Association between quantum paradoxes based on weak values and a realistic interpretation of quantum measurements

Alice M. Aredes[1,2] and Pablo L. Saldanha[1,*]
[1]*Departamento de Física, Universidade Federal de Minas Gerais, Belo Horizonte, MG 31270-901, Brazil*
[2]*Instituto de Física, Universidade Federal do Rio de Janeiro, Caixa Postal 68528, Rio de Janeiro, RJ 21941-972, Brazil*

(Received 11 September 2023; accepted 7 February 2024; published 26 February 2024)

Many quantum paradoxes based on a realistic view of weak values were discussed in the last decades. They lead to astonishing conclusions such as the measurement of a spin component of a spin-$1/2$ particle resulting in $100\hbar$, the separation of a photon from its polarization, and the possibility of having three particles in two boxes without any two particles being in the same box, among others. Here, we show that the realistic view of the weak values present in these (and other) works is equivalent to a realistic (and highly controversial) view of quantum measurements, where a measurement reveals the underlying reality of the measured quantity. We discuss that all quantum paradoxes based on weak values simply disappear if we deny these realistic views of quantum measurements and weak values. Our work thus aims to demonstrate the strong assumptions and the corresponding problems present in the interpretation of these quantum paradoxes.

DOI: 10.1103/PhysRevA.109.022238

I. INTRODUCTION

The concept of weak value was introduced by Aharonov *et al.* in a seminal paper in 1988 [1]. This concept has found many important applications. In the field of quantum metrology, it is associated to the weak value amplification, a technique that can amplify the perturbation one wants to measure, making it detectable by existing detector schemes [2–5]. The measurement of weak values is also useful for directly obtaining complex quantities of quantum systems, such as wave functions [6–8] and geometrical phases [9,10], among other applications [11–13].

Besides the many applications using the weak value concept, many quantum paradoxes based on it were discussed in the past decades. The title of the original paper states one of these paradoxes: "How the result of a measurement of a component of the spin of a spin-$1/2$ particle can turn out to be $100\hbar$" [1]. In the three-box paradox [14,15], the authors say that there may be a situation with one quantum particle and three boxes where, "in spite of the fact that we have only one particle in the above situation, we find this particle with probability one in any one of the first two boxes" [14]. In the past-of-a-quantum-particle paradox [16,17], the authors state that "the photons tell us that they have been in the parts of the interferometer through which they could not pass" [17]. In the quantum Cheshire cat paradox [18,19], the authors state that "in the curious way of quantum mechanics, photon polarization may exist where there is no photon at all" [18]. In a sequence of the quantum Cheshire cat paradox [20,21], it was stated that it would be possible to "decouple two photons from their respective polarizations and then interchange them during recombination" [20]. In the quantum violation of the pigeonhole principle paradox [22–25], the authors say that they "find instances when three quantum particles are put in two boxes, yet no two particles are in the same box" [22].

Something that is not always clearly stated in many of the papers dealing with quantum paradoxes based on weak values is that their paradoxical conclusions depend on a realistic interpretation of the weak values. In other (nonrealistic) views, the experimental predictions and experimental results can be understood as simple quantum interference effects. Simple interferometric descriptions, free from paradoxical conclusions, were presented for the paradoxes of the spin measurement resulting in $100\hbar$ [26], of the past of a quantum particle [27–29], of the quantum Cheshire cat [30,31] and its sequence [32], and of the quantum violation of the pigeonhole principle [32], while the three-box paradox was experimentally implemented with classical light [15], thus also having an interferometric explanation.

The main objective of the present paper is to present a single argument to criticize all quantum paradoxes cited above, among others (including one proposed here), associating a realistic view of the weak values to the following realistic interpretation of a quantum measurement [33]: *A measurement performed on a quantum system reveals the underlying ontological value of the measured quantity, that continues the same after the measurement is performed*. This interpretation is highly controversial, generating numerous paradoxes. For instance, if a measurement reveals a preexisting value for the measured quantity, this value must depend on which other compatible observers are simultaneously measured, due to quantum contextuality [34]. In the present work, we conclude that all the cited paradoxes disappear if we simply deny this realistic and controversial interpretation of a quantum measurement, also denying a realistic view of the weak values. In this sense, the cited paradoxes can be seen as demonstrations that realistic interpretations of quantum measurements and of

*saldanha@fisica.ufmg.br

weak values lead to inconsistencies, not as demonstrations of astonishing behaviors of nature.

II. GENERAL ARGUMENT

In a weak measurement procedure, a quantum system is preselected in a state $|\psi_i\rangle$ and postselected in a state $|\psi_f\rangle$. Between the pre- and postselection, the system interacts with a probe (which is also a quantum system) that extracts some information. The weakness of the interaction means that the quantum system induces a small change on the probe state. So, considering the probe system as a pointer, the central position of the pointer wave function is displaced by an amount much smaller than its initial quantum uncertainty. This displacement is proportional to the real part of the weak value $\langle O\rangle_w$ of the observable O from the system that rules the system-pointer interaction, defined as [1,11]

$$\langle O\rangle_w = \frac{\langle\psi_f|O|\psi_i\rangle}{\langle\psi_f|\psi_i\rangle}. \tag{1}$$

The weak value $\langle O\rangle_w$ may assume values outside the eigenvalue spectrum of the observable O, including complex values. Since the real part of $\langle O\rangle_w$ is associated to the shift of the measuring device pointer, $\langle O\rangle_w$ is sometimes considered to be a quantity associated to the result of a measurement. However, since in a weak measurement procedure the shift of the pointer must be much smaller than its quantum uncertainty, many repetitions of the protocol are necessary to experimentally extract the weak value [1,11]. In fact, for an accurate determination of the weak value, infinite repetitions are necessary [35].

Our method for associating a realistic interpretation of the weak values to the cited realistic interpretation of quantum measurements applies to observables O that can be written as the sum of an observable P from which the preselected state is an eigenvector with eigenvalue p and an observable Q from which the postselected state is an eigenvector with eigenvalue q:

$$O = P + Q, \text{ with } \quad P|\psi_i\rangle = p|\psi_i\rangle, \quad Q|\psi_f\rangle = q|\psi_f\rangle. \tag{2}$$

For observables O that obey Eq. (2), the weak value can be written as

$$\langle O\rangle_w = \langle P\rangle_w + \langle Q\rangle_w = \frac{\langle\psi_f|P|\psi_i\rangle}{\langle\psi_f|\psi_i\rangle} + \frac{\langle\psi_f|Q|\psi_i\rangle}{\langle\psi_f|\psi_i\rangle} = p + q, \tag{3}$$

where Eq. (2) and the fact that q is real were used. In our argument, the operators P and Q may commute or not. For most of the observables O relevant in the quantum paradoxes to be treated in the next section, their decomposition as in Eq. (2) are done with operators P and Q that do not commute.

In this work we use tildes to represent physical quantities with an object reality, under the realistic assumptions we discuss. For instance, if S_z is the operator representing the z component of the spin of a spin-1/2 particle, by writing $\tilde{S}_z = \hbar/2$ we are assuming that the z component of the particle spin has the ontological value $\hbar/2$ at that time. The realistic assumption that leads to all the quantum paradoxes cited before [1,14–25] is that the weak value $\langle O\rangle_w$ of an operator O reveals the objective reality of the physical quantity $\tilde{O}$ associated to this operator at a time between the pre- and postselections. Under this realistic view of the weak values, for observables that can be written as in Eq. (2) we have $\tilde{P} = \langle P\rangle_w = p$, $\tilde{Q} = \langle Q\rangle_w = q$, and $\tilde{O} = \langle O\rangle_w = p + q$.

Let us now consider the cited realistic interpretation of quantum measurements in this situation. Since it is assumed that a measurement reveals the underlying ontological value of the measured quantity, considering that the postselection of the state $|\psi_f\rangle$ includes a measurement of the observable Q resulting in the eigenvalue q, we conclude that we have an objective value $\tilde{Q} = q$ for this physical quantity before this measurement. With the assumption that the objective value of the measured quantity continues the same after the measurement is performed, considering that the preselection of the state $|\psi_i\rangle$ includes a measurement of the observable P resulting in the eigenvalue p, we conclude that $\tilde{P} = p$ after this measurement. So, at a time between the pre- and postselection, we have $\tilde{P} = p$, $\tilde{Q} = q$, and, consequently, $\tilde{O} = \tilde{P} + \tilde{Q} = p + q$. Note that the equality $\tilde{O} = \tilde{P} + \tilde{Q}$ only holds because both physical quantities $\tilde{P}$ and $\tilde{Q}$ have definite values at the same time in this example under our assumptions. We thus see that, when the observable O can be written as in Eq. (2), the attribution of a physical reality to the weak value $\langle O\rangle_w$ is equivalent to adopting the cited realistic interpretation of quantum measurements, since both assumptions lead to the same objective value for the physical quantity $\tilde{O}$: $\tilde{O} = p + q$.

In the following, we show that all observables used in the cited quantum paradoxes based on weak values [1,14–25] can be written as in Eq. (2). Since the cited realistic interpretation of quantum measurements is highly controversial, a reasonable way to avoid all the cited quantum paradoxes is to deny this realistic interpretation of quantum measurements, also denying the realistic view of the weak values.

III. QUANTUM PARADOXES REVISITED

A. Measurement of spin resulting in 100$\hbar$

Let us discuss the paradox of the original paper from Aharonov *et al.* dealing with a spin-1/2 particle [1]. The particle is preselected by a Stern-Gerlach apparatus with the magnetic field in a direction ξ in the xz plane that makes an angle α with the x direction, obtaining a spin component $+\hbar/2$ in this direction ξ. The postselection is made by a Stern-Gerlach apparatus with magnetic field in the x direction, obtaining a spin component $+\hbar/2$ in the direction x. The weak value of the z component of spin can be computed by Eq. (1) and the result is $\langle S_z\rangle_w = (\hbar/2)\tan(\alpha/2)$ [1]. For $\alpha = 179.43°$, we have $\langle S_z\rangle_w \approx 100\hbar$. If, between the pre- and postselection, the quantum particle interacts with a small nonuniform magnetic field in the z direction, its average momentum gain will be the same as the one of a particle with spin 100$\hbar$ (and the same gyromagnetic ratio) [1]. This fact may lead to the attribution of a physical reality to the z component of the particle spin as being $\tilde{S}_z = \langle S_z\rangle_w \approx 100\hbar$. In this realistic view, the measurement of the particle momentum deviation would be a measurement of the particle spin component in the z direction between the two Stern-Geralch apparatuses. However, this *weak measurement* procedure only works if the momentum deviation is much smaller than the initial

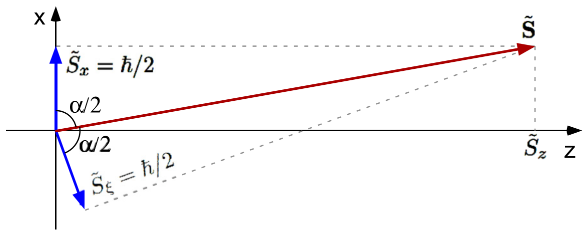

FIG. 1. Realistic view of the spin components in the paradox of a spin measurement resulting in $100\hbar$ [1].

momentum quantum uncertainty of the particle in the z direction, such that the result can be completely understood in interferometric terms [26]. The postselection selects a portion of the initial momentum wave function with higher values of momentum, such that the average momentum of the selected wave function has a relatively large value, but only contains momentum components that were already present in the initial distribution. So, there is no need to attribute a physical reality to the weak value of the z component of the particle spin [26].

Let us now see what are the conclusions about the value of the z component of the particle spin between the Stern Gerlach measurements using the cited realistic interpretation of quantum measurements. Under the pre- and postselection, we attribute objective values to the spin components in the directions of the apparatuses' magnetic fields at a time between the measurements $\tilde{S}_\xi = \hbar/2$, $\tilde{S}_x = \hbar/2$. As depicted in Fig. 1, these values for these spin components imply a z component of the spin $\tilde{S}_z = (\hbar/2)\tan(\alpha/2)$ equal to $\langle S_z\rangle_w$. The equivalence between $\tilde{S}_z$ and $\langle S_z\rangle_w$ occurs because we have $S_z = [S_\xi - \cos(\alpha)S_x]/\sin(\alpha)$, which is in the form of Eq. (2) with $O = S_z$, $P = S_\xi/\sin(\alpha)$, and $Q = -S_x/\tan(\alpha)$. The realistic interpretation of the particle spin as the vectors represented in Fig. 1 is certainly controversial, and leads to the same conclusions as the realistic view of the weak values, with the z component of spin having the value $\tilde{S}_z = (\hbar/2)\tan(\alpha/2)$ at a time between the measurements. Since both realistic interpretations lead to the same conclusions here and in the other cases we treat, we say they are equivalent.

B. Three-box paradox

Let us now consider the three-box paradox, where a realistic view of the weak values leads to the conclusion that a quantum particle can be found at two different boxes with probability 1 in each box [14]. A quantum particle is prepared in a superposition state of being in three orthogonal states $|A\rangle$, $|B\rangle$, and $|C\rangle$, which can be considered as the states of the particle inside boxes labeled A, B, and C, respectively. The preselected state is $|\psi_i\rangle = (|A\rangle + |B\rangle + |C\rangle)/\sqrt{3}$ and the postselected state is $|\psi_f\rangle = (|A\rangle + |B\rangle - |C\rangle)/\sqrt{3}$. Projectors $\Pi_j = |j\rangle\langle j|$ (with $j = \{A, B, C\}$) are associated to the presence of the particle in box j, with an eigenvalue 1 representing the presence and an eigenvalue 0 the absence of the particle in the corresponding box. For this configuration, we compute the following weak values using Eq. (1): $\langle\Pi_A\rangle_w = 1$, $\langle\Pi_B\rangle_w = 1$, $\langle\Pi_C\rangle_w = -1$. By adopting a realistic view of the weak values, the authors conclude that the probability of the particle to be found in box A is 1 (since $\langle\Pi_A\rangle_w = 1$), as well as the probability that the particle to be found in box B (since $\langle\Pi_B\rangle_w = 1$) [14].

There are many different ways to decompose the operators Π_A, Π_B, and Π_C of the three-box paradox in the form of Eq. (2). We only show one possible decomposition for each operator below, in the basis $\{|A\rangle, |B\rangle, |C\rangle\}$ in matrix form. For Π_A, with $O = \Pi_A$, we can write

$$P = \begin{pmatrix} 5/3 & -7/6 & 0 \\ -7/6 & 7/6 & 1/2 \\ 0 & 1/2 & 0 \end{pmatrix},$$

$$Q = \begin{pmatrix} -2/3 & 7/6 & 0 \\ 7/6 & -7/6 & -1/2 \\ 0 & -1/2 & 0 \end{pmatrix},$$

$p = q = 1/2$. The physical interpretation of the above observables is not as simple as in the previous case that considered spin components, but the idea is the same. By assuming the cited realistic interpretation of quantum measurements, we obtain an ontological value for the presence of the particle in box A given by $\tilde{\Pi}_A = \tilde{P} + \tilde{Q} = p + q = 1$, in the same way as by assuming a realistic view of the weak value $\langle\Pi_A\rangle_w = 1$. For Π_B, with $O = \Pi_B$ in Eq. (2), we can write

$$P = \begin{pmatrix} -1/2 & 1 & 1/2 \\ 1 & 0 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix}, \quad Q = \begin{pmatrix} 1/2 & -1 & -1/2 \\ -1 & 1 & 0 \\ -1/2 & 0 & -1/2 \end{pmatrix},$$

$p = 1$, $q = 0$. By assuming the cited realistic interpretation of quantum measurements, we obtain $\tilde{\Pi}_B = p + q = 1 = \langle\Pi_B\rangle_w$. For Π_C, with $O = \Pi_C$ in Eq. (2), we can write

$$P = \begin{pmatrix} 1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 \\ -1/2 & -1/2 & 3/2 \end{pmatrix},$$

$$Q = \begin{pmatrix} -1/2 & -1/2 & 1/2 \\ -1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \end{pmatrix},$$

$p = 1/2$, $q = -3/2$. We obtain $\tilde{\Pi}_C = p + q = -1 = \langle\Pi_C\rangle_w$ under the cited assumptions. We see that the cited realistic interpretation of quantum measurements leads to the same conclusions as the realistic view of the weak values.

As mentioned before, the relevant operators for the paradox can be decomposed as in Eq. (2) in different ways. Consider that an operator O can be decomposed as in Eq. (2), but also as $O = P' + Q'$, with $P'|\psi_i\rangle = p'|\psi_i\rangle$ and $Q'|\psi_f\rangle = q'|\psi_f\rangle$. Obviously, we must have $p + q = p' + q'$. We can consider that the preselection is performed by the measurement of the observable P, such that we have $\tilde{P} = p$ under the realistic view of quantum measurements we consider in this work, or by the measurement of the operator P', such that we have $\tilde{P}' = p'$, or by both measurements, associating ontological values to both $\tilde{P}$ and $\tilde{P}'$. Similar considerations can be done for the postselection. In any case, we have $\tilde{O} = \langle O\rangle_w = \mathrm{p} + \mathrm{q} = \mathrm{p}' + \mathrm{q}'$, such that realistic interpretation of quantum measurements leads to the same conclusion as the realistic view of the weak values.

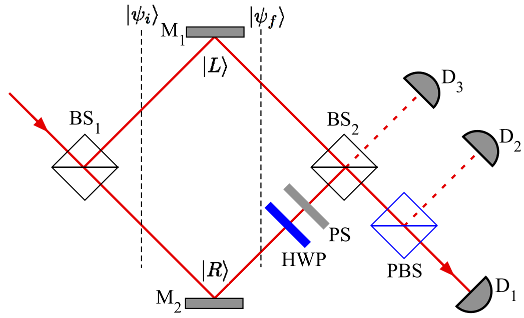

FIG. 2. Scheme of the quantum Cheshire cat paradox [18]. BS_1 and BS_2 are beam splitters, M_1 and M_2 are mirrors, PBS is a polarizing beam splitter, PS is a phase shifter, HWP is a half-wave plate, and D_1, D_2, and D_3 are photon detectors.

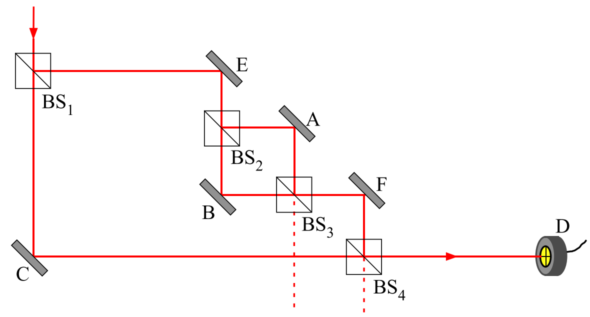

FIG. 3. Scheme of the paradox involving the past of a quantum particle [17]. BS_1, BS_2, BS_3, and BS_4 are beam splitters. A, B, C, E, and F are mirrors. D is a photon detector.

C. Quantum Cheshire cat

The quantum Cheshire cat paradox considers the interferometer depicted in Fig. 2 [18]. A photon with horizontal polarization is sent to the interferometer, such that after the beam splitter BS_1 the preselected state is $|\psi_i\rangle = (i|L\rangle + |R\rangle)|H\rangle/\sqrt{2}$, where $|L\rangle$ and $|R\rangle$ represent the possible photon paths in the interferometer, as depicted in the figure, and $|H\rangle$ and $|V\rangle$ represent horizontal and vertical linear polarizations for the photon, respectively. The phase shifter PS includes a phase $\pi/2$ in the corresponding path and the half-wave plate HWP rotates the polarization from $|H\rangle$ to $|V\rangle$, such that a photon detection by detector D_1 corresponds to the postselection of the state $|\psi_f\rangle = (|L\rangle|H\rangle + |R\rangle|V\rangle)/\sqrt{2}$ inside the interferometer. The presence of the photon in each path of the interferometer is associated to the observables $\Pi_L = |L\rangle\langle L|$ and $\Pi_R = |R\rangle\langle R|$. The polarization in the circular basis $|\pm\rangle = (|H\rangle \pm i|V\rangle)$ is associated to the observable $\sigma_z = |+\rangle\langle +| - |-\rangle\langle -|$. The circular polarization in each path is associated to the observables $\sigma_z^{(L)} = \Pi_L \otimes \sigma_z$ and $\sigma_z^{(R)} = \Pi_R \otimes \sigma_z$. The following weak values are found with the use of Eq. (1): $\langle \Pi_L\rangle_w = 1$, $\langle \Pi_R\rangle_w = 0$, $\langle \sigma_z^{(L)}\rangle_w = 0$, $\langle \sigma_z^{(R)}\rangle_w = 1$. By adopting a realistic view of the weak values, the authors conclude that the photon propagates through path L (since $\langle \Pi_L\rangle_w = 1$ and $\langle \Pi_R\rangle_w = 0$), but its polarization propagates through path R (since $\langle \sigma_z^{(L)}\rangle_w = 0$ and $\langle \sigma_z^{(R)}\rangle_w = 1$) [18]. But we reinforce that the theoretical predictions [18] and experimental results [19] of the quantum Cheshire cat effect can be explained as simple quantum interference effects [30,31].

To show that the cited realistic interpretation of quantum measurements leads to the same conclusions as the realistic view of the weak values in the quantum Cheshire cat paradox, such that these realistic assumptions may be considered equivalent, we only need to show that each of the observables Π_L, Π_R, $\sigma_z^{(L)}$, and $\sigma_z^{(R)}$ can be decomposed as in Eq. (2). We do this in Appendix A.

D. Past of a quantum particle

The paradox regarding the past of a quantum particle uses a scheme like the one depicted in Fig. 3, with a nested Mach-Zehnder interferometer [16,17]. A photon is sent to the interferometer with beam splitters projected such that its preselected state inside the interferometer is $\psi_i = (|A\rangle + i|B\rangle + |C\rangle)/\sqrt{3}$. A state $|j\rangle$ represents the photon in a path that includes mirror j. A photon detection by the detector D postselects the quantum state $\psi_f = (|A\rangle - i|B\rangle + |C\rangle)/\sqrt{3}$. This configuration implies that there is destructive interference in the inner interferometer for light exiting in the direction of mirror F. The weak values of the projectors $\Pi_A = |A\rangle\langle A|$, $\Pi_B = |B\rangle\langle B|$, and $\Pi_E = |E\rangle\langle E|$ are found to be $\langle \Pi_A\rangle_w = 1$, $\langle \Pi_B\rangle_w = -1$, $\langle \Pi_E\rangle_w = 0$. By using a realistic view of the weak values, the authors conclude that the photon passes through path A (since $\langle \Pi_A\rangle_w = 1$), but not through path E (since $\langle \Pi_E\rangle_w = 0$), which is impossible. It is important to stress that both the theoretical prediction [16] and the experimental implementation [17] of this paradox can be described as simple interference effects [27–29].

In Appendix B we show that each of the observables Π_A, Π_B, and Π_E of the paradox involving the past of a quantum particle can be decomposed as in Eq. (2). In this way, the cited realistic assumption regarding quantum measurements leads to the same conclusions as the realistic view of the weak values, so that these realistic assumptions are equivalent.

E. Quantum pigeonhole paradox

In the quantum violation of the pigeonhole principle [22], there is a system of three particles, each of which can be in orthogonal states $|L\rangle$ or $|R\rangle$, associated to the presence in one of two boxes L and R, as well as in superposition states. The preselected state is $\psi_i = |x\rangle_1|x\rangle_2|x\rangle_3$, with $|x\rangle = (|L\rangle + |R\rangle)/\sqrt{2}$. The postselected state is $\psi_f = |+\rangle_1|+\rangle_2|+\rangle_3$, with $|+\rangle = (|L\rangle + i|R\rangle)/\sqrt{2}$. The observable $\Pi_{1,2}^{\text{same}} = |L\rangle_1|L\rangle_2\langle L|_1\langle L|_2 + |R\rangle_1|R\rangle_2\langle R|_1\langle R|_2$ is associated to the presence of the particles 1 and 2 in the same box. The eigenvalue 1 corresponds to the two particles being in the same box, while the eigenvalue 0 corresponds to the two particles being in different boxes. Using Eq. (1), the weak value of this observable is $\langle \Pi_{1,2}^{\text{same}}\rangle_w = 0$. Since the pre- and postselected states are symmetric under the exchange of any two particles, we also have $\langle \Pi_{1,3}^{\text{same}}\rangle_w = \langle \Pi_{2,3}^{\text{same}}\rangle_w = 0$, with obvious notation. So, a realistic view of the weak values implies that no two particles are in the same box at times between the pre- and postselection, even if we have three particles and two boxes. But it is important to stress that the theoretical predictions [22] and experimental results [23,24] regarding this paradox can be understood as simple interference effects, without such paradoxical conclusions [32].

To show that the cited realistic interpretation of quantum measurements leads to the same conclusions as the realistic view of the weak values in the quantum violation of the pigeonhole principle, so that these realistic assumptions may be considered equivalent, we only need to show that each of the observables $\Pi_{1,2}^{\mathrm{same}}$, $\Pi_{1,3}^{\mathrm{same}}$, and $\Pi_{2,3}^{\mathrm{same}}$ can be decomposed as in Eq. (2). We do this in Appendix C.

F. A different paradox with a spin-2 particle

In this subsection we propose and criticize a different paradox based on a realistic view of the weak values. The paradox is inspired in the situation used in Ref. [33] to criticize the realistic interpretation of quantum measurements we are discussing here. Consider that a spin-2 particle is preselected in the state $|2\hbar\rangle_x$, an eigenstate of S_x with eigenvalue $2\hbar$, and post-selected in the state $|2\hbar\rangle_z$, an eigenstate of S_z with eigenvalue $2\hbar$ (S_i is the i component of the particle spin). It can be readily shown that the weak value of S_y^2 is given by $\langle S_y^2\rangle_w = -2\hbar^2$. We could argue that in this situation the square of the y component of the particle spin has a negative value, demonstrating an astonishing behavior of nature and contradicting the notion that the square of a real quantity must be positive or zero. But this conclusion would be based on a realistic view of the weak values that we can simply deny.

The same paradoxical conclusion is achieved with the cited realistic interpretation of quantum measurements, as discussed in Ref. [33]. We can write $S_y^2 = S^2 - S_x^2 - S_z^2$, with any state of a spin-2 particle being an eigenstate of S^2 with eigenvalue $6\hbar^2$. The operator S_y^2 can be decomposed as in Eq. (2), with $O = S_y^2$, $P = S^2 - S_x^2$, and $Q = -S_z^2$. So, if we simultaneously attribute physical reality to the quantities $\tilde{S}^2 = 6\hbar^2$, $\tilde{S}_x^2 = 4\hbar^2$, and $\tilde{S}_z^2 = 4\hbar^2$ at a time between the pre- and postselection, due to the results of the performed measurements, we conclude that $\tilde{S}_y^2 = \tilde{S}^2 - \tilde{S}_x^2 - \tilde{S}_z^2 = -2\hbar^2$.

IV. CONCLUSION

The assumption that a quantum measurement reveals the underlying reality of the measured quantity is certainly controversial, incapable of describing quantum phenomena without generating numerous paradoxes [33]. We have shown that this realistic assumption regarding quantum measurements leads to the same conclusions as the realistic view of the weak values in many quantum paradoxes described in the literature [1,14–25] and in one proposed here. So, instead of assuming the validity of the bizarre behaviors of nature described in these works, one may simply deny the cited realistic interpretation of a quantum measurement, also denying the realistic view of the weak values. Quantum mechanics is a very rich, complex, and difficult subject. We believe that the numerous quantum paradoxes based on a realistic view of the weak values present in the literature tend to obscure, rather than to clarify, the understanding of quantum phenomena. After all, we can interpret these paradoxes as showing that these realistic views of quantum measurements and weak values are not reasonable. We hope our work contributes to reinforce this point of view.

ACKNOWLEDGMENTS

This work was supported by the Brazilian agencies CNPq (Conselho Nacional de Desenvolvimento Científico e Tecnológico), CAPES (Coordenação de Aperfeiçoamento de Pessoal de Nível Superior), and FAPEMIG (Fundação de Amparo à Pesquisa do Estado de Minas Gerais).

APPENDIX A: OBSERVABLES OF THE QUANTUM CHESHIRE CAT PARADOX

Here, we show that each of the observables Π_L, Π_R, $\sigma_z^{(L)}$, and $\sigma_z^{(R)}$ of the quantum Cheshire cat paradox can be decomposed as in Eq. (2), in the basis $\{|L\rangle|H\rangle, |L\rangle|V\rangle, |R\rangle|H\rangle, |R\rangle|V\rangle\}$ in matrix form. One possible decomposition for Π_L, with $O = \Pi_L$ in Eq. (2), is

$$P = \begin{pmatrix} 2 & 1 & -i & -1 \\ 1 & 1 & -i & -1 \\ i & i & 2 & -i \\ -1 & -1 & i & 1 \end{pmatrix},$$

$$Q = \begin{pmatrix} -1 & -1 & i & 1 \\ -1 & 0 & i & 1 \\ -i & -i & -2 & i \\ 1 & 1 & -i & -1 \end{pmatrix},$$

$p = 1$, $q = 0$, which results in $\tilde{\Pi}_L = 1 = \langle \Pi_L\rangle_w$ with the cited realistic interpretation of quantum measurements. For Π_R, with $O = \Pi_R$ in Eq. (2), we can write

$$P = \begin{pmatrix} 0 & 1 & i & 1 \\ 1 & 1 & -i & -1 \\ -i & i & 0 & i \\ 1 & -1 & -i & 1 \end{pmatrix},$$

$$Q = \begin{pmatrix} 0 & -1 & -i & -1 \\ -1 & -1 & i & 1 \\ i & -i & 1 & -i \\ -1 & 1 & i & 0 \end{pmatrix},$$

$p = 1$, $q = -1$, which results in $\tilde{\Pi}_R = 0 = \langle \Pi_R\rangle_w$ under the realistic assumptions. For $\sigma_z^{(L)}$, with $O = \sigma_z^{(L)}$ in Eq. (2), we can write

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & i \\ 0 & 0 & 1 & 0 \\ 0 & -i & 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} -1 & -i & 0 & 0 \\ i & -1 & 0 & -i \\ 0 & 0 & -1 & 0 \\ 0 & i & 0 & -1 \end{pmatrix},$$

$p = 1, q = -1$, which results in $\tilde{\sigma}_z^{(L)} = 0 = \langle \sigma_z^{(L)}\rangle_w$ under the realistic assumptions. For $\sigma_z^{(R)}$, with $O = \sigma_z^{(R)}$ in Eq. (2), we can write

$$P = \begin{pmatrix} 1 & 1 & 0 & -1 \\ 1 & 1 & -i & -1 \\ 0 & i & 1 & -i \\ -1 & -1 & i & 1 \end{pmatrix},$$

$$Q = \begin{pmatrix} -1 & -1 & 0 & 1 \\ -1 & -1 & i & 1 \\ 0 & -i & -1 & 0 \\ 1 & 1 & 0 & -1 \end{pmatrix},$$

$p = 1$, $q = 0$, what results in $\tilde{\sigma}_z^{(R)} = 1 = \langle \sigma_z^{(R)} \rangle_w$ under the realistic assumptions.

APPENDIX B: OBSERVABLES OF THE PARADOX REGARDING THE PAST OF A QUANTUM PARTICLE

Here, we show that the relevant observables of the paradox regarding the past of a quantum particle (Π_A, Π_B, and Π_E) can be decomposed as in Eq. (2), in the basis $\{|A\rangle, |B\rangle, |C\rangle\}$ in matrix form. One possible decomposition for Π_A, with $O = \Pi_A$ in Eq. (2), is

$$P = \frac{1}{6}\begin{pmatrix} 2 & 0 & 1 \\ 0 & 0 & 3i \\ 1 & -3i & -1 \end{pmatrix}, \quad Q = \frac{1}{6}\begin{pmatrix} 4 & 0 & -1 \\ 0 & 0 & -3i \\ -1 & 3i & 1 \end{pmatrix},$$

$p = q = 1/2$, which results in $\tilde{\Pi}_A = 1 = \langle \Pi_A \rangle_w$ with the cited realistic interpretation of quantum measurements. For Π_B, with $O = \Pi_B$ in Eq. (2), we can write

$$P = \frac{1}{6}\begin{pmatrix} 1 & 3i & -1 \\ -3i & 3 & -3i \\ -1 & 3i & 1 \end{pmatrix}, \quad Q = \frac{1}{6}\begin{pmatrix} -1 & -3i & 1 \\ 3i & 3 & 3i \\ 1 & -3i & -1 \end{pmatrix},$$

$p = q = -1/2$, which results in $\tilde{\Pi}_B = -1 = \langle \Pi_B \rangle_w$ with the realistic interpretation of quantum measurements. Π_E is an eigenstate of $|\psi_f\rangle$ with eigevalue 0, so that it can be readily written in the form of Eq. (2) with $O = Q = \Pi_E$, $P = 0$, $p = q = 0$, which results in $\tilde{\Pi}_E = 0 = \langle \Pi_E \rangle_w$ under the realistic assumptions.

APPENDIX C: OBSERVABLES OF THE QUANTUM PIGEONHOLE PARADOX

Here, we show that each of the observables $\Pi_{i,j}^{\text{same}}$ of the quantum pigeonhole paradox can be decomposed as in Eq. (2). All these operators have the same form in the subspace of particles i and j in the basis $\{|L\rangle_i|L\rangle_j, |L\rangle_i|R\rangle_j, |R\rangle_i|L\rangle_j, |R\rangle_i|R\rangle_j\}$ in matrix form. One possible decomposition, with $O = \Pi_{i,j}^{\text{same}}$ in Eq. (2), is

$$P = \frac{1}{4}\begin{pmatrix} 5 & 5 & -4-i & -2+i \\ 5 & 5 & -6+5i & -5i \\ -4+i & -6-5i & 13 & 1+4i \\ -2-i & 5i & 1-4i & 5 \end{pmatrix},$$

$$Q = \frac{1}{4}\begin{pmatrix} -1 & -5 & 4+i & 2-i \\ -5 & -5 & 6-5i & 5i \\ 4-i & 6+5i & -13 & -1-4i \\ 2+i & -5i & -1+4i & -1 \end{pmatrix},$$

$p = 1$, $q = -1$, which results in $\tilde{\Pi}_{i,j}^{\text{same}} = 0 = \langle \Pi_{i,j}^{\text{same}} \rangle_w$ with the cited realistic interpretation of quantum measurements.

[1] Y. Aharonov, D. Z. Albert, and L. Vaidman, How the result of a measurement of a component of the spin of a spin-1/2 particle can turn out to be 100, Phys. Rev. Lett. **60**, 1351 (1988).

[2] O. Hosten and P. Kwiat, Observation of the spin Hall effect of light via weak measurements, Science **319**, 787 (2008).

[3] P. B. Dixon, D. J. Starling, A. N. Jordan, and J. C. Howell, Ultrasensitive beam deflection measurement via interferometric weak value amplification, Phys. Rev. Lett. **102**, 173601 (2009).

[4] G. B. Alves, B. M. Escher, R. L. de Matos Filho, N. Zagury, and L. Davidovich, Weak-value amplification as an optimal metrological protocol, Phys. Rev. A **91**, 062107 (2015).

[5] Y. Kim, S.-Y. Yoo, and Y.-H. Kim, Heisenberg-limited metrology via weak-value amplification without using entangled resources, Phys. Rev. Lett. **128**, 040503 (2022).

[6] J. S. Lundeen, B. Sutherland, A. Patel, C. Stewart, and C. Bamber, Direct measurement of the quantum wavefunction, Nature (London) **474**, 188 (2011).

[7] M. Mirhosseini, O. S. Magaña-Loaiza, S. M. Hashemi Rafsanjani, and R. W. Boyd, Compressive direct measurement of the quantum wave function, Phys. Rev. Lett. **113**, 090402 (2014).

[8] W.-W. Pan, X.-Y. Xu, Y. Kedem, Q.-Q. Wang, Z. Chen, M. Jan, K. Sun, J.-S. Xu, Y.-J. Han, C.-F. Li, and G.-C. Guo, Direct measurement of a nonlocal entangled quantum state, Phys. Rev. Lett. **123**, 150402 (2019).

[9] E. Sjöqvist, Geometric phase in weak measurements, Phys. Lett. A **359**, 187 (2006).

[10] Y.-W. Cho, Y. Kim, Y.-H. Choi, Y.-S. Kim, S.-W. Han, S.-Y. Lee, S. Moon, and Y.-Ho. Kim, Emergence of the geometric phase from quantum measurement back-action, Nat. Phys. **15**, 665 (2019).

[11] J. Dressel, M. Malik, F. M. Miatto, A. N. Jordan, and R. W. Boyd, *Colloquium*: Understanding quantum weak values: Basics and applications, Rev. Mod. Phys. **86**, 307 (2014).

[12] M. Huang, R.-K. Lee, L. Zhang, S.-M. Fei, and J. Wu, Simulating broken $\mathcal{PT}$-symmetric Hamiltonian systems by weak measurement, Phys. Rev. Lett. **123**, 080404 (2019).

[13] R. Wagner, W. Kersten, A. Danner, H. Lemmel, A. K. Pan, and S. Sponar, Direct experimental test of commutation relation via imaginary weak value, Phys. Rev. Res. **3**, 023243 (2021).

[14] Y. Aharonov and L. Vaidman, Complete description of a quantum system at a given time, J. Phys. A: Math. Gen. **24**, 2315 (1991).

[15] K. J. Resch, J. S. Lundeen, and A. M. Steinberg, Experimental realization of the quantum box problem, Phys. Lett. A **324**, 125 (2004).

[16] L. Vaidman, Past of a quantum particle, Phys. Rev. A **87**, 052104 (2013).

[17] A. Danan, D. Farfurnik, S. Bar-Ad, and L. Vaidman, Asking photons where they have been, Phys. Rev. Lett. **111**, 240402 (2013).

[18] Y. Aharonov, S. Popescu, D. Rohrlich, and P. Skrzypczyk, Quantum Cheshire cats, New J. Phys. **15**, 113015 (2013).

[19] T. Denkmayr, H. Geppert, S. Sponar, H. Lemmel, A. Matzkin, J. Tollaksen, and Y. Hasegawa, Observation of a quantum Cheshire cat in a matter-wave interferometer experiment, Nat. Commun. **5**, 4492 (2014).

[20] D. Das and A. K. Pati, Can two quantum Cheshire cats exchange grins? New J. Phys. **22**, 063032 (2020).

[21] Z.-H. Liu, W.-W. Pan, X.-Y. Xu, M. Yang, J. Zhou, Z.-Y. Luo, K. Sun, J.-L. Chen, J.-S. Xu, C.-F. Li, and G.-C. Guo, Experimental exchange of grins between quantum Cheshire cats, Nat. Commun. **11**, 3006 (2020).

[22] Y. Aharonov, F. Colombo, S. Popescu, I. Sabadini, D. C. Struppa, and J. Tollaksen, Quantum violation of the pigeonhole principle and the nature of quantum correlations, Proc. Natl. Acad. Sci. USA **113**, 532 (2016).

[23] M. Waegell, T. Denkmayr, H. Geppert, D. Ebner, T. Jenke, Y. Hasegawa, S. Sponar, J. Dressel, and J. Tollaksen, Confined contextuality in neutron interferometry: Observing the quantum pigeonhole effect, Phys. Rev. A **96**, 052131 (2017).

[24] M.-C. Chen, C. Liu, Y.-H. Luo, H.-L. Huang, B.-Y. Wang, X.-L. Wang, L. Li, N.-L. Liu, C.-Y. Lu, and J.-W. Pan, Experimental demonstration of quantum pigeonhole paradox, Proc. Natl. Acad. Sci. USA **116**, 1549 (2019).

[25] G. Reznik, S. Bagchi, J. Dressel, and L. Vaidman, Footprints of quantum pigeons, Phys. Rev. Res. **2**, 023004 (2020).

[26] I. M. Duck, P. M. Stevenson, and E. C. G. Sudarshan, The sense in which a “weak measurement” of a spin-1/2 particle’s spin component yields a value 100, Phys. Rev. D **40**, 2112 (1989).

[27] P. L. Saldanha, Interpreting a nested Mach-Zehnder interferometer with classical optics, Phys. Rev. A **89**, 033825 (2014).

[28] K. Bartkiewicz, A. Černoch, D. Javůrek, K. Lemr, J. Soubusta, and J. Svozilík, One-state vector formalism for the evolution of a quantum state through nested Mach-Zehnder interferometers, Phys. Rev. A **91**, 012103 (2015).

[29] B.-G. Englert, K. Horia, J. Dai, Y. L. Len, and H. K. Ng, Past of a quantum particle revisited, Phys. Rev. A **96**, 022126 (2017).

[30] R. Corrêa, M. F. Santos, C. H. Monken, and P. L. Saldanha, ‘Quantum Cheshire cat’ as simple quantum interference, New J. Phys. **17**, 053042 (2015).

[31] D. P. Atherton, G. Ranjit, A. A. Geraci, and J. D. Weinstein, Observation of a classical Cheshire cat in an optical interferometer, Opt. Lett. **40**, 879 (2015).

[32] R. Corrêa and P. L. Saldanha, Apparent quantum paradoxes as simple interference: Quantum violation of the pigeonhole principle and exchange of properties between quantum particles, Phys. Rev. A **104**, 012212 (2021).

[33] P. L. Saldanha, Inconsistency of a realistic interpretation of quantum measurements: A simple example, Braz. J. Phys. **50**, 438 (2020).

[34] C. Budroni, A. Cabello, O. Gühne, M. Kleinmann, and J.-Å. Larsson, Kochen-Specker contextuality, Rev. Mod. Phys. **94**, 045007 (2022).

[35] E. Haapasalo, P. Lahti, and J. Schultz, Weak versus approximate values in quantum state determination, Phys. Rev. A **84**, 052107 (2011).