Exact quantum revivals for the Dirac equation

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In the present work, the results obtained by Strange [Phys. Rev. Lett. **104**, 120403 (2010)] about the revivals of a relativistic fermion wave function on a torus are considerably expanded. In fact, all the possible quantum states exhibiting revivals are fully characterized. The revivals are exact, that is, are true revivals without taking any particular limit such as the nonrelativistic one. The present results are of interest since they generalize the Talbot effect and the revivals of the Schrödinger equation to a relativistic situation with nonzero mass. This makes the problem nontrivial, as the dispersion relation is modified and is not linear. The present results are obtained by the use of arithmetic tools, which are described in certain detail. In addition, several plots of the revivals are presented, which are useful for exemplifying the procedure proposed in the present paper.

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I. INTRODUCTION

One of the historical situations where the revivals of an initial state was of particular interest is the Talbot effect. This replication phenomenon appears when a plane electromagnetic wave is incident upon a periodic diffraction grating and the image of the grating is repeated at regular distances away from the grating plane [1–9]. This effect, of course, is related to wave equations, which are nondispersive.

The existence of revivals in the context of quantum mechanics is instead more surprising. The Schrödinger equation is an example of a dispersive equation and it implies that when time evolves the solution is expected to randomize or spread out. This can be intuitively visualized as follows. Given an equation of the form $\partial_t\Psi + P(\partial_x)\Psi = 0$, with $P(x)$ a polynomial, if a solution of the form $\Psi = \exp(i\omega t - ikx)$ is assumed, this is consistent if and only if the dispersion relation $\omega(k) = iP(ik)$ is satisfied. This implies that the velocity for a wave number k is given by $v(k) = k^{-1}\omega(k)$ and is k dependent, unless $P(x) = x$; this last case corresponds to the wave equation. In particular, if an initial localized wave packet Ψ_0 is expressed as a Fourier transform $\Psi_0 = \int e^{ikx}\Psi(k)dk$, then for the wave equation the resulting evolved modes corresponding to different k values travel at the same speed. The result is a traveling wave which does not change its shape. Instead, for other equations, the Fourier modes for different values of k travel at different speeds and they start to cancel by destructive interference. The original initial solution Ψ_0 spreads out. For the particular case of the Schrödinger equation, a basic explicit example in quantum mechanics describing a dispersive behavior is the evolution of an initial Gaussian packet, which at a time t is given by

$$\Psi(\mathbf{r}, t) = \frac{\exp\left[-\frac{1}{2}(a + i\hbar t/m)^{-1}\mathbf{r}\cdot\mathbf{r}\right]}{(\pi a)^{3/4}(1 + i\hbar t/ma)^{3/2}}.$$

It is clear that the modulus of this wave function spreads out with time. In the quantum mechanics context, this is interpreted as a manifestation of the uncertainty principle.

Despite the considerations given above about dispersive behavior, there are exceptions to the rule. In fact, in some situations there is a coherence between the phase velocities causing quantum revivals. This is an exact or approximate periodicity in time. For instance, if the energy spectrum is discrete, say, $\{E_n\}_{n=1}^{\infty}$, and there exists $s \in \mathbb{R}^+$ such that $\{sE_n\}_{n=1}^{\infty} \subset \mathbb{Z}$, then the solutions of the Schrödinger equation are of the form

$$\Psi(\mathbf{r}, t) = \sum_{n=1}^{\infty} a_n\psi_n(\mathbf{r})e^{-iE_n t/\hbar}$$

and an exact revival $\Psi(\mathbf{r}, t) = \Psi(\mathbf{r}, t + T_{\text{rev}})$ appears, with $T_{\text{rev}} = \frac{2\pi s}{\hbar}$ the revival time. A simple example of this situation is the infinite quantum well. The changes suffered by the initial state at a fractional multiple of the revival time and its fractalization for irrational multiples have been studied for several quantum systems in many works [1–3,5,10,11], establishing some interesting connections with the classic Talbot effect in optics and certain topics in number theory [6].

There exist simulations related to quantum particles in boxes [7,8] which might suggest that, by taking into account relativistic effects, the coherence of the phase velocities is lost and one can only expect approximate quantum revivals (as it occurs in many models; cf. [9]). However, Ref. [12] shows a different truth. In this reference, a fermionic Dirac particle confined to a circle of radius R is studied and it is shown that if the dimensionless quantity $q = \frac{McR}{\hbar}$, where M is the mass, is a positive even integer, then one can construct a finite number of plane-wave spinor solutions presenting exact quantum revivals. An elegant part of the argument is that these plane waves are associated with Pythagorean triples. These are integral solutions of the algebraic equation $x^2 + y^2 = z^2$

(see Sec. 13.2 in [13]) which constitute a very old topic in pure mathematics. This observation introduces a connection between quantum revivals and number theory, to be explored further in the present paper.

With the methods developed in [12], it follows that for $q \in 2\mathbb{Z}^+$ there are only a few possible plane-wave solutions which exhibit quantum revivals. These solutions are related to the divisors of q [on average, this number goes like $\log q$ (see Sec. 18.2 in [13]). This fact suggests that there is little freedom to construct examples following those lines. In this paper this model is analyzed with more advanced arithmetic tools. In these terms, it turns out that for each q satisfying a less restrictive condition, apart from some trivial cases, there are infinitely many plane-wave solutions such that any linear combination of them presents exact revivals. In fact, for each valid q all the states showing these revivals are characterized in an algorithmic way. The exponential growth of the energies explains some fractal-like quantum carpets (which are density plots of the state along a period [14]) and connects with some topics in fractal geometry. The case of particles confined in a square flat torus is also studied. The number of plane waves serving as building blocks to obtain the states possessing exact quantum revivals is still infinite for suitable fixed values of q in a dense set, but in this two-dimensional case the possible energies present a mild growth, allowing high degeneracies and more freedom to choose the initial conditions. This will be described in detail throughout the text and involves an interesting connection with Fuchsian groups.

The present work is organized as follows. In Sec. II the defining equation and the representation employed throughout the text are clarified. Also, the main problem of characterization of revivals is stated. In Sec. III the possible revivals in a one-dimensional torus are characterized in terms of solutions of Pell's equations and the presence of an infinite number of states exhibiting quantum revivals is pointed out. In Sec. IV the two-dimensional case is analyzed. This situation is more complex and its analysis relies on arithmetic topics related to Fuchsian groups. Section V contains several examples with numerical simulation, which are aimed at clarifying the main procedure described throughout the text. Section VI discusses the results presented.

II. BASIC MODEL

In the following, the plane-wave solutions of the Dirac equation in one time and two spatial dimensions $i\gamma^\mu\partial_\mu\Psi - m\Psi = 0$, under the representation $\gamma^0 = \sigma^0$, $\gamma^1 = i\sigma^1$, and $\gamma^2 = i\sigma^2$, with σ^i the standard Pauli matrices, will be considered, following standard textbooks such as [15]. The particles will be assumed to be confined in a flat two-dimensional torus obtained by identifying the opposite sides of $[2\pi R_1, 0] \times [0, 2\pi R_2]$. In this situation the Cartesian coordinates may be parametrized as $x = R_1\phi_1$ and $y = R_2\phi_2$, with ϕ_1 and ϕ_2 two 2π-periodic angles. This corresponds to periodic boundary conditions on the wave function, which correspond to relativistic Bloch states $u\exp(-\frac{ik\cdot x}{\hbar})$, and imposes the following quantization of the momentum:

$$\vec{k} = \left(\frac{\hbar n_1}{R_1}, \frac{\hbar n_2}{R_2}\right) \quad \text{with } \vec{n} = (n_1, n_2) \in \mathbb{Z}^2.$$

Let $\omega_{\vec{n}} = \hbar^{-1}E_{\vec{n}} > 0$ be the angular frequency for each $\vec{n}$ and consider the complex quantity

$$z_{\vec{n}} = \frac{\hbar n_1}{R_1} + \frac{i\hbar n_2}{R_2},$$

associated with the momentum. The generic energy equation $E^2 = M^2c^4 + c^2\vec{k}^2$ implies

$$\omega_{\vec{n}} = \frac{c}{\hbar}\sqrt{M^2c^2 + |z_{\vec{n}}|^2},$$

which in particular gives the useful relation

$$\left(\frac{\hbar\omega_{\vec{n}}}{c} + Mc\right)^2 + |z_{\vec{n}}|^2 = \frac{2\hbar\omega_{\vec{n}}}{c}\left(\frac{\hbar\omega_{\vec{n}}}{c} + Mc\right).$$

The normalized solution is in these terms

$$\Psi_{\vec{n}}(\phi_1, \phi_2, t) = \sqrt{\frac{E_{\vec{n}} + Mc^2}{2E_{\vec{n}}}}\begin{pmatrix}\frac{z^*_{\vec{n}}}{E_{\vec{n}}+Mc^2}\\ 1\end{pmatrix}e^{i(n_1\phi_1+n_2\phi_2-\omega_{\vec{n}}t)}.$$

We restrict ourselves to the fermionic case since similar considerations hold for antifermions.

Some particular limits are in order. The one-dimensional case corresponds to letting $R_1 \to \infty$, so the periodicity in the x coordinate is eliminated, $k^1 = 0$, and the particle is confined to move in a (flat) circle of radius $R = R_2$ given by a $2\pi R$-periodic y coordinate. The two-dimensional case corresponds to the model above and choosing $R = R_1 = R_2$ (a square flat torus). Therefore, $z_{\vec{n}} = \frac{i\hbar n_2}{R}$ in the first case and $z_{\vec{n}} = \frac{\hbar(n_1+in_2)}{R}$ in the second one.

Both cases above corresponding to $R_1 = R_2 = R$ can be written in a unified way by introducing the dimensionless positive quantity $q = \frac{McR}{\hbar}$, from which $\omega_{\vec{n}} = \frac{c}{R}\sqrt{\vec{n}^2 + q^2}$, and the formula

$$\Psi_{\vec{n}}(\phi_1, \phi_2, t) = \frac{1}{\sqrt{2}}\left(1 + \frac{q}{\sqrt{\vec{n}^2 + q^2}}\right)^{1/2} \times \begin{pmatrix}\frac{n_1 - in_2}{q+\sqrt{\vec{n}^2+q^2}}\\ 1\end{pmatrix}e^{i(n_1\phi_1+n_2\phi_2-\omega_{\vec{n}}t)} \tag{2.1}$$

follows. The one-dimensional case corresponds to setting $n_1 = 0$ (so the state does not depend on ϕ_1) and the two-dimensional case to leaving it free. The $\frac{1}{2\pi}$ factor is taken apart from the normalization in order to avoid conflicts with the choice of the dimension.

In order to make a comparison with [12], note that Strange considers a wave function with four components, and the corresponding γ^μ matrices are of order 4×4. This is different from the present case. However, the revival issues, which are of central importance in the present paper, are sensitive only to the phase and not to the normalization factors or the number of components of the spinor.

The spinor given above is an energy eigenfunction. A generic state having a discrete, finite, and non-negative energy spectrum is a superposition of the form

$$\Phi = \sum_{\vec{n}\in\mathcal{N}} c_{\vec{n}}\Psi_{\vec{n}} \quad \text{with } c_{\vec{n}} \in \mathbb{C} - \{0\}, \tag{2.2}$$

where $\mathcal{N} = \{\vec{n}_0, \vec{n}_1, \ldots, \vec{n}_N\}$ contains N vectors, $\vec{n}_j = (k_j, l_j) \in \mathbb{Z}^2$, and $k_j = 0$ in the one-dimensional model. The energy spectrum is related to the squared norm of the $\vec{n}_j$. It can be described as the set

$$\mathcal{E} = \left\{ \frac{c\hbar}{R}\sqrt{k_j^2 + \ell_j^2 + q^2} : (k_j, \ell_j) \in \mathcal{N} \right\}.$$

The cardinality $\#\mathcal{E}$ of this set may be different from N, as there is a possible degeneracy of states with the same energy value. At this point, it is important to state the following assertion, which is a direct and simple consequence of the spinor formulas given above.

Proposition 1. The state described by the formulas (2.1) and (2.2) is periodic in time, and thus exhibits quantum revival, if and only if for each $0 \leqslant j \leqslant N$ there exists an irreducible fraction $\frac{a_j}{b_j} > 0$ such that

$$k_j^2 + \ell_j^2 + q^2 = \frac{a_j^2}{b_j^2}\left(k_0^2 + \ell_0^2 + q^2\right). \tag{2.3}$$

Here N is the number of elements of the state (2.2). The revival time is given by the formula

$$T_{\rm rev} = \frac{2\pi L}{\omega_{\vec{n}_0}} \quad \text{with } L = \operatorname{lcm}\left(\{b_j\}_{j=0}^N\right), \tag{2.4}$$

where, as usual, lcm denotes the least common multiple.

Proof. This proposition follows directly from the formulas just stated. In fact, consider a generic wave function (2.2) expanded in terms of the eigenfunctions (2.1). For fixed values of ϕ_1 and ϕ_2, recalling the formula for $\omega_{\vec{n}}$, there will be a revival if for all the $0 \leqslant j \leqslant N$ there is a time $T_{\rm rev}$, the revival time, such that

$$\sqrt{\vec{n}_j^2 + q^2}\, c T_{\rm rev} = 2\pi R m_j, \quad m_j \in \mathbb{Z}.$$

The timelike phase of the state (2.2) in this case will be a multiple of 2π and will reproduce the initial wave function at $t = 0$. Now, as the preceding formula follows from every j, it follows that

$$\left(\vec{n}_j^2 + q^2\right) = \frac{m_j^2}{m_0^2}\left(\vec{n}_0^2 + q^2\right).$$

This is of the form (2.3) after simplifying the common factors between m_0 and m_j. We have $\frac{\omega_{\vec{n}_0}}{2\pi} T_{\rm rev} = m_j \frac{\omega_{\vec{n}_0}}{\omega_{\vec{n}_j}} = m_j \frac{b_j}{a_j}$. Since m_j is a multiple of a_j and $T_{\rm rev}$ is defined as the minimal period, $\frac{\omega_{\vec{n}_0}}{2\pi} T_{\rm rev}$ equals L, as stated in (2.4). ■

The revival problem has been reduced to the problem of solving Eq. (2.3). The corresponding solutions are not necessarily trivial from the arithmetic point of view. In fact, as it will be shown below, they are described in terms of the generalized Pell equations in the one-dimensional case and they relate to some Fuchsian groups in the two-dimensional (2D) case.

The philosophy that will guide the present paper is perhaps different from the standard one. Instead of looking for particular states which exhibit quantum revivals, the intention is to characterize all of them, which is achieved by fixing a specific vector state $\vec{n}_0$ and considering all possible Φ in (2.2) containing $\Psi_{\vec{n}_0}$.

Philosophy. Consider a pivot eigenstate $\Psi_{\vec{n}_0}$. The idea is to characterize the possible sets $\mathcal{N}$ containing $\vec{n}_0$ such that quantum revivals appear. More concretely, given $\vec{n}_0$, we want to determine the maximal set of quantum numbers $\mathcal{N}_0$ including $\vec{n}_0$ such that the wave function

$$\Phi = \sum_{\vec{n} \in \mathcal{N}} c_{\vec{n}} \Psi_{\vec{n}} \tag{2.5}$$

verifies

$$\Phi \text{ is periodic in } t \Leftrightarrow \mathcal{N} \text{ is a finite subset of } \mathcal{N}_0. \tag{2.6}$$

Note that, once the revival set $\mathcal{N}_i$ for an arbitrary pivot eigenstate $\Psi_{\vec{n}_i}$ is determined, given a composite wave function

$$\Phi_c = \sum_i c_i \Psi_{\vec{n}_i},$$

its most general revival wave function is

$$\Phi_r = \sum_{\vec{n} \in \mathcal{N}_I} c_{\vec{n}} \Psi_{\vec{n}},$$

where $\mathcal{N}_I$ is the intersection of all the sets $\mathcal{N}_i$. In addition, it is not difficult to see that $\mathcal{N}_0$ is not uniformly bounded. This can be exemplified even in the simplest case, in which one forces all the states to have the same energy, that is, $\#\mathcal{E} = 1$. In the one-dimensional situation $\Psi_{(0,\ell_0)}$ can be accompanied by only $\Psi_{(0,-\ell_0)}$, giving obvious revivals in (2.2) and $\mathcal{N}_0 = 2$. This situation is uninteresting. However, in the 2D case the situation is less trivial because $\#\mathcal{N}$ can be arbitrarily large while keeping $\#\mathcal{E} = 1$ because the function giving the number of representations as a sum of two squares is unbounded.[1] So $\mathcal{N}_0$ is finite but not uniformly bounded in terms of $\vec{n}_0$. For instance, $\vec{n}_0 = (178, 19)$ can be complemented with 63 other integral pairs to get the same $\frac{R\omega_{\vec{n}_j}}{c} = \sqrt{\vec{n}_0^2 + q^2} = \sqrt{32\,045 + q^2}$. Examples of these pairs are

$$\begin{aligned} 178^2 + 19^2 &= 166^2 + 67^2 = 157^2 + 86^2 = 179^2 + 2^2 \\ &= 142^2 + 109^2 = 163^2 + 74^2 = \cdots . \end{aligned}$$

This phenomenon of unbounded degeneracy of the energy leads typically to intricate density probabilities for a fixed time (see Sec. V) and it is linked to some unsolved problems about the structure of the nodal lines [16,17].

The rest of the present work is aimed at showing the complexity of the set $\mathcal{N}_0$ without restricting the attention to just one energy eigenvalue.

III. ONE-DIMENSIONAL CASE

Recall that in the one-dimensional case the first coordinate of $\vec{n}$ is set to 0 and the necessary and sufficient condition (2.3) for having quantum revivals becomes

$$\ell_j^2 + q^2 = \frac{a_j^2}{b_j^2}\left(\ell_0^2 + q^2\right) \quad \text{for } 0 \leqslant j \leqslant N. \tag{3.1}$$

Since l_0, l_j, a_j, and b_j are integers, it is clear that q^2 should be rational. In fact, if $q^2 \notin \mathbb{Q}^+$ then $\frac{a_j}{b_j} = 1$ and $\#\mathcal{E} = 1$, leading

[1] However, it is π on average, as is easily checked by approximating the number of integer points in a circle by its area (see Theorem 339 in [13]).

to a trivial case in which the wave function is composed of two eigenstates corresponding to l_0 and $-l_0$, as mentioned before. Therefore, it can be safely assumed that $q^2 \in \mathbb{Q}^+$. In fact, it is convenient to focus on $q^2 \in \mathbb{Z}^+$ because it is somewhat simpler and contains all the ingredients appearing in the general situation.

Some concepts of arithmetic are in order. First, the standard number theory notation $a|b$ will be employed to express that the integer a divides the integer b. In addition, recall that a positive integer is said to be square-free if it is not divisible by a square different from 1. Each positive integer m admits a unique factorization as a product of a square-free and a square integer in the form $m = Dr^2$ such that $\sqrt{D} \notin \mathbb{Q}$ and $r \in \mathbb{Z}$. Consider such a decomposition for $\ell_0^2 + q^2$,

$$\ell_0^2 + q^2 = Ds^2, \tag{3.2}$$

with D square-free. Then (3.1) can be arranged as

$$\ell_j - D\left(\frac{a_j s}{b_j}\right)^2 = -q^2. \tag{3.3}$$

As D is square-free and q^2 is an integer, $\frac{a_j s}{b_j}$ is an integer too. So all the possibilities for ℓ_j and the corresponding $\frac{a_j}{b_j}$ are in a one-to-one correspondence with the integer solutions of the generalized Pell equation

$$x^2 - Dy^2 = -q^2. \tag{3.4}$$

In these terms, Proposition 1 of the preceding section, adapted to the one-dimensional case, is the following.

Proposition 2. Given $\ell_0 \in \mathbb{Z}$ and $q^2 \in \mathbb{Z}^+$ and defining D and s as in (3.2), the set $\mathcal{N}_0$ of quantum numbers to have (2.6) when $(0, \ell_0) \in \mathcal{N}$ is

$$\mathcal{N}_0 = \{(0, x) : x^2 - Dy^2 = -q^2 \text{ with } (x, y) \in \mathbb{Z}^2\}. \tag{3.5}$$

Moreover, the revival time is given by

$$T_{\text{rev}} = \frac{2\pi RL}{cs\sqrt{D}} = \frac{2\pi L}{\omega_{\vec{n}_0}},$$

where L is the lcm of the denominators of

$$\frac{a_j}{b_j} = \frac{1}{s}\sqrt{\frac{\ell_j^2 + q^2}{D}}$$

for $(0, \ell_j) \in \mathcal{N}$.

In view of the last proposition, it is mandatory to review the solutions of the generalized Pell equation (3.4) that allows us to describe the structure of $\mathcal{N}_0$ for any choice of $\ell_0 \in \mathbb{Z}$ and $q^2 \in \mathbb{Z}^+$. Some studies about this equations can be found in [18].

The case $D = 1$ is somewhat trivial because $x^2 - Dy^2 = (x - y)(x + y) = -q^2$ implies that $x - y$ and $x + y$ are divisors of q^2, leaving only a finite number of possibilities. In other words, if $\ell_0^2 + q^2$ is a square, $\mathcal{N}_0$ is finite. This includes the case covered in [12] using Pythagorean triples. If $D \neq 1$, $\mathcal{N}_0$ is instead an infinite set, which can be seen as follows. For the standard Pell equation

$$x^2 - Dy^2 = 1, \tag{3.6}$$

a classic result due to Lagrange is of fundamental importance for finding the solution.

Lagrange algorithm. If the constant $D \in \mathbb{Z}^+$ is not a square (this is ensured in the present case since D is square-free and $D \neq 1$) then Eq. (3.6) always has infinitely many solutions $(x, y) \in \mathbb{Z}^+ \times \mathbb{Z}^+$. These can be expressed in a synthetic way as

$$x + y\sqrt{D} = (x_p + y_p\sqrt{D})^n \quad \text{with } n \in \mathbb{Z}^+, \tag{3.7}$$

where (x_p, y_p) is the minimal positive solution, namely, the one having x_p and hence y_p as small as possible.

It can be checked directly that the preceding formula gives solutions by simply multiplying by the conjugate because

$$(x_p + y_p\sqrt{D})(x_p - y_p\sqrt{D}) = x_p^2 - Dy_p^2 = 1.$$

In the present case, it is deduced from (3.2) that (ℓ_0, s) is an integer solution of (3.4). The same argument just employed, using the conjugate, shows that

$$x + y\sqrt{D} = \pm(\ell_0 + s\sqrt{D})(x_p + y_p\sqrt{D})^n \quad \text{with } n \in \mathbb{Z} \tag{3.8}$$

are infinitely many integer solutions of (3.4), proving that $\mathcal{N}_0$ is infinite. The $\pm$ and the extension of the range of n from $\mathbb{Z}^+$ to $\mathbb{Z}$ are only to take into consideration different combinations of signs. Note that $x_p - y_p\sqrt{D} = (x_p + y_p\sqrt{D})^{-1}$.

By defining the nth family of solutions as $x_n + y_n\sqrt{D} = (x_p + y_p\sqrt{D})^n$, it follows that (3.7) can be written as

$$x_{n+1} + y_{n+1}\sqrt{D} = (x_p + y_p\sqrt{D})(x_n + y_n\sqrt{D}).$$

This leads to the first-order recurrence

$$x_{n+1} = x_n x_p + Dy_n y_p, \quad y_{n+1} = x_p y_n + x_n y_p.$$

Repeating this step once more leads to an expression for x_{n+2} as a linear combination of x_n and y_n, and similarly for x_{n+1}. By eliminating the y_n variable with these two equations, we obtain a second-order linear recurrence for x_n of the generic form

$$x_{n+2} = \alpha x_{n+1} + \beta x_n,$$

with α and β constants depending on D, x_p, and y_p. The variable x_n is the fundamental quantity describing $\mathcal{N}_0$. A concrete example is given in the following sections, more precisely in the formula (5.1).

A complication of the theory (related to deep topics in number theory such as unique factorization and class number [19]) is that a finite number of similar families of solutions may exist. More specifically, in our case, families of integer solutions may appear that replace ℓ_0 and s by a and b with $(x, y) = (a, b)$ satisfying (3.4); however, it can be proved that $0 < a < q(1 + \sqrt{2x_p})/2$ [20] and then there is only a finite number of possibilities to explore. All the possible families obey the same recurrence law because they always come from the multiplication by $x_p + y_p\sqrt{D}$.

Note that (3.8) always can be translated in one-sided recurrence formulas by dividing into $n \geqslant n_0$ and $n < n_0$. With this remark and the previous considerations, the following is deduced.

Proposition 3. There exists a finite number of sequences $c_n^{(1)}, \ldots, c_n^{(J)}$ satisfying a second-order linear recurrence such that

$$\mathcal{N}_0 = \big\{\big(0, \pm c_n^{(j)}\big) \text{ for } 1 \leqslant j \leqslant J,\ n \geqslant 0\big\}.$$

Let us illustrate the situation with an example. If $\ell_0 = 5$ and $q = \sqrt{2}$ then (3.2) produces $D = 3$ and $s = 3$. A direct search shows that $(x_p, y_p) = (2, 1)$ is the minimal positive solution of $x^2 - 3y^2 = 1$. The values $n = -1, 0, 1, 2$ in (3.8) give

$$x + y\sqrt{3} = \pm(1+\sqrt{3}), \pm(5+3\sqrt{3}), \pm(19+11\sqrt{3}), \pm(71+41\sqrt{3}).$$

Hence $(0, \pm 1)$, $(0, -5)$, $(0, \pm 19)$, and $(0, \pm 71)$ are in the $\mathcal{N}_0$ corresponding to $(0, \ell_0) = (0, 5)$. For this example, it can be proved that (see Sec. V)

$$\mathcal{N}_0 = \{(0, \pm c_n) \text{ where } c_{n+2} = 4c_{n+1} - c_n \text{ with } c_0 = 1,\ c_1 = 5\}. \tag{3.9}$$

There are not more families because the bound $0 < a < q(1 + \sqrt{2x_p})/2$ only allows $(a, b) = (1, \pm 1)$, which belong to the same family as $(\ell_0, s) = (5, 3)$ because $1 + \sqrt{3} = (5 + 3\sqrt{3})(2 + \sqrt{3})^{-1}$ and $1 - \sqrt{3} = -(5 + 3\sqrt{3})(2 + \sqrt{3})^{-2}$.

The sequence, due to (3.8), has exponential growth

$$\{c_n\} = \{1, 5, 19, 71, 265, 989, 3691, 13\,775, 51\,409, 191\,861, 716\,035, 2\,672\,279, 9\,973\,081, \ldots\}.$$

Depending on the elements chosen to compose $\mathcal{N}_0$, the revival time can be at most $\frac{6\pi}{\omega_{\vec{n}_0}}$, because $\frac{a_j s}{b_j} = \frac{3a_j}{b_j}$ must be an integer and hence L divides 3. For an exhaustive description of the generalized Pell equation, some interesting references are [19–22].

The next task is to show how the previous ideas can be modified to cover the remaining cases $q^2 \in \mathbb{Q}^+ - \mathbb{Z}$ attaining the same result. This will be done briefly. In this situation, Eq. (3.2) is replaced by the square-free–square decomposition of the numerator and the denominator of $\ell_0^2 + q^2$ to get

$$\ell_0^2 + q^2 = \frac{Ds^2}{D_* s_*^2} \quad \text{with } D \text{ and } D_* \text{ square-free.}$$

Clearing denominators in (3.3), it is obtained that $(\ell_j, \frac{a_j s}{b_j})$ is a solution of

$$D_* s_*^2 x^2 - Dy^2 = -q^2 D_* s_*^2.$$

In fact, it is an integral solution because $-q^2 D_* s_*^2 \in \mathbb{Z}$ and the tentative denominator of $(\frac{a_j s}{b_j})^2$ could not be canceled with D because the latter is square-free. So, to cover the full rational case in the affirmation above, $\mathcal{N}_0$ must be generalized to

$$\mathcal{N}_0 = \{(0, x) : D_* s_*^2 x^2 - Dy^2 = -q^2 D_* s_*^2 \text{ with } (x, y) \in \mathbb{Z}^2\}. \tag{3.10}$$

Note that this is actually a generalization because $D_* s_*^2 = 1$ if $q^2 \in \mathbb{Z}$.

The case $D = D_* = 1$ leads, as before, to a finite number of possibilities because $s_* x + y$ and $s_* x - y$ are divisors of $-q^2 s_*^2$. The explicit description of $\mathcal{N}_0$ is

$$\mathcal{N}_0 = \left\{\left(0, \frac{q^2 s_*^2 - d^2}{2 s_* d}\right) : d \,\middle|\, q^2 s_*^2 \text{ and } \frac{q^2 s_*^2}{d} - d \text{ multiple of } 2s_*\right\}.$$

Increasing s_*, being a multiple of $2s_*$, imposes a strong condition and $\mathcal{N}_0$ might reduce to the trivial set $\{(0, \pm\ell_0)\}$ even if $q^2 s_*^2$ has nontrivial divisors. An example of this situation is $\ell_0 = 2$ and $q = \frac{2}{5}\sqrt{299}$.

In the rest of the cases (D and D_* not simultaneously 1), a modification of the arguments for $q^2 \in \mathbb{Z}$ applies. Recall that (ℓ_0, s) is a solution of Eq. (3.10). Reasoning as in (3.8) and multiplying by the conjugate, it is deduced that

$$x s_* \sqrt{D_*} + y s \sqrt{D} = \pm(\ell_0 s_* \sqrt{D_*} + s\sqrt{D})(x_p + y_p s s_* \sqrt{D D_*})^n \quad \text{with } n \in \mathbb{Z}$$

gives infinitely many integer solutions where (x_p, y_p) is the minimal positive solution of the Pell equation

$$x^2 - s^2 s_*^2 D D_* y^2 = 1.$$

Again, a finite number of other families may appear.

Summing up, for $q^2 \in \mathbb{Q}$ the set $\mathcal{N}_0$ is infinite except in the case in which $\ell_0^2 + q^2$ is the square of an irreducible fraction.

IV. TWO-DIMENSIONAL CASE

The two-dimensional case is much more involved because, as it will be shown below, the families of eigenstates presenting exact quantum revivals are parametrized Fuchsian groups i.e., discrete subgroups of $\mathrm{PSL}_2(\mathbb{R})$. Recall that (2.3) implies $q^2 \in \mathbb{Q}$ except in the case $\#\mathcal{E} = 1$ in which $\frac{a_j}{b_j} = 1$. As in the one-dimensional case, the focus will be on the case $q^2 \in \mathbb{Z}$ and some conclusions about the fractional case will be made later on. Some considerations about $\#\mathcal{E} = 1$ were included above and will be expanded in the next section.

Given $q^2 \in \mathbb{Z}$, consider the decomposition

$$k_0^2 + \ell_0^2 + q^2 = Ds^2 \quad \text{with } D \text{ square-free.} \tag{4.1}$$

The condition (2.3) reads

$$k_j^2 + \ell_j^2 + q^2 = D\left(\frac{a_j s}{b_j}\right)^2,$$

where $\frac{a_j s}{b_j}$ must be an integer[2] and it follows a natural analog of the one-dimensional result.

Proposition 4. Given $(k_0, \ell_0) \in \mathbb{Z}^2$ and $q^2 \in \mathbb{Z}^+$ and then defining D and s as in (4.1), the set $\mathcal{N}_0$ of quantum numbers needed for (2.6) with $(k_0, \ell_0) \in \mathcal{N}$ to hold is

$$\mathcal{N}_0 = \{(y, z) : Dx^2 - y^2 - z^2 = q^2 \text{ with } (x, y, z) \in \mathbb{Z}^3\}.$$

Moreover, the revival time is given by

$$T_{\text{rev}} = \frac{2\pi RL}{cs\sqrt{D}} = \frac{2\pi L}{\omega_{\vec{n}_0}},$$

[2] This is because D is square-free and $k_j^2, \ell_j^2, q^2 \in \mathbb{Z}$.

where L is the lcm of the denominators of $\frac{1}{s}\sqrt{\frac{k_j^2+\ell_j^2+q^2}{D}}$ for $(k_j, \ell_j) \in \mathcal{N}$.

This statement suggests that the study of the group that leaves invariant the quadratic form $Q_{2D} = Dx^2 - y^2 - z^2$ and which applies integer solutions to integer solutions is of importance, as it leads to an algorithm generating new solutions. The resulting group is related to the Lorentz group, but it is rather clear that it does not contain all the elements. The task is to determine which elements are to be included. This group is not present in the one-dimensional case.

It may be convenient to explain the difference between the one- and two-dimensional cases from an abstract point of view. As reviewed above, the structure of the solutions in the one-dimensional case can be summarized by saying that there are finitely many families of solutions and each family is given by the action of the discrete group $\{(x_p + y_p\sqrt{D})^n\}_{n\in\mathbb{Z}}$, which is isomorphic to $\mathbb{Z}$, producing sparse solutions because of the exponential growth. In the two-dimensional case, the group is non-Abelian and gives a denser set of solutions.

In order to study the above-mentioned group, let us start with the case $D = 1$, which is no longer trivial. It is known (see, for instance, [23]) that the map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} \frac{a^2+b^2+c^2+d^2}{2} & \frac{-a^2+b^2-c^2+d^2}{2} & -ab-cd \\ \frac{-a^2-b^2+c^2+d^2}{2} & \frac{a^2-b^2-c^2+d^2}{2} & ab-cd \\ -ac-bd & ac-bd & ad+bc \end{pmatrix} \tag{4.2}$$

establishes an isomorphism between $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{\pm I\}$ and the proper Lorentz group $\mathrm{SO}^+(1,2)$ of linear transformations, leaving invariant the quadratic form

$$Q_2(x, y) = x^2 - y^2 - z^2,$$

up to a global change of sign. In general, only the matrices in $\mathrm{SO}^+(1,2) \cap \mathcal{M}_3(\mathbb{Z})$, with $\mathcal{M}_3(\mathbb{Z})$ the set of integral 3×3 matrices, apply integral solutions to integral solutions. In the jargon of the quadratic form theory, these matrices are called integral automorphs.

Clearly, the fundamental problem is to characterize the preimage of $\mathrm{SO}^+(1,2) \cap \mathcal{M}_3(\mathbb{Z})$ by the isomorphism (4.2) in a simple way. The fact that the 3×3 on the right-hand side of the isomorphism (4.2) is an integral automorph does not, at first sight, imply that the matrix on the left-hand side has integer entries a, b, c, and d. The nature of those entries has to be clarified further. First, let A be the matrix in the image of (4.2). If $A \in \mathcal{M}_3(\mathbb{Z})$, as its entries are integers, it is seen that

$$2a^2 = a_{11} + a_{22} - a_{12} - a_{21} \in \mathbb{Z}.$$

Other choices of the signs prove that in general $2a^2, 2b^2, 2c^2, 2d^2 \in \mathbb{Z}$. On the other hand, using the rest of the entries and the determinant equation $ad - bc = 1$, it is seen that $2ab = a_{23} - a_{13} \in \mathbb{Z}$ and, similarly, $2ac, 2ad, 2bc, 2bd, 2cd \in \mathbb{Z}$. Considering $a_{ij} \pm a_{i'j'}$ with $i, j, i', j' \in \{1, 2\}$, it is deduced that $2a^2$, $2b^2$, $2c^2$, and $2d^2$ are all even or all odd. For instance, $a_{11} + a_{22}$ leads to

$$\frac{2a^2 + 2d^2}{2} \in \mathbb{N};$$

thus $2a^2$ and $2d^2$ are simultaneously odd or even. The same line of reasoning applies for the remaining pairs.

The entries a, b, c, and d can be characterized further. Consider first the even case. If one selects a nonzero variable, say, a, then $2a^2$ is an even integer and thus a^2 is an integer. As reviewed in the preceding section, every positive integer admits a decomposition as a square-free and a square integer; therefore $a = s_a\sqrt{R_a}$, with R_a a square-free integer. This last condition means in particular that if $R_a \neq 1$ the number a is not an integer. On the other hand, as $2ab, 2ac, 2ad \in \mathbb{Z}$ it follows, for instance, that $2s_a s_b\sqrt{R_a R_b} \in \mathbb{Z}$. Here $b = s_b\sqrt{R_b}$ and so on. As $2s_a s_b$ is an integer, so is $\sqrt{R_a R_b}$. This last requirement implies that $R_a R_b$ is the square of an integer. On the other hand, the product of two square-free integers $R_a R_b$ gives a square integer only if $R_a = R_b$. After a bit of reasoning, it follows that $R_a = R_b = R_c = R_d = R$. In other words, b, c, and d are integers when divided by $\sqrt{R}$. Finally, the determinant equation $ad - bc = 1$ implies $R = 1$ and it is concluded that $a, b, c, d \in \mathbb{Z}$.

The conditions given above still are not enough. The point is that the quantities on the right-hand side of (4.2) should be such that $a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{Z}$, and this is not necessarily true even when all the conditions above take place. The task is to derive this further requirement; the result is that $2 \mid a + b + c + d$. This can be seen as follows. Note that $\pm n^2 - n$ is even for any $n \in \mathbb{Z}$ and any choice of the sign; then $a_{ij} - \frac{1}{2}(a + b + c + d) \in \mathbb{Z}$ for $i, j \in \{1, 2\}$ and hence $a_{ij} \in \mathbb{Z}$ requires $a + b + c + d$ to be even.

In these terms, the searched discrete group is composed of the elements of the so-called θ group

$$\Gamma_\theta = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}) : 2 \mid a + b + c + d\right\}.$$

This covers the case in which $2a^2$, $2b^2$, $2c^2$, and $2d^2$ are even integers.

For the odd case, one may consider the quantities $2(a\sqrt{2})^2$, $2(b\sqrt{2})^2$, $2(c\sqrt{2})^2$, and $2(d\sqrt{2})^2$, which are clearly even. By repeating the argument it is deduced that $a\sqrt{2}$, $b\sqrt{2}$, $c\sqrt{2}$, and $d\sqrt{2}$ are odd integers. In this case, the elements of the group

$$C_\theta = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\Gamma_\theta$$

are of the searched type since the elements of this coset are of the form

$$\begin{pmatrix} \frac{(a-c)\sqrt{2}}{2} & \frac{(b-d)\sqrt{2}}{2} \\ \frac{(a+c)\sqrt{2}}{2} & \frac{(b+d)\sqrt{2}}{2} \end{pmatrix}.$$

If $a\sqrt{2}$, $b\sqrt{2}$, $c\sqrt{2}$, and $d\sqrt{2}$ are odd integers, it is clear that the entries A_{ij} of these matrices are even integers which satisfy $2 \mid A_{11} + A_{12} + A_{21} + A_{22}$. The isomorphism (4.2) then maps from integer solutions in $\mathcal{N}_0$ one into other. Conversely, a bit of reasoning shows that any matrix with this property has the form given above. Therefore, the isomorphism

$$F : \Gamma \longrightarrow \mathrm{SO}^+(1,2) \cap \mathcal{M}_3(\mathbb{Z})$$

holds, where

$$\Gamma = \Gamma_\theta \cup C_\theta \quad \text{with } C_\theta = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\Gamma_\theta. \tag{4.3}$$

In these terms, the following revival generating algorithm takes place.

Revival algorithm for $D = 1$. Given an element of

$$\mathcal{N}_0 = \{(y, z) : Dx^2 - y^2 - z^2 = q^2 \text{ with } (x, y, z) \in \mathbb{Z}^3\},$$

with $D = 1$, Eq. (4.1) implies that $x = s$, $y = k_0$, and $z = \ell_0$ are a valid solution of $Dx^2 - y^2 - z^2 = q^2$ and it follows that

$$F(\gamma)\begin{pmatrix} s \\ k_0 \\ \ell_0 \end{pmatrix} \quad \text{with } \gamma \in \Gamma \tag{4.4}$$

gives a family of infinitely many solutions in $\mathcal{N}_0$. The group Γ is described in (4.3). Generating elements of Γ is a simple task because the extended Euclidean algorithm easily produces integer solutions of $aX - bY = 1$ (even under a parity condition) to obtain matrices in Γ_θ composed of a, b, $c = Y$, and $d = X$.

To give an example of the above procedure, consider that the choice of $q = \sqrt{6}$, $k_0 = 1$, and $\ell_0 = 3$ gives $D = 1$ and $s = 4$ in (4.1). By applying $F(\gamma)$ with γ each of the 12 matrices in Γ with $|a|, |b|, |c|, |d| \leqslant 2$, the following valid values (k_j, ℓ_j) satisfying (2.3) are obtained: $\pm(1, 3)$, $\pm(1, -3)$, $\pm(3, -1)$, $\pm(3, 7)$, $\pm(13, 9)$, and $\pm(15, -13)$.

The case $D > 1$ is more difficult, though it runs along similar lines. Fortunately, there exists literature about the subject that dates from more than a century ago [24]. The initial idea is that the change of variables $x \mapsto \frac{x}{\sqrt{D}}$ passes the quadratic form

$$Q_{2D} = Dx^2 - y^2 - z^2 \longrightarrow Q_2 = x^2 - y^2 - z^2.$$

This induces a change in the image of (4.2) and now the isomorphism between $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{\pm I\}$ and the proper Lorentz group preserving $Q_{2D} = Dx^2 - y^2 - z^2$, except for a global change of sign, becomes

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} \frac{1}{2}(a^2+b^2+c^2+d^2) & \frac{1}{2\sqrt{D}}(-a^2+b^2-c^2+d^2) & -\frac{1}{\sqrt{D}}(ab+cd) \\ \frac{\sqrt{D}}{2}(-a^2-b^2+c^2+d^2) & \frac{1}{2}(a^2-b^2-c^2+d^2) & ab-cd \\ -(ac+bd)\sqrt{D} & ac-bd & bc+ad \end{pmatrix}.$$

The crucial question is again how to characterize a, b, c, and d in the initial matrix having an integral matrix as the image. A simplification is reached, reparametrizing the initial matrix as

$$M = \begin{pmatrix} -a - c\sqrt{D} & -b - d\sqrt{D} \\ b - d\sqrt{D} & -a + c\sqrt{D} \end{pmatrix}, \tag{4.5}$$

because it rules out the square roots in the image, namely, the previous isomorphism becomes

$$F : M \longmapsto \begin{pmatrix} a^2+b^2+D(c^2+d^2) & -2ac+2bd & -2bc-2ad \\ -2D(ac+bd) & a^2-b^2+D(c^2-d^2) & 2ab+2Dcd \\ 2D(bc-ad) & -2ab+2Dcd & a^2-b^2-D(c^2-d^2) \end{pmatrix}. \tag{4.6}$$

In these terms it is plain to see that for $a, b, c, d \in \mathbb{Z}$ an integral matrix is obtained. In fact, as all the entries are quadratic, one can expect that it is possible to introduce $\sqrt{2}$ denominators in M if the parity conditions cancel the $1/2$ in the diagonal entries of the image, as above. Elaborating this idea, it can be proved [24] (cf. [25]) that the case $4 \mid D - 1$ parallels the case $D = 1$, namely, the group is composed of two parts that reduce to Γ_θ and the companion coset C_θ for $D = 1$. Specifically,

$$\Gamma = \Gamma'_\theta \cup C'_\theta,$$

where

$$\Gamma'_\theta = \{M \in \mathrm{PSL}_2(\mathbb{R}) : a, b, c, d \in \mathbb{Z}\},$$

$$C'_\theta = \left\{\frac{1}{\sqrt{2}}M \in \mathrm{PSL}_2(\mathbb{R}) : a, b, c, d \in \mathbb{Z}\right\}.$$

Here M is like in (4.5).

If instead $4 \nmid D - 1$ then it is necessary to add a new set, which is a kind of semi-integral version of Γ'_θ, namely,

$$\Gamma = \Gamma'_\theta \cup C'_\theta \cup C''_\theta,$$

where Γ'_θ and C'_θ are as before and

$$C''_\theta = \left\{\tfrac{1}{2}M \in \mathrm{PSL}_2(\mathbb{R}) : a, b, c, d \text{ odd integers}\right\}.$$

In fact, the distinction between $4 \mid D - 1$ and $4 \nmid D - 1$ is a little artificial (only to emphasize the analogy of the former case with $D = 1$) because C''_θ is empty for $4 \mid D - 1$. It is also empty for D even because, by computing the determinant, $\frac{1}{2}M \in \mathrm{PSL}_2(\mathbb{R})$ implies $\frac{1}{2}(a^2 + b^2) = 2 + \frac{1}{2}D(c^2 + d^2)$. If a, b, c, and d are odd then the left-hand side is odd and the right-hand side is even.

In any case, Eq. (4.4) with F given by (4.6) produces infinitely many solutions of $Dx^2 - y^2 - z^2 = q^2$ showing that $\mathcal{N}_0$ is infinite. As in the one-dimensional case, the theory ensures [26] that there is a finite number of infinite families of solutions, that is to say, with a finite number of choices of (s, k_0, ℓ_0), all the solutions needed to construct $\mathcal{N}_0$ are reached with (4.4).

Let us close this section with some comments about $q^2 \in \mathbb{Q} - \mathbb{Z}$. Instead of delving deeper into the theory, we only show a shortcut to deduce $\#\mathcal{N}_0 = \infty$ from our previous knowledge about $q^2 \in \mathbb{Z}$.

If $q^2 \in \mathbb{Q} - \mathbb{Z}$, Eq. (4.1) must be replaced by

$$k_0^2 + \ell_0^2 + q^2 = \frac{Ds^2}{D_* s_*^2} \text{ with } D \text{ and } D_* \text{ square-free and coprime.}$$

Clearing the denominator and recalling (2.3), the following is deduced:

$$D\left(\frac{a_j s}{b_j}\right)^2 - D_* s_*^2 k_j^2 - D_* s_*^2 \ell_j^2 = q^2 s_*^2.$$

Hence

$$\mathcal{N}_0 = \{(y, z) : Dx^2 - D_* s_*^2 y^2 - D_* s_*^2 z^2 = q^2 s_*^2 \text{ with } (x, y, z) \in \mathbb{Z}^3\}.$$

In principle, one could study the group, leaving invariant the quadratic form

$$Q_{2DD_*} = Dx^2 - D_* s_*^2 y^2 - D_* s_*^2 z^2,$$

and proceed as before [24], but some technical issues appear because $D_* s_*^2$ is not square-free if $s_* > 1$. A trick to overcome this problem is to rewrite the equation as

$$DD_* x^2 - (D_* s_* y)^2 - (D_* s_* z)^2 = q^2 s_*^2.$$

Then one has to look for integral solutions of $DD_* x^2 - Y^2 - Z^2 = q^2 s_*^2$ and furthermore to restrict Y and Z to be multiples of $D_* s_*$. Note that DD_* is square-free and therefore the action (4.4) with F obtained from (4.6) with the replacement $D \to DD_*$ and $(s, k_0, \ell_0) \to (s, D_* s_* k_0, D_* s_* \ell_0)$ produces infinitely many solutions (x, Y, Z). The main issue is to prove that infinitely many of them verify that their two last coordinates are multiples of $D_* s_*$. This can be done as follows. The 3×3 matrices of determinant 1 modulo $D_* s_*$ form a finite group, which implies that there exists $N \in \mathbb{Z}^+$ such that

$$[F(\gamma)]^N = F(\gamma^N) \equiv \text{Id} \mod D_* s_*.$$

Hence

$$F(\gamma^N)\begin{pmatrix} s \\ D_* s_* k_0 \\ D_* s_* \ell_0 \end{pmatrix} \equiv \begin{pmatrix} s \\ 0 \\ 0 \end{pmatrix} \mod D_* s_*.$$

As $\{\gamma^N : \gamma \in \Gamma\}$ is infinite, there are infinite solutions with $D_* s_* \mid Y, Z$ and $\mathcal{N}_0$ is infinite too.

Although it is possible to obtain an explicit formula for N, for moderate values of $D_* s_*$ it may be more useful in practice to compute the image of successive powers of γ until its image by F fulfills the divisibility conditions.

V. NUMERICAL SIMULATIONS

The purpose of this section is to explain some numerical examples in detail and to plot some quantum carpets.[3] The latter in the 1D case corresponds to density plots of $|\Phi(\phi_2, t)|$, the square root of the probability density for the state in (2.2) with t on the horizontal axis and ϕ_2 on the vertical axis. The only reason to prefer $|\Phi|$ instead of the more natural $|\Phi|^2$ is to avoid the saturation giving less appealing images. These plots involve a choice of the coefficients $c_{\vec{n}}$ indicated in each case. The angle is considered in the range $[-\pi, \pi]$, as mentioned in the Introduction. On the other hand, the natural timescale is T_{rev}, the revival time. Hence [0,1] in the plot must be interpreted as $[0, T_{\text{rev}}]$ in usual units. In the 2D case a density plot of $|\Phi(\phi_1, \phi_2, t)|$ is not possible because it would require three dimensions. It will replaced below by plots for close fixed values of t to exemplify the evolution of the state.

The first example corresponds to a 1D case having a finite $\mathcal{N}_0$, corresponding to the values $\ell_0 = 3$ and $q = \frac{3}{2}\sqrt{21}$. Then $\ell_0^2 + q^2 = \frac{15^2}{2^2}$, which corresponds in (3.10) to $D = D_* = 1$ and $s_* = 3$. Using the explicit formula for $\mathcal{N}_0$ in terms of the divisors of $q^2 s_*^2 = 189$ results in

$$\mathcal{N}_0 = \{(0, \pm 3), (0, \pm 5), (0, \pm 15), (0, \pm 47)\}.$$

Here all the divisors of 189 satisfy the extra condition of $\frac{189}{d} - d$ being a multiple of 4. By computing

$$\frac{a_j}{b_j} = \frac{\sqrt{\ell_j^2 + q^2}}{\sqrt{\ell_0^2 + q^2}},$$

where the last identity follows according to (3.1), the following is obtained:

$$\frac{a_1}{b_1} = 1, \quad \frac{a_2}{b_2} = \frac{17}{15}, \quad \frac{a_3}{b_3} = \frac{11}{5}, \quad \frac{a_4}{b_4} = \frac{19}{3}.$$

Hence $L = 15$ and the revival time is $T_{\text{rev}} = \frac{30\pi}{\omega_{\vec{n}_0}}$. This time remains invariant when $(0, \pm 47)$ is omitted. Figure 1 plots the corresponding quantum carpets in these situations with coefficients 1.

As a second example, the values $\ell_0 = 5$ and $q = \sqrt{2}$ will be considered in order to show how to obtain the result claimed in (3.9). Recall that in this situation $s = D = 3$. It was already shown in previous sections that there is only a family of solutions, so $\mathcal{N}_0 = \{(0, x_n) : n \in \mathbb{Z}\}$ where, according to (3.8),

$$x_n + y_n\sqrt{3} = \pm(5 + 3\sqrt{3})(2 + \sqrt{3})^n.$$

Taking apart the $\pm$, this recurrence produces $x_{n+1} + y_{n+1}\sqrt{3} = (2 + \sqrt{3})(x_n + y_n\sqrt{3})$, a procedure leading to the first-order recurrence

$$(x_{n+1}, y_{n+1}) = (2x_n + 3y_n, x_n + 2y_n). \tag{5.1}$$

By making a further recurrence step and eliminating y_n and y_{n+1}, the second-order recurrence is obtained:

$$x_{n+2} = 4x_{n+1} - x_n \quad \text{with } x_0 = 5, \ x_1 = 19 \quad \text{for } n \in \mathbb{Z}.$$

It is clear that $x_{-3-n} = -x_{-n}$ for $n = 0, 1$, and this identity generalizes for n. Hence, by defining $c_n = x_{n-1}$ for $n \geqslant 0$, the formula (3.9) follows. In the same way, $c'_n = y_{n-1}$ satisfies $c'_{n+2} = 4c'_{n+1} - c'_n$ with starting values $c'_0 = 1$ and $c'_1 = 2$. An easy inductive argument using the recurrence shows that c'_n and c'_{n+1} are coprime (in particular, at least one is not divisible by 3). Therefore, for any $\mathcal{N}$ containing $(0, c_n)$ and $(0, c_{n+1})$

[3] The MATLAB or OCTAVE code generating the figures and the full size color images are available from [27] along with more related material.

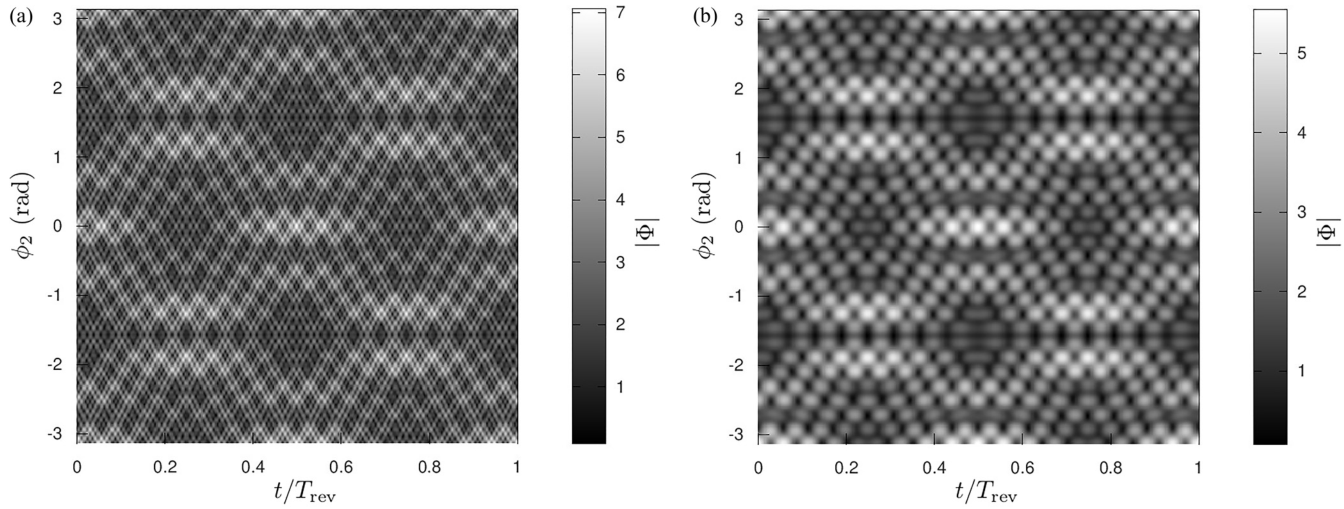

FIG. 1. Density plot of the state (2.2) for $q = \frac{3}{2}\sqrt{21}$ with unit coefficients $c_{\vec{n}} = 1$. (a) For $\mathcal{N}$ the full $\mathcal{N}_0$ is $\{(0, \pm 3), (0, \pm 5), (0, \pm 15), (0, \pm 47)\}$. (b) Same plot as in (a) but omitting $(0, \pm 47)$ from $\mathcal{N}$. In both cases the revival time is $T_{\rm rev} = 30\pi/\omega_{\vec{n}_0}$.

for some n we have $L = 3$ and the revival time is

$$T_{\rm rev} = \frac{2\pi R}{c\sqrt{3}} = \frac{6\pi}{\omega_{\vec{n}_0}}.$$

Due to the exponential growth of c_n, even for fairly small cardinalities of $\mathcal{N}$, the state (2.2) shows large variations for tiny changes of ϕ_2 and t and the density plot is close to being a cloud of random points. In this context it is natural to let $c_{\vec{n}}$ decay with $|\vec{n}|$. Figures 2(a) and 2(b) show examples of ℓ_j chosen with the smallest possible values and $c_{\vec{n}} = |\vec{n}|^{-1/4} = |\ell_j|^{-1/4}$. As a matter of fact, for each t fixed, when $\#\mathcal{N} \to \infty$ a lacunary series is obtained and according to known facts in fractal geometry [28] the graphs of the real and imaginary parts give rise to a fractal, a curve with fractional box dimension.

In this and other examples, an underlying structure is apparent based on interwoven straight lines with two different slopes. The explanation is that in (2.2), due to the exponential growth, $\frac{\sqrt{\vec{n}^2+q^2}}{n_2}$ is initially close to the sign of n_2 and the pairs of lines correspond to the light cones. In fact, if all the ℓ_j are chosen to be positive, commonly the density plot looks like a dull collection of oblique parallel bands.

In order to mask the light cones, a possibility is to take q^2 large in order to avoid

$$\frac{\sqrt{\ell_j^2 + q^2}}{|\ell_j|} \sim 1,$$

at least for the first few values of ℓ_j. With this idea in mind and to illustrate the appearance of several families of solutions, consider the example

$$\vec{n}_0 = (0, 3) \quad \text{with } q = \sqrt{791}.$$

From (3.2), $D = 2$ and $s = 20$. Consequently, the generalized Pell equation is $x^2 - 2y^2 = -791$ and by (3.8) it is seen that

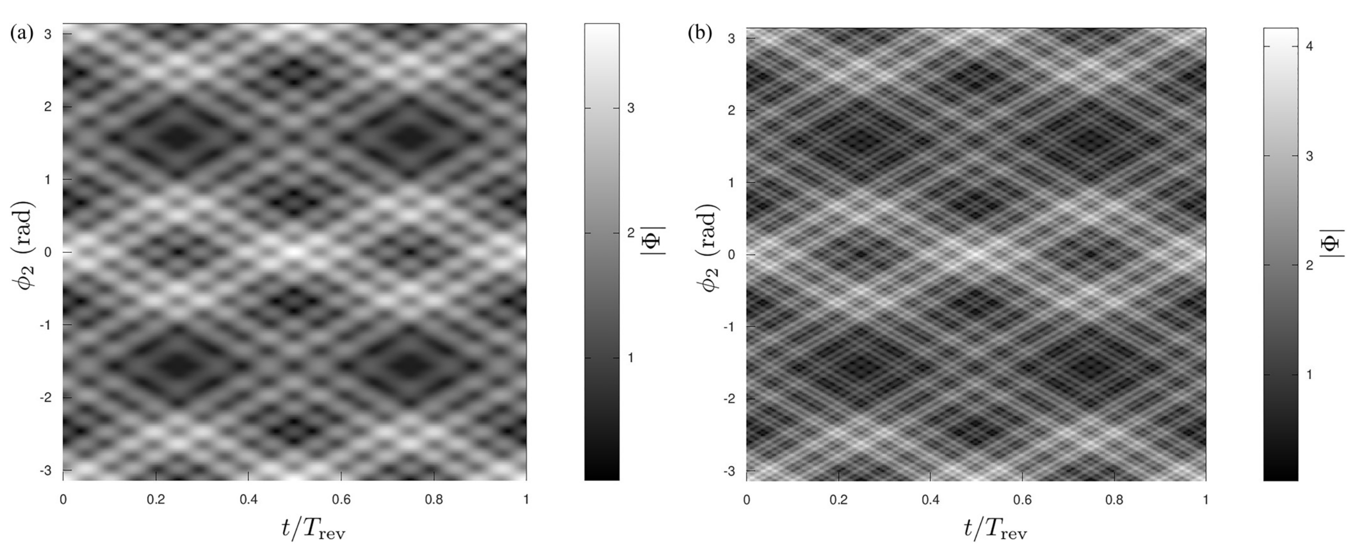

FIG. 2. Density plot of the state (2.2) for $q = \sqrt{2}$ with (a) $\vec{n} \in \{(0, \pm 1), (0, \pm 5), (0, \pm 19)\}$ and $c_{\vec{n}} = |\vec{n}|^{-1/4}$ and (b) $\vec{n} \in \{(0, \pm 1), (0, \pm 5), (0, \pm 19), (0, \pm 71)\}$ and $c_{\vec{n}} = |\vec{n}|^{-1/4}$. The revival time is $T_{\rm rev} = 6\pi/\omega_{\vec{n}_0}$.

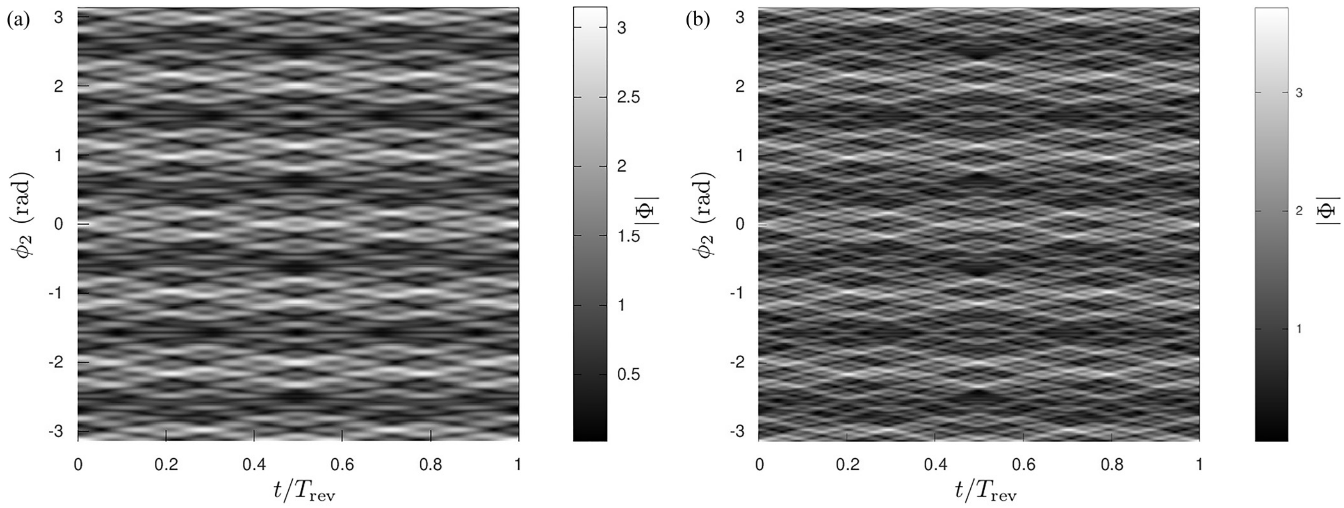

FIG. 3. Density plot of the state (2.2) for $q = \sqrt{791}$ (a) with $\vec{n} \in \{(0, \pm 3), (0, \pm 19), (0, \pm 39)\}$ and $c_{\vec{n}} = |\vec{n}|^{-1/4}$ and (b) $\vec{n} \in \{(0, \pm 3), (0, \pm 19), (0, \pm 39), (0, \pm 71)\}$ and $c_{\vec{n}} = |\vec{n}|^{-1/4}$. In both cases the revival time is $T_{\rm rev} = 20\pi/\omega_{\vec{n}_0}$.

it admits a family of solutions described by

$$x + y\sqrt{2} = \pm(3 + 20\sqrt{2})(3 + 2\sqrt{2})^n \quad \text{with } n \in \mathbb{Z}$$

because (3,2) is a minimal positive solution of $x^2 - 2y^2 = 1$. However, solutions (x, y) in other families may exist. The condition $0 < 2a < 791(1 + 2\sqrt{2 \times 3})$ for a positive solution (a, b) to be the seed of a new family, by direct calculations, only gives (19,24). So the second family is obtained as

$$x + y\sqrt{2} = \pm(19 + 24\sqrt{2})(3 + 2\sqrt{2})^n \quad \text{with } n \in \mathbb{Z}$$

and any solution of $x^2 - 2y^2 = -791$ belongs to one of them.

Taking $n \geqslant 0$, the first positive values for x in the first family are 3, 89, 531, 3097, 18 051, 105 209, 613 203, . . . , which satisfy the recurrence $c_{k+1} = 6c_k - c_{k-1}$. For $n < 0$ the values are 71, 429, 2503, 14 589, 85 031, 495 597, 2 888 551, . . . , which naturally satisfy the same recurrence with different starting values. In the same way, for the second family it is obtained for $n \geqslant 019$, 153, 899, 5241, 30 547, 178 041, 1 037 699, . . . and for $n < 039$, 253, 1479, 8621, 50 247, 292 861, 1 706 919, . . . still satisfying the same recurrence. Then

$$\mathcal{N}_0 = \left\{\left(0, \pm c_k^{(j)}\right) : 1 \leqslant j \leqslant 4,\ k \in \mathbb{Z}_{\geqslant 0}\right\},$$

where $c_{k+1}^{(j)} = 6c_k^{(j)} - c_{k-1}^{(j)}$ with the following starting values:

	$j = 1$	$j = 2$	$j = 3$	$j = 4$
$\left(c_k^{(0)}, c_k^{(1)}\right)$	(3,89)	(71,429)	(19,153)	(39,253)

Figure 3(a) considers the subset

$$\mathcal{N} = \{(0, \pm 3), (0, \pm 19), (0, \pm 39)\}$$

with the coefficients $c_{\vec{n}} = |\vec{n}|^{-1/4}$, as before, and similarly in Fig. 3(b) but with the addition of $(0, \pm 71)$. In both cases the revival time is $T_{\rm rev} = \frac{20\pi}{\omega_{\vec{n}_0}}$ because the $\frac{a_j}{b_j}$ are 1, $\frac{6}{5}$, and $\frac{17}{10}$ and $(0, \pm 71)$ adds $\frac{27}{10}$, which does not change the value of L, which is 10.

Note that, as expected, the clear line structure in the previous example coming from the light cones has been blurred, especially in Fig. 1.

The examples given above correspond to a 1D case. It is a good point to exemplify the two-dimensional situation. Recall that in the 2D setting a way to escape from the condition $q^2 \in \mathbb{Q}$ keeping $\mathcal{N}_0$ nontrivial was to consider numbers with many representations as a sum of two squares. According to the standard theory (see Sec. 16.9 in [13]), if n is a product of k distinct primes of the form $4m + 1$ then the number of representations of n as a sum of two squares is 2^{k+2}. By choosing $k_0^2 + \ell_0^2 = n$ it follows that $\#\mathcal{N}_0 = 2^{k+2}$ for any $q^2 \notin \mathbb{Q}$ while its energy set reduces to the single element $E = \frac{c\hbar\sqrt{n+q^2}}{R}$. Many representations are obtained by applying the eight obvious symmetries

$$(n_1, n_2) \mapsto (\pm n_1, \pm n_2), \quad (n_1, n_2) \mapsto (\pm n_2, \pm n_1).$$

For instance, for $n = 1105 = 5 \times 13 \times 17$, the $2^{3+2} = 32$ representations come from applying the symmetries to the four pairs (4,33), (9,32), (12,31), and (23,24). The value of q^2 does not affect $\mathcal{N}_0$ and letting it grow, only the first coordinate in (2.1) is relevant, giving the limit

$$\sum \Psi^2_{(\pm n_1, \pm n_2)} + \sum \Psi^2_{(\pm n_2, \pm n_1)}$$
$$= 4e^{-iEt/\hbar}[\cos(n_1\phi_1)\cos(n_2\phi_2) + \cos(n_2\phi_1)\cos(n_1\phi_2)],$$

where the sum is over all sign combinations. Although this may seem very simple, the chaotic distribution of n_1 and n_2 for n large leads to some hard questions in mathematical physics, namely, even in this toral situation, some general conjectures [17] about the geometry of the nodal lines (the zero level sets) of the states corresponding to the given energy remain unsolved.

To illustrate the richness of the situation, Fig. 4(a) plots the nodal lines of the first coordinate of the state (2.2) for $\mathcal{N}_0$ as in the previous example with $k_0^2 + \ell_0^2 = 1105$ and $c_{\vec{n}} = 1$ in the range $\phi_1, \phi_2 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. The image in Fig. 4(b) is the corresponding density plot of $|\Phi|^{0.5}$. As $e^{-iEt/\hbar}$ is constant,

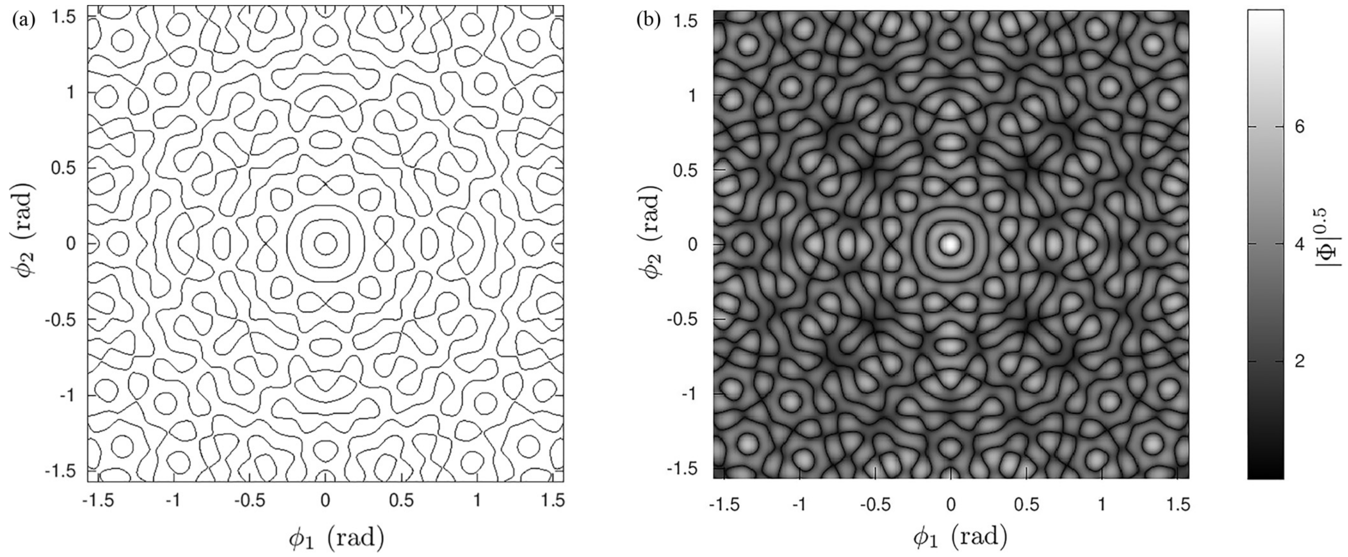

FIG. 4. (a) Nodal lines of $\sum_{\vec{n}}[\cos(n_1\phi_1)\cos(n_2\phi_2)+\cos(n_2\phi_1)\cos(n_1\phi_2)]$ when $\vec{n}$ runs over the couples (4,33), (9,32), (12,31), and (23,24). This expression appears as a limit of (2.2) with $c_{\vec{n}}=1$ when $q\to\infty$ for $\mathcal{N}$ composed of these couples changing the signs and the order of their coordinates in all possible ways. (b) Density plot of $|\Phi|^{0.5}$. It is more informative than that of $|\Phi|$ due to the large central peak.

$|\Phi|$ does not depend on t. The exponent 0.5 is only for visualization purposes. It has been introduced to mitigate the effect of the peak at $\phi_1=\phi_2=0$.

Consider now the example $(k_0,\ell_0)=(1,2)$ with $q^2=3$. The decomposition (4.1) gives $D=s=2$ and $\mathcal{N}_0$ is composed of the (y,z) pairs obtained from the solutions of $2x^2-y^2-z^2=3$. If one only wants to compute a small subset of $\mathcal{N}_0$ then one can proceed by direct inspection instead of using the parametrization by the Fuchsian groups. Let us take

$$\mathcal{N}=\{(\pm1,\pm2),\ (\pm2,\pm1),\ (\pm2,\pm5),\ (\pm5,\pm2),\\ (\pm2,\pm11),\ (\pm11,\pm2),\ (\pm5,\pm10),\ (\pm10,\pm5)\},$$

which comes from $\{(1,2),(2,5),(2,11),(5,10)\}$ applying the symmetries. The resulting sequence $\frac{a_j}{b_j}=\sqrt{\frac{(k_j^2+\ell_j^2+3)}{8}}$ is 1, 2, 4, 4. All the denominators are 1 and then the revival time is $T_{\rm rev}=\frac{2\pi}{\omega_{\vec{n}_0}}$.

As mentioned before, the density plot of the associated state Φ is not feasible because there are three variables (ϕ_1,ϕ_2,t). To illustrate the situation, Fig. 5 is composed of four images showing the density plots in (ϕ_1,ϕ_2) for $t=0$, $\frac{T_{\rm rev}}{10}$, $\frac{T_{\rm rev}}{5}$, and $\frac{T_{\rm rev}}{10}$. Playing with different values, it seems that the sensitivity in t is not uniform. For instance, the density plots for $t=0$ and $t=\frac{T_{\rm rev}}{20}$ are quite similar whereas there are noticeable differences between $t=\frac{T_{\rm rev}}{2}$ and $t=\frac{T_{\rm rev}}{2}-\frac{T_{\rm rev}}{20}$. Probably part of the explanation is the large peak corresponding to $\phi_1=\phi_2=t=0$ which masks relative differences.

VI. DISCUSSION OF THE RESULTS

In the present paper, the possible revivals for a relativistic free fermion ruled by the Dirac equation in a one-dimensional and a two-dimensional torus were characterized. This was achieved by using arithmetic tools such as Pell's equations or Fuchsian groups. The revivals shown here are exact, that is, there is no need to take a relativistic limit in order to detect them. These results generalize those found in [12], indicating revivals related to Pythagorean triples. In the present

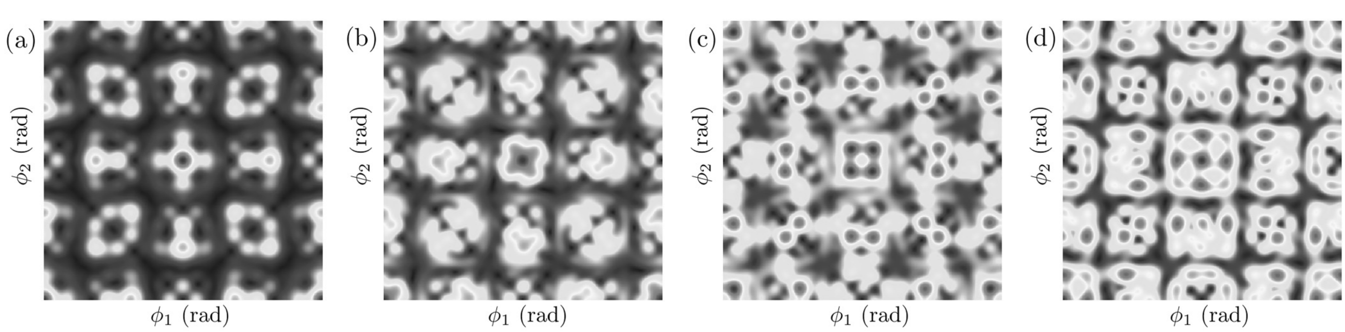

FIG. 5. Density plot of $|\Phi(\phi_1,\phi_2,t)|$ for $\phi_1,\phi_2\in[-\pi,\pi]$ at different times (a) $t=0$, (b) $t=T_{\rm rev}/10$, (c) $t=T_{\rm rev}/5$, and (d) $t=3T_{\rm rev}/10$, where Φ is like in (2.2) with $q=\sqrt{3}$, $c_{\vec{n}}=1$, and $\vec{n}$ running over the couples (1,2), (2,5), (2,11), and (5,10) changing the signs and the order of their coordinates in all possible ways. The revival time is $T_{\rm rev}=2\pi/\omega_{\vec{n}_0}$.

paper, all the possible states exhibiting revivals were characterized. The interesting feature of these revivals is that they are known to hold for massless wave equations or for the standard Schrödinger equation. However, in the present context the dispersion relation is modified to $E = \sqrt{m^2 + p^2}$ and still the revivals are exact. Although the torus topology may seem unrealistic at first sight, it should be remarked that the Dirac equation in several topologies may have applications in solid-state physics. The study of revivals in this context is encouraging and perhaps may lead to interesting experimental results such as the ones reported in [9]. Finally, it may be possible and interesting to generalize the results presented herein to three dimensions, when the spatial coordinates are periodic. However it is likely that the analysis will be more involved. This could be an interesting task to be studied further.

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