Assessing non-Markovian dynamics through moments of the Choi state

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Non-Markovian effects in open quantum system dynamics usually manifest the backflow of information from the environment to the system, indicating complete-positive divisibility breaking of the dynamics. We provide a criterion for witnessing such non-Markovian dynamics exhibiting information backflow, based on partial moments of Choi matrices. The moment condition determined by the positive semi-definiteness of a matrix does not hold for a Choi state describing non-Markovian dynamics. We then present some explicit examples in support of our proposed non-Markovianity detection scheme. Finally, a moment-based measure of non-Markovianity for unital dynamics is formulated.

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I. INTRODUCTION

According to the postulates of quantum mechanics, closed systems evolve unitarily. However, due to the inevitable interaction with noisy environments, the system undergoes irreversible phenomena such as dissipation and decoherence. The theory of open quantum systems provides adequate tools for studying such dynamics comprised of system-environment interactions [1–8]. System-environment interactions are often assumed to be *Markovian* where the environment does not keep the memory of past interactions with the system and the interaction is considered to be sufficiently weak. However, in realistic scenarios, when the system-environment coupling is not sufficiently weak and the environment has some finite memory, the description of open quantum systems by the Markovian model may fall short leading to the requirement of the *non-Markovian* paradigm [9–13]. Unlike Markovian dynamics (i.e., the dynamics without memory effects), non-Markovian dynamics usually contains a backflow of information from the environment to the system providing a unique signature [6,7].

In recent times, much effort has been devoted to the study of quantum non-Markovian dynamics, which provides advantages in several quantum information processing tasks such as perfect teleportation with mixed states [14], efficient work extraction from Otto cycle [15], efficient quantum control [16], entangled state preparation [17,18], quantum metrology [19], quantum evolution speedup [20], and so on. Experimental realization of non-Markovianity has been achieved in trapped-ion, nuclear magnetic resonance and photonic systems indicating a potential resource for executing quantum information processing tasks in real systems [21–29].

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Despite several interesting applications of non-Markovianity, a fundamental and important question is to assess whether the underlying quantum dynamics is non-Markovian at all, so that one can utilize it as a resource in legitimate quantum information processing tasks. Therefore, identifying whether a dynamics provides non-Markovian traits is a substantial task for advancement of quantum technologies. Several methods have been proposed to date from different perspectives and utilizing different properties of non-Markovian dynamics [9,11–13,30–37]. In this work, we provide an adequate technique to efficiently detect non-Markovian dynamics entailed with environmental memory. Our approach is based on the determination of partial moments of the Choi state and does not require full process tomography, thereby making it easier to realize in a real experiment.

Our proposal utilizing the moment criterion requires the evaluation of simple functionals which can be efficiently estimated using an experimental technique called shadow tomography [38–40]. It is based on a recently proposed methodology for the simultaneous evaluation of several quantities for noisy intermediate scale quantum (NISQ) devices and is more efficient than the usual tomography. Moreover, it may be noted here that our criterion is state independent unlike the witness-based detection scheme for which prior information about the quantum state is necessary. We further provide two explicit examples in support of our detection scheme for non-Markovian evolution.

In addition to the task of detecting a non-Markovian dynamics, another important task is to provide a quantitative measure of non-Markovianity. However, non-Markovianity can be manifested in several ways indicating that there exists no common or general way of comparing non-Markovian dynamics for different physical models. Two measures of non-Markovianity proposed earlier were based on the concept of divisibility of the dynamical map (RHP measure) [5,9,41] and distinguishability of quantum states (BLP measure) [6,10,11]. In this work, we define a measure based on partial moments of the Choi matrix to quantify non-Markovianity.

The paper is organized as follows. In Sec. II, we provide a brief overview of the essential mathematical preliminaries concerning the dynamics of open quantum systems, as well as the moment criteria proposed in earlier works for entanglement detection. In Sec. III we present our framework for the detection of non-Markovianity along with some explicit examples. A measure of non-Markovianity is proposed in Sec. IV where we also compare our proposed measure with the RHP measure for the pure dephasing channel with Ohmic spectral density. Finally, in Sec. V, we summarize our main findings.

II. PRELIMINARIES

A. Dynamics of open quantum system

Isolated systems undergo unitary evolution. However, a general quantum evolution (or a quantum channel) can be represented by a completely positive trace-reserving (CPTP) map $[\Lambda(t,t_0)]$, which maps an element $[\rho(t_0)]$ of the set of density operators $[B(\mathcal{H})]$ to another element of the set i.e., $\Lambda(t,t_0):\rho(t_0)\mapsto\rho(t)$. The set of all such CPTP maps can be represented as D. We assume that the inverse $\Lambda^{-1}(t,t_0)$ exists for all time from t_0 to t. One can thus write the dynamical map for any $t\geqslant s\geqslant t_0$ into a composition

$$\Lambda(t,t_0)=\Lambda(t,s)\circ\Lambda(s,t_0). \tag{1}$$

Even though $\Lambda(t,t_0)$ is always completely positive since it must correspond to a physically legitimate dynamics [and hence $\Lambda^{-1}(t,t_0)$ is well defined] and $\Lambda(s,t_0)$ is completely positive, the map $\Lambda(t,s)$, however, need not be completely positive. A dynamics acting on the system of interest is said to be *divisible* iff it can be written as Eq. (1) for any time $t\geqslant s\geqslant t_0$, where t_0 is the initial time of dynamics and $\circ$ represents the composition between two maps. The dynamics $\Lambda(t,t_0)$ is said to be *positive divisible* (P divisible) if $\Lambda(t,s)$ is a positive map for every $t\geqslant s\geqslant t_0$ satisfying the composition law. The dynamics $\Lambda(t,t_0)$ is said to be *completely positive divisible* (CP divisible) if $\Lambda(t,s)$ is a CPTP map for every $t\geqslant s\geqslant t_0$ and satisfies the composition law.

The above mathematical characterization of a dynamical map $\Lambda(t,t_0)$ in terms of "divisibility" describing a memoryless evolution as a composition of physical maps leads to the definition of quantum Markovianity. According to the RHP criterion, a dynamics is said to be non-Markovian if it is not CP divisible [9]. Another way of characterizing non-Markovian dynamics is provided by Breuer *et al.* [10,11] where the distinguishability of quantum states after the action of the dynamical map is considered. Due to the interaction of a quantum system with the noisy environment, two quantum states lose their state distinguishability gradually with time. However, if at any instant of time, the distinguishability increases, then there is backflow of information from the environment to the system leading to the signature of non-Markovianity. The former way of representing a dynamics to be non-Markovian is known as RHP-type non-Markovianity [5,9], whereas the latter one is known as BLP-type non-Markovianity [6,10]. A dynamics which is Markovian in the RHP sense is also Markovian in the BLP sense, but the converse is not true in general. Therefore, CP divisibility breaking is a necessary, but not sufficient, condition for information backflow from the environment to the system. In this paper we adopt CP divisibility as the sole property of quantum Markovianity, and any deviation from CP divisibility (indivisible) will be considered as the benchmark of non-Markovianity.

Now, for each of the dynamical maps $\Lambda(t,t_0)\in D$, one can find a one-to-one correspondence to a state, called the Choi state $\mathcal{C}_\Lambda(t,t_0)\in\mathcal{F}$ (where $\mathcal{F}$ is the set of all Choi states) via channel-state duality where the Choi state [42] is defined as

$$\mathcal{C}_\Lambda(t,t_0)=(\mathbb{I}\otimes\Lambda(t,t_0))\,|\phi\rangle\,\langle\phi|\,, \tag{2}$$

with $|\phi\rangle\,\langle\phi|$ being a maximally entangled bipartite state of dimension $d\times d$. According to the Choi-Jamiolkowski isomorphism [42,43], for checking the complete-positivity of $\Lambda(t,t_0)$, it is sufficient to check the positive semi-definiteness of the corresponding Choi state $[\mathcal{C}_\Lambda(t,t_0)]$.

B. Partial moment criterion

In the bipartite scenario, one of the most well-known detection schemes of entanglement is based on the PPT criterion which examines whether the partial transposed state $\rho_{AB}^{T_A}$ (where partial transposition is taken with regard to subsystem A) is positive semi-definite (all eigenvalues are nonnegative) or not. Violation of this criterion implies that the given state ρ_{AB} is entangled. This criterion has been shown to be a necessary and sufficient condition for $2\otimes2$, $2\otimes3$, and $3\otimes2$ systems and has many applications in theoretical works [44–49]. However, the transposition map not being a physical one, it is impossible to implement exactly in an experimental scenario. A useful measure using this PPT criterion is the *negativity* measure [50], defined as

$$\mathbb{N}=||\rho_{AB}^{T_A}||=\sum_i|\lambda_i|,$$

where λ_i's are the negative eigenvalues of $\rho_{AB}^{T_A}$. However, this again requires an access to the full spectrum of $\rho_{AB}^{T_A}$, which is not obtainable through an experimental setup. To overcome this issue, the idea of moments of the partially transposed density matrix (PT moments) was introduced to study the correlations in many-body systems in relativistic quantum field theory by Calabrese *et al.* in 2012 [51].

For a bipartite state ρ_{AB}, these PT moments are given by

$$P_n=\mathrm{Tr}[({\rho_{AB}}^{T_A})^n], \tag{3}$$

for $n=1,2,3,\ldots$. One may note that $P_1=\mathrm{Tr}[\rho_{AB}^{T_A}]=1$ while $P_2=\mathrm{Tr}[(\rho_{AB}^{T_A})^2]$ is related to the purity of the state. Therefore, P_3 is the first nontrivial moment which is necessary to capture additional information related to the partial transposition. Using only these first three PT moments, a simple but powerful entanglement detection criterion was proposed in Ref. [52]. This suggests that, if a state ρ_{AB} is PPT, then ${P_2}^2\leqslant P_3P_1$. Therefore, from the contrapositivity of this statement, it follows that if a state ρ_{AB} violates this condition, then it must be entangled which is the p_3-PPT criterion. Just like the PPT condition, this p_3-PPT condition is also applicable to mixed states and is a state-independent criterion unlike entanglement witness [53,54]. While entanglement witness provides a stronger criterion for entanglement detection, some prior knowledge about the state is required for the implementation of entanglement witness.

Even though this p_3-PPT criterion is weaker than the general PPT criterion, the first involves simple functionals which are easy to realize in a real experiment by a method called shadow tomography [38–40]. For Werner states, the p_3-PPT criterion and the full PPT criterion are equivalent, and hence, the p_3-PPT criterion is a necessary and sufficient criteria for bipartite entanglement of Werner states. The PT moments can be obtained experimentally with the help of shadow tomography without actually performing full state tomography, thus making it more efficient in terms of resources consumed. For a detailed discussion on shadow tomography and its advantage over general tomography, interested readers are referred to Refs. [38–40,52]. The technique of PT moments offers unparalleled advantage in NISQ and in many-body systems where a single qubit is used as a control and many distinct PT moments can be estimated from the same data unlike using random global unitaries for randomized measurements [40,52]. Furthermore, the p_3 moment, in addition to detecting mixed-state entanglement [52,55,56] is also used to study entanglement dynamics in many-body quantum systems [57,58].

It might be noted here that the p_3-PPT condition provides a necessary condition for separability. However, each higher-order moment ($n \geqslant 4$) gives rise to an independent and different entanglement detection criterion, and evaluating all the higher-order moments provides a necessary and sufficient criterion for NPT entanglement [55]. However, this is very challenging from an experimental point of view, and hence, moments up to third order are used to provide an entanglement detection criteria and this simplifies the task. Motivated by the above considerations, in the next section we explore whether a moment-based detection scheme can be developed for non-Markovian dynamics since, in realistic scenarios, many different categories of dynamics consist of non-Markovian memory. We define Λ moments (r_n), and based on it we develop, a formalism for detection of non-Markovian dynamics characterized via indivisibility.

III. DETECTION OF NON-MARKOVIANITY

Definition 1. Let $\Lambda(t,s)$ be a trace preserving, linear map that satisfies the composition law (1). We define the nth -order Λ-moments (r_n) as

$$r_n = \mathrm{Tr}\{\,[\mathcal{C}_\Lambda(t,s)]^n\}, \tag{4}$$

with n being an integer and $\mathcal{C}_\Lambda(t,s)$ is the Choi state (defined earlier) corresponding to the dynamics acting on the system between the time intervals s and t, such that $s \leqslant t$. With the above definition, we are now ready to propose our criterion for detecting non-Markovian dynamics.

Theorem 1. If a dynamics is Markovian then

$$r_2{}^2 \leqslant r_3, \tag{5}$$

where r_2 and r_3 are defined in Eq. (4).

Proof. If t_0 be the initial time of a dynamics, then the time evolution of an open quantum system is governed by a family of completely positive trace-preserving maps $\{\Lambda(t,t_0)\}_{t\geqslant t_0}$ satisfying the composition law (1). Considering the concept of Choi-Jamiolkowski isomorphism [42,43], we shall henceforth use the Choi operator ($\mathcal{C}_\Lambda$) corresponding to the map $\Lambda(t,s)$.

Let us now consider the Schatten-p norms for $p \geqslant 1$, which are defined as

$$||X||_p = \left(\sum_{i=1}^n |\chi_i|^p\right)^{\frac{1}{p}} = [\mathrm{Tr}(|X|^p)]^{\frac{1}{p}}, \tag{6}$$

where X is a $n\times n$ Hermitian matrix having eigenvalue decomposition $X = \sum_{i=1}^n \chi_i \,|x_i\rangle\langle x_i|$. Replacing X by the Choi matrix $\mathcal{C}_\Lambda$ in Eq. (6), the Schatten-p norms for the Choi matrix are analogously defined. Further, the l_p norm of the vector of eigenvalues of $\mathcal{C}_\Lambda$ corresponding to each Schatten-p norm is defined by

$$||\lambda||_{l_p} := \left(\sum_{i=1}^n |\lambda_i|^p\right)^{\frac{1}{p}}, \tag{7}$$

where $\{\lambda_i\}_{i=1}^n$ is the spectrum of $\mathcal{C}_\Lambda$. The inner product corresponding to an n vector is defined as

$$\langle u, v\rangle := \sum_{i=1}^n u_i v_i, \tag{8}$$

for $u, v \in \mathbb{R}^n$. Now, from Hoelder's inequality for vector norms, we know that for $p, q \geqslant 1$ and $\frac{1}{p}+\frac{1}{q}=1$, the following relation holds:

$$|\langle u, v\rangle| \leqslant \sum_{i=1}^n |u_i v_i| \leqslant ||u||_{l_p}||v||_{l_q}. \tag{9}$$

Putting $p=3$ and $q=\frac{3}{2}$ in Eq. (9), we obtain

$$\mathrm{Tr}[(\mathcal{C}_\Lambda)^2] = \langle\lambda,\lambda\rangle \leqslant ||\lambda||_{l_3}||\lambda||_{l_{\frac{3}{2}}} = ||\mathcal{C}_\Lambda||_3||\lambda||_{l_{\frac{3}{2}}}. \tag{10}$$

We next apply the Cauchy-Schwarz inequality which is obtained by putting $p=\frac{1}{2}$ and $q=\frac{1}{2}$ in Hoelder's inequality. Therefore,

$$\begin{aligned} ||\mathcal{C}_\Lambda||_2{}^2 &= \mathrm{Tr}[(\mathcal{C}_\Lambda)^2]||\mathcal{C}_\Lambda||_3||\lambda||_{l_{\frac{3}{2}}} \\ &= ||\mathcal{C}_\Lambda||_3\left(\sum_{i=1}^n |\lambda_i|^{\frac{3}{2}}\right)^{\frac{2}{3}} \\ &= ||\mathcal{C}_\Lambda||_3\left(\sum_{i=1}^n |\lambda_i||\lambda_i|^{\frac{1}{2}}\right)^{\frac{2}{3}} \\ &\leqslant ||\mathcal{C}_\Lambda||_3\left[\left(\sum_{i=1}^n |\lambda_i|^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^n |\lambda_i|\right)^{\frac{1}{2}}\right]^{\frac{2}{3}} \\ &= ||\mathcal{C}_\Lambda||_3||\mathcal{C}_\Lambda||_2{}^{\frac{2}{3}}||\mathcal{C}_\Lambda||_1{}^{\frac{1}{3}}. \end{aligned} \tag{11}$$

Taking the third power of Eq. (11), we obtain

$$||\mathcal{C}_\Lambda||_2{}^4 \leqslant ||\mathcal{C}_\Lambda||_3{}^3||\mathcal{C}_\Lambda||_1. \tag{12}$$

Since Λ is a trace-preserving map, $||\mathcal{C}_\Lambda||_1 = 1$, and hence Eq. (12) reduces to

$$||\mathcal{C}_\Lambda||_2{}^4 \leqslant ||\mathcal{C}_\Lambda||_3{}^3, \tag{13}$$

i.e.,

$$(\mathrm{Tr}[(\mathcal{C}_\Lambda)^2])^2 \leqslant \mathrm{Tr}[(\mathcal{C}_\Lambda)^3], \tag{14}$$

which completes the proof. ■

The above theorem indicates that condition (5) is necessary for a dynamics to be Markovian. Violation of above theorem is therefore sufficient to conclude that the underlying dynamics is actually CP indivisible and hence non-Markovian. Below, we present some explicit examples of non-Markovian dynamics which can be detected by the condition mentioned above.

A. Examples

We would now like to present two examples in support of Theorem 1. It may be noted that here we will consider the set of operations which have Lindblad-type generators. For system density matrix ρ, the Lindblad master equation can be written as

$$\frac{d\rho}{dt} = -\frac{\iota}{\hbar}[H,\rho] + \sum_i \gamma_i \left(L_i \rho L_i^\dagger - \frac{1}{2}(L_i^\dagger L_i \rho + \rho L_i^\dagger L_i) \right), \tag{15}$$

where the unitary aspects of the dynamics is described by the Hamiltonian H, γ_i are the Lindblad coefficients, and L_i are the Lindblad operators which describe the dissipative part of the dynamics [2].

1. Example 1

We consider a qubit system that interacts with an amplitude-damping environment which is modeled by another qubit system. The non-Markovian character of this model studied earlier [59] is motivated by the experimental realization of such non-Markovian dynamics through the violation of temporal Bell-like inequalities in a controllable nuclear magnetic resonance system [60]. The master equation is given by

$$\frac{d\rho}{dt} = \mathcal{L}(\rho) = \gamma_1(\sigma_x\rho\sigma_x - \rho) + \gamma_2(\sigma_y\rho\sigma_y - \rho) + \gamma_3(\sigma_z\rho\sigma_z - \rho). \tag{16}$$

The Lindblad coefficients are taken as $\gamma_1 = \gamma_2 = \gamma_3 = e^{-t'}\cos t'$ (for all i) and $t' = kt$, with k being a constant having the dimension of $[T^{-1}]$. The corresponding dynamical map is given by $\Lambda(\rho) = e^{\mathcal{L}}(\rho)$. For small time approximation (i.e., $|\epsilon\gamma_i| << 1$), the Choi state corresponding to $\Lambda(\rho)$ is given by

$$\mathcal{C}_\Lambda = (\mathbb{I} \otimes (\mathbb{I} + \epsilon\mathcal{L}))\,|\phi^+\rangle\,\langle\phi^+|, \tag{17}$$

with $|\phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$. Here, we consider $\epsilon = 0.001$ and $k = 1$. It is known that the above dynamics shows its non-Markovian nature when $\gamma_i < 0$. Therefore, for $\gamma_i < 0$, we should have $r_2^2 - r_3 > 0$, which is evident from Fig. 1.

2. Example 2

As a second example, we consider a pure dephasing non-Markovian dynamics. A qubit system interacts with a thermal reservoir which is modeled by an infinite set of harmonic oscillators in the vacuum state. The Hamiltonian corresponding to the system-reservoir interaction is

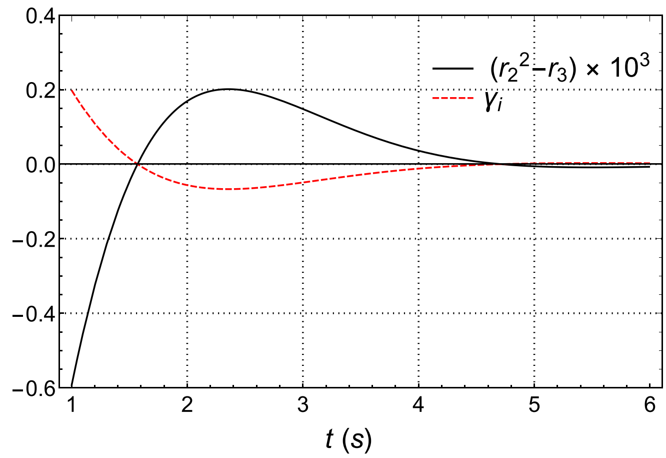

FIG. 1. Non-Markovian behavior of the dynamics given by Eq. (16), $r_2^2 - r_3 > 0$, is exhibited for $\gamma < 0$.

$$H = \omega_\sigma\sigma_z + \sum_k \omega_k a_k^\dagger a_k + \sum_k \alpha_k\sigma_z[a_k\exp(i\theta_k) + a_k^\dagger\exp(-i\theta_k)],$$

where ω_σ, ω_k are the energy gap of the system and the frequency of the kth mode of the reservoir, respectively, $a_k^\dagger(a_k)$ are the creation (annihilation) operators of the harmonic oscillator, α_k is the coupling constant for the kth mode, and θ_k is the corresponding phase. The master equation is given by

$$\frac{d\rho}{dt} = \mathcal{L}(\rho) = \gamma[\sigma_z\rho(t)\sigma_z - \rho(t)]. \tag{18}$$

The time-dependent decay rate γ corresponding to a Lorentzian spectral density is [59,61]

$$\gamma = \frac{2\lambda\gamma_0\sinh(t'g/2)}{g\cosh(t'g/2) + \lambda\sinh(t'g/2)}, \tag{19}$$

with $g = \sqrt{\lambda^2 - 2\gamma_0\lambda}$, $t' = kt$. Here k is a constant having the dimension of $[T^{-1}]$, λ is the spectral width, and γ_0 is the coupling strength.

In the small time approximation (i.e., $|\epsilon\gamma| << 1$), the Choi state corresponding to the dynamical map $\Lambda(\rho) = e^{\mathcal{L}}(\rho)$ is given by $\mathcal{C}_\Lambda = (\mathbb{I} \otimes (\mathbb{I} + \epsilon\mathcal{L}))\,|\phi^+\rangle\,\langle\phi^+|$ where $|\phi^+\rangle$ is the maximally entangled state defined earlier. It is known that the above dynamics shows its non-Markovian nature when $\gamma < 0$, which is possible only when $\gamma_0 > \lambda/2$ [59]. So, for $\gamma < 0$, we should have $r_2^2 - r_3 > 0$ which is again evident from Fig. 2. We consider here $\lambda = 1.5$, $\gamma_0 = 1$, and $k = 1$ for the figure.

IV. MEASURE OF NON-MARKOVIANITY

In this section, we would like to define a quantitative measure of non-Markovianity. Using Schatten-p norms for $p = 2$ and $p = 3$ we define a measure of non-Markovianity. Let us first denote

$$f(t) = \lim_{\epsilon \to 0} \frac{M_\epsilon}{\epsilon}, \tag{20}$$

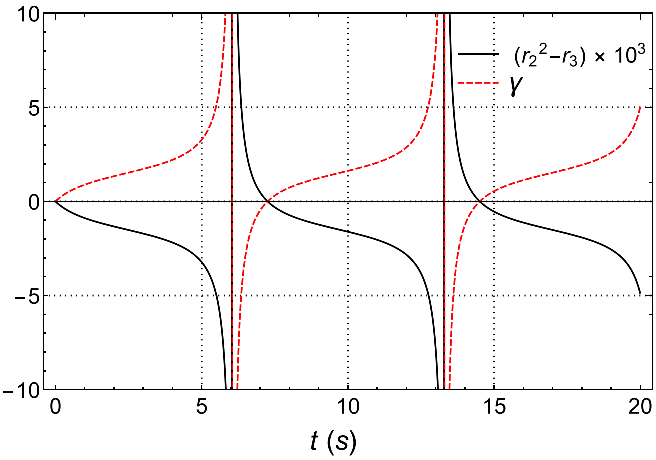

FIG. 2. Non-Markovian behavior of the dynamics given by Eq. (18), $r_2^2 - r_3 > 0$, is exhibited for $\gamma < 0$.

where

$$\begin{aligned} M_\epsilon &= M_T(t+\epsilon, t) \\ &= \max\{0, (||\mathcal{C}_\Lambda(t+\epsilon, t)||_2{}^2)^2 - ||\mathcal{C}_\Lambda(t+\epsilon, t)||_3{}^3\}. \end{aligned} \tag{21}$$

We define

$$\mathcal{M} = \int_0^\infty f(t)\, dt \tag{22}$$

as a measure of non-Markovianity.

Below we will show that $\mathcal{M}$ can be used as a measure of non-Markovianity for unital dynamics. To show that $\mathcal{M}$ as a measure of non-Markovianity for unital dynamics, we need to show that $\mathcal{M} = 0$ for all Markovian dynamics and $\mathcal{M}$ is monotone under divisible unital dynamics. It was shown earlier that for all unital dynamical maps having corresponding Lindblad generators, the Lindblad operators are normal [62]. Therefore, to prove the monotonicity of our measure we will consider that the Lindblad operators are normal.

Lemma 1. If the α-Renyi entropy defined by

$$S_\alpha(t) := \frac{1}{1-\alpha} \log_2 \mathrm{Tr}[X^\alpha(t)], \quad (\alpha > 0)$$

evolves under a Lindblad-type divisible operation having normal Lindblad operator, then

$$\frac{d}{dt} S_\alpha(t) \geqslant 0. \tag{23}$$

Proof. In Refs. [62–67], it was shown that

$$\frac{d}{dt} S_\alpha(t) = 2 \sum_i \gamma_i(t) \chi_i(t), \tag{24}$$

where

$$\chi_i(t) = \frac{\alpha}{1-\alpha} \frac{1}{\mathrm{Tr}[X^\alpha(t)]} \mathrm{Tr}[X^{\alpha-1}(t) L_i X(t) L_i^\dagger - X^\alpha(t) L_i^\dagger L_i],$$

with L_i being the Lindblad operator and γ_i are the Lindblad coefficients. Further [63]

$$\chi_i(t) > \langle [L_i^\dagger, L_i] \rangle_\alpha, \tag{25}$$

where $\langle A \rangle_\alpha = \mathrm{Tr}[A X^\alpha(t)]/\mathrm{Tr}[X^\alpha(t)]$. For the normal Lindblad operator $[L_i^\dagger, L_i] = 0$. Therefore, from Eq. (25), $\chi_i(t) > 0$. Also, if the dynamics is divisible, i.e., all $\gamma_i > 0$, then from Eq. (24) it immediately follows that $\frac{d}{dt} S_\alpha(t) \geqslant 0$. ■

Lemma 2. If the Schatten-p norms of a positive, Hermitian operator X evolve under divisible operation having Lindblad type generators, and the Lindblad operators are normal, then

$$||X(t+\delta t)||_p \leqslant ||X(t)||_p, \tag{26}$$

where $||X(t+\delta t)||_p$, $||X(t)||_p$ are the Schatten-p norms of X at times $t+\delta t$ and t, respectively.

Proof. From Lemma 1 it follows that for Lindblad-type divisible operation having normal Lindblad operator

$$S_\alpha(t+\delta t) - S_\alpha(t) \geqslant 0$$

$$\Rightarrow \frac{1}{1-\alpha} \log_2 \mathrm{Tr}[X^\alpha(t+\delta t)] \geqslant \frac{1}{1-\alpha} \log_2 \mathrm{Tr}[X^\alpha(t)].$$

Now, if $\alpha > 1$, then $\frac{1}{1-\alpha} < 0$ and hence

$$\begin{aligned} -\log_2 \mathrm{Tr}[X^\alpha(t+\delta t)] &\geqslant -\log_2 \mathrm{Tr}[X^\alpha(t)] \\ \Rightarrow \alpha \log_2 \mathrm{Tr}[X^\alpha(t+\delta t)] &\leqslant \alpha \log_2 \mathrm{Tr}[X^\alpha(t)] \\ \Rightarrow \mathrm{Tr}[X^\alpha(t+\delta t)]^\alpha &\leqslant \mathrm{Tr}[X^\alpha(t)]^\alpha. \end{aligned} \tag{27}$$

Since $\alpha > 0$, by taking the αth root of both sides, we obtain

$$\mathrm{Tr}[X^\alpha(t+\delta t)] \leqslant \mathrm{Tr}[X^\alpha(t)]. \tag{28}$$

From the definition of Schatten-p norms, it follows

$$||X(t+\delta t)||_\alpha{}^\alpha \leqslant ||X(t)||_\alpha{}^\alpha. \tag{29}$$

Again, taking the αth root of both sides we get

$$||X(t+\delta t)||_\alpha \leqslant ||X(t)||_\alpha. \tag{30}$$

Now, replacing α by p, it follows that

$$||X(t+\delta t)||_p \leqslant ||X(t)||_p. \tag{31}$$

■

The above two lemmas imply the following proposition.

Proposition 1. For unital dynamics having Lindblad-type generators, $\mathcal{M}$ is monotone under divisible operation.

Proof. To show $\mathcal{M}$ to be *monotone* under unital divisible dynamics, we have to prove that $\mathcal{M}$ satisfies

$$\begin{aligned} &(\mathrm{Tr}\{[\mathcal{C}_\Lambda(t+\delta t)]^2\})^2 - \mathrm{Tr}\{[\mathcal{C}_\Lambda(t+\delta t)]^3\} \\ &\leqslant (\mathrm{Tr}\{[\mathcal{C}_\Lambda(t)]^2\})^2 - \mathrm{Tr}\{[\mathcal{C}_\Lambda(t)]^3\}. \end{aligned} \tag{32}$$

It is known that for unital dynamics having Lindblad-type generators, the Lindblad operators are normal [62]. Therefore, replacing X by $\mathcal{C}_\Lambda$, lemma 2 implies that for $p = 2$ [it follows from Eq. (26)]

$$||\mathcal{C}_\Lambda(t+\delta t)||_2{}^4 \leqslant ||\mathcal{C}_\Lambda(t)||_2{}^4. \tag{33}$$

Using Eq. (33), we get

$$\begin{aligned} &||\mathcal{C}_\Lambda(t+\delta t)||_2{}^4 - ||\mathcal{C}_\Lambda(t+\delta t)||_3{}^3 \\ &= (\mathrm{Tr}[(\mathcal{C}_\Lambda(t+\delta t))^2])^2 - \mathrm{Tr}[(\mathcal{C}_\Lambda(t+\delta t))^3] \\ &\leqslant (\mathrm{Tr}[(\mathcal{C}_\Lambda(t))^2])^2 - \mathrm{Tr}(\mathcal{C}_\Lambda(t+\delta t))^3]. \end{aligned} \tag{34}$$

For any quantum state ρ, evolving under the Lindblad-type generators $\mathcal{L}$,

$$\frac{d\rho}{dt} = e^{\mathcal{L}}\rho, \tag{35}$$

it follows from Eq. (34) that

$$\begin{aligned}
&\{\mathrm{Tr}[\mathcal{C}_\Lambda^2(t+\delta t)]\}^2 - \mathrm{Tr}[\mathcal{C}_\Lambda^3(t+\delta t)] \\
&\quad \leqslant \{\mathrm{Tr}[\mathcal{C}_\Lambda^2(t)]\}^2 - \mathrm{Tr}[(I+\epsilon\mathcal{L}+\epsilon^2\mathcal{L}^2+\cdots)\mathcal{C}_\Lambda^3(t)] \\
&\quad \leqslant \{\mathrm{Tr}[\mathcal{C}_\Lambda^2(t)]\}^2 - \mathrm{Tr}[\mathcal{C}_\Lambda^3(t)].
\end{aligned} \tag{36}$$

Using Eq. (36), it can be seen that

$$\mathcal{M}(t+\delta t) \leqslant \mathcal{M}(t), \tag{37}$$

which proves that $\mathcal{M}$ is monotone under divisible unital dynamics. This completes the proof. ■

Theorem 2. $\mathcal{M}$ is a measure of unital Lindblad-type non-Markovian dynamics having normal Lindblad operators.

Proof. By the definition of $\mathcal{M}$ and from the proof of Theorem 1, it is clear that $\mathcal{M}=0$ for all Markovian dynamics. Furthermore, from Proposition 1 it follows that $\mathcal{M}$ is monotone under the divisible operation for Lindblad-type dynamics having normal Lindblad operators. Therefore, $\mathcal{M}$ satisfies the necessary properties of a measure and can be taken to be a measure of non-Markovianity for unital dynamics. ■

Comparison between our measure and the RHP measure

For a detailed comparison between our proposed measure ($\mathcal{M}$) and the RHP measure ($\mathcal{I}$), we consider a pure dephasing channel as presented in Example 2. The time-dependent dephasing rate is given by [61]

$$\gamma(t) = \int \frac{J(\omega)\coth(\hbar\omega/2k_BT)\sin(\omega t)}{\omega} d\omega, \tag{38}$$

where ω and $J(\omega)$ represent the frequency of the reservoir modes and spectral function of the reservoir, respectively. For Ohmic spectral density, the spectral function is given by

$$J(\omega) = \omega \exp\frac{-\omega}{\omega_c}, \tag{39}$$

where ω_c is the cutoff frequency of the reservoir.

In the small time approximation (i.e., $|\epsilon\gamma| \ll 1$), the Choi state corresponding to the dynamical map $\Lambda(\rho)=e^{\mathcal{L}}(\rho)$ is given by $\mathcal{C}_\Lambda = [\mathbb{I}\otimes(\mathbb{I}+\epsilon\mathcal{L})]\,|\phi^+\rangle\langle\phi^+|$ where $|\phi^+\rangle$ is the maximally entangled state.

RHP Measure. From the Choi-Jamiolkowski isomorphism, we know that the dynamical map $\Lambda_\epsilon = \Lambda(t+\epsilon, t)$ is CP iff $(\mathbb{I}\otimes\Lambda_\epsilon)\,|\phi^+\rangle\langle\phi^+| \geqslant 0\ \forall\epsilon$. From the trace-preserving condition, $||(\mathbb{I}\otimes\Lambda_\epsilon)\,|\phi^+\rangle\langle\phi^+|\,||_1 = 1$ iff Λ_ϵ is completely positive and $||(\mathbb{I}\otimes\Lambda_\epsilon)\,|\phi^+\rangle\langle\phi^+|\,||_1 > 1$, if Λ_ϵ is not completely positive. Considering these facts, one can define

$$g(t) = \lim_{\epsilon\to 0}\frac{||(\mathbb{I}\otimes\Lambda_\epsilon)\,|\phi\rangle\langle\phi|\,||_1 - 1}{\epsilon}, \tag{40}$$

where $\Lambda_\epsilon = (\mathbb{I}+\epsilon\mathcal{L})$. It is clear that $g(t)\geqslant 0$ and $g(t)=0$ iff Λ_ϵ is CP. The integral

$$\mathcal{I} = \int_0^\infty g(t)\,dt \tag{41}$$

is the RHP measure [5,9].

In the case of the pure dephasing channel with Ohmic spectral density, $\mathcal{C}_\Lambda = (\mathbb{I}\otimes(\mathbb{I}+\epsilon\mathcal{L}))\,|\phi^+\rangle\langle\phi^+|$ has eigenvalues $\lambda_1=0$, $\lambda_2=0$, $\lambda_3=\gamma\epsilon$, $\lambda_4=1-\gamma\epsilon$.

Therefore,

$$g(t) = \begin{cases} 0 & \text{when } \gamma(t)\geqslant 0, \\ -2\gamma & \text{when } \gamma(t)<0. \end{cases} \tag{42}$$

and

$$\mathcal{I} = \int_{\gamma<0} -2\gamma\,dt. \tag{43}$$

Our Proposed Measure. For the same example, the value of $f(t)$ defined in Eq. (20) turns out to be

$$f(t) = \begin{cases} 0 & \text{when } \gamma(t)\geqslant 0 \\ -\gamma & \text{when } \gamma(t)<0. \end{cases} \tag{44}$$

Therefore, our proposed measure becomes

$$\mathcal{M} = \int_{\gamma<0} -\gamma\,dt. \tag{45}$$

Comparing Eqs. (43) and (45), it turns out that the two measures are related by $\mathcal{I}=2\mathcal{M}$.

It is worth noting that a straightforward calculation for other well-known channels, such as the depolarizing channel, reveals that the RHP measure is two times that of our proposed measure. However, establishing a general relationship between the two measures for arbitrary channels remains a subject of further investigation.

V. CONCLUSION

Non-Markovianity of open quantum system dynamics has already been established as a useful resource for several information processing tasks [14–18]. However, prior to incorporating it into any information processing task, it is of foremost importance to detect the signature of such a resource. In this work, we developed a methodology to assess whether a dynamics attributes non-Markovian traits. Our proposal is based on the moment criterion of Choi matrices, which can be efficiently demonstrated in an experimental setup. We presented two explicit examples to illustrate our detection scheme. Further, we proposed a measure of non-Markovianity for unital dynamics which is again based on the partial moment criterion. Interestingly, our proposed measure turns out to be just half of the RHP measure for some of the well-known channels. However, the exploration of a general relationship between these two measures for arbitrary channels requires further analysis.

Our proposed non-Markovianity detection protocol requires computation of simple functionals without necessitating the evaluation of the full spectrum of the evolution, and hence, is a lot more efficient than full-process tomography [52]. Moreover, the protocol being state independent, no prior information about the dynamics is required unlike witness-based detection schemes [40]. Furthermore, since our proposed measure relies on moments of the Choi state, it is amenable for implementation in experiments utilizing the tools of shadow tomography [40]. The relevance of our proposed detection criterion should be further evident in the case of non-Markovian dynamics involving multiqubit systems wherein an exponentially lesser number of samples would be required. Extension of the moments-based criterion for non-Markovianity detection of multiqubit systems is therefore recommended as a natural off-shoot of our present analysis.

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