Quantum complementarity from a measurement-based perspective

Shan Huang,[1,2,3] Wen-Bo Liu,[1] Yundu Zhao,[1,2] Hua-Lei Yin,[4,1,*] Zeng-Bing Chen,[1,†] and Shengjun Wu[1,2,3,‡]

[1]*National Laboratory of Solid State Microstructures and School of Physics, Collaborative Innovation Center of Advanced Microstructures, Nanjing University, Nanjing 210093, China*

[2]*Institute for Brain Sciences and Kuang Yaming Honors School, Nanjing University, Nanjing 210023, China*

[3]*Hefei National Laboratory, University of Science and Technology of China, Hefei 230088, China*

[4]*Department of Physics and Beijing Key Laboratory of Opto-Electronic Functional Materials and Micro-Nano Devices, Key Laboratory of Quantum State Construction and Manipulation (Ministry of Education), Renmin University of China, Beijing 100872, China*

(Received 15 July 2023; revised 28 September 2023; accepted 1 February 2024; published 26 February 2024)

One of the most remarkable features of quantum physics is that attributes of quantum objects, such as the wavelike and particlelike behaviors of single photons, can be complementary in the sense that they are equally real but cannot be observed simultaneously. Quantum measurements, serving as windows providing views into the abstract edifice of quantum theory, are basic tools for manifesting the intrinsic behaviors of quantum objects. However, a quantitative formulation of complementarity that highlights its manifestations in general measurement scenarios remains elusive. Here we develop a general framework for demonstrating quantum complementarity in the form of information exclusion relations (IERs), which incorporates the wave-particle duality relations as particular examples. Moreover, we explore the applications of our theory in entanglement detection and elucidate that our IERs lead to an extended form of entropic uncertainty relations, providing insights into the connection between quantum complementarity and the preparation uncertainty.

DOI: 10.1103/PhysRevA.109.022235

I. INTRODUCTION

Quantum mechanics imposes fundamental limits on an observer's information gain in complementary measurements. In the light of Bohr's complementarity principle [1], quantum systems possess mutually exclusive properties that are equally real, and a measurement to reveal one property would inevitably preclude all the complementary ones. Characterizing this subtle relationship between measurement strategy and information gain is significant for the sophisticated manipulation of quantum measurements in various tasks, from demonstrating genuine nonclassical features of quantum objects to general quantum information processing.

Wootters and Zurek [2] proposed the first quantitative statement of complementarity relation by taking an information-theoretical perspective into the competitive trade-off between the wavelike and particlelike behaviors of single photons. This kind of wave-particle duality relations (WPDRs) is currently expressed in a concise inequality form [3–6] for photons within the Mach-Zehnder interferometer (MZI; see Fig. 3). For example, Jaeger *et al.* [4] and Englert [6] obtained the duality relation $\mathcal{V}^2 + \mathcal{D}^2 \leqslant 1$ between fringe visibility (wave property) $\mathcal{V}$ and path distinguishability (particle property) $\mathcal{D}$. It is thus obvious that better which-way information implies less wave information, and vice versa.

Heisenberg's uncertainty principle [7] is another fundamental concept in quantum mechanics which captures similar underlying physics of complementarity. It states that specific quantum observables, such as position and momentum of single particles, cannot be known with arbitrary precision simultaneously or both measured with certainty. Modern formulations of the uncertainty principle typically use entropic uncertainty measures due to their operational significance [8,9], known as entropic uncertainty relations (EURs). EURs have widespread applications in quantum information processing tasks [10], e.g., the security analysis of quantum protocols [11–13]. The connections and contrasts between uncertainty and complementarity have been intensively debated [14–20]. It has been wondered whether novel complementarity relations can be derived directly from the well-studied and already-proven EURs. Particularly, Coles *et al.* [20–23] proved that several WPDRs can be equivalently reformulated as EURs for measuring complementary observables, i.e., measurements in mutually unbiased bases (MUBs). Two fundamental concepts of quantum mechanics are thus unified in this simple case.

Nevertheless, entropy is a natural measure of lack of information regarding only observation-independent properties and becomes conceptually inadequate [24] for quantum properties which are contextual and do not exist prior to measurements [25,26]. To avoid this dilemma, Brukner and Zeilinger proposed an operationally invariant information measure of quantum systems [27]. This measure is naturally aligned with the concept of complementarity as being elegantly defined as the sum of individual measures of information gain over a complete set of MUBs (CMUBs) [28–31]. Furthermore, it is independent of particular choices of CMUBs and invariant under any unitary time evolution. These intriguing properties inspired a series of insightful

*hlyin@ruc.edu.cn

†zbchen@nju.edu.cn

‡sjwu@nju.edu.cn

investigations [32–39], including quantum state estimation [32,33] and uncertainty relations for MUBs [37,38].

In this paper, we adopt the operationally invariant measure [27] of complete information content contained in quantum systems and develop a general framework for characterizing quantum complementarity beyond WPDRs, in terms of basic limits on one's ability to gain information about quantum systems under certain measurement setups, i.e., information exclusion relations (IERs). In contrast to the IERs [40–44] for nondegenerate observables in terms of Shannon entropic mutual information or deriving complementarity relations from EURs [20–23], our framework applies to generalized measurements. We emphasize that when considering generalized measurements, identifying certainty of outcome statistics with information gain or visibility of physical property faces conceptual challenge: an outcome predictable with 100% certainty not necessarily reflects the complete information of the measured system. We thus introduce a measure of information gain in individual measurements which well captures the complete information of quantum systems as conserved quantities comprised of complementary pieces. On the way, we establish IERs applicable to much more general measurement scenarios compared with previous ones, and show how they lead to tight WPDRs in the MZI arrangement. Aside from the significance of our IERs in interpreting quantum complementarity, we also explore their applications in entanglement detection and derive from them an extended form of EURs.

This paper is structured as follows. In Sec. II A, we introduce some preliminary notations. In Sec. II B, we propose a measure of information gain in individual measurements while formalizing the concept of complementary information. In Secs. II C and II D, we proceed to establish IERs which restrict one's weighted sum of information gains over multiple measurements, with and without quantum memory, respectively. In Sec. II E, we show how our IERs lead to tight WPDRs. In Sec. II F, we explore practical applications of our IERs. Finally, we briefly conclude this work in Sec. III.

II. RESULTS

A. Preliminary

On a d-dimensional Hilbert space $\mathcal{H}_d$, each generalized measurement, i.e., positive-operator-valued measure (POVM), is a collection of positive-semidefinite operators (called effects) $\mathcal{M} = \{M_i\}$ that sum up to the identity operator: $M_i \geqslant 0$ and $\sum_i M_i = \mathbb{1}_d$. In particular, the measurement of a nondegenerate observable is described by rank-1 projectors onto its eigenvectors, i.e., rank-1 projective measurement. When a quantum state ρ is measured, the outcome probabilities are given by Born's rule, $p_i = \mathrm{tr}(M_i\rho)$.

The Choi-Jamiołkowski isomorphism [45] allows us to elegantly represent each operator O on $\mathcal{H}_d$ as a vector $|O\rangle$ in the product space $\mathcal{H}_d^{\otimes 2}$:

$$|O\rangle = \sqrt{d} O \otimes \mathbb{1}_d|\psi_d\rangle = \sum_{i,j=0}^{d-1} O_{i,j}|i\rangle \otimes |j\rangle^*,$$
$$O = \sqrt{d}\,\mathrm{tr}_2(|O\rangle\langle\psi_d|), \tag{1}$$

where $*$ denotes the complex conjugate and $|\psi_d\rangle = \frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}|i\rangle \otimes |i\rangle^*$ is the maximally entangled isotropic state. Meanwhile, $\mathrm{tr}_2(\cdot)$ denotes the partial trace over the second space.

A useful property of Eq. (1) that will be exploited is that $\langle O_1|O_2\rangle = \mathrm{tr}(O_1^\dagger O_2)$ holds for any two operators O_1 and O_2 on $\mathcal{H}_d$. With the notations above, it is convenient to restate Born's rule as $p_i = \langle M_i|\rho\rangle$. Observe here the probability p_i is exactly the expansion coefficient of the vector $|\rho\rangle$ in the basis $|M_i\rangle$. This provides us an intuitive picture on how the information of a density operator is encoded in the outcome probabilities under a certain measurement, which enables us to formalize our concept of complementary information in the subsequent subsection.

B. Measure of information gain

In the light of Kochen-Specker's theorem [25] (see also Ref. [26]), it is impossible to assign a definite value to every quantum observable without specifying the measurement arrangement. During a measurement, all that an observer has is the probabilistic occurrence of one outcome (labeled truth value 1), which simultaneously negates the occurrence of other outcomes (labeled truth values 0). The information content of quantum systems is thus reflected in the statistics of these binary strings.

Consider an experimental setup to perform the measurement $\mathcal{M} = \{M_i\}$ on individual copies of a quantum state that is unknown to the experimenter. Each time the ith outcome occurs, the experimenter gets a squared deviation $[1 - \mathrm{tr}(M_i)/d]^2$ from the expectation $\mathrm{tr}(M_i)/d$ for the completely mixed state (least information state) or gets $[0 - \mathrm{tr}(M_i)/d]^2$ otherwise. After repeating the experiments large enough N times, the total squared deviation is $D_i^2 = N\{p_i[1 - \mathrm{tr}(M_i)/d]^2 + (1-p_i)[\mathrm{tr}(M_i)/d]^2\}$, which consists of two contributions $D_i^2 = \Delta_i^2 + B_i^2$. Wherein $\Delta_i^2 = N[p_i(1-p_i)]$ is the total uncertainty (variance), which determines the width $2\Delta_i/N$ of the confidence interval $[p_i - \frac{1}{N}\Delta_i, p_i + \frac{1}{N}\Delta_i]$ for estimating the outcome probabilities $\{p_i\}$.

What truly discriminates the measured state from the completely mixed state, on the other hand, is the total squared bias $B_i^2 = N[p_i - \mathrm{tr}(M_i)/d]^2$. We suggest the measure of information gain on the state ρ in each individual trial of the measurement $\mathcal{M} = \{M_i\}$ to be the sum of mean-squared bias over all outcomes

$$G(\mathcal{M})_\rho = \sum_i [p_i - \mathrm{tr}(M_i)/d]^2 =: \langle\rho|\hat{G}(\mathcal{M})|\rho\rangle. \tag{2}$$

In the above, we leverage the isomorphism (1) to define the view operator of a measurement $\mathcal{M}$ as

$$\hat{G}(\mathcal{M}) = \sum_i |\tilde{M}_i\rangle\langle\tilde{M}_i|, \tag{3}$$

where $\tilde{M}_i = M_i - \frac{1}{d}\mathrm{tr}(M_i)\mathbb{1}_d$ is traceless or, equivalently, $|\tilde{M}_i\rangle = |M_i\rangle - |\psi_d\rangle\langle\psi_d|M_i\rangle$ is orthogonal to $|\psi_d\rangle$. View operators are positive semidefinite, $\hat{G} \geqslant 0$ on the (d^2-1)-dimensional subspace $\mathcal{H}_{\perp\psi_d}$ of $\mathcal{H}_d^{\otimes 2}$ orthogonal to $|\psi_d\rangle$, and vanish for trivial POVMs whose effects are all proportional to the identity $M_i = \frac{1}{d}\mathrm{tr}(M_i)\mathbb{1}_d$.

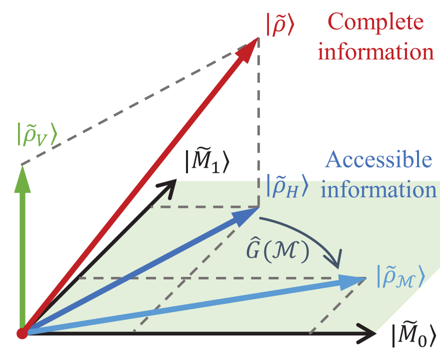

FIG. 1. Illustration of the information complementarity, where the vector $|\tilde{\rho}\rangle$ encodes the complete information of the state ρ. For a two-outcome measurement $\mathcal{M}=\{M_i\}_{i=0}^{1}$, the vectors $\{|\tilde{M}_i\rangle\}$ span a two-dimensional space (colored horizontal plane), on which the view operator $\hat{G}(\mathcal{M})$ is a bijective transform. While the horizontal component $|\tilde{\rho}_H\rangle$ of $|\tilde{\rho}\rangle$ can be reconstructed from the vector $|\tilde{\rho}_{\mathcal{M}}\rangle$ encoding the outcome statistics, the vertical component $|\tilde{\rho}_V\rangle$ contains only information complementary to what is accessible through $\mathcal{M}$.

Now we are able to formalize our idea of complementary information. Let $\tilde{\rho}=\rho-\mathbb{1}_d/d$, observe that the outcome probabilities of a measurement $\mathcal{M}$ on the state ρ are encoded in the expansion coefficients of the vector $|\tilde{\rho}_{\mathcal{M}}\rangle=\hat{G}(\mathcal{M})|\rho\rangle=\hat{G}(\mathcal{M})|\tilde{\rho}\rangle=\sum_i[p_i-\mathrm{tr}(M_i)/d]|\tilde{M}_i\rangle$ under the basis $\{|\tilde{M}_i\rangle\}$. The vector $|\tilde{\rho}_{\mathcal{M}}\rangle$ encodes the complete information of ρ if $|\tilde{\rho}\rangle$ lies in the subspace of $\mathcal{H}_{\perp\psi_d}$ on which the view operator $\hat{G}(\mathcal{M})$ is invertible, whereas if $|\tilde{\rho}\rangle$ is orthogonal to that space, $|\tilde{\rho}_{\mathcal{M}}\rangle$ vanishes and $\mathcal{M}$ cannot be employed to distinguish ρ from the completely mixed state (see Fig. 1 for an illustration of the geometric relations between the above vectors). In the sense above, two nontrivial measurements $\mathcal{M}_1$ and $\mathcal{M}_2$ satisfying

$$\hat{G}(\mathcal{M}_1)\cdot\hat{G}(\mathcal{M}_2)=0 \tag{4}$$

are complementary since, if the complete information of ρ is accessible through $\mathcal{M}_1$, then no information gain (2) is accessible through $\mathcal{M}_2$, and vice versa. We prove in Appendix A that measurements in MUBs [28–31] are mutually complementary.

It is worth mentioning that the combined view operator $\hat{G}=\sum_\theta\hat{G}(\mathcal{M}_\theta)$ associated with a set of POVMs $\mathscr{M}=\{M_\theta\}$ on $\mathcal{H}_d$ can be positive definite (invertible) on $\mathcal{H}_{\perp\psi_d}$. In this case, no POVM can be complementary to all POVMs of $\mathscr{M}$ simultaneously. This means that $\mathscr{M}$ is informationally complete and $\hat{G}$ offers a complete view to all d-dimensional quantum states. Utilizing the isomorphism (1), arbitrary unknown state ρ can then be reconstructed from the vector $\hat{G}|\tilde{\rho}\rangle=|\tilde{\rho}_{\mathscr{M}}\rangle$ encoding the outcome statistics as follows:

$$\rho=\sqrt{d}\ \mathrm{tr}_2(\hat{G}^{-1}|\tilde{\rho}_{\mathscr{M}}\rangle\langle\psi_d|)+\mathbb{1}_d/d. \tag{5}$$

For further readings on the topic of state estimation, we recommend Refs. [46,47].

Interestingly, the combined view operator associated with CMUBs in $\mathcal{H}_d$, i.e., $d+1$ MUBs [28–31], is simply the identity operator $\mathbb{1}_{\perp\psi_d}=\mathbb{1}_d\otimes\mathbb{1}_d-|\psi_d\rangle\langle\psi_d|$ on $\mathcal{H}_{\perp\psi_d}$ (see Appendix A). Thus, the operationally invariant measure [27] of complete information content contained in quantum states can be restated in our language as

$$I_{\mathrm{com}}(\rho)=\langle\rho|\mathbb{1}_{\perp\psi_d}|\rho\rangle=\mathrm{tr}(\rho^2)-1/d. \tag{6}$$

This measure naturally coincides with Bohr's idea [1] that only the totality of complementary properties together exhausts the complete information of objects.

C. Local information exclusion relations

To formulate quantum complementarity into information exclusion relations, next we focus on the measurement scenarios where distinct measurements on individual quantum systems in the same state are selected with biased (nonuniform) probabilities.

Theorem 1. For a set of measurements $\{\mathcal{M}_\theta\}$ with selection probabilities $\{w_\theta\}$, the average information gain on the state ρ satisfies

$$\sum_\theta w_\theta G(\mathcal{M}_\theta)_\rho=\langle\rho|\hat{g}|\rho\rangle\leqslant\|\hat{g}\|\cdot I_{\mathrm{com}}(\rho), \tag{7}$$

where $\hat{g}=\sum_\theta w_\theta\hat{G}(\mathcal{M}_\theta)$ is the average view operator and $\|\cdot\|$ denotes the operator norm, i.e., the largest eigenvalue of an operator.

Proof. According to Eqs. (1) and (3), for any density operator ρ on $\mathcal{H}_d$ there is $\langle\psi_d|\rho\rangle\langle\rho|\psi_d\rangle=1/d$, $\langle\rho|\rho\rangle=\mathrm{tr}(\rho^2)$, and $\langle\rho|\tilde{M}_{i|\theta}\rangle=p_{i|\theta}-\mathrm{tr}(M_{i|\theta})/d$. Hence, we have $\sum_{i,\theta}w_\theta[p_{i|\theta}-\mathrm{tr}(M_{i|\theta})/d]^2=\langle\rho|\hat{g}|\rho\rangle\leqslant\|\hat{g}\|\cdot\langle\rho|\mathbb{1}_{\perp\psi_d}|\rho\rangle=\|\hat{g}\|\cdot[\mathrm{tr}(\rho^2)-1/d]$. ■

Theorem 1 limits an observer's weighted average information gain over multiple measurements to be less than a proportion $\|\hat{g}\|$ of the complete information content (6) contained in quantum states. We show in Appendix A that $\frac{1}{\Theta}\leqslant\|\hat{g}\|\leqslant 1$ for a number Θ of rank-1 projective measurements. Specifically, for nondegenerate observables with one or more common eigenstates, we have $\|\hat{g}\|=1$ and the rightmost side of Eq. (7) is achieved by density operators whose eigenvectors corresponding to positive eigenvalues form a subset of the common eigenstates of observables, which means that no state-independent information exclusion exists. In contrast, we have $\|\hat{g}\|=\max_\theta\{w_\theta\}\leqslant 1$ for MUBs. Particularly, for random measurements in one of Θ MUBs, $w_1=\cdots=w_\Theta=\frac{1}{\Theta}$, thereby $\|\hat{g}\|=\frac{1}{\Theta}$. We therefore see that the average information gain is rather limited with an increasing number of MUBs.

Example 1. For random measurements on a qubit in one of two bases $\{|i_1\rangle\}$ and $\{|j_2\rangle\}$, Eq. (7) gives $\langle\rho|\hat{g}|\rho\rangle\leqslant c_{\max}I_{\mathrm{com}}(\rho)$. Here, $c_{\max}=\max_{i,j}\{|\langle i_1|j_2\rangle|^2\}$ is the maximal overlap between bases and in this simple example $\frac{1}{2}\leqslant c_{\max}\leqslant 1$. By definition, $c_{\max}=\frac{1}{2}$ holds for MUBs, while for compatible bases $c_{\max}=1$.

We remark that for those measurement strategies with which the associated view operator $\hat{g}\propto\mathbb{1}_{\perp\psi_d}$, the rightmost side of Eq. (7) can be achieved by any density operator on $\mathcal{H}_d$. Typical examples include random measurements in CMUBs, random selection of measurements from a complete set of mutually unbiased measurements [48], and other design-structured POVMs [49–54] (see Appendix A for details).

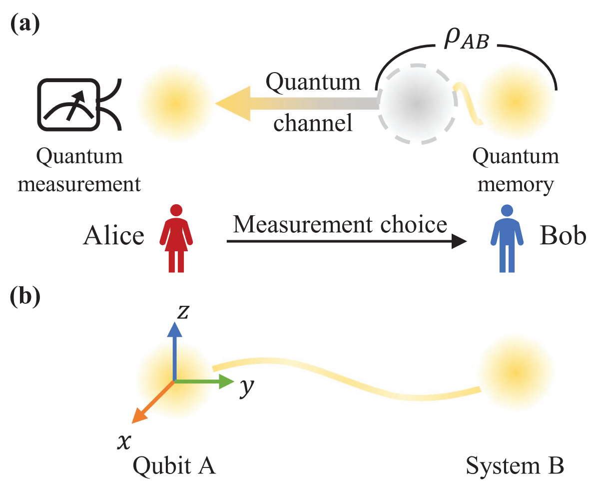

FIG. 2. (a) Sketch of the proposal. (b) When Alice chooses to measure a qubit in one of three orthogonal directions, Bob's total lack of information (uncertainty) about Alice's measurement outcomes is negative linearly related to the recoverable entanglement fidelity $F^{\rm pg}(A|B)$ of the initial state, which is time invariant if there exits no information exchange with environments or between subsystems A and B.

D. Information exclusion relations with memory

We move on to investigate the basic limits on an observer's information with respect to measurements on a distant quantum system, given access to another system (called memory). To illustrate, let us consider the guessing game [12] involving two participants, Alice and Bob. As depicted in Fig. 2(a), in the beginning, Bob prepares a bipartite system in the state ρ_{AB}, and sends subsystem A to Alice. Upon receiving subsystem A, Alice chooses a measurement according to the value θ of a random variable drawn from the probability distribution $\{w_\theta\}$, and announces her choice to Bob. Bob's win condition is to guess the final state on Alice's side correctly.

To quantify Bob's lack of information about system A while possessing a memory system B, we define the conditional linear entropy as below

$$S_L(A|B) = 1 - dF^{\rm pg}(A|B). \tag{8}$$

Here, $F^{\rm pg}(A|B) = \frac{1}{d}{\rm tr}\{[(\mathbb{1}_A \otimes \rho_B^{-1/4})\rho_{AB}(\mathbb{1}_A \otimes \rho_B^{-1/4})]^2\}$ is the recoverable entanglement fidelity with which ρ_{AB} can be transformed into a maximally entangled state through the pretty good recovery operation on system B [55,56], and d denotes the dimension of system A. In the case of a product state $\rho_{AB} = \rho_A \otimes \rho_B$, system B offers no side information about system A and Eq. (8) reduces to the linearized entropy $S_L(\rho_A) = 1 - {\rm tr}(\rho_A^2)$, i.e., the complement of the information content (6) contained in the state ρ_A. More generally, according to the data-processing inequality [57,58] we have $S_L(A|B) \leqslant S_L(\rho_A)$, thereby a memory helps to reduce Bob's ignorance. Further, ρ_{AB} is necessarily entangled if $S_L(A|B) < S_L(\sqrt{\rho_B}) \equiv 0$ since one's ignorance about the overall system in a separable state does not increase with the removal of any its local subsystem [59,60].

For brevity, we will focus on rank-1 projective measurements. Bob has no direct access to system A once it is sent to Alice, his understanding of the overall system when Alice chooses the θth measurement is described by the classical-quantum state

$$\rho_{\mathcal{M}_\theta B} = \sum_i |i\rangle\langle i| \otimes {\rm tr}_A[(M_{i|\theta} \otimes \mathbb{1}_B)\rho_{AB}], \tag{9}$$

where $M_{i|\theta}$ denotes the ith effect of the θth POVM $\mathcal{M}_\theta$ and $\{|i\rangle\langle i|\}$ are the measurement outcomes stored in a classical register. Then, the conditional linearized entropy (8) evaluated on the classical-quantum state (9), denoted $S_L(\mathcal{M}_\theta|B) = 1 - dF^{\rm pg}(\mathcal{M}_\theta|B)$, measures Bob's ignorance about Alice's measurement outcomes. Indeed, $F^{\rm pg}(\mathcal{M}_\theta|B)$ is now precisely the probability for Bob to correctly guess Alice's measurement outcome by performing the pretty good measurement on system B [61,62].

Theorem 2. Suppose ρ_{AB} describes a bipartite system and $\{\mathcal{M}_\theta\}$ are rank-1 projective measurements on system A with selection probabilities $\{w_\theta\}$. The average conditional linearized entropy is bounded below by

$$\sum_\theta w_\theta S_L(\mathcal{M}_\theta|B) \geqslant (1 - \|\hat{g}\|) \cdot [1 - F^{\rm pg}(A|B)]. \tag{10}$$

We prove in Appendix B a result that is valid for more general measurements. Like the memoryless IER (7), Eq. (10) becomes an equality saturated by arbitrary bipartite state if the equality $\hat{g} = \|\hat{g}\| \cdot \mathbb{1}_{\perp\psi_d}$ holds. Consequently, in the absence of information exchange with environments or between systems A and B, Bob's total information with respect to measurements on system A in CMUBs, as well as other design-structured measurements [49–54], is time invariant.

Impressively, the right-hand side of Eq. (10) is a product of two independent terms controlled by Alice and Bob, respectively. The first term, $1 - \|\hat{g}\| =: \mathcal{X}$, is a state-independent signature of information exclusion and Alice is free to manipulate it through her measurement strategy. It varies in the range $\mathcal{X} \in [0, 1 - \frac{1}{\Theta}]$ when the number of observables under consideration is Θ. To keep her measurement outcomes secret, Alice should avoid measuring observables that share a common eigenstate ($\mathcal{X} = 0$), as Bob can completely eliminate his uncertainty by preparing system A precisely in that eigenstate. In contrast, Bob's uncertainty will be maximized if Alice randomly selects one of Θ MUBs ($\mathcal{X} = 1 - \frac{1}{\Theta}$). The special case when Alice chooses to measure the Pauli observables of a qubit is illustrated in Fig. 2(b). We need to mention here that a set of Θ MUBs may not exist for sufficiently large Θ, and numerical methods can be utilized to maximize the exclusivity $\mathcal{X}$ in such cases.

As for the second term, it decreases monotonically with the recoverable entanglement fidelity $F^{\rm pg}(A|B)$ of the initial state ρ_{AB}. Bob's pretty good guessing probability [61,62] $F^{\rm pg}(\mathcal{M}_\theta|B)$ would be less than 1 whenever $F^{\rm pg}(A|B) < 1$. However, he can prepare an appropriate entangled state such that this fidelity enables him to guess the outcomes of measurements on system A with high probability. Indeed, it is well known that maximally entangled states provide perfect side information. For example, two systems in the state $|\psi_d\rangle = \frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}|i\rangle_A \otimes |i\rangle_B^*$ are perfectly correlated with no local

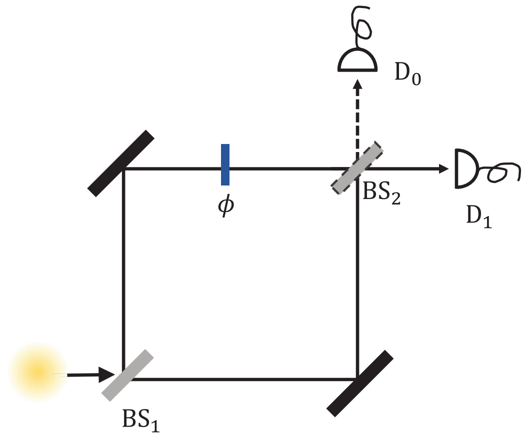

FIG. 3. Mach-Zehnder interferometer. Each input photon is directed into two paths by an asymmetric beam splitter (BS_1) and then is recombined on a 50:50 beam splitter (BS_2) to trigger two detectors (D). Modulating the phase shift $\phi \in [0, 2\pi]$ in the upper path, the phenomenon that the probability of clicking in each detector oscillates periodically reflects the interference pattern of path amplitudes, which is a signature of wave property. When BS_2 is removed, the detection probabilities then reflect the path information of photons, which are independent of the local phase shift ϕ.

information content at all, $I_{\rm com}(\rho_B) = I_{\rm com}(\rho_A) = 0$, whereas the joint information content $I_{\rm com}(\rho_{AB}) = 1 - 1/d^2$ is maximal. This leads to $F^{\rm pg}(A|B) = 1$, namely, the correlation between A and B is strong enough to completely remove Bob's uncertainty. Just as is mentioned in Refs. [24,27], the information content of a maximally entangled state is "exhausted in defining the joint properties" and "none is left for individual systems."

E. Origin of tight WPDRs

We argue that the tight WPDRs are particular examples of the IERs (7) and (10) for measuring complementary observables. To see this, let us consider two complementary setups of the Mach-Zehnder interferometer depicted in Fig. 3: (i) the second beam splitter is removed to gain the path information of single photons inside the interferometer (let $\sigma^{\rm p}$ denote the associated path observable with binary outcomes $+1$ and -1, corresponding to clicks in detectors D_0 and D_1, respectively); (ii) BS_2 is inserted in and the phase shift ϕ is adjustable to reveal wave properties of photons (let $\sigma^{\rm w}_\phi$ denote the associated wave observable with binary outcomes ±1). It takes some calculation (see Appendix C) to see that Eq. (7) leads to the equality

$$G(\sigma^{\rm w}_\phi)_\rho + G(\sigma^{\rm w}_{\phi'})_\rho = \cos(\phi' - \phi)\langle\sigma^{\rm w}_\phi\rangle\langle\sigma^{\rm w}_{\phi'}\rangle + [I_{\rm com}(\rho) - G(\sigma^{\rm p})_\rho]\sin^2(\phi' - \phi), \tag{11}$$

where $\langle\sigma\rangle = {\rm tr}(\sigma\rho)$ denotes the average of observable σ, and $G(\sigma^{\rm p})_\rho = \frac{1}{2}\langle\sigma^{\rm p}\rangle^2$ and $G(\sigma^{\rm w}_\phi)_\rho = \frac{1}{2}\langle\sigma^{\rm w}_\phi\rangle^2$ are the respective information gains (2) for measuring the path and wave observables in the qubit ρ.

Observe that the information gain regarding an individual wave observable oscillates as the phase shift ϕ varies, Eq. (11) essentially depicts an interference pattern of the wave information. To make it clearer, let $\vec{\phi}$ and $\vec{\phi}'$ be two real unit vectors at an angle of $\phi' - \phi$. Equation (11) can then be restated as

$$\left|\langle\sigma^{\rm w}_\phi\rangle\vec{\phi} + e^{i\pi}\langle\sigma^{\rm w}_{\phi'}\rangle\vec{\phi}'\right|^2 = 2[I_{\rm com}(\rho) - G(\sigma^{\rm p})_\rho]\sin^2(\phi' - \phi). \tag{12}$$

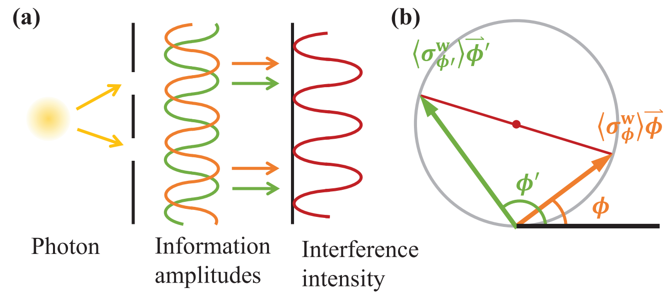

FIG. 4. Interference pattern of information amplitude. (a) In the double-slit experiment, filtering out the component containing path information from photon's density operator, the remaining components lead to a fringe with 100% contrast on the screen. The intensity varies periodically at different locations, corresponding to the intensity oscillation in the MZI as the phase shift ϕ varies. (b) The information gains on single photons in the MZI when two complementary wave observables ($\phi' - \phi = \pi/2$) are measured constitute the complete description of the wavelike behavior. The fringe visibility is given by the diameter of the gray circle.

It is interesting to note that the average values of wave observables behave like the "amplitudes of wave information" and interfere with each other [see Fig. 4(a)]. Notably, the average interference intensity $\mathcal{I} = I_{\rm com}(\rho) - G(\sigma^{\rm p})_\rho$ on the right-hand side of Eq. (12) disappears if the photon exhibits particle property only: the complete information content of ρ is accessible through measuring the path observable or, formally, $I_{\rm com}(\rho) = G(\sigma^{\rm p})_\rho$. In this view, the average intensity $\mathcal{I} = G(\sigma^{\rm w}_\phi)_\rho + G(\sigma^{\rm w}_{\phi+\pi/2})_\rho$ [see the case $\phi' - \phi = \pm\frac{\pi}{2}$ in Eq. (12)] emerges as a measure of wave property which can be determined by measuring two complementary wave observables.

Conventionally, the wave property is frequently quantified by the fringe visibility [3–6]

$$\mathcal{V} = \max_\phi |p^0_\phi - p^1_\phi|, \tag{13}$$

where p^i_ϕ is the probability that the ith detector clicks when the observable $\sigma^{\rm w}_\phi$ is measured. We remark here that the average interference intensity is precisely half of the fringe visibility squared, i.e., $\mathcal{V} = \max_\phi|\langle\sigma^{\rm w}_\phi\rangle| = \sqrt{2\mathcal{I}}$ [see also Fig. 4(b) for an illustration]. Combined with the squared path distinguishability $\mathcal{D}^2 = \langle\sigma^{\rm p}\rangle^2 = 2G(\sigma^{\rm p})_\rho$, we then arrive at the WPDR $\mathcal{V}^2 + \mathcal{D}^2 = 2{\rm tr}(\rho^2) - 1$ [63]. We therefore see that the WPDR originates from the IER $[G(\sigma^{\rm w}_\phi)_\rho + G(\sigma^{\rm w}_{\phi+\pi/2})_\rho] + G(\sigma^{\rm p})_\rho = I_{\rm com}(\rho)$ for three complementary observables, including the path observable and two wave observables with phase difference satisfying $\phi' - \phi = \pm\pi/2$.

Example 2. Suppose the first beam splitter in the MZI (see Fig. 3) is so arranged such that it implements the transformation $|0\rangle \rightarrow |\psi\rangle = \cos\gamma|0\rangle + \sin\gamma|1\rangle$ for some

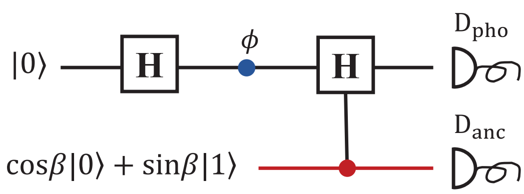

FIG. 5. Delayed choice experiment controlled by an ancilla qubit (red line) in the state $\cos\beta|0\rangle + \sin\beta|1\rangle$. The beam splitters in the two-way interferometer are now equivalently represented by the Hadamard gates H: $H|0\rangle \to (|0\rangle + |1\rangle)/\sqrt{2}$ and $H|1\rangle \to (|0\rangle - |1\rangle)/\sqrt{2}$.

$\gamma \in [0, \pi/2]$. With BS_2 removed, the two detectors click, respectively, with probabilities $\cos^2\gamma$ and $\sin^2\gamma$ in response to each input photon, and an observer's path information gain is $G(\sigma^{p})_{|\psi\rangle} = (\cos^2\gamma - \frac{1}{2})^2 + (\sin^2\gamma - \frac{1}{2})^2$. Obviously, when $\gamma = \pi/4$ the detectors click at random, from which no path information can be gained. With BS_2 inserted in, on the other hand, the detectors click, respectively, with probabilities $\frac{1}{2}|e^{i\phi}\cos\gamma + \sin\gamma|^2$ and $\frac{1}{2}|e^{i\phi}\cos\gamma - \sin\gamma|^2$, and an observer's wave information gain is $G(\sigma^{w}_{\phi})_{|\psi\rangle} = \frac{1}{2}\sin(2\gamma)^2\cos^2\phi$. Hence, the amount of information for describing a photon's wave property, defined as the sum of information gains over two complementary wave observables, is $G(\sigma^{w}_{\phi})_{|\psi\rangle} + G(\sigma^{w}_{\phi+\pi/2})_{|\psi\rangle} = \frac{1}{2}\sin(2\gamma)^2$, which equals to half of the squared fringe visibility (13) $\mathcal{V}^2 = \sin^2(2\gamma)$ and is maximized when the corresponding path information vanishes ($\gamma = \pi/4$). Combining the wave and path information then leads us to the complete information $I_{com}(|\psi\rangle) = \frac{1}{2}\sin(2\gamma)^2 + (\cos^2\gamma - \frac{1}{2})^2 + (\sin^2\gamma - \frac{1}{2})^2 = \frac{1}{2}$.

Theorem 1 applies to more sophisticated measurement setups, including the quantum delayed-choice experiment [64] where complementary properties of photons are measured in a single experimental setup. As shown in Fig. 5, the presence of BS_2 is controlled by an ancilla qubit, the value of which determines whether to reveal the wave property or particle property. In this case, Eq. (7) limits an observer's weighted average information gain about three complementary observables, with (unnormalized) weights $w_1 = \cos^2\beta$ for the path observable and $w_2 = w_3 = \frac{1}{2}\sin^2\beta$ for the wave observables. We therefore see that, although quantum complementarity does not prohibit the observation of complementary properties in a single measurement setup, it restricts one's average information gain about complementary properties in individual measurements. Similar analyses apply also to the quantum-controlled reality experiment (QCRE) [65] where the presence of BS_1, instead of BS_2, is controlled by an ancilla qubit.

Example 3. Let us consider a two-way interferometer with each beam splitter controlled by an ancilla qubit and suppose the one-photon state before entering the interferometer is $\rho_0 = |0\rangle\langle 0|$. In this setup, an input photon, depending on the presence of BS_1, either travels along its initial direction like a particle or evolves into the state $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ which has wave property only after entering the interferometer. Meanwhile, the overall state of the photon and the ancilla qubit controlling BS_1 becomes $\cos\beta|0\rangle|0\rangle_{anc} + \sin\beta|+\rangle|1\rangle_{anc}$, a superposition between two "complementary realities" [65]. First, the whole setup above can be viewed as a device to probabilistically perform measurements on input photons. Let σ^{ab}_{ϕ} denote the observable when the phase shift is ϕ, with $a, b \in \{0, 1\}$ labeling the presence of BS_1 and BS_2, respectively. For two complementary observables $\{\sigma^{0b}_{\phi}, \sigma^{1b}_{\phi}\}$, Theorem 1 then requires an observer's average information gain in individual measurements to satisfy $\langle\rho_0|\hat{g}|\rho_0\rangle \leqslant \|\hat{g}\| \cdot I_{com}(\rho_0)$, where $\hat{g} = \cos^2\beta\hat{G}(\sigma^{0b}_{\phi}) + \sin^2\beta\hat{G}(\sigma^{1b}_{\phi})$. Second, one might be interested in the photon state inside the interferometer and BS_1 can be viewed as a device to prepare a photon in one of the two states $\{\rho_0, \rho_1 = |+\rangle\langle +|\}$ probabilistically. Considering the equivalence between a unitary transformation of measurement bases (view operators) and that of states $\langle\rho|H^{+}\otimes H\hat{G}H\otimes H^{+}|\rho\rangle = \langle H\rho H^{+}|\hat{G}|H\rho H^{+}\rangle$, Theorem 1 then constrains an observer's average information gain about two "complementary realities" when measuring a single observable, $\cos^2\beta\langle\rho_0|\hat{G}(\sigma^{0b}_{\phi})|\rho_0\rangle + \sin^2\beta\langle\rho_1|\hat{G}(\sigma^{0b}_{\phi})|\rho_1\rangle \leqslant \|\hat{g}\| \cdot I_{com}(\rho_0)$.

Another meaningful issue concerns the WPDRs when an observer has side information about single photons in the MZI, but without direct access to them. Let us consider two photons in the bipartite state ρ_{AB}. As a measure of information about photon A conditioned on photon B, we turn to the complement of the conditional linearized entropy (8) below:

$$I(A|B) = dF^{pg}(A|B) - 1/d. \tag{14}$$

This is non-negative and reduces to the complete information of the reduced state ρ_A, $I(A|B) = I_{com}(\rho_A)$ when $\rho_{AB} = \rho_A \otimes \rho_B$ is a product state.

We derive in Appendix C the following generalization of Eq. (12):

$$\mathrm{tr}[(\vec{\rho}_{\phi B} + e^{i\pi}\vec{\rho}_{\phi' B})^2] = 2[I(A|B) - I(\sigma^{p}|B)]\sin^2(\phi - \phi'). \tag{15}$$

Here, $\vec{\rho}_{\phi B} = \mathrm{tr}_A[(\rho_B^{-1/4}\rho_{AB}\rho_B^{-1/4})(\sigma^{w}_{\phi}\otimes\mathbb{1}_B)]\ \vec{\phi}$ is the "amplitude of conditional information" which connects to the conditional information $I(\sigma^{w}_{\phi}|B)$ through its squared modulus $\mathrm{tr}(\vec{\rho}^{\,2}_{\phi B}) = 2I(\sigma^{w}_{\phi}|B)$.

Equation (15) manifests the interference pattern of conditional information amplitude, with the right-hand side of it being the interference intensity. Combining the average intensity (wave property) $I(A|B) - I(\sigma^{p}|B) = I(\sigma^{w}_{\phi}|B) + I(\sigma^{w}_{\phi+\pi/2}|B)$ with the conditional which-way information (particle property) $I(\sigma^{p}|B)$, we then obtain the WPDR $[I(\sigma^{w}_{\phi}|B) + I(\sigma^{w}_{\phi+\pi/2}|B)] + I(\sigma^{p}|B) = I(A|B)$. Again, we see that a tight WPDR saturated by all bipartite systems with dimension $d_A = 2$ arises from an IER for three complementary observables, wherein two complementary wave observables constitute the complete description of wave property.

F. Applications

Our theory for characterizing information complementarity from a measurement-based perspective enables us to analyze the behaviors of quantum systems through their manifestations in versatile measurement setups, without delving into the exhaustive calculations with quantum state parameters. As two examples, we explore the implications of our IERs (7) and (10) for entanglement detection and EURs, respectively.

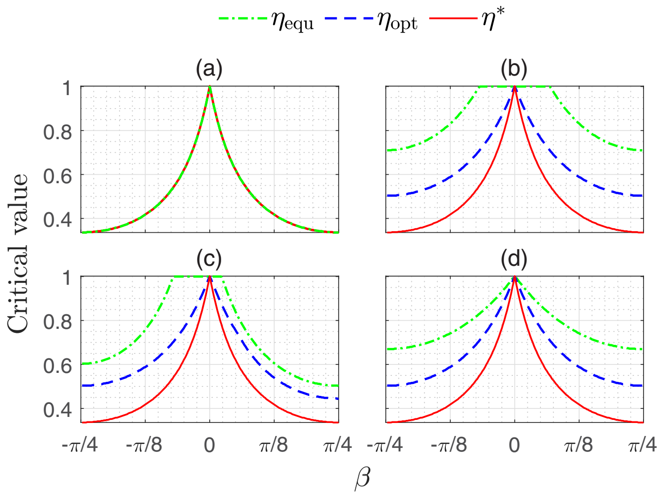

FIG. 6. Numerical comparison of the critical value of η for the state (17) to be entangled (denoted η^*) and that to violate Eq. (16) under four different choices of three local observables, with equal weights and optimized weights, respectively (denoted $\eta_{\rm equ}$ and $\eta_{\rm opt}$). The three local observables considered here are $\sigma_y \otimes \sigma_y$, $\sigma_z \otimes \sigma_z$ and (a) $\sigma_x \otimes \sigma_x$, (b) $\sigma_z \otimes \sigma_x$, (c) $(\frac{1}{2}\sigma_z + \frac{\sqrt{3}}{2}\sigma_x) \otimes (\frac{1}{2}\sigma_z + \frac{\sqrt{3}}{2}\sigma_x)$, (d) $\sigma_z \otimes \sigma_z$.

1. Entanglement detection

Quantum correlation tends to suppress the local information content contained in individual subsystems. For example, a pair of maximally entangled qubits possess only joint properties in the sense that each single qubit is in the completely mixed state. We introduce the correlation measure $J(\rho_{AB}) = \sum_{i,\theta} w_\theta |\mathrm{tr}(J_{i|\theta}\rho_{AB})|$ for local measurements $\{\mathcal{M}_\theta^A \otimes \mathcal{M}_\theta^B\}$ on individual copies of the bipartite state ρ_{AB}, where $M_{i|\theta}$ denotes the ith effect of the θth measurement and $J_{i|\theta} = [M_{i|\theta}^A - \frac{1}{d_A}\mathrm{tr}(M_{i|\theta}^A)\mathbb{1}_A] \otimes [M_{i|\theta}^B - \frac{1}{d_B}\mathrm{tr}(M_{i|\theta}^B)\mathbb{1}_B]$ are the correlation detection operators. We show in Appendix D the following.

Theorem 3. For any bipartite separate state ρ_{AB}, it holds that

$$J(\rho_{AB}) \leqslant \sqrt{L_A L_B}, \tag{16}$$

where $L = \|\hat{g}\|(1 - 1/d)$ is the state-independent upper bound on local information gain given by Eq. (7).

Consequently, a violation of Eq. (16) necessarily indicates the presence of entanglement. As a concrete example, we can apply Eq. (16) to the mixture of a pure two-qubit state $|\psi(\beta)\rangle = \cos\beta|00\rangle + \sin\beta|11\rangle$ $(-\frac{\pi}{4} \leqslant \beta \leqslant \frac{\pi}{4})$ and white noise:

$$\rho_{\eta,\beta} = \eta|\psi(\beta)\rangle\langle\psi(\beta)| + (1-\eta)\mathbb{1}_4/4 \ \ (0 \leqslant \eta \leqslant 1). \tag{17}$$

Note that the noiseless state $|\psi(\beta)\rangle$ is entangled as long as $\beta \neq 0$. Now the question is how much noise it can resist from being separable, i.e., the critical value η^* of η below which $\rho_{\eta,\beta}$ ceases to be entangled. In Fig. 6, we present numerical results regarding the critical values $\eta_{\rm equ}$ and $\eta_{\rm opt}$ for the state (17) to violate Eq. (16), under measurements with equal weights and optimized weights, respectively. As depicted, three complementary observables with equal weights are enough to detect all the entanglement ($\eta_{\rm equ} = \eta_{\rm opt} = \eta^*$). For more general observables $\eta_{\rm opt} \leqslant \eta_{\rm equ}$, an optimization over the weights $\{w_\theta\}$ yields better performance.

2. Implications for EURs

Entropic uncertainty relations (EURs) that take into account information leakage from a memory system play a crucial role in various aspects of quantum information processing [10], particularly in the security analysis of quantum protocols [12]. However, existing EURs [10] are thus far limited since they are restricted to providing lower bounds on simply entropy sums. On a conceptual level, there is no reason to assign equal weights, instead of biased weights, to different measurements. Based on Theorem 2, we have the following lower bounds on the weighted sum of entropies over multiple measurements (see the proof in Appendix. E).

Theorem 4. Suppose ρ_{AB} describes a bipartite system and $\{\mathcal{M}_\theta\}$ are rank-1 projective measurements to be performed on system A with selection probabilities $\{w_\theta\}$. The smooth minimum entropy evaluated on the state (9) satisfies $\sum_\theta w_\theta H_{\rm min}^\varepsilon(\mathcal{M}_\theta|B) \geqslant q_{\rm min}^\varepsilon$, where

$$q_{\rm min}^\varepsilon = -\log_2[\|\hat{g}\| + F^{\rm pg}(A|B)(1 - \|\hat{g}\|)] - \log_2\frac{2}{\varepsilon^2}. \tag{18}$$

The conditional smooth minimum entropy [66] (see also Ref. [10]) is a fundamental tool for the security analysis of quantum protocols. In quantum cryptographic protocols where an eavesdropper aims to know an experimenter's measurement outcomes by probing a memory system, the weighted EURs we introduced provide guidance for adjusting the probabilities of selecting distinct measurements to minimize potential information leakage. It is conceivable that equal selection probabilities are not optimal for biased measurements. Optimized selection probabilities are thus crucial for elaborating the measurement strategies to enhance security and achieve stronger levels of protection. Importantly, this optimization does not require additional quantum costs and can be easily done on a classical computer.

III. DISCUSSIONS

Quantum complementarity is a topic of intense debates and fruitful insights. Early investigations mainly focus on the WPDRs [3–6] for single photons in a two-way interferometer. Since then, more and more exquisite proposals have been designed to provide in-depth interpretations of complementarity from diverse perspectives [65,67–72]. Our basic idea is that every meaningful feature of physical systems must be testable under specific experiments and quantum complementarity necessitates a description in terms of measurement outcomes only.

We have developed a general approach to formulate the complementarity principle quantitatively in the form of basic limits on one's ability to gain information on quantum systems under versatile measurement setups, with and without memory, respectively. In contrast to previous IERs [40–44] which describe upper bounds on the sum of an observer's Shannon entropic mutual information over multiple measurement bases, we adopt the operationally invariant information measure [27] of quantum systems and obtain upper bounds on the weighted sum of information gains over generalized measurements. Our IERs thus enable us to address much more general measurement scenarios where distinct measurements are selected with biased probabilities. Further, we showed that the tight WPDRs for photons in a two-way interferometer are particular examples of IERs for which-way measurement and two wave measurements.

We remark that formulating complementarity from a measurement-based perspective naturally circumvents the exhaustive calculations with quantum state parameters, highlighting how complementarity manifests itself differently with respect to different measurement strategies. As applications, we first showed that our IERs can be utilized to certify genuine quantum features of physical systems, such as entanglement based on local measurement outcomes. We also introduced an extended form of EURs, which turn out to be advantageous in practical quantum information processing [73].

ACKNOWLEDGMENTS

This work is supported by the National Natural Science Foundation of China (Grants No. 12175104 and No. 12274223), the National Key Research and development Program of China (Grant No. 2023YFC2205802), the Innovation Program for Quantum Science and Technology (Grant No. 2021ZD0301701), the Natural Science Foundation of Jiangsu Province (Grant No. BK20211145), the Fundamental Research Funds for the Central Universities (Grant No. 020414380182), the Key Research and Development Program of Nanjing Jiangbei New Area (Grant No. ZDYD20210101), the Program for Innovative Talents and Entrepreneurs in Jiangsu (Grant No. JSSCRC2021484), and the Program of Song Shan Laboratory (Included in the management of Major Science and Technology Program of Henan Province) (Grant No. 221100210800-02).

APPENDIX A: VIEW OPERATOR AND PROPERTIES

Consider a set of POVMs $\{M_{i|\theta}\}$ assigned with weights $\{w_\theta\}$ ($w_\theta \geqslant 0$, $\sum_\theta w_\theta = 1$). We define the associated average view operator to be

$$\hat{g} = \sum_{i,\theta} w_\theta \hat{G}(\mathcal{M}_\theta) = \sum_{i,\theta} w_\theta |\tilde{M}_{i|\theta}\rangle\langle\tilde{M}_{i|\theta}|, \tag{A1}$$

where $\tilde{M}_{i|\theta} = M_{i|\theta} - \frac{1}{d}\mathrm{tr}(M_{i|\theta})\mathbb{1}_d$ is traceless or, equivalently, $|\tilde{M}_{i|\theta}\rangle = |M_{i|\theta}\rangle - |\psi_d\rangle\langle\psi_d|M_{i|\theta}\rangle$ is orthogonal to $|\psi_d\rangle$. View operators are positive semidefinite, $\hat{G} \geqslant 0$ on the (d^2-1)-dimensional subspace $\mathcal{H}_{\perp\psi_d}$ of $\mathcal{H}_d^{\otimes 2}$ orthogonal to $|\psi_d\rangle$, and vanish for trivial POVMs whose effects satisfy $M_{i|\theta} = \frac{1}{d}\mathrm{tr}(M_{i|\theta})\mathbb{1}_d$.

The matrix representation of $\hat{g}$ under an orthonormal basis $\{|a\rangle\}$ of $\mathcal{H}_{\perp\psi_d}$ takes the form

$$g_{a,a'} = \sum_{i,\theta} w_\theta \langle a|\tilde{M}_{i|\theta}\rangle\langle\tilde{M}_{i|\theta}|a'\rangle = (RR^+)_{a,a'}. \tag{A2}$$

Here, the matrix elements of R are given by $R_{a,b(i,\theta)} = \sqrt{w_\theta}\langle a|\tilde{M}_{i|\theta}\rangle$, with b being a bijection from the labels $\{(i,\theta)\}$ of POVM effects to the labels $\{a\}$ of the basis vectors $\{|a\rangle\}$. Note that the positive eigenvalues of $g = RR^+$ are identical to those of the Gram matrix for the vectors $\{\sqrt{w_\theta}|\tilde{M}_{i|\theta}\rangle\}$, that is, $\bar{g} = R^+R$. To obtain eigenvalues of a view operator $\hat{g}$, it will be enough to deal with the Gram matrix $\bar{g}$, whose elements are $\bar{g}_{b(i,\theta),b(j,\theta')} = \sqrt{w_\theta w_{\theta'}}\langle\tilde{M}_{i|\theta}|\tilde{M}_{j|\theta'}\rangle$.

Claim 1. POVMs that form a design structure are mutually complementary.

Claim 2. The combined view operator associated with a complete set of design-structured POVMs is proportional to the identity operator on $H_{\perp\psi_d}$.

Claim 3. The average view operator of a set of MUBs with weights $\{w_\theta\}$ satisfies $\|\hat{g}\| = \max_\theta\{w_\theta\}$.

Proof. Design-structured measurements include complete sets of mutually unbiased measurements (MUMs) [48], general symmetric informationally complete POVMs [52–54], POVMs from equiangular tight frames [51], and POVMs from general quantum designs [49,50]. Without loss of generality, we prove the above claims for MUMs. MUMs [48] are d-outcome POVMs satisfying $\mathrm{tr}(M_{i|\theta}) = 1$, $\mathrm{tr}(M_{i|\theta}M_{j|\theta'}) = \frac{1}{d}$, and $\mathrm{tr}(M_{i|\theta}M_{j|\theta}) = \kappa\delta_{ij} + \frac{1-\kappa}{d-1}(1-\delta_{ij})$ for all $i, j = 0, \dots, d-1$ and $\theta \neq \theta'$. Here $\kappa \in (\frac{1}{d}, 1]$ is called the efficiency parameter, wherein $\kappa = 1$ corresponds to projective measurements in MUBs [28–31]. ■

Consider the view operator $\hat{G}_{\mathrm{mum}} = \sum_\theta \hat{G}(\mathcal{M}_\theta)$ associated with a set of MUMs [48] on $\mathcal{H}_d$, according to Eq. (A2) the corresponding Gram matrix $\bar{G}$ is given as

$$\begin{aligned}\bar{G}_{b(i,\theta),b(j,\theta')} &= \langle\tilde{M}_{i|\theta}|\tilde{M}_{j|\theta'}\rangle = \mathrm{tr}(M_{i|\theta}M_{j|\theta'}) - \frac{1}{d}\\ &= \delta_{\theta\theta'}\left[\frac{\kappa d-1}{d-1}\delta_{ij} + \frac{1-\kappa d}{d(d-1)}\right].\end{aligned} \tag{A3}$$

According to Eq. (A3), obviously two MUMs are complementary since $\hat{G}(\mathcal{M}_\theta)\cdot\hat{G}(\mathcal{M}_{\theta'}) = 0$ whenever $\theta \neq \theta'$. Next, let us focus on the $d \times d$ submatrix

$$\bar{G}_{b(i,1),b(j,1)} = \frac{\kappa d-1}{d-1}\mathbb{1}_d - \frac{\kappa d-1}{d(d-1)}Q, \tag{A4}$$

where Q denotes the matrix satisfying $Q_{i,j} = 1$ for all $i, j = 0, \dots, d-1$. This submatrix (A4) has $d-1$ identical nonzero eigenvalues $(\kappa d-1)/(d-1)$, thus, the view operator of a complete set of $d+1$ MUMs (CMUMs) has $(d+1)(d-1) = d^2-1$ identical nonzero eigenvalues. In other words, $\hat{G}_{\mathrm{CMUM}} = \frac{\kappa d-1}{d-1}\mathbb{1}_{\perp\psi_d}$, with $\mathbb{1}_{\perp\psi_d} = \mathbb{1}_{d\times d} - |\psi_d\rangle\langle\psi_d|$ being the identity operator on the (d^2-1)-dimensional space $H_{\perp\psi_d}$. Claim 3 follows from the fact that MUBs (i.e., MUMs with efficient parameter $\kappa = 1$) are complementary, thus $\|\hat{g}\| = \max_\theta\{w_\theta\|\hat{G}(\mathcal{M}_\theta)\|\} = \max_\theta\{w_\theta\}$.

Claim 4. For arbitrary d-outcome POVMs $\mathcal{M} = \{M_i\}$ on $\mathcal{H}_d$ that consists of equal-trace effects (ETE-POVMs), i.e., $\mathrm{tr}(M_0) = \cdots = \mathrm{tr}(M_{d-1})$, we have $\|\hat{G}(\mathcal{M})\| \leqslant 1$.

Claim 5. For any set of d-outcome ETE-POVMs $\{M_\theta\}$ on $\mathcal{H}_d$, $\|\hat{g}\| = \|\sum_\theta w_\theta\hat{G}(\mathcal{M}_\theta)\| \leqslant \sum_\theta w_\theta\|\hat{G}(\mathcal{M}_\theta)\| \leqslant 1$.

Claim 6. For a number Θ of d-outcome ETE-POVMs $\{M_\theta\}$ on $\mathcal{H}_d$ with equal weights, $\|\hat{g}\| = \frac{1}{\Theta}\|\sum_\theta\hat{G}(\mathcal{M}_\theta)\| = 1$ iff the overlap matrix W, defined as $W_{b(i,\theta),b(j,\theta')} = \mathrm{tr}(M_{i|\theta}M_{j|\theta'})$, is reducible.

Proof. Consider the Gram matrix $\bar{G}_{i,j} = \langle\tilde{M}_i|\tilde{M}_j\rangle = \mathrm{tr}(M_iM_j) - \frac{1}{d}$. We can rewrite it as $\bar{G} = W - Q/d$, where $W_{i,j} = \mathrm{tr}(M_iM_j)$ is referred to as the overlap matrix, and $Q_{i,j} = 1$ for all i, j. Note W is doubly stochastic, i.e., $\sum_i W_{i,j} = \sum_j W_{i,j} = 1$, its first eigenvalue (arranged in descending order) must be $\lambda_1(W) = 1$. Moreover, the corresponding eigenvector $v_1 = (1,\dots,1)^{\mathrm{T}}$ is also an eigenvector of Q which corresponds to the unique nonzero eigenvalue d

of Q. Immediately $\bar{G}v_1 = 0$, and $\|\hat{G}(\mathcal{E})\| = \|\bar{G}\| = \lambda_2(W) \leqslant \lambda_1(W) = 1$. Claim 5 follows directly from Claim 4. Further, considering that the matrix $\frac{1}{\Theta}W$ is doubly stochastic, according to Theorem 3.1 of Ref. [74] we have $\lambda_2(\frac{1}{\Theta}W) = 1$ iff W is reducible. ■

APPENDIX B: PROOF OF THEOREM 2

Let $\{M_{i|\theta}\}$ be a set of generalized measurements such that the POVM effects of each measurement are equal-trace, i.e., $\mathrm{tr}(M_{0|\theta}) = \cdots = \mathrm{tr}(M_{l_\theta-1|\theta}) = d/l_\theta$, where l_θ denotes the number of effects in the θth POVM. After Alice performed the θth measurement on system A, Bob's understanding of the overall system is then described by the classical-quantum state

$$\rho_{\mathcal{M}_\theta B} = \sum_{i=0}^{l_\theta-1} |i\rangle\langle i| \otimes (K_{i|\theta} \otimes \mathbb{1}_B)\rho_{AB}(K_{i|\theta}^{+} \otimes \mathbb{1}_B). \tag{B1}$$

Here, $\{K_{i|\theta}\}$ are the Kraus operators [75] which satisfy $K_{i|\theta}^{+}K_{i|\theta} = M_{i|\theta}$ by definition.

To prove Theorem 2, we only need to show the operator

$$\hat{\Gamma}_{AB} = \|\hat{g}\|\mathbb{1}_A \otimes \rho_B^{1/2} + \left(\sum_\theta \frac{w_\theta}{l_\theta} - \frac{1}{d}\|\hat{g}\|\right)\bar{\rho}_{AB} - \sum_{\theta,i,x,x'} w_\theta K_{i|\theta}^{+}|x\rangle_A\langle x'|K_{i|\theta} \otimes {}_A\langle x|K_{i|\theta}\bar{\rho}_{AB}K_{i|\theta}^{+}|x'\rangle_A \tag{B2}$$

is positive semidefinite on the space $\mathcal{H}_A \otimes \mathcal{H}_B$, where $\{|x\rangle\}_{x=0}^{d-1}$ is an orthonormal basis of $\mathcal{H}_A$ and $\bar{\rho}_{AB} = (\mathbb{1}_A \otimes \rho_B^{-1/4})\rho_{AB}(\mathbb{1}_A \otimes \rho_B^{-1/4})$. Notice that the measurement-induced local transformation $\rho_{AB} \to \rho_{\mathcal{M}_\theta B}$ commutes with the map $\rho_{AB} \to \bar{\rho}_{AB}$, from $\hat{\Gamma}_{AB} \geqslant 0$ we have

$$\begin{aligned}\mathrm{tr}(\hat{\Gamma}_{AB}\bar{\rho}_{AB}) &= \|\hat{g}\| + \left(\sum_\theta w_\theta/l_\theta - \|\hat{g}\|/d\right)\mathrm{tr}(\bar{\rho}_{AB}^2)\\ &\geqslant \sum_{i,\theta} w_\theta \mathrm{tr}[K_{i|\theta}\bar{\rho}_{AB}K_{i|\theta}^{+}K_{i|\theta}\bar{\rho}_{AB}K_{i|\theta}^{+}]\\ &= \sum_\theta w_\theta \mathrm{tr}(\bar{\rho}_{\mathcal{M}_\theta B}^2),\end{aligned} \tag{B3}$$

where $\bar{\rho}_{\mathcal{M}_\theta B} = (\mathbb{1}_A \otimes \rho_B^{-1/4})\rho_{\mathcal{M}_\theta B}(\mathbb{1}_A \otimes \rho_B^{-1/4})$. This leads us to

$$\sum_\theta w_\theta S_L(\mathcal{M}_\theta|B) \geqslant 1 - \|\hat{g}\| - \left(\sum_\theta w_\theta/l_\theta - \|\hat{g}\|/d\right) \times [1 - S_L(A|B)]. \tag{B4}$$

In the case of rank-1 projective measurements, $l_1 = \cdots = l_\Theta$ are equal to the dimension d of system A. With Eq. (B4), Theorem 2 is already obvious.

Next, we proceed to show $\hat{\Gamma} \geqslant 0$. Observe the operator below is positive semidefinite:

$$\begin{aligned}\hat{\Omega} &= \|\hat{g}\| \cdot \left(\mathbb{1}_d^{\otimes 2} - |\psi_d\rangle\langle\psi_d|\right) - \hat{g}\\ &= \|\hat{g}\| \cdot \mathbb{1}_d^{\otimes 2} - \sum_{i,\theta} w_\theta |M_{i|\theta}\rangle\langle M_{i|\theta}|\\ &\quad + \left(\sum_\theta w_\theta d/l_\theta - \|\hat{g}\|\right)|\psi_d\rangle\langle\psi_d| \geqslant 0,\end{aligned} \tag{B5}$$

and, accordingly, so does its partial transpose over the second space

$$\hat{\Omega}^{T_2} = \|\hat{g}\| \cdot \mathbb{1}_d^{\otimes 2} - \sum_{i,\theta} w_\theta(|M_{i|\theta}\rangle\langle M_{i|\theta}|)^{T_2} + \left(\sum_\theta w_\theta d/l_\theta - \|\hat{g}\|\right)\hat{F} \geqslant 0. \tag{B6}$$

In the above

$$\hat{F} = (|\psi_d\rangle\langle\psi_d|)^{T_2} = \frac{1}{d}\sum_{i,j=0}^{d-1} |i\rangle\langle j| \otimes |j\rangle\langle i|, \tag{B7}$$

and

$$\begin{aligned}&(|M_{i|\theta}\rangle\langle M_{i|\theta}|)^{T_2}\\ &= d(K_{i|\theta}^{+}K_{i|\theta} \otimes \mathbb{1}_d|\psi_d\rangle\langle\psi_d|K_{i|\theta}^{+}K_{i|\theta} \otimes \mathbb{1}_d)^{T_2}\\ &= \sum_{x,x'} K_{i|\theta}^{+}|x\rangle\langle x'|K_{i|\theta} \otimes \sum_{y,y'}(K_{i|\theta})_{xy}|y'\rangle\langle y|(K_{i|\theta}^{+})_{y'x'}.\end{aligned} \tag{B8}$$

Let $\hat{\Omega}_{AC}$ be the operator $\hat{\Omega}$ when defined on the space $\mathcal{H}_A \otimes \mathcal{H}_C$. Similarly, ρ_{CB} and ρ_{AB} denote the same density operator ρ but defined on different spaces. Then, with T_C denoting to the partial transpose over the space $\mathcal{H}_C$, it can be checked that

$$\hat{\Gamma}_{AB} = \mathrm{tr}_C\left(\hat{\Omega}_{AC}^{T_C}\bar{\rho}_{CB}\right). \tag{B9}$$

As a positive-semidefinite Hermitian operator, $\hat{\Omega}$ can be written as the sum of (unnormalized) rank-1 projectors $\hat{\Omega} = \sum_x |\pi_x\rangle\langle\pi_x|$, thereby

$$\begin{aligned}\hat{\Gamma}_{AB} &= \sum_x \mathrm{tr}_C[(|\pi_x\rangle_{AC}\langle\pi_x|)^{T_C}\bar{\rho}_{CB}]\\ &= \sum_x \mathrm{tr}_C[\sqrt{\bar{\rho}_{CB}}(|\pi_x\rangle_{AC})^{T_C}({}_{AC}\langle\pi_x|)^{T_C}\sqrt{\bar{\rho}_{CB}}]\\ &= \sum_x \mathrm{tr}_C(\Pi_x^{+}\Pi_x),\end{aligned} \tag{B10}$$

where $\Pi_x = \sqrt{\bar{\rho}_{CB}}(|\pi_x\rangle_{AC})^{T_C}$. Considering that $\Pi_x^{+}\Pi_x \geqslant 0$ are positive-semidefinite operators on the space $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$, immediately we have $\hat{\Gamma}_{AB} \geqslant 0$. This completes the proof of Theorem 2.

For design-structured measurements, the corresponding combined view operators are proportional to $\mathbb{1}_{\perp\psi_d} = \mathbb{1}_d^{\otimes 2} - |\psi_d\rangle\langle\psi_d|$, then $\hat{\Omega} = \hat{\Gamma}_{AB} = 0$ and Eq. (B3) becomes an equality saturated by arbitrary state ρ_{AB} on $\mathcal{H}_A \otimes \mathcal{H}_C$.

APPENDIX C: INTERFERENCE PATTERN OF INFORMATION AMPLITUDE

We denote by $\{|i_\phi\rangle\}_{i=0,1}$ the measurement basis with respect to the experimental setup where BS_2 of the two-way interferometer (see Fig. 3) is inserted in and the phase shift is ϕ. Then, the associated view operator is

$$\begin{aligned}\hat{G}^{\rm w}_\phi &= \sum_{i=0,1} |i_\phi\rangle\langle i_\phi| \otimes |i_\phi\rangle^*\langle i_\phi| - |\psi_2\rangle\langle\psi_2| \\ &= \frac{1}{2}\sum_{i,j=0,1} (-1)^{i+j}|i_\phi\rangle\langle j_\phi| \otimes |i_\phi\rangle^*\langle j_\phi| \\ &= \frac{1}{2}\big|\sigma^{\rm w}_\phi\big\rangle\!\big\langle\sigma^{\rm w}_\phi\big|,\end{aligned} \tag{C1}$$

where $|\sigma^{\rm w}_\phi\rangle = |0_\phi\rangle \otimes |0_\phi\rangle^* - |1_\phi\rangle \otimes |1_\phi\rangle^*$ is the vector representation of the wave observable $\sigma^{\rm w}_\phi = |0_\phi\rangle\langle 0_\phi| - |1_\phi\rangle\langle 1_\phi|$ given by the isomorphism (1). Similarly, the view operator associated with the path observable $\sigma^{\rm p}$ is given as $\hat{G}^{\rm p} = \frac{1}{2}|\sigma^{\rm p}\rangle\langle\sigma^{\rm p}|$.

Recall that the path observable is complementary to wave observables and, consequently, the view operators $\{\hat{G}^{\rm p}, \hat{G}^{\rm w}_\phi, \hat{G}^{\rm w}_{\phi+\frac{\pi}{2}}\}$ are mutually orthogonal and satisfy

$$\hat{G}^{\rm p} + \hat{G}^{\rm w}_\phi + \hat{G}^{\rm w}_{\phi+\frac{\pi}{2}} \equiv \mathbb{1}_{\perp\psi_2}. \tag{C2}$$

Moreover, for arbitrary two-wave observables $\sigma^{\rm w}_{\phi'}$ and $\sigma^{\rm w}_\phi$, it can be easily checked that

$$\big|\sigma^{\rm w}_{\phi'}\big\rangle = \cos(\phi'-\phi)\big|\sigma^{\rm w}_\phi\big\rangle + \sin(\phi'-\phi)\big|\sigma^{\rm w}_{\phi+\frac{\pi}{2}}\big\rangle. \tag{C3}$$

This leads us to

$$\begin{aligned}\sin^2(\phi'-\phi)\hat{G}^{\rm w}_{\phi+\frac{\pi}{2}} &= \sin(\phi'-\phi)^2\big|\sigma^{\rm w}_{\phi+\frac{\pi}{2}}\big\rangle\!\big\langle\sigma^{\rm w}_{\phi+\frac{\pi}{2}}\big| \\ &= \hat{G}^{\rm w}_{\phi'} + \hat{G}^{\rm w}_\phi\hat{G}^{\rm w}_{\phi'}\hat{G}^{\rm w}_\phi - \hat{G}^{\rm w}_\phi\hat{G}^{\rm w}_{\phi'} - \hat{G}^{\rm w}_{\phi'}\hat{G}^{\rm w}_\phi.\end{aligned} \tag{C4}$$

Combining Eqs. (C4) and (C2) we have for any qubit density operator ρ

$$\begin{aligned}&\sin^2(\phi'-\phi)\langle\rho|\hat{G}^{\rm p} + \hat{G}^{\rm w}_\phi + \hat{G}^{\rm w}_{\phi+\frac{\pi}{2}}|\rho\rangle \\ &\quad= \sin^2(\phi'-\phi)\big[G(\sigma^{\rm p})_\rho + G\big(\sigma^{\rm w}_\phi\big)_\rho\big] + G\big(\sigma^{\rm w}_{\phi'}\big)_\rho \\ &\qquad+ \cos^2(\phi'-\phi)G\big(\sigma^{\rm w}_\phi\big)_\rho - \cos(\phi'-\phi)\big\langle\sigma^{\rm w}_\phi\big\rangle\!\big\langle\sigma^{\rm w}_{\phi'}\big\rangle \\ &\quad= \sin^2(\phi'-\phi)\langle\rho|\mathbb{1}_{\perp\psi_2}|\rho\rangle = \sin^2(\phi'-\phi)I_{\rm com}(\rho),\end{aligned} \tag{C5}$$

which completes the proof of Eq. (11).

To derive the interference pattern of the "amplitude of conditional information" as given in Eq. (15), let us consider the equality

$$\begin{aligned}\hat{\Delta}_{AC} &= \hat{G}^{\rm w}_\phi + \hat{G}^{\rm w}_{\phi'} - \hat{G}^{\rm w}_\phi\hat{G}^{\rm w}_{\phi'} - \hat{G}^{\rm w}_{\phi'}\hat{G}^{\rm w}_\phi \\ &= \sin^2(\phi'-\phi)[\mathbb{1}_{\perp\psi_2} - \hat{G}^{\rm p}].\end{aligned} \tag{C6}$$

From the proof of Theorem 2 we have

$$\begin{aligned}{\rm tr}_{ABC}\big(\bar{\rho}_{AB}\hat{\Delta}^{T_C}_{AC}\bar{\rho}_{CB}\big) &= \sin^2(\phi'-\phi)\big[1 - {\rm tr}\big(\bar{\rho}^2_{\sigma^{\rm p}B}\big)\big] \\ &= {\rm tr}\big(\bar{\rho}^2_{\sigma^{\rm w}_\phi B}\big) + {\rm tr}\big(\bar{\rho}^2_{\sigma^{\rm w}_{\phi'} B}\big) - {\rm tr}\big(\bar{\rho}^2_{AB}\big) \\ &\quad- T(\phi'-\phi),\end{aligned} \tag{C7}$$

where $\rho_{\sigma B}$ denotes the classical-quantum state (B1) after measuring the observable σ and

$$\begin{aligned}T(\phi'-\phi) &= 2\,{\rm tr}_{ABC}\big[\bar{\rho}_{CB}\bar{\rho}_{AB}\big(\hat{G}^{\rm w}_\phi\hat{G}^{\rm w}_{\phi'}\big)^{T_C}_{AC}\big] \\ &= 2\cos(\phi'-\phi){\rm tr}_{ABC}\big[\bar{\rho}_{CB}\bar{\rho}_{AB}\big(\big|\sigma^{\rm w}_\phi\big\rangle\!\big\langle\sigma^{\rm w}_{\phi'}\big|\big)^{T_C}_{AC}\big] \\ &= \cos(\phi'-\phi)\sum_{i,j=0,1}(-1)^{i+j}{\rm tr}_{ABC} \\ &\quad\times[\bar{\rho}_{AB}|i_\phi\rangle_A\langle j_{\phi'}| \otimes |j_{\phi'}\rangle_C\langle i_\phi|\bar{\rho}_{CB}] \\ &= \cos(\phi'-\phi)\sum_{i,j=0,1}(-1)^{i+j}{\rm tr}_B \\ &\quad\times[{}_A\langle j_{\phi'}|\bar{\rho}_{AB}|i_\phi\rangle_A\langle i_\phi|\bar{\rho}_{AB}|j_{\phi'}\rangle_A] \\ &= \cos(\phi'-\phi){\rm tr}_{AB}\big(\sigma^{\rm w}_{\phi'}\bar{\rho}_{AB}\sigma^{\rm w}_\phi\bar{\rho}_{AB}\big) \\ &= \cos(\phi'-\phi){\rm tr}_B\big[{\rm tr}_A\big(\bar{\rho}_{AB}\sigma^{\rm w}_\phi \otimes \mathbb{1}_B\big) \\ &\quad\times{\rm tr}_A\big(\bar{\rho}_{AB}\sigma^{\rm w}_{\phi'} \otimes \mathbb{1}_B\big)\big] \\ &\quad- \cos^2(\phi'-\phi)\big[{\rm tr}\big(\bar{\rho}^2_{AB}\big) - 1\big].\end{aligned} \tag{C8}$$

Observe Eq. (C7) can be rewritten as

$$\begin{aligned}&{\rm tr}\big(\bar{\rho}^2_{\sigma^{\rm w}_\phi B}\big) - 1/2 + {\rm tr}\big(\bar{\rho}^2_{\sigma^{\rm w}_{\phi'} B}\big) - 1/2 - \cos(\phi'-\phi){\rm tr}_B \\ &\quad\times\big[{\rm tr}_A\big(\bar{\rho}_{AB}\sigma^{\rm w}_\phi \otimes \mathbb{1}_B\big){\rm tr}_A\big(\bar{\rho}_{AB}\sigma^{\rm w}_{\phi'} \otimes \mathbb{1}_B\big)\big] \\ &\quad= \sin^2(\phi'-\phi)\big[{\rm tr}\big(\bar{\rho}^2_{AB}\big) - {\rm tr}\big(\bar{\rho}^2_{\sigma^{\rm p}B}\big)\big].\end{aligned} \tag{C9}$$

Let $\vec{\rho}_{\phi B} = \bar{\rho}_{\phi B}$ $\vec{\phi} = {\rm tr}_A(\bar{\rho}_{AB}\sigma^{\rm w}_\phi \otimes \mathbb{1}_B)$ $\vec{\phi}$, apparently ${\rm tr}(\vec{\rho}^{\,2}_{\phi B}) = {\rm tr}(\bar{\rho}^{\,2}_{\phi B}) = 2\,{\rm tr}(\bar{\rho}^{\,2}_{\sigma_\phi B}) - 1 = 2I(\sigma^{\rm w}_\phi|B)$. Equation (C9) thus completes the proof of Eq. (15).

APPENDIX D: PROOF OF THEOREM 3

This proof is inspired by the works [76,77] on entanglement detection with MUMs [48]. By definition any bipartite separable state can be written as a linear combination of product states in the form $\rho_{AB} = \sum_k p_k\rho_{A_k} \otimes \rho_{B_k}$ ($p_k > 0$, $\sum_k p_k = 1$). For a product state $\rho_A \otimes \rho_B$, obviously ${\rm tr}(J_{i|\theta}\rho_A \otimes \rho_B) = [p^A_{i|\theta} - \frac{1}{d_A}{\rm tr}(M^A_{i|\theta})][p^B_{i|\theta} - \frac{1}{d_B}{\rm tr}(M^B_{i|\theta})]$. Then we have

$$\begin{aligned}&J(\rho_A \otimes \rho_B) \\ &\quad= \sum_{i,\theta}\sqrt{w_\theta}\Big|p^A_{i|\theta} - {\rm tr}\big(M^A_{i|\theta}\big)\frac{1}{d_A}\Big|\sqrt{w_\theta}\Big|p^B_{i|\theta} - {\rm tr}\big(M^B_{i|\theta}\big)\frac{1}{d_B}\Big| \\ &\quad\leqslant \Bigg[\sum_{i,\theta} w_\theta\big[p^A_{i|\theta} - {\rm tr}\big(M^A_{i|\theta}\big)/d_A\big]^2\Bigg]^{1/2} \\ &\qquad\times\Bigg[\sum_{i,\theta} w_\theta\big[p^B_{i|\theta} - {\rm tr}\big(M^B_{i|\theta}\big)/d_B\big]^2\Bigg]^{1/2} \\ &\quad\leqslant \sqrt{\|\hat{g}_A\|I_{\rm com}(\rho_A)\cdot\|\hat{g}_B\|I_{\rm com}(\rho_B)} \leqslant \sqrt{L_AL_B},\end{aligned}$$

with $L_A = \|\hat{g}_A\|(1 - 1/d_A)$ and $L_B = \|\hat{g}_B\|(1 - 1/d_B)$ being state-independent upper bounds on local information gains, and the first inequality above exploits the Cauchy-Schwarz inequality. Therefore, for bipartite separable states there must be $J(\rho_{AB}) = \sum_{i,\theta} w_\theta |\mathrm{tr}(J_{i|\theta} \sum_k p_k \rho_{A_k} \otimes \rho_{B_k})| \leqslant \sum_k p_k \sum_{i,\theta} w_\theta |\mathrm{tr}(J_{i|\theta} \rho_{A_k} \otimes \rho_{B_k})| = \sum_k p_k J(\rho_{A_k} \otimes \rho_{B_k}) \leqslant \sum_k p_k \sqrt{L_A L_B} = \sqrt{L_A L_B}$.

APPENDIX E: PROOF OF THEOREM 4

Observe that in the case of rank-1 projective measurements Eq. (B3) becomes

$$\|\hat{g}\| + (1 - \|\hat{g}\|) F^{\mathrm{pg}}(A|B) \geqslant \sum_\theta w_\theta \mathrm{tr}\big(\bar{\rho}^2_{\mathcal{M}_\theta B}\big). \tag{E1}$$

Considering that $H^\varepsilon_{\min}(\mathcal{M}_\theta|B) \geqslant -\log_2[\mathrm{tr}(\bar{\rho}^2_{\mathcal{M}_\theta B})] - \log_2 \frac{2}{\varepsilon^2}$ (see Lemma 19 of Ref. [78] and Theorem 7 of Ref. [79]), immediately

$$\begin{aligned}\sum_\theta w_\theta H^\varepsilon_{\min}(\mathcal{M}_\theta|B) &\geqslant -\log_2\left[\sum_\theta w_\theta \,\mathrm{tr}\big(\bar{\rho}^2_{\mathcal{M}_\theta B}\big)\right] - \log_2 \frac{2}{\varepsilon^2} \\ &\geqslant -\log_2[\|\hat{g}\| + (1 - \|\hat{g}\|)F^{\mathrm{pg}}(A|B)] \\ &\quad - \log_2 \frac{2}{\varepsilon^2}.\end{aligned} \tag{E2}$$

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