantum correlations do not violate the CHSH inequality to its largest possible value 4, and this value can be attained by some nonsignaling box such as the Popescu-Rohrlich (PR) box model [11] without superluminal communication. A natural question arises as to which quantum features determine the degree of quantum nonlocality or even the boundaries of quantum correlations.

The challenge has been met from two different approaches, for instance, one is the axiomatic approach [12–20], which constructs an axiom to single out quantum correlations, however, most of them are based on bipartite information concepts and not able to determine the set of quantum correlations [21]. The other one is a numerical approach, like semi-definite programming (SDP) [22,23], which provides general numerical nonlocality bounds at the cost of requiring the construction of an infinite semi-definite matrix for an exact result. Unfortunately, neither of these approaches offers an analytical and comprehensive solution, until Oppenheim and Wehner provide yet another response to the above challenge: the uncertainty principle as quantified by fine-grained uncertainty relations (FGURs) [24].

Contrary to the coarse-grained uncertainty relation, which means measuring uncertainty with some kinds of functions of the probability distribution of measurement outcomes, like entropic uncertainty relations [25–27], majorization [28,29], as well as by measurement uncertainty relations [30–33],

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Oppenheim and Wehner first proposed a fine-grained uncertainty relation involving the whole probability distributions of measurement outcomes, trying to account for the degree of quantum nonlocality. They offered a formalized fine-grained uncertainty relation and established a general connection between uncertainty, steering, and nonlocality. Later on, some explicit FGURs with equal weights were proposed and found applications in contextuality [34] and nonlocality [35], and some FGURs in different physical situations like FGUR under relativistic motion [36] and quasi-FGUR [37] appropriate for quantum memory were proposed. Based on these pioneering works, we show that the Tsirelson bounds in various scenarios ranging from bipartite to tripartite to multipartite can be determined by FGURS alone. Notably, in the simplest Bell scenario, one can obtain the entire correlation boundary due to Tsirelson, Landau, and Masanes (TLM) [38–40] from FGURs.

We at first prove an explicit weighted FGUR for two general observables and a weighted FGUR for multiple observables for a qubit system in Sec. II. Then we propose a reformulation of Bell inequalities in such a way that our FGUR can be readily applied to various Bell scenarios to bound quantum nonlocality in Sec. III. To show the effectiveness of our method, we apply our method to some different Bell scenarios to obtain the exact Tsirelson bounds for Bell inequalities. In Sec. IV we derive the Tsirelson bounds for a general Bell inequality for correlations, which gives rise to the exact TLM correlation boundary, and the chained Bell inequality [41]. In Sec. V, we consider a tripartite scenario and show that the Tsirelson bounds for 32 out of 46 tight tripartite Bell inequalities [42,43] are determined by FGUR. We also provide the quantum nonlocality bound for a general tripartite Bell inequality for correlations. In Sec. VI we consider a multipartite scenario and show the complete set of correlation Bell inequalities due to Werner, Wolf, Żukowski, and Brukner (WWZB) [44,45] is also determined by FGUR. We conclude with some discussions in Sec. VII.

II. WEIGHTED FINE-GRAINED UNCERTAINTY RELATIONS

Consider a fixed set of m measurements, labeled with $x = 1, 2, \dots, m$, performed on a given (local) quantum system prepared in some state ρ according to *a priori* probability distribution $\{p_x\}$. For each combination $\boldsymbol{a} = (a_1, \dots, a_m)$ of possible outcomes, the fine-grained uncertainty relation proposed by Oppenheim and Wehner reads

$$\sum_{x=1}^{m} p_x p(a_x|x;\rho) \leqslant \zeta_{\boldsymbol{a}} := \max_{\rho} \sum_{x=1}^{m} p_x p(a_x|x;\rho), \tag{2}$$

where $p(a_x|x;\rho)$ denotes the probability of obtaining outcome a_x by performing measurement x on the state ρ. Previously, the probability distribution p_x is commonly taken to be completely random, i.e., independent of x, and here we shall consider a weighted FGUR. When ranging over all possible weights $\{p_x\}$ FGUR gives exactly the numerical algebraic range [46–48] of the given set of observables. The upper bound $\zeta_{\boldsymbol{a}}$ implies the tradeoff in their predictabilities of the m probability distributions and the smaller $\zeta_{\boldsymbol{a}}$, the more predictable it is [49,50].

As the upper bound $\zeta_{\boldsymbol{a}}$ involves the optimization over all possible states, the fine-grained uncertainty relation cannot be used directly to bound quantum nonlocality. In what follows, we shall derive at first an explicit fine-grained uncertainty relation with general weights about two projective measurements of arbitrary dimensions. Then we apply our fine-grained uncertainty relation to some particular cases, and we will obtain some specific fine-grained uncertainty relations. FGURs with equal weights were derived in other articles [51].

Consider two von Neumann projective measurements $A = \{|a_j\rangle\}_{j=1}^d$ and $B = \{|b_k\rangle\}_{k=1}^d$ corresponding to two orthonormal bases. Let p_A and p_B be the probabilities of performing measurements A, B and the resulting probabilities read $p(a_i|A) = \langle a_i|\rho|a_i\rangle$ and $p(b_j|B) = \langle b_j|\rho|b_j\rangle$ with $i, j = 1, 2, \dots, d$. For any combination of outcome i, j, our explicit fine-grained uncertainty relation for A, B reads

$$\zeta_{ij} = \frac{p_A + p_B + \sqrt{(p_A - p_B)^2 + 4p_A p_B |\langle a_i|b_j\rangle|^2}}{2}. \tag{3}$$

In fact, the weighted probability $p_A p(a_i|A) + p_B p(b_j|B)$ can be formulated as an expectation value $\mathrm{Tr}\rho\Pi_{ij}$ of some Hermitian operator $\Pi_{ij} = p_A|a_i\rangle\langle a_i| + p_B|b_j\rangle\langle b_j|$ and thus the nonlocality bound ζ_{ij}, i.e., the maximal value of the weighted probability over all possible states, is the largest eigenvalue of Π_{ij} as given in Eq. (3). In the two-dimensional subspace spanned by $|a_i\rangle$ and $|b_j\rangle$, we can choose an orthogonal basis $\{|a_i\rangle, |a_i^\perp\rangle\}$ in which $|b_j\rangle = |a_i\rangle\langle a_i|b_j\rangle + |a_i^\perp\rangle\langle a_i^\perp|b_j\rangle$ with $|\langle a_i|b_j\rangle|^2 + |\langle a_i^\perp|b_j\rangle|^2 = 1$. In this basis, we have

$$\Pi_{ij} = \begin{pmatrix} p_A + p_B|\langle a_i|b_j\rangle|^2 & p_B\langle a_i|b_j\rangle\langle b_j|a_i^\perp\rangle \\ p_B\langle a_i^\perp|b_j\rangle\langle b_j|a_i\rangle & p_B|\langle a_i^\perp|b_j\rangle|^2 \end{pmatrix}$$

whose largest eigenvalue is given by the right-hand side (r.h.s.) of Eq. (3). We note here that the coefficients p_A and p_B can be arbitrary, not necessarily normalized. As an example, we consider two mutually unbiased bases (MUBs) [52,53] satisfying $|\langle a_i|b_j\rangle| = 1/\sqrt{d}$ for all i, j. By setting $p_A = p_B = 1$, we reproduce the FGUR for MUBs:

$$p(a_i|A) + p(b_j|B) \leqslant 1 + \frac{1}{\sqrt{d}}. \tag{4}$$

Now we consider a qubit and perform m measurements specified by Bloch vectors $\{\boldsymbol{r}_x\}$ according to *a priori* probability $\{p_x\}$, which needs not to be normalized and we denote $p = \sum_x p_x$. Let $\{p(a_x|x,\rho) = \langle a_x|\rho|a_x\rangle$ be the probability distribution for the xth observable and the weighted probability $\sum_x p_x p(a_x|x,\rho)$ is the expectation value of the following Hermitian operator:

$$\sum_x p_x \frac{I + (-1)^{a_x}\vec{r}_x \cdot \vec{\sigma}}{2} = \frac{p}{2} + \frac{\vec{\sigma}}{2}\sum_x p_x(-1)^{a_x}\vec{r}_x$$

whose maximal eigenvalue gives the following FGUR:

$$\zeta_{\mathbf{a}} = \frac{p}{2} + \frac{1}{2}\sqrt{\sum_x p_x^2 + \sum_{x\neq y} p_x p_y (-1)^{a_i + a_j}\cos\theta_{xy}}, \tag{5}$$

where $\cos\theta_{xy} = \boldsymbol{r}_x\boldsymbol{r}_y$. Specifically, in the case of two qubit observables, we have only one independent probability each and the explicit fine-grained uncertainty relation reads

$$\zeta_{ab} = \frac{p_A + p_B + \sqrt{p_A^2 + 2p_A p_B(-1)^{a+b}\cos\theta + p_B^2}}{2}. \quad (6)$$

We note that all the fine-grained uncertainty relations above are tight as the corresponding maximums are attainable.

III. QUANTUM NONLOCALITY BOUNDS FROM FGUR

We consider a general Bell scenario $(n, m, 2)$ of n space-like parties labeled with $N = \{1, \ldots, n\}$ with each observer performing m dichotomic measurements labeled with $M = \{1, \ldots, m\}$. We denote by $\mathcal{M} = \cup_{\alpha\subseteq N} M^\alpha$ all possible measurement settings and by $\mathcal{A} = \cup_{\alpha\subseteq N}\{0, 1\}^\alpha$ the corresponding possible outcomes with resulting joint probability being $p(\boldsymbol{a}|\boldsymbol{x}) = p^n(a_1, a_2, \ldots, a_n|x_1, x_2, \ldots, x_n)$. The superscript represents the number of variates. A most general Bell inequality reads

$$B = \sum_{\boldsymbol{x}} (-1)^{\omega_{\boldsymbol{x}}} s_{\boldsymbol{x}} E_{\boldsymbol{x}} = \sum_{\boldsymbol{a},\boldsymbol{x}} (-1)^{|\boldsymbol{a}|+\omega_{\boldsymbol{x}}} s_{\boldsymbol{x}} p(\boldsymbol{a}|\boldsymbol{x}), \quad (7)$$

where $|\boldsymbol{a}| = \sum_i a_i$, $s_{\boldsymbol{x}} \geqslant 0$, $\omega_{\boldsymbol{x}} \in \{0, 1\}$, and the summation is over all possible measurement settings $\boldsymbol{x} \in \mathcal{M}$ and corresponding outcomes $\boldsymbol{a} \in \mathcal{A}$. In addition, the outcomes $\boldsymbol{a}$ depend on measurement settings $\boldsymbol{x}$. We note that, commonly for dichotomic measurements, the Bell inequalities are given in terms of correlations $E_{\boldsymbol{x}} = \sum_{\boldsymbol{a}} (-1)^{|\boldsymbol{a}|} p(\boldsymbol{a}|\boldsymbol{x})$ and it is clear that the above form is the most general in this Bell scenario.

In a realistic and local model, the Bell inequality is bounded from above by some real value, and quantum correlations may give rise to some larger values and the Tsirelson bound is the maximal violation allowed by quantum theory. The boundary of quantum correlation is therefore delineated by all the Tsirelson bounds of all possible Bell inequalities in the given scenario. The search for the maximal violation of Bell inequality over all the quantum states and measurements is, however, a hard problem [54] as there are too many free parameters. Some other more efficient numerical methods such as SDP can reach the exact value only in the asymptotic limit. Here, we propose a general method to derive the Tsirelson bounds of various kinds of Bell inequalities of dichotomic measurements by using FGUR.

We note that, in its original formulation Eq. (7), half of the coefficients are negative. To apply our FGUR with positive weights, our first step is to reformulate the given Bell inequality in terms of positive coefficients. This is always possible as there are only two outcomes for each local measurement and the probabilities are normalized for all measurement settings. In fact, by denoting $H_{\boldsymbol{x}} = \{\boldsymbol{a} \in \mathcal{A} \mid |\boldsymbol{a}| = \omega_{\boldsymbol{x}}\}$ for each measurement setting $\boldsymbol{x} \in \mathcal{M}$, we can rewrite

$$\begin{aligned} B &= \sum_{\boldsymbol{x}} s_{\boldsymbol{x}} \big(\textstyle\sum_{\boldsymbol{a}\in H_{\boldsymbol{x}}} p(\boldsymbol{a}|\boldsymbol{x}) - \sum_{\boldsymbol{a}\notin H_{\boldsymbol{x}}} p(\boldsymbol{a}|\boldsymbol{x}) \big) \\ &= 2\sum_{\boldsymbol{x}} \sum_{\boldsymbol{a}\in H_{\boldsymbol{x}}} s_{\boldsymbol{x}} p(\boldsymbol{a}|\boldsymbol{x}) - \sum_{\boldsymbol{x}} s_{\boldsymbol{x}} \\ &:= 2b - c, \end{aligned} \quad (8)$$

where for a specific Bell inequality coefficients $s_{\boldsymbol{x}}$ are given so that the second term c is a constant. One has only to bound b, and as all coefficients are nonnegative, we can apply our weighted FGUR.

The joint probabilities in b can be split into two parts: the kth party and the remaining $\bar{k}$ parties. Then, we can recast b as

$$\begin{aligned} b &= \sum_{\boldsymbol{x}} \sum_{\boldsymbol{a}\in H_{\boldsymbol{x}}} s_{\boldsymbol{x}} p(\boldsymbol{a}|\boldsymbol{x}) \\ &= \sum_{\boldsymbol{x}_{\bar{k}}} \sum_{x_k=1}^{m} \sum_{\boldsymbol{a}\in H_{\boldsymbol{x}}} s_{\boldsymbol{x}} p_{\bar{k}}(a_k|x_k) p(\boldsymbol{a}_{\bar{k}}|\boldsymbol{x}_{\bar{k}}) \\ &= \sum_{\boldsymbol{x}_{\bar{k}},\boldsymbol{a}_{\bar{k}}} \left(\sum_{x_k=1}^{m} s_{\boldsymbol{x}} p_{\bar{k}}(|\boldsymbol{a}_{\bar{k}}| + \omega_{\boldsymbol{x}}|x_k) \right) p(\boldsymbol{a}_{\bar{k}}|\boldsymbol{x}_{\bar{k}}). \end{aligned} \quad (9)$$

First, we can rewrite $p(\boldsymbol{a}|\boldsymbol{x}) = p_{\bar{k}}(a_k|x_k) p(\boldsymbol{a}_{\bar{k}}|\boldsymbol{x}_{\bar{k}})$ as a product of the kth conditional probablity $p_{\bar{k}}(a_k|x_k) = p^1(a_k|x_k, x_1 = a \ldots x_{k-1} = a_{k-1}, x_{k+1} = a_{k+1}, \ldots, x_n = a_n) = p(\boldsymbol{a}|\boldsymbol{x})/p(\boldsymbol{a}_{\bar{k}}|\boldsymbol{x}_{\bar{k}})$, which, measuring x_k on the kth conditional state that produced after the remaining parties measured, and the remaining $\bar{k}$ joint probability $p(\boldsymbol{a}_{\bar{k}}|\boldsymbol{x}_{\bar{k}}) = p^{n-1}(a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n|x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n) = \sum_{a_k} p(\boldsymbol{a}|\boldsymbol{x})$, which is the probabilities that measuring $\boldsymbol{x}_{\bar{k}}$ on the original state ρ and obtaining outcome $\boldsymbol{a}_{\bar{k}}$. Because of the nonsignaling condition of the n space-like parties, the split is always possible, and we can always sum the kth party and the $\bar{k}$ parties separately. Second, in the third line, we split a_k from $\sum_{\boldsymbol{a}\in H_{\boldsymbol{x}}}$. On account of $\boldsymbol{a} \in H_{\boldsymbol{x}}$, for any $\boldsymbol{a}_{\bar{k}}$, we can solve $a_k = |\boldsymbol{a}_{\bar{k}}| + \omega_{\boldsymbol{x}}$ from the condition in $H_{\boldsymbol{x}}$.

Then, we bound the sum of the kth probability in the round bracket with FGUR,

$$\begin{aligned} b &\leqslant \sum_{\boldsymbol{x}_{\bar{k}},\boldsymbol{a}_{\bar{k}}} \zeta_{\boldsymbol{a}_k}(\{s_{\boldsymbol{x}}\}_{x_k}, \{\theta_{ij}\}) p(\boldsymbol{a}_{\bar{k}}|\boldsymbol{x}_{\bar{k}}) \\ &\leqslant \sum_{\boldsymbol{x}_{\bar{k}}} \max_{\boldsymbol{a}_{\bar{k}}} \zeta_{\boldsymbol{a}_k}(\{s_{\boldsymbol{x}}\}_{x_k}, \{\theta_{ij}\}) \\ &\leqslant \max_{\{\theta_{ij}\}} \sum_{\boldsymbol{x}_{\bar{k}}} \max_{\boldsymbol{a}_{\bar{k}}} \zeta_{\boldsymbol{a}_k}(\{s_{\boldsymbol{x}}\}_{x_k}, \{\theta_{ij}\}) := \zeta_{\text{sup}}. \end{aligned} \quad (10)$$

The first inequality comes from weighted FGUR Eq. (5) with $\zeta_{\boldsymbol{a}_k}(\{s_{\boldsymbol{x}}\}_{x_k}, \{\theta_{ij}\})$ being the upper bound showing explicitly its dependence on the weights $\{s_{\boldsymbol{x}} = s_{\boldsymbol{x}_{\bar{k}}x_k}\}_{x_k}$ and angles among measurements of the kth party. We use the subscript $\boldsymbol{a}_k = \{a_k\}$ to denote the outcome vector of measuring x_k. For different x_k, we have different a_k which depends on $\boldsymbol{a}_{\bar{k}}$ and all the a_k form a vector $\boldsymbol{a}_k = \{a_k\}$. For most canonical Bell inequalities, the $\zeta_{\boldsymbol{a}_k}$ are equal for all the $p(\boldsymbol{a}_{\bar{k}}|\boldsymbol{x}_{\bar{k}})$. However, for the very few extremely special Bell inequalities, the $\zeta_{\boldsymbol{a}_k}$ are unequal, so we relax it with the maximal $\zeta_{\boldsymbol{a}_k}$ in the second inequality. Then, we only need to solve the maximum ζ_{sup} about the function of $\{\theta_{ij}\}$ in the last line. Finally, we get a nonlocality bound $B \leqslant 2\zeta_{\text{sup}} - c$.

Some remarks are in order. First, for a canonical Bell inequality of n parties, it may contain correlations with less than n like $E_{s,s+1,\ldots,t-1,t}$ and $1 < s < t < n$. While the split choice of party k is arbitrary, we assume that party k belongs to the support of all correlations in the above derivation. If there are correlations not involving k, for instance, $k > t$ or $k < s$, we can choose another party $s < k_1 < t$ and bound similarly the rest terms in B with Eqs. (9) and (10). This process can go on until we can cover all the correlations. Second, we might obtain the boundary of quantum correlation by ranging over all possible coefficients $s_{\boldsymbol{x}}$. In the next section, we will derive the boundary of the (2,2,2) scenario for demonstration. Third, our method above can be generalized to scenarios involving measurements with three or more outcomes, provided that we have as effective FGURs as qubit case. However, reaching the exact upper bound is challenging for the high-dimensional Bell inequalities because the fine-grained uncertainty relations of high-dimensional systems might not be tight. Our method is general, in principle, only if you can replace the $\zeta_{\boldsymbol{a}_k}$ in different scenarios.

IV. BIPARTITE SCENARIO

As the first application, we consider the simplest Bell scenario (2,2,2) in which two observers A, B performing two local dichotomic measurements $A_{0,1}, B_{0,1}$, respectively. Although CHSH inequalities are complete in this scenario, other Bell inequalities such as tilted CHSH inequality [55] found important applications. Here we consider the following generalized Bell inequality:

$$B_s = s_{00}A_0B_0 + s_{01}A_0B_1 + s_{10}A_1B_0 - A_1B_1, \tag{11}$$

with $s_{xy} > 0$. We note that the relevant measurement settings are $\{A_xB_y\}$ and the corresponding coefficients are $s_{\boldsymbol{x}} = \{s_{xy}\}$ and, for convenience, we denote $s_{11} = 1$ together with signs $\omega_{\boldsymbol{x}} = xy$. To apply FGUR to the party A, we at first rewrite $B_s = 2b_s - c_s$ according to Eq. (8) with $c_s = \sum_{xy} s_{xy}$ and we can bound

$$\begin{aligned}
b_s &= \sum_{b,y}\sum_{x} s_{xy}p(b+xy, b|xy) \\
&= \sum_{b,y} p(b|y)\sum_{x} s_{xy}p_{b|y}(b+xy|x) \\
&\leqslant \sum_{b,y} p(b|y)\zeta_{(b,b+y)}(\{s_{0y}, s_{1y}\}, \theta) \\
&= \sum_{y} \zeta_{(0,y)}(\{s_{0y}, s_{1y}\}, \theta) \\
&= \frac{c_s}{2} + \frac{1}{2}\sum_{y}\sqrt{s_{0y}^2 + s_{1y}^2 + 2s_{0y}s_{1y}(-1)^y\cos\theta} \\
&= \frac{c_s}{2} + \frac{1}{2}\sum_{y}\sqrt{s_{0y}s_{1y}}\sqrt{\tfrac{s_{0y}^2+s_{1y}^2}{s_{0y}s_{1y}} + 2(-1)^y\cos\theta} \\
&\leqslant \frac{c_s}{2} + \frac{1}{2}\sqrt{\textstyle\sum_y s_{0y}s_{1y}}\sqrt{\textstyle\sum_y \tfrac{s_{0y}^2+s_{1y}^2}{s_{0y}s_{1y}}} \\
&:= \frac{c_s + T_s}{2},
\end{aligned}$$

where the first inequality is due to FGUR on local system A while the second inequality is due to the Cauchy-Schwarz inequality. As a result, we have nonlocality bound $B_s = 2b_s - c_s \leqslant T_s$ with

$$T_s := \sqrt{\textstyle\sum_{xy} s_{xy}^2 + \left(\prod_{xy} s_{xy}\right)\left(\sum_{xy} s_{xy}^{-2}\right)} \tag{12}$$

being exactly the Tsirelson bound in this case, i.e., the largest possible eigenvalue of the corresponding Bell operator (see Appendix A). In the case of $s_{10} = 1$ and $s_{01} = s_{00} = \alpha$ we reproduce the Tsirelson bound $T_s = 2\sqrt{1+\alpha^2}$ for a family of tilted Bell inequalities [55].

In the above derivation, we note that, for the Cauchy-Schwarz inequality to be saturated, there should be some θ such that

$$2\cos\theta\frac{s_{00}s_{10} + s_{01}s_{11}}{s_{00}s_{10}s_{01}s_{11}} = \frac{1}{s_{01}^2} + \frac{1}{s_{11}^2} - \frac{1}{s_{00}^2} - \frac{1}{s_{10}^2}.$$

As $|\cos\theta| \leqslant 1$ we see that T_s is the Tsirelson bound for the generalized Bell inequality iff

$$\min_{x,y} s_{xy}\sum_{xy}\frac{1}{s_{xy}} \geqslant 2. \tag{13}$$

Otherwise, the maximal value of b_s is attained at compatible measurements, i.e., $\theta = 0, \pi$, which means that the corresponding Bell inequality has no quantum advantages (see Appendix B).

Bipartite full correlations in the (2,2,2) scenario are completely characterized by the TLM inequality. In what follows we shall show that the joint numerical range of four correlations $E_{xy} = \langle A_xB_y\rangle$ will result in exactly the same TLM boundary. In the four-dimensional parametric space with four correlations E_{xy} as coordinates, the Tsirelson bound $\max B_s = T_s$ are hyperplanes touching the correlation boundary. The envelope of the these hyperplanes reads

$$E_{xy}^{\circ} = \frac{\partial T_s}{\partial s_{xy}} = \frac{1}{T_s}\left(s_{xy} - \frac{s}{s_{xy}^3} + \frac{s}{2s_{xy}}\sum_{x'y'}\frac{1}{s_{x'y'}^2}\right),$$

which $s := \prod_{xy} s_{xy}$. Moreover, for a nontrivial Bell inequality, the condition Eq. (13) must hold and therefore there exist four angles $0 < \theta_{xy} < \pi$ with $\sum_{xy}\theta_{xy} = 2\theta_{11}$ and a constant λ such that $s_{xy} = \lambda/\sin\theta_{xy}$. In this case, the correlation boundary become $E_{xy}^{\circ} = (-1)^{xy}\cos\theta_{xy}$, which coincides with the TLM boundary.

As the next application, we consider a generalization of the CHSH inequality to the Bell scenario $(2, m, 2)$ with two parties measuring m dichotomic observables each, denoted by A_k, B_k. In this case, the chained Bell inequality reads

$$B_m = \sum_{i=1}^{m} A_iB_i + \sum_{i=1}^{m-1} A_{i+1}B_i - A_1B_m. \tag{14}$$

Wehner generalized the CHSH inequality to the multimeasurement inequality and derived its Tsirelson bound $T_m = 2m\cos\frac{\pi}{2m}$ with semi-definite programming [41]. As the first step, we rewrite $B_m = 2b_m - c_m$ with $c_m = 2m$ and for the chained Bell inequality, we have

$$\begin{aligned}
b_m &= \sum_{i=1}^{m-1}\sum_{a}[(aa|ii) + p(aa|i+1, i)] \\
&\quad + \sum_{a} p(aa|mm) + p(a\bar{a}|1m) \\
&= \sum_{i=1}^{m-1}\sum_{a=0}^{1}[p_{a|i}(a|i) + p_{a|i}(a|i+1)]p_B(a|i)
\end{aligned}$$

$$+\sum_{a=0}^{1}[p_{a|m}(a|1)+p_{a|m}(\bar{a}|n)]p_B(a|m)$$
$$\leqslant m+\sum_{i=1}^{n-1}\sqrt{(1+\cos\theta_i)/2}+\sqrt{(1-\cos\theta_m)/2}$$
$$\leqslant m+\frac{T_m}{2}. \quad (15)$$

According to Eqs. (9) and (10), the first step is the expression of the b_m. Then, we split the joint probabilities and regroup them in the second step. Finally, we use the weighted FGUR to constrain the conditional probabilities in the round brackets. It is worth noting that θ_i is the angle between A_i and A_{i+1} actually, and we assume that all measurement directions are in a plane and the angle between A_1 and A_m is $\theta_m=\sum_{i=1}^{n-1}\theta_i$. In this multimeasurement scenario, b_m is a function of m different angles θ_i as we have m different measurements, and its maximal values over all possible angles can be found analytically (see Appendix C) to reach the Tsirelson bound $B_m\leqslant T_m$ in the case of all the $\theta_i=\pi/m$ being equal.

V. TRIPARTITE SCENARIO

In the Bell scenario (3,2,2), three parties A, B, C measure two dichotomic measurements each and the local correlations are completely characterized by 46 tight Bell inequalities as documented by Śliwa [42], with their maximum quantum violation found numerically in [43]. Regarding the applications of FGUR, all the 46 inequalities can be divided into three classes. The first class includes six Bell inequalities numbered {3, 9, 11, 13, 14, 17} for which the Tsirelson bounds are determined by FGUR using our analytical approach. As an example of the first class, we consider the 17th Bell inequality in [43]

$$B_{17}=A_0+A_1+A_0B_0+A_1B_0+A_0C_0+A_1C_0$$
$$-A_0B_0C_0-A_1B_0C_0+2A_0B_1C_1-2A_1B_1C_1. \quad (16)$$

In this case the relevant measurement settings are $\{A_x\}$, $\{A_xB_0, A_xC_0\}$, and $\{A_xB_yC_y\}$ and coefficients s_x and signs $\omega_{\boldsymbol{x}}$ can be read off straightforwardly. To apply FGUR on party A we at first rewrite $B_{17}=2b_{17}-c_{17}$ with $c_{17}=12$ and we can bound

$$b_{17}=\sum_{x,b,c=0}^{1}p(0bc|x00)+p(bbc|x00)+p(cbc|x00)$$
$$+p(b+c+1,b,c|x00)+2p(x+b+c,b,c|x11)$$
$$=2+2\sum_{b,c=0}^{1}\left(p_{\bar{A}}(bc|0)+p_{\bar{A}}(bc|1)\right)p(bc|00)$$
$$+2\sum_{b,c=0}^{1}(p_{\bar{A}}(b+c|0)+p_{\bar{A}}(1+b+c|1))p(bc|11)$$
$$\leqslant 6+2(\sqrt{(1+\cos\theta_A)/2}+\sqrt{(1-\cos\theta_A)/2})$$
$$\leqslant 6+2\sqrt{2}. \quad (17)$$

In b_{17}, there are two group of joint probablities which are $\sum_{a,b,c}p(abc|000)$ and $\sum_{a,b,c}p(abc|100)$ and they can be summed to 1. Thus we have accounted for the Tsirelson bound $B_{17}\leqslant 4\sqrt{2}$ as shown numerically in [43] by using FGUR.

Some remarks are in order. For the measurement settings with support 1, e.g., A_0, one can have the freedom of choosing which observables are measured alongside, e.g., $p(a|x)=\sum_{bc}p(abc|xyz)$ with y, z being arbitrary due to the nonsignaling condition and for the measurement setting with support on two parties we still have the freedom to choose the measurement setting of the rest party, e.g., $p(ab|xy)=\sum_c p(abc|xyz)$ with z being arbitrary. Different choices may lead to different upper bounds for quantum nonlocality. For example, we can also expand the probabilities with a different measurement setting, such as

$$b_{17}=\sum_{x,b,c=0}^{1}p(0bc|x11)+p(bbc|x01)+p(|bcb|x10)$$
$$+p(b,c,1+b+c|x00)+2p(b,c,x+b+c|x11)$$
$$\leqslant 6+(2\sqrt{(1+\cos\theta_A)/2}+\sqrt{(1-\cos\theta_A)/2})$$
$$\leqslant 6+2\sqrt{5}, \quad (18)$$

giving rise to the upper bound $4\sqrt{5}$, which is greater than the Tsirelson bound $4\sqrt{2}$. As a result, we need to explore different possible expansions and we should select the optimal expansion for a better nonlocality bound.

FGURs are able to determine the exact Tsirelson bounds for many more tight tripartite Bell inequalities beyond the general and analytical approach proposed in Sec. III. With the help of numerical search, e.g., the sequential quadratic programming (SQP), we can numerically calculate the Tsirelson bounds with the constraints given by the fine-grained uncertainty relations. The second class includes those tight tripartite Bell inequalities numbered {4–6,8,12,15,16,18–20,22,24,26,28–30,33,36–39,42,44,45}. A typical example is the eighth inequality, which reads

$$B_8=A_0B_0+A_1B_0+A_0B_1+A_1B_1+2A_0B_0C_0$$
$$-2A_1B_1C_0+A_0B_0C_1-A_0B_1C_0-A_0B_1C_1$$
$$+A_1B_1C_1. \quad (19)$$

In this case, we cannot pair off all the items with one partition and a direct application of the method proposed in this section fails to give the optimal result 20/3.

There are still 14 untight tripartite Bell inequalities left, {7, 10, 21, 23, 25, 27, 31, 32, 34, 35, 40, 41, 43, 46}. In this class, even aided with the numerical method, the fine-grained uncertainty relations alone are insufficient to determine the Tsirelson bound. A typical example is the seventh inequality

$$B_7=3A_0B_0C_0+A_1B_0C_0+A_0B_1C_0-A_1B_1C_0$$
$$+A_0B_0C_1-A_1B_0C_1-A_0B_1C_1+A_1B_1C_1. \quad (20)$$

A direct application of the FGUR method will lead to the bound $b_7\leqslant\sqrt{(5+3\cos\theta)/2}+3\sqrt{(1-\cos\theta)/2}+5\leqslant 9$ and $c_7=10$. Thus we have $B_7\leqslant 8$ with the actual Tsirelson bound being 20/3. The optimal upper bound by the numerical search via SQP is 7.21. This is because that quantum correlation cannot saturate the upper bound of each fine-grained uncertainty relation in this case since each uncertainty relation constraints on a conditional state and all the conditional states originated from a specific original quantum state $|\psi\rangle$. Only parts of the fine-grained uncertainty relations can reach the upper bound.

To explore the boundary of tripartite quantum correlations, we consider the following most general Bell inequality for correlations:

$$B_p = \sum_{x,y,z=0}^{1} s_{xyz}(-1)^{\omega_{xyz}} A_x B_y C_z. \tag{21}$$

When the coefficients s_{xyz} range over all possible values, we might reproduce the partial boundary of tripartite quantum correlation by finding their Tsirelson bounds.

For ease of calculation, we can set $s_{xyz} = s_{xyz+100}$. Based on the symmetry of signs and the cyclic permutational symmetry of A, B, C, we only need to consider four different cases about $\{\omega_{xyz}\}$ depending on how many 1's are there. The first case is $\omega_{xyz} = 0$ for all x, y, z, which is equivalent to all the $\omega_{xyz} = 1$ based on symmetry. This case is trivial as both the quantum bound and classical bound are equal to $\sum_{x,y,z} s_{xyz}$.

For the last three cases, we set (i) $\omega_{011} = 1$; (ii) $\omega_{010} = \omega_{011} = 1$; (iii) $\omega_{001} = \omega_{010} = \omega_{011} = 1$ with all the other $\omega_{xyz} = 0$. For convenience, we set $s_{xyz} = s_{yz}$ to be independent of the measurement setting x of A. Then we have

$$B_p \leqslant \sum_{\pm} S_\pm \sqrt{\frac{1 \pm \cos\theta}{2}} \leqslant \sqrt{S_+^2 + S_-^2}, \tag{22}$$

with $S_\pm = \sum_{yz}[1 + (-1)^{\omega_{0yz}}]s_{yz}$ which is attained by $\tan\frac{\theta}{2} = S_-/S_+$.

VI. WWZB INEQUALITY

Now we consider the quantum correlations in a general Bell scenario $(n, 2, 2)$ where there are n observers labeled with $N = \{1, 2, \ldots, n\}$ and measure two alternative dichotomic observables each. The set of all correlations in this scenario is completely characterized by a complete set of Bell inequalities [44]

$$B_{\mathrm{WWZB}} = \sum_{\boldsymbol{a},\boldsymbol{x}} \sum_{\boldsymbol{y}} S_y (-1)^{\boldsymbol{x}\cdot\boldsymbol{y}+|\boldsymbol{a}|} p(\boldsymbol{a}|\boldsymbol{x}), \tag{23}$$

with 2^n given independent signs $S_{\boldsymbol{y}} = \pm 1$ as parameters, where $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a} \in \{0, 1\}^n$ are n-dimensional binary vectors with addition modular 2 and $|\boldsymbol{a}| = \sum_{i\in N} a_i$. For later use we introduce the Fourier transformation $\tilde{S}_{\boldsymbol{x}} = \sum_{\boldsymbol{y}} S_y (-1)^{\boldsymbol{x}\cdot\boldsymbol{y}}$ of S_x and denote $\tilde{S}_{\boldsymbol{x}} = |\tilde{S}_{\boldsymbol{x}}|(-1)^{\omega_x}$. By rewriting the Bell inequality $B_{\mathrm{WWZB}} = 2b - c$ according to Eq. (8) with $c = \sum_{\boldsymbol{x}} |\tilde{S}_{\boldsymbol{x}}|$ and

$$\begin{aligned} b_{\mathrm{WWZB}} &= \sum_{\boldsymbol{x}} |\tilde{S}_{\boldsymbol{x}}| \sum_{|\boldsymbol{a}|+\omega_{\boldsymbol{x}}=0} p(a|x) = \sum_{x_1,x'} |\tilde{S}_{x_1x'}| \sum_{a'} p(|a'| + \omega_{x_1x'}, a'|x_1, x') \\ &= \sum_{a',x'} p(a'|x') \sum_{x_1} |\tilde{S}_{x_1x'}|\, p_{a'|x'}(|a'| + \omega_{x_1x'}|x_1) \\ &\leqslant \sum_{a',x'} p(a'|x') \frac{|\tilde{S}_{0x'}| + |\tilde{S}_{1x'}| + \sqrt{\tilde{S}_{0x'}^2 + \tilde{S}_{1x'}^2 + 2\tilde{S}_{0x'}\tilde{S}_{1x'}\cos\theta}}{2} \\ &= \sum_{x'} \frac{|\tilde{S}_{0x'}| + |\tilde{S}_{1x'}| + \sqrt{\tilde{S}_{0x'}^2 + \tilde{S}_{1x'}^2 + 2\tilde{S}_{0x'}\tilde{S}_{1x'}\cos\theta}}{2} \\ &\leqslant \frac{1}{2}\sum_x |\tilde{S}_x| + \frac{1}{2}\sqrt{\textstyle\sum_{x'} 1}\sqrt{\textstyle\sum_x \tilde{S}_x^2} = \frac{c}{2} + \frac{1}{2} \times 2^{\frac{3n-1}{2}}. \end{aligned} \tag{24}$$

Here, we choose the first subsystem on which FGUR will be applied and write $x = (x_1, x')$ with x' being the measurement setting for the rest parties with the corresponding outcome denoted by a'. In the first inequality, we apply weighted FGUR on the first subsystem with weights $\{|\tilde{S}_{0,x'}|, |\tilde{S}_{1,x'}|\}$ and outcomes $\{|a'| + \omega_{0x'}, |a'| + \omega_{1x'}\}$ with the upper bounds being independent of a' for all measurement settings x'. The second inequality is due to the Cauchy-Schwarz inequality and we take into the orthogonality

$$\begin{aligned} \sum_{x'} \tilde{S}_{0x'}\tilde{S}_{1x'} &= \sum_{x',y,z} S_{y_1y'} S_{z_1z'} (-1)^{z_1+x'(y'+z')} \\ &= 2^{n-1} \sum_{y'} (S_{0y'} + S_{1y'})(S_{0y'} - S_{1y'}) = 0. \end{aligned}$$

Thus we obtain the Tsirelson bound $B_{\mathrm{WWZB}} \leqslant 2^{\frac{3n-1}{2}}$ in this general scenario.

VII. CONCLUSION

In this work, we at first prove an analytical weighted fine-grained uncertainty relation and propose a reformulation of Bell inequalities involving dichotomic observables in such a way that the FGURs can be readily applied. As applications, we show that the Tsirelson bounds in various Bell scenarios are completely determined by the uncertainty principle. These scenarios include the most general Bell inequality for correlations as well as a chained Bell inequality in bipartite systems, 32 out 46 tight Bell inequalities in tripartite systems, and a complete set of Bell inequalities for correlations. Our proposed method is also applicable in determining the boundary of quantum correlation, e.g., the TLM correlation boundary. It should be emphasized that all the Tsirelson bounds obtained here, although we consider only FGURs for incompatible qubit measurements, are actually device independent. This is because, in the $(n, 2, 2)$ scenario, all quantum extreme points are achievable by measuring n-qubit pure states with projective measurements [56] and the maximal violation of multipartite Bell inequalities can self-test the corresponding quantum states and measurements [57].

Conceptually, two main assumptions in our derivations of Tsirelson bounds are local fine-grained uncertainty relations with general weight and nonsignaling condition, which is implicitly used in our reformulation of Bell inequalities and numerical search based on FGURs. In the bipartite scenario, not only the Tsirelson bound, but also the exact boundary can be obtained based on these two assumptions. Although in the tripartite scenarios, the Tsirelson bounds for most of the tight Bell inequalities, there are 14 tight Bell inequalities whose largest violations remain unaccounted for. This might call for some genuine multipartite quantum features to account for the degree of quantum nonlocality in all cases.

Finally, we emphasize that our approach can be readily applied to high-dimensional scenarios. We first use probabilities to reexpress the high-dimensional Bell inequalities and then replace the weighted FGUR Eq. (5) with a specific high-dimensional FGUR, e.g., as proposed in [34]. However, exact Tsirelson bounds might not follow from this approach as the existing high-dimensional FGURs are not tight. This presents us with the challenge of finding more effective high-dimensional FGURs.

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APPENDIX A: QUANTUM BOUND OF BELL INEQUALITY EQ. (11)

In this Appendix, we shall find the maximal eigenvalue of the Bell operator Eq. (11) of a most general Bell inequality for correlations. More generally, we consider the following Bell expression:

$$\hat{B}_s = \hat{A}_0(s_{00}\hat{B}_0 + s_{01}\hat{B}_1) + \hat{A}_1(s_{10}\hat{B}_0 + s_{11}\hat{B}_1),$$

by assuming $s_{11} = -1$. For convenience we denote

$$X_\pm = s_{00}s_{01} \mp s_{10}s_{11}, \quad Y_\pm = s_{00}s_{10} \mp s_{01}s_{11},$$
$$Z_\pm = -s_{00}s_{11} \pm s_{01}s_{10}, \quad s = s_{00}s_{11}s_{01}s_{10},$$

and consider a two-qubit system on which two local qubit measurements $\hat{A}_i = \vec{a}_i \cdot \vec{\sigma}_A$ and $\hat{B}_j = \vec{b}_j \cdot \vec{\sigma}_B$ are performed, with $a = \vec{a}_0 \cdot \vec{a}_1$ and $b = \vec{b}_0 \cdot \vec{b}_1$ as well as $\bar{a} = \sqrt{1-a^2}$ and $\bar{b} = \sqrt{1-b^2}$. Let $\hat{Y}_A$ and $\hat{Y}_B$ be the ideal qubit observable along the directions orthogonal to $\vec{a}_{0,1}$ and $\vec{b}_{0,1}$ respectively, and it holds $\hat{A}_0\hat{A}_1 = a + i\bar{a}\hat{Y}_A$ and $\hat{B}_0\hat{B}_1 = b + i\bar{b}\hat{Y}_B$. To calculate the eigenvalues of the Bell expression $\hat{B}_s$ we consider its square

$$\begin{aligned}
\hat{B}_s^2 &= \hat{A}_0^2 B_{s+}^2 + A_1^2 B_{s-}^2 + A_0A_1B_{s+}B_{s-} + A_1A_0B_{s-}B_{s+} \\
&= \sum_{ij} s_{ij}^2 + 2X_+b + a\{B_{s+}, B_{s-}\} + i\bar{a}\hat{Y}_A[B_{s+}, B_{s-}] \\
&= \sum_{ij} s_{ij}^2 + 2X_+b + 2a(Y_+ + Z_+b) - 2\bar{a}\bar{b}Z_-\hat{Y}_A\hat{Y}_B \\
&\leqslant \sum_{ij} s_{ij}^2 + 2X_+b + 2a(Y_+ - Z_+b) + 2\bar{a}\bar{b}|Z_-|
\end{aligned}$$

as operators $\hat{Y}_A, \hat{Y}_B$ are commuting and have eigenvalues ± 1. Applications of Cauchy inequality lead to

$$\begin{aligned}
\hat{B}_s^2 &\leqslant \sum_{ij} s_{ij}^2 + 2X_+b + 2\sqrt{(Y_+ + Z_+b)^2 + \bar{b}^2Z_-^2} \\
&= \sum_{ij} s_{ij}^2 + \frac{X_+Y_+Z_+}{2s} + \frac{X_+}{\sqrt{s}}\left(2b\sqrt{s} - \frac{Y_+Z_+}{2\sqrt{s}}\right) \\
&\quad + 2\sqrt{\frac{Y_-^2Z_-^2}{4s} - \left(2b\sqrt{s} - \frac{Y_+Z_+}{2\sqrt{s}}\right)^2} \\
&\leqslant \sum_{ij} s_{ij}^2 + \frac{X_+Y_+Z_+}{2s} + \sqrt{4 + \frac{X_+^2}{s}} \times \frac{|Y_-Z_-|}{2\sqrt{s}} \\
&= \sum_{ij} s_{ij}^2 + \frac{X_+Y_+Z_+}{2s} + \frac{|X_-Y_-Z_-|}{2s} \\
&= \sum_{ij} s_{ij}^2 + \frac{s}{s_{ij}^2} \\
&= T_s^2.
\end{aligned}$$

That is, the maximal eigenvalue of $\hat{B}_s$ over all possible states and local measurements is T_s.

APPENDIX B: CONDITION FOR QUANTUM ADVANTAGES

Lemma. For a set of four positive numbers $\{s_{xy}\}$ there exists positive λ and four angles θ_{xy} satisfying

$$\sum_{xy}(-1)^{xy}\theta_{xy} = 0, \quad 0 < \theta_{xy} < \pi,$$

such that $s_{xy} = \lambda/\sin\theta_{xy}$ iff

$$\min_{x,y} s_{xy} \sum_{xy} \frac{1}{s_{xy}} > 2. \tag{B1}$$

Proof. Obviously we can define three angles

$$\sin\theta_{xy} = \frac{\lambda}{s_{xy}}, \quad (x, y) \neq (1, 1)$$

for some suitably chosen λ. Then the condition $\theta_{11} = \theta_{00} + \theta_{01} + \theta_{10}$ imposes a constraint on possible λ

$$\sin(\theta_{00} + \theta_{01} + \theta_{10}) = \sin\theta_{11} = \frac{\lambda}{s_{11}}, \tag{B2}$$

which reads

$$\sin\theta_{00}\cos\theta_{01}\cos\theta_{10}+\sin\theta_{01}\cos\theta_{00}\cos\theta_{10}+\sin\theta_{10}\cos\theta_{00}\cos\theta_{01}-\sin\theta_{00}\sin\theta_{01}\sin\theta_{10}=\frac{\lambda}{s_{11}},$$

with $\cos\theta_{xy}=(-1)^{\omega_{xy}}\sqrt{1-\frac{\lambda^2}{s_{xy}^2}}$ for some $\omega_{xy}=0,1$ for $(x,y)\neq(1,1)$. Taking into account $\lambda\neq 0$ and by squaring the constraint above, we have

$$\left(1-\frac{\lambda^2}{s_{10}^2}\right)\left(\frac{1-\frac{\lambda^2}{s_{01}^2}}{s_{00}^2}+\frac{1-\frac{\lambda^2}{s_{00}^2}}{s_{01}^2}+\frac{2\cos\theta_{00}\cos\theta_{01}}{s_{00}s_{01}}\right)$$
$$=\left(\frac{1}{s_{11}}+\frac{\lambda^2}{s_{01}s_{10}s_{00}}\right)^2+\frac{1}{s_{10}^2}\left(1-\frac{\lambda^2}{s_{00}^2}\right)\left(1-\frac{\lambda^2}{s_{01}^2}\right)-2\left(\frac{1}{s_{11}}+\frac{\lambda^2}{s_{01}s_{10}s_{00}}\right)\frac{\cos\theta_{00}\cos\theta_{01}}{s_{10}},$$

which simplifies into

$$2\left(\frac{1}{s_{11}s_{10}}+\frac{1}{s_{01}s_{00}}\right)\cos\theta_{00}\cos\theta_{01}=\frac{1}{s_{11}^2}+\frac{1}{s_{10}^2}-\frac{1}{s_{00}^2}-\frac{1}{s_{01}^2}+\frac{2\lambda^2}{s_{00}s_{01}}\left(\frac{1}{s_{11}s_{10}}+\frac{1}{s_{01}s_{00}}\right)$$

from which it follows, by squaring again,

$$4\left(\frac{1}{s_{11}s_{10}}+\frac{1}{s_{01}s_{00}}\right)^2\left(1-\frac{\lambda^2}{s_{00}^2}-\frac{\lambda^2}{s_{01}^2}\right)=\left(\frac{1}{s_{11}^2}+\frac{1}{s_{10}^2}-\frac{1}{s_{00}^2}-\frac{1}{s_{01}^2}\right)^2+\frac{4\lambda^2}{s_{00}s_{01}}\left(\frac{1}{s_{11}s_{10}}+\frac{1}{s_{01}s_{00}}\right)\left(\frac{1}{s_{11}^2}+\frac{1}{s_{10}^2}-\frac{1}{s_{00}^2}-\frac{1}{s_{01}^2}\right),$$

giving rise to possible λ as

$$\lambda^2=\frac{s^2}{4T_s^2}\prod_{u,v=0}^{1}\sum_{x,y}\frac{(-1)^{(u+x)(v+y)}}{s_{xy}}. \tag{B3}$$

As λ^2 must be greater than 0, we have equivalently

$$\sum_{xy}\frac{1}{s_{xy}}\geqslant\max_{x,y}\frac{2}{s_{xy}}, \tag{B4}$$

which is exactly condition Eq. (B1). One can also check

$$s_{xy}^2-\lambda^2=\frac{s^3}{4T_s^2}\left(\sum_{u,v}\frac{1}{s_{uv}^2}-\frac{2}{s_{xy}^2}+\frac{2s_{xy}^2}{s}\right)^2\geqslant 0$$

for all $x,y=0,1$, i.e., $|\lambda|\leqslant\min_{x,y}s_{xy}$. However, if we start with condition Eq. (B1) then we can introduce a positive λ as defined in Eq. (B3) from which four angles $0<\theta_{xy}<\pi$ are well defined by $\sin\theta_{xy}=\lambda/s_{xy}$ with $x,y=0,1$. By working backward from the process giving rise to Eq. (B3), we can obtain Eq. (B2) with suitably chosen ω_{xy}.

APPENDIX C: CHAINED BELL INEQUALITY B_m

In this Appendix, we solve the analytical Tsirelson bound of B_m. For the chained Bell inequality

$$B_m=\sum_{i=1}^{m}A_iB_i+\sum_{i=1}^{m-1}A_{i+1}B_i-A_1B_m$$

we have, as previously derived in Eq. (15),

$$b_m-m\leqslant\sum_{i=1}^{m-1}\sqrt{\frac{1+\cos\theta_i}{2}}+\sqrt{\frac{1-\cos\theta_m}{2}}:=t_m(\boldsymbol{\theta}).$$

Here θ_i is the angle between A_i and A_{i+1} and the angle between A_1 and A_m is $\theta_m=\sum_{i=1}^{m-1}\theta_i$ as all measurements are assumed to be in the same plane. The partial derivative with respect to θ_i gives

$$\frac{\partial t_m}{\partial\theta_i}=\frac{-\sin\theta_i}{2\sqrt{2(1+\cos\theta_i)}}+\frac{\sin\theta_m}{2\sqrt{2(1-\cos\theta_m)}}=0 \tag{C1}$$

from which it follows that

$$\frac{-\sin\theta_i}{2\sqrt{2(1+\cos\theta_i)}}=\frac{-\sin\theta_j}{2\sqrt{2(1+\cos\theta_j)}}$$

for any $i\neq j$. As a result all the θ_i must be equal. Then, we can set $\theta=\theta_i$ and plug it into Eq. (C1) and solve the equation, we can get $\theta=\frac{\pi}{m}$, $t_m(\frac{\pi}{m})=m\cos\frac{\pi}{2m}$, and $B_m\leqslant 2m\cos\frac{\pi}{2m}$.

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