Advantages of quantum communication revealed by the reexamination of hyperbit theory limitations

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Pawłowski and Winter's hyperbit theory, proposed in 2012, presented itself as an alternative to quantum theory, suggesting novel ways of redefining entanglement and classical communication paradigms. This research undertakes a meticulous reevaluation of hyperbit theory, uncovering significant operational constraints that question its equivalence with quantum mechanics. Crucially, the supposition that hyperbit theory and quantum theory are equivalent relies on the receiver having unattainable additional knowledge about the sender's laboratory, indicating that the work by Pawłowski and Winter is incorrect. This study accentuates the constraints of hyperbits in information processing and sheds light on the superiority of quantum communication, thereby advancing the investigation at the intersection of classical and quantum communication.

I. INTRODUCTION

Quantum resources can enhance information processing beyond classical limitations. In a communication scenario involving two distant parties, Alice and Bob, whose chosen settings of $\vec{x}$ and $\vec{y}$ produce classical output s and s', then the joint probabilities $p(s, s'|\vec{x}, \vec{y})$ characterize the correlations between them. Two types of correlations can be considered: one where Alice and Bob share prior entanglement but do not communicate and another where they do not share any prior entanglement but communicate directly (in its simplest case, one-way communication from Alice to Bob). The latter case corresponds to the prepare and measure scenario [1–3]. By combining these two cases, a more general scenario can be obtained, where Alice and Bob both share an entangled state and communicate with each other. This results in new types of quantum correlation, enabling new protocols like quantum teleportation [4] and quantum dense coding [5].

From both a conceptual and applied standpoint it is crucial to examine whether entanglement-assisted classical communication [6] and another kind of transmission are equivalent resources in communication tasks. Many quantum communication tasks can be completed using this equivalent and novel scenario [7,8].

In 2012, Pawłowski and Winter in Ref. [9] introduced a resource equivalent to entanglement and classical communication using the same communication protocols. This resource, called *hyperbit*, was introduced as an information quasiparticle. Quasiparticles are mathematical artifacts useful to describe the effective behavior of complex systems and make predictions by simplifying the calculations for the

system as a whole. Examples are phonons in the collective behavior of atoms, vacancies in solid-state physics, or as mediators, such as plasmons, magnons, polarons, and gluons, just to mention a few of them [10–14].

The motivation behind the work of Pawłowski and Winter comes from this background. Information theory, like all other physical theories, has its particles, bits, and qubits, and their quasiparticles are the hyperbits, so called because they are elements of a hyperball. In analogy with quantum theory, where states and measurements are vectors in some Euclidean ball [15], as it is easy to see for qubits in the Bloch sphere, there are difficulties in imagining how it works in higher dimensions.

For higher dimensions, generalized Bloch spheres have been defined [16–20]. Within these frameworks, the formula for probabilities of experimental outcomes mirrors that for qubits. Here, the state is denoted by a vector $\vec{v}$, of length $|\vec{v}|$, in a d-dimensional Euclidean space, and the measurement outcomes $X = \pm 1$ are depicted by the pair of unit vectors $\pm\vec{w}$. The expectation value is given by $\langle X \rangle = \langle \vec{w}, \vec{v} \rangle$. The primary distinction from the Bloch sphere lies in the dimension of the vectors signifying states and measurements. Therefore, a hyperbit corresponds to the vector representing a finite-dimensional system state.

On a broader note, such a hyperball embodies the set of preparations, while its dual signifies the measurement set. The choice of a rule for probability computation facilitates the construction of a probabilistic theory within a preparation-measurement scenario.

In this article, we discuss the shortcomings of the original theorem mentioned in Ref. [9]. A counterexample that disrupts the equivalence between hyperbit theory and quantum theory was previously identified in Ref. [2]. Here, we elucidate why that theorem is incorrect by highlighting physical

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constraints that were not initially considered. Our objective is to demonstrate that the operational limitations restrict the equivalence between hyperbit theory and quantum theory to merely suboptimal communication protocols.

The paper is organized as follows. In Sec. II we revisit the aforementioned theorem within the theory of hyperbit (with its operational and diagrammatic procedures in the Appendix). In Sec. III we show the unimposed physical restrictions of the original theorem. We clarify the limitation of the equivalence in Sec. IV, subsequently highlighting the limits to use hyperbits in Sec. IV A. Finally, we offer a summation of our findings in the conclusions, paving the way for fresh interpretations of quantum superiority in communication tasks.

II. HYPERBITS FOR COMMUNICATIONS

The main theorem of Ref. [9] establishes the equivalence of quantum theory and hyperbit theory[1] in communication tasks:

For tasks where Bob gives binary answers, sending one hyperbit from Alice to Bob is equivalent to sharing any amount of entanglement and sending one classical bit.

The theorem posits the existence of a theory, referred to as hyperbit theory $\mathcal{H}$, which has the capability to predict event probabilities analogous to quantum theory $\mathcal{Q}$, which employs the Born rule.

The hyperbit theory $\mathcal{H}$ finds its application particularly beneficial in scenarios where Alice and Bob, operating in separate laboratories, employ input n–bitstrings $\vec{a}, \vec{b} \in \{0, 1\}^n$. Alice utilizes these to choose one from her set of 2^n dichotomic measurements. Bob responds with $s \in \{-1, +1\}$ based on his communication with Alice and his input bitstring $\vec{b}$. Therefore, the theorem stipulates that the following expectation value can be achieved by both the theories $\mathcal{Q}$ and $\mathcal{H}$

$$\langle B(\vec{a}, \vec{b}, r, \rho) \rangle_{\mathcal{Q}} = \langle B(\vec{a}, \vec{b}, r, \vec{x}_{\vec{a}}) \rangle_{\mathcal{H}}, \tag{1}$$

where $\langle \cdot \rangle_{\mathcal{Q}}$ stands for the Born rule in $\mathcal{Q}$ and $\langle \cdot \rangle_{\mathcal{H}}$ is its analog rule in $\mathcal{H}$.[2] Equation (1) yields

$$\sum_{s \in \{-1, +1\}} s P_{\mathcal{Q}}(s|\vec{a}, \vec{b}, r, \rho) = \sum_{s \in \{-1, +1\}} s P_{\mathcal{H}}(s|\vec{a}, \vec{b}, r\vec{x}_{\vec{a}}), \tag{2}$$

where r in $\mathcal{H}$ is an unbiased dichotomic random variable shared by Alice and Bob, but in $\mathcal{Q}$ it corresponds to the classical bit of Alice's outcome that she sends to Bob[3]; $\vec{x}_{\vec{a}}$ is

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[1]Here we recall hyperbit theory, for simplicity, specifically is only a fragment of a theory, since we describe only the set of preparations and effects, and the rule to compute probabilities without specifies other properties that fully define a theory which are the composition rule and the update state rule after the interaction with the effects.

[2]In general probabilistic theories (GPTs), specifically within the framework of preparation-measurement scenarios, a fragment of a theory is characterized by its set of states, effects, and the rule employed to determine the probability of obtaining outcomes and expectation values (e.g., $\langle \cdot \rangle_{\mathcal{H}}$) as a result of the effects exerted on the states [21,22].

[3]Although r serves different roles in the two theories—as Alice's output in quantum theory and as shared randomness in the hyperbit

her prepared hyperbit which depends on the input bitstring $\vec{a}$ multiplied by r.[4]

Defined by Eq. (2), we can say that $\mathcal{Q}$ is *weak* or *distributionally* equivalent to $\mathcal{H}$, i.e.,

$$P_{\mathcal{Q}}(s|\vec{a}, \vec{b}, r, \rho) \overset{w}{\simeq} P_{\mathcal{H}}(s|\vec{a}, \vec{b}, r\vec{x}_{\vec{a}}). \tag{3}$$

The left-hand side (LHS) calculates the probability according to the Born rule. Meanwhile, the right-hand side (RHS) signifies the probability rule we intend to define for the hyperbit theory. Regardless of this rule's specifics, the theorem restated in Eqs. (1), (2), and (3) can be expressed also via *Penrose's graphical notation* [23,24],

$$\tag{4}$$
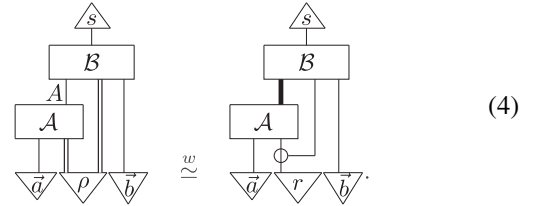
Here, the boxes labeled by $\mathcal{A}$ and $\mathcal{B}$ encapsulate the internal operational procedures that Alice and Bob undertake in their spatially separated laboratories (refer to the Appendix for further details).

The LHS represents the probability of Bob obtaining outcome s when measuring the observable $B_{\vec{b},r}$ with PVM elements $\Pi_{\vec{b},r}^{\pm}$, chosen based on the input bitstring $\vec{b}$ and Alice's outcome r. Alice measures $A_{\vec{a}}$ with PVM elements $P_{\vec{a}}^{\pm}$,

$$P_{\mathcal{Q}}(s|\vec{a}, \vec{b}, r, \rho) = \text{Tr}\left(P_{\vec{a}}^{A} \otimes \Pi_{\vec{b},r}^{s} \rho \right). \tag{5}$$

The double wire in the LHS denotes that Alice and Bob share the quantum state ρ and Alice sends the classical bit r to Bob.

In contrast, on the RHS, Alice and Bob share only a classical copy r and the thicker wire represents the hyperbit sent from Alice to Bob. We proceed now to define the analog of Born rule to compute $P_{\mathcal{H}}$ for the hyperbit theory.

A. Tsirelson isomorphism

In communication tasks where Bob's answer $s \in \{-1, +1\}$ is a dichotomic outcome, its expectation value $E[s]$ fully characterizes the probability distribution $P_{\mathcal{H}}(s = \pm 1|\vec{a}, \vec{b}, r\vec{x}_{\vec{a}}) \equiv \pi_{\vec{b},r}^{\pm}$,

$$\pi_{\vec{b},r}^{\pm} = \frac{1 \pm E[s]}{2}. \tag{6}$$

Notice that, in $\mathcal{Q}$, $\pi_{\vec{b},r}^{\pm} = \text{Tr} \rho \, \Pi_{\vec{b},r}^{\pm}$, but in $\mathcal{H}$ we have not defined yet the analogous of the Born rule. In order to do

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theory—this distinction will facilitate the introduction of a potential equivalence between the hyperbit and quantum theories. Then, the narrative in Eqs. (14) and (18) will be simplified.

[4]Notice that, in $\mathcal{H}$, r is a free-cost random variable with vanishing expectation. Here, a copied random variable with vanishing expectation value does not carry on any information from Alice to Bob.

that, we introduce Tsirelson's isomorphism $\mathcal{J}_\rho$ [25] which, in essence, maps observables into vectors:

$$\mathcal{J}_\rho(A_k, B_m) = (\vec{x}_k, \vec{y}_m), \tag{7}$$

$$(A_k, B_m) = \mathcal{J}_\rho^{-1}(\vec{x}_k, \vec{y}_m), \tag{8}$$

where A_k, B_m are Hermitian operators satisfying $-\mathbf{1} \leqslant A_k, B_m \leqslant \mathbf{1}$, $[A_k, B_m] = 0$, and the vectors satisfy $|\vec{x}_k|, |\vec{y}_m| \leqslant 1$. This relationship is explicitly characterized by

$$\langle \vec{x}_k | \vec{y}_m \rangle = \mathrm{Tr}(A_k \otimes B_m \rho). \tag{9}$$

In this context the aforementioned equivalence is formulated: Alice's *steering measurement* and a single classical bit of communication are equivalent to Bob receiving the hyperbit $\vec{x}_k$. The isomorphism then defines the rule for calculating probabilities and expectation values in $\mathcal{H}$, akin to the Born rule in $\mathcal{Q}$. Hence the expectation value of the probability distribution in Eq. (6) is opportunely calculated as $E[s] = \langle \vec{x}_k | \vec{y}_m \rangle$ represents the expectation value of measurement $\vec{y}_m$ given $\vec{x}_k$.

In other words, for $\rho = T^\dagger T$, the isomorphism can be visualized as

$$\tag{10}$$

In this representation, the cyclic wire signifies the contraction of indices, i.e., the trace operation. The lower triangle depicts the input state, while the upper triangle represents the result of the effect. The dotted line helps to distinguish the preparation (lower side) from the measurement (upper side).

1. Properties of the Tsirelson isomorphism

The homomorphism (perfect correlated input choices) maps the identity to the identity,

$$1 = \mathrm{Tr}(\mathbf{1} \otimes \mathbf{1}\rho) = \langle \vec{x}_1 | \vec{y}_1 \rangle = |\vec{x}_1||\vec{y}_1| \cos \theta$$
$$\implies \vec{x}_1 = \vec{y}_1, \quad |\vec{x}_1| = 1. \tag{11}$$

For dichotomic measurements with equal probability on the outcomes related to $\{P_{\vec{a}}^{+1}, P_{\vec{a}}^{-1}\}$ we have

$$\begin{cases} A_{\vec{a}} = P_{\vec{a}}^{+1} - P_{\vec{a}}^{-1}, \\ \mathbf{1} = P_{\vec{a}}^{+1} + P_{\vec{a}}^{-1}, \end{cases} \xrightarrow{\mathcal{J}_\rho(A_{\vec{a}}, \cdot)} \begin{cases} \vec{x}_{\vec{a}} = \vec{x}_{\vec{a},+1} - \vec{x}_{\vec{a},-1}, \\ \vec{x}_1 = \vec{x}_{\vec{a},+1} + \vec{x}_{\vec{a},-1}. \end{cases} \tag{12}$$

Without any loss of generality, suppose that Alice has unbiased expectation value $\langle A_{\vec{a}} \rangle = 0$; then, going through the isomorphism, the sum of two vectors, e.g., $\vec{x}_{\vec{a},\pm 1}$, with the same module is always orthogonal to the difference, as in Fig. 1, and with Eq. (11) it yields

$$\langle \vec{x}_{\vec{a}} | \vec{y}_1 \rangle = 0 \implies \langle \vec{x}_{\vec{a},+1} | \vec{y}_1 \rangle = \langle \vec{x}_{\vec{a},-1} | \vec{y}_1 \rangle,$$
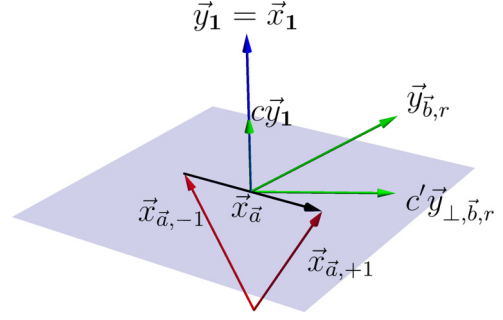$$|\vec{x}_{\vec{a},+1}| \cos \theta_+ = |\vec{x}_{\vec{a},-1}| \cos \theta_-. \tag{13}$$

FIG. 1. Geometrical representation in the affine space for the Tsirelson isomorphism for Alice's choice of $A_{\vec{a}}$ related to $\{\vec{x}_{\vec{a},+1}, \vec{x}_{\vec{a},-1}\}$ with norm of the sum $|\vec{x}_{\vec{a}}| \leqslant 1$ and Bob's choice $\vec{y}_{\vec{b},r}$ with $|\vec{y}_{\vec{b},r}| \leqslant 1$ decomposed along $\vec{y}_1$, and $\vec{y}_{\perp,\vec{b},r}$ orthogonal to $\vec{y}_1$.

Therefore, $\theta_- = \theta_+$. Then

$$\mathrm{Tr}(P_{\vec{a}}^r \otimes \mathbf{1}\rho) = \mathrm{Tr}\left(\frac{\mathbf{1} + rA_{\vec{a}}}{2} \otimes \mathbf{1}\rho\right) = \frac{1}{2} + r\langle \vec{x}_{\vec{a}} | \vec{y}_1 \rangle = \frac{1}{2}, \tag{14}$$

and combined with Eq. (13) gives

$$\langle \vec{x}_{\vec{a},1} | \vec{y}_1 \rangle + \langle \vec{x}_{\vec{a},-1} | \vec{y}_1 \rangle = 2\langle \vec{x}_{\vec{a},\pm 1} | \vec{y}_1 \rangle = 1 \implies |\vec{x}_{\vec{a},\pm 1}| \cos \theta = \frac{1}{2}. \tag{15}$$

In the case of Bob's choices the probability distribution of the outcomes is not uniform as in Eq. (12); therefore,

$$\begin{cases} y_{\vec{b},r} = \alpha y_{\vec{b},r,+1} - \beta y_{\vec{b},r,-1} \\ \vec{y}_1 = y_{\vec{b},r,+1} + y_{\vec{b},r,-1} \end{cases} \longrightarrow \langle y_{\vec{b},r} | \vec{y}_1 \rangle = \alpha - \beta. \tag{16}$$

In Fig. 1, $\vec{y}_{\vec{b},r}$ is decomposed in $c\,\vec{y}_1$ along the axes parallel to $\vec{y}_1$ with $0 \leqslant c \leqslant 1$ and $c'N\vec{y}_{\perp,\vec{b},r}$ orthogonal to $\vec{y}_1$. The crucial assumption for our communication tasks is that all possible choices of Alice $A_{\vec{a}}$ must be with equal probability outcomes[5] as commented along Eq. (12) and the distribution of Bob's probability outcomes depend on the values of α and β in Eq. (16) going from -1 to 1. This assumption allows copying and sharing, as a free resource, a random bit r, equal to Alice's output with vanishing expectation value in the hyperbit theory as represented by a circle on the RHS of Eq. (4).

B. Expectation value in quantum theory

Now we are ready to compute the LHS of Eq. (1), the expectation value of Bob's answer in quantum theory $\mathcal{Q}$, as the expectation value of the observable $B_{\vec{b},r}$ on the steering state $\rho_{\vec{a},r}$ via $A_{\vec{a}} = P_{\vec{a}}^{+1} - P_{\vec{a}}^{-1}$. With Eq. (14) and tracing out

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[5]It should be noted that this condition comes from the assumption of the expectation value of the Alice outcome $r \in \{-1, +1\}$ being zero. This condition does not reduce the generality of the protocol since it is obtainable by some postprocessing (in Alice's case) or preprocessing (in Bob's) and can be incorporated into the measurement operators.

Alice's Hilbert space we found

$$\rho_{\vec{a},r} = \frac{\mathrm{Tr}_A(P_{\vec{a}}^r \otimes \mathbf{1}\rho)}{\mathrm{Tr}(P_{\vec{a}}^r \otimes \mathbf{1}\rho)} = 2\,\mathrm{Tr}_A\big(P_{\vec{a}}^r \otimes \mathbf{1}\rho\big), \qquad (17)$$

such that, summing on Eq. (12) and then using Tsirelson isomorphism,

$$
\begin{aligned}
\langle B(\vec{a}, \vec{b}, r, \rho)\rangle_{\mathcal{Q}} &= \mathrm{Tr}_B B_{\vec{b},r}\rho_{\vec{a},r} = 2\,\mathrm{Tr}P_{\vec{a}}^r \otimes B_{\vec{b},r}\rho \\
&= \mathrm{Tr}[(\mathbf{1} + r\hat{A}_{\vec{a}}) \otimes B_{\vec{b},r}\rho] \\
&= \langle x\mathbf{1} + r\vec{x}_{\vec{a}}|\vec{y}_{\vec{b},r}\rangle \\
&= \langle x\mathbf{1} + r\vec{x}_{\vec{a}}|c(\vec{b},r)y\mathbf{1} + c'(\vec{b},r)\vec{y}_{\perp\vec{b},r}\rangle \\
&= c(\vec{b},r) + c'(\vec{b},r)N(\vec{b},r)\langle r\vec{x}_{\vec{a}}|\vec{\tilde{y}}_{\perp\vec{b},r}\rangle. \quad (18)
\end{aligned}
$$

It is important to note that the vector $x\mathbf{1} + A\vec{x}_{\vec{a}}$ possesses a *length* exceeding 1. Consequently, it does not qualify as a hyperbit, as its length precludes the possibility of measurement with normalized outcome probabilities. This necessitates the application of postprocessing procedures to ensure that hyperbits attain a normalized length suitable for measurement and necessary for the equivalence $\mathcal{H} \equiv \mathcal{Q}$. Moreover, it is particularly evident from the second to the third line of Eq. (18) the application of the Tsirelson isomorphism that represents the passage from $\mathcal{Q}$ to $\mathcal{H}$, hence the change of the meaning of the variable r, as Alice's output to shared randomness [see Eq. (14)], despite the equivalence at the operational level. This approaches a potential equivalence for these theories, despite their different interpretations.

Now, according to Fig. 1, we defined $c(\vec{b}, r) = \langle y\mathbf{1}|\vec{y}_{\vec{b},r}\rangle$ and $c'(\vec{b}, r)N(\vec{b}, r) = \langle \vec{\tilde{y}}_{\perp\vec{b},r}|\vec{y}_{\vec{b},r}\rangle$ with $\vec{\tilde{y}}_{\perp\vec{b},r}$ the normalized $\vec{y}_{\perp\vec{b},r}$, which is the projection of $\vec{y}_{\vec{b},r}$ on the subspace orthogonal to $\vec{y}\mathbf{1}$ and $N(\vec{b}, r)$ the normalization factor. We expressed c, c', and N with the dependence on $\vec{b}$ and the random bit r to stress the dependence on Bob's resources. In addition to that, the constraint from the Tsirelson isomorphism gives

$$|\vec{y}_{\vec{b},r}| \leqslant 1 \implies [c'(\vec{b},r)N(\vec{b},r)]^2 + c^2(\vec{b},r) \leqslant 1. \quad (19)$$

For a lighter notation we denoted

$$x = c'(\vec{b},r)N(\vec{b},r), \quad y = c(\vec{b},r), \quad z = \langle r\vec{x}_{\vec{a}}|\vec{\tilde{y}}_{\perp\vec{b},r}\rangle. \quad (20)$$

Equation (18) reads as a three-variable function t:

$$\langle B(\vec{a}, \vec{b}, r, \rho)\rangle_{\mathcal{Q}} = t(x, y, z) = y + xz \qquad (21)$$

in the domain

$$\mathcal{C} = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \leqslant 1, |t(x, y, z)| \leqslant 1, |z| \leqslant 1\}. \quad (22)$$

Notice that Bob's answer is -1 or $+1$; therefore, this induces the constraint $|t(x, y, z)| \leqslant 1$. In conclusion, the expectation value of Bob's answer in $\mathcal{Q}$ is represented by the function t in its domain $\mathcal{C}$.

C. Expectation value in hyperbit theory

To show the equivalence between the quantum theory $\mathcal{Q}$ and the hyperbit theory $\mathcal{H}$, each value of the function $t(x, y, z)$ for any selection of $(x, y, z) \in \mathcal{C}$, derived from the Born rule in $\mathcal{Q}$, must be also obtained within the hyperbit theory through meaningful operational procedures. First,

the Tsirelson isomorphism defines the operational procedure to obtain the expectation value of Bob's answer as $E[s] = \langle r\vec{x}_{\vec{a}}|\vec{\tilde{y}}_{\perp\vec{b},r}\rangle$, which must correspond to the expectation value in Eq. (6) of the output s distributed in $\{-1, +1\}$ by $\pi_{\vec{b},r}^{\pm 1}$ as

$$E[s] = \pi_{\vec{b},r}^{+1} - \pi_{\vec{b},r}^{-1}, \quad \text{with} \sum_{s=\pm 1} \pi_{\vec{b},r}^{s} = 1. \quad (23)$$

The expectation value of Bob's answer s is obtained, similarly to the Born rule in $\mathcal{Q}$, by measuring with the effect $\vec{\tilde{y}}_{\perp\vec{b},A}$ on the received hyperbit $r\vec{x}_{\vec{a}}$, as shown in the lighter notation in Eq. (20); thus $z = E[s]$. Now, we are ready to reformulate the equivalence.

$\mathcal{H}$ is equivalent to $\mathcal{Q}$ if and only if the most general postprocessing operational procedures on Bob's random answer s admit a function $g = t$ for all $(x, y, z) \in \mathcal{C}$.

Notice that, being s random, there is no *a priori* information about its expectation value z. To characterize the most general postprocessing procedure we observe that the function g can be represented as a point of a tetrahedron obtainable as a convex combination of all the possible deterministic functions which are

$$f_1(s) = +1, \quad f_2(s) = -1, \quad f_3(s) = s, \quad f_4(s) = -s, \quad (24)$$

where f_1 and f_2 are the deterministic discarding operations that replace the outcome into $s \to +1$ and $s \to -1$, respectively; f_3 is *do nothing* operation and f_4 flips Bob's outcome. Therefore, the most general operational procedure that constructs $g = \langle B(\vec{a}, \vec{b}, r, \vec{x}_{\vec{a}})\rangle_{\mathcal{H}}$ is

$$g = \sum_{s=\pm 1}\sum_{i=1}^{4} k_i f_i(s)\pi_{\vec{b},r}^{s}, \quad \sum_{i=1}^{4} k_i = 1,\ k_i \geqslant 0, \quad (25)$$

which yields, from Eqs. (23) and (24),

$$g(\vec{k}, z) = k_1 - k_2 + (k_3 - k_4)z. \quad (26)$$

In conclusion, Eq. (26) represents the expectation value of Bob's answer in $\mathcal{H}$ as a function g of the weights $\vec{z} = (k_1, \ldots, k_4)$ and the expectation value z.

Before showing the limitations concerning the equivalence of Eq. (1), let us revisit the protocol in Ref. [9].

III. PAWŁOWSKI-WINTER PROTOCOL

Pawłowski and Winter in Ref. [9] formulate the following operational procedure in hyperbit theory $\mathcal{H}$ intending to demonstrate its equivalence to the shared entanglement scenario in the quantum theory $\mathcal{Q}$.

PW protocol.

(1) Alice and Bob copy and share a random bit r with vanishing expectation value.

(2) Alice receives the input string $\vec{a}$ and selects the measurement $x_{\vec{a}}$ and sends to Bob the hyperbit multiplied by r, that is, $\vec{x}_{\vec{a},r} = r\vec{x}_{\vec{a}}$.

(3) Bob receives the input bitstring $\vec{b}$ and then measures and computes the expectation value on the hyperbit with $\vec{\tilde{y}}_{\perp\vec{b},r}$.

(4) The expectation value $\langle B(\vec{a}, \vec{b}, r, \vec{x}_{\vec{a}})\rangle_{\mathcal{H}}$ can be obtained by the following postprocessing.

(4.1) With probability $|c(\vec{b}, r)| = |y|$, as in Eq. (20), Bob discards his outcome and outputs $\mathrm{sgn}[c(\vec{b}, r)]$.

(4.2) With probability $1 - |c(\vec{b}, r)|$ he tosses a coin that has probability q for the outcome "head" and probability $1 - q$ for the outcome "tail."

(4.3) If the outcome of the coin is head, then Bob flips his outcome; otherwise, with tail, his outcome remains invariant.

Now the protocol that realizes $g = \langle B(\vec{a}, \vec{b}, A, \vec{x}_{\vec{a}}) \rangle_{\mathcal{H}}$ mathematically reads

$$\langle B(\vec{a}, \vec{b}, r, \vec{x}_{\vec{a}}) \rangle_{\mathcal{H}}$$

$$= \sum_{s=\pm 1} \pi^s_{\vec{b}, r} \left\{ |y| \left[\frac{1 + y/|y|}{2} f_1(s) + \left(1 - \frac{1 + y/|y|}{2} \right) f_2(s) \right] \right.$$

$$\left. + (1 - |y|)[q f_4(s) + (1 - q) f_3(s)] \right\} \tag{27}$$

or, equivalently, introducing the *probabilistic discarding operation* $\mathbb{D}_{c(r,N)}$ with probability $c(A, N) = |y|$ (in lighter notation) that replaces s with $\mathrm{sgn}(y)$ and with probability $1 - |y|$, s is not discarded; then the *probabilistic flipping operation* $\mathbb{F}_q$ happens with probability q,

$$\langle B(\vec{a}, \vec{b}, r, \vec{x}_{\vec{a}}) \rangle_{\mathcal{H}} = (\mathbb{F}_q \circ \mathbb{D}_{|y|} \circ E)(s)$$

$$= \mathbb{F}_q[y + (1 - |y|)z]$$

$$= \mathbb{F}_q[y + (1 - |y|)z]$$

$$= y + (1 - |y|)z(1 - 2q). \tag{28}$$

Explicitly, first it acts as $E(s) = \pi^+_{\vec{b}, r} - \pi^-_{\vec{b}, r} = z$. Then we apply the probabilistic discarding operation, such that with probability $|y|$ gives $\mathrm{sgn}(y)$ and with probability $1 - |y|$ we toss a coin where we apply the probabilistic flip operation; therefore, we obtain

$$|y|\mathrm{sgn}(y) + (1 - |y|)\mathbb{F}_q(z).$$

Now, with probability q, we have $z \mapsto -z$ and with probability $1 - q$, $z \mapsto z$. Therefore, $\mathbb{F}_q(z) = q(-z) + (1 - q)z = (1 - 2q)z$. Finally, $y = |y|\mathrm{sgn}(y)$. Now, by imposing Eq. (1), and with Eqs. (20)–(23), one has

$$q = \frac{1}{2} \left(1 - \frac{x}{1 - |y|} \right). \tag{29}$$

We observe that the PW protocol makes sense if and only if the flipping probability lies $0 \leqslant q \leqslant 1$, i.e.,

$$|c| \pm Nc' \leqslant 1, \quad \langle r\vec{x}_{\vec{a}}, \vec{\bar{y}}_{\vec{b}, r} \rangle \in [-1, 1] \tag{30}$$

that with lighter notation is

$$\mathcal{D} = \{(x, y, z) \in \mathbb{R}^3 \mid -1 \leqslant z \leqslant 1, |y| \pm x \leqslant 1\}. \tag{31}$$

In its current formulation, the hyperbit theory is equivalent to the quantum theory only within the subregion $\mathcal{D} \subset \mathcal{C}$ as depicted in Fig. 2. For points in $\mathcal{C} \setminus \mathcal{D}$ the equivalence of Eq. (1), i.e., $g = t$ iff $q \notin [0, 1]$, which is impossible, as noticed in Ref. [2]. Therefore, to establish an equivalence between quantum theory and hyperbit theory for communication tasks, other operational procedures $g = \langle B(\vec{a}, \vec{b}, r, \vec{x}_{\vec{a}}) \rangle_{\mathcal{H}}$ must be considered. These new procedures from Eq. (25) must satisfy (1); hence $g(\{k_i, q_i, y_i\}, z) = t(x, y, z)$, since $t(x, y, z) = \langle B(\vec{a}, \vec{b}, A, \rho) \rangle_{\mathcal{Q}}$ also for all $(x, y, z) \in \mathcal{C} \setminus \mathcal{D}$.
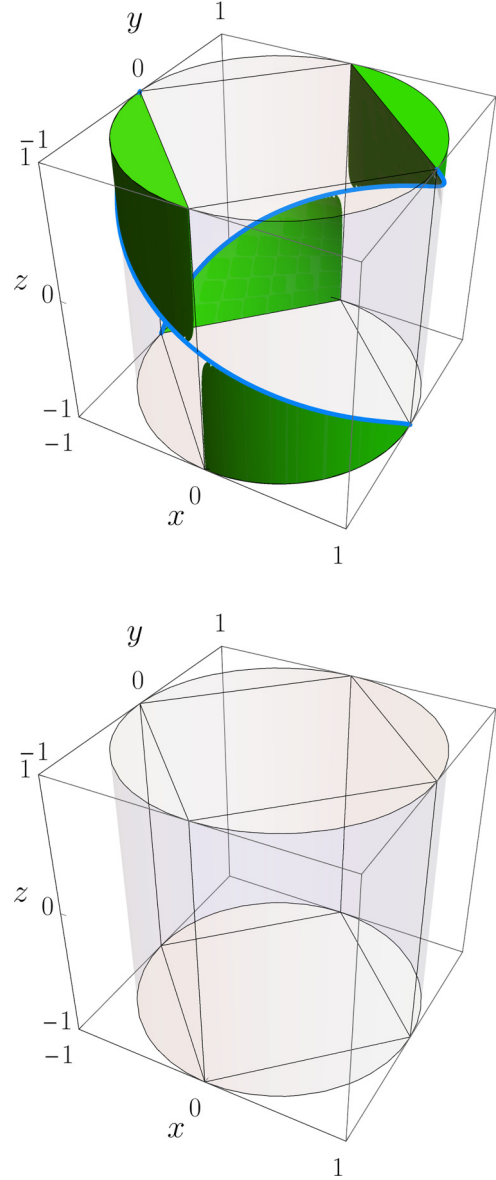

FIG. 2. On the lower panel: a white parallelepiped representing the region $\mathcal{D}$ and a white cylinder. On the upper panel: by removing the four green regions, characterized by $1 \leqslant |t(x, y, z)| \leqslant \sqrt{2}$, from the white cylinder the region $\mathcal{C}$ explicitly written in Eq. (22) is obtained. It contains $\mathcal{D}$. The blue helices on the surface of the cylinder and parametric equation $\{\cos \tau, \sin(\pm\tau), \pm\frac{1 - \sin \tau}{\cos \tau}\}$ for $\tau \in [0, \pi]$ correspond to the points for $t = 1$. These helices illustrate that, for each point z, it is possible to identify a violation of the equivalence as the PW protocol, as well as its widest generalization being valid only in the region $\mathcal{D}$.

The purpose of this paper is to show that even by considering all possible operational procedures in Eq. (25) obtained by fixing k_i's the equivalence $\mathcal{Q} \equiv \mathcal{H}$ is unattainable under the assumption that Bob has no access to Alice's laboratory.

IV. MAIN RESULT

We show the lack of equivalence between $\mathcal{Q}$ and $\mathcal{H}$ and the operational limitation of using hyperbits starting from the
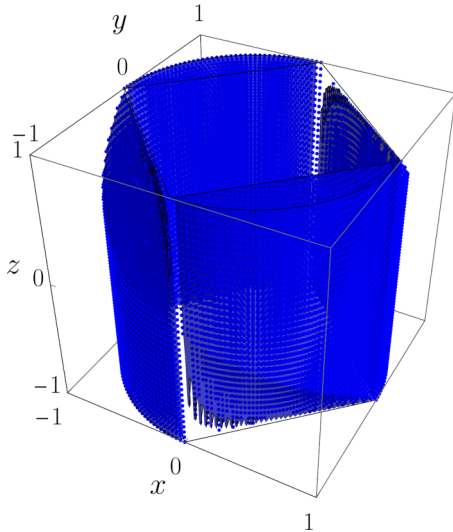
FIG. 3. Bob's *a priori* knowledge of z allows Bob to adjust this strategy such that $g = t$ for all $(x, y, z) \in \mathcal{C}/\mathcal{D}$.

following proposition that corrects the main theorem stated in Sec. II of Ref. [9].

Proposition IV.1. For tasks where Bob gives binary answers, sending one hyperbit from Alice to Bob is equivalent to sharing any amount of entanglement and sending one classical bit if and only if Bob can adjust his strategy by knowing *a priori* the expectation value on the hyperbit.

Proof. Let us consider the reformulation of the equivalence $\mathcal{H} \simeq \mathcal{Q}$ as in the paragraph above Eq. (24). In other words, within the framework of $\mathcal{H}$ the most general Bob's postprocessing procedure admits a function $g = t$ for all $(x, y, z) \in \mathcal{C}$, where g, t, and $\mathcal{C}$ are respectively defined in Eqs. (26), (21), and (22). Therefore, $g = t$, from Eqs. (26) and (21), reads as

$$k_1 - k_2 + (k_3 - k_4)z = y + xz, \quad \forall (x, y, z) \in \mathcal{C}. \quad (32)$$

Now if $\mathcal{H} \simeq \mathcal{Q}$ is trivial to show that Bob knows *a priori* the expectation value of the hyperbit z, indeed from Eq. (32) immediately he can determine

$$k_1 - k_2 - y + (k_3 - k_4 - x)z = 0 \implies z = \frac{y - k_1 + k_2}{k_3 - k_4 - x}. \quad (33)$$

The other direction is more interesting. To prove it, we simply observe from Eq. (20) that x and y are functions of Bob's input $\vec{b}$ and r; hence if Bob also knows the value of z he can freely select k_i's values (with $\sum_i k_i = 1$ and $k_i \geqslant 0$) as part of his strategy such that Eq. (32) holds; thus $\mathcal{H} \simeq \mathcal{Q}$. For $(x, y, z) \in \mathcal{D}$ using the PW protocol once obtain $g = t$. For $(x, y, z) \in \mathcal{C}/\mathcal{D}$ a numerical evidence is in Fig. 3. ∎

At this point, we are confronted with the question of whether the equivalence $\mathcal{Q} \simeq \mathcal{H}$ established in proposition IV is operationally meaningful or is required to drop the hypothesis of Bob's *a priori* knowledge of the expectation value of the hyperbit. The answer hinges on understanding which resources are operationally accessible to Bob. From Eqs. (20), it is evident that the variables x and y depend solely on $\vec{b}$ and the random variable r, both of which are readily accessible resources in Bob's laboratory. However, for Bob to know the

value of z, he requires *a priori* knowledge of the outcome from the effect $\vec{\tilde{y}}_{\perp, \vec{b}, r}$ on the preparation $r\vec{x}_{\vec{a}}$. This knowledge enables him to adjust his strategy by selecting appropriate k_i values to satisfy Eq. (1).

Clearly, $\vec{\tilde{y}}_{\perp, \vec{b}, r}$ is known to Bob, as it is for x and y, but he needs to fix the postprocessing procedure (depicted in the box $\mathcal{B}$) even before receiving the hyperbit $r\vec{x}_{\vec{a}}$. However, once the strategy is fixed, it must satisfy Eq. (32) for all the values of z. In other words, the equivalence between quantum and hyperbit theory must always hold regardless of the specific value of z (the expectation value of the hyperbit Bob receives). This creates an obstacle to establishing equivalence between hyperbit theory and quantum theory without the *a priori* knowledge of z. It means that Eq. (32) must be satisfied as follows (subject to the postprocessing probability constraints):

$$k_1 - k_2 = y, \quad k_3 - k_4 = x, \quad \sum_i k_i = 1, \quad k_i \geqslant 0. \quad (34)$$

Then, we observe that the equivalence $\mathcal{Q} \simeq \mathcal{H}$ is only possible within the $\mathcal{D}$ region.[6] The operational limitation necessitates reexamining and adjusting a kind of negation of Proposition IV. Specifically, if Bob does not know the value of z (meaning the k_i values are chosen in advance to fix his strategy), then there will be instances where $t(x, y, z) \neq g(k_i, z)$, as it happens within the region $\mathcal{C}/\mathcal{D}$ shown in Fig. 2.

The subsequent proposition is the kind of negation we are looking for. It formally states this discussion highlighting the importance of distinguishing between the resources that are operationally accessible *a priori* to Bob.

Proposition IV.2. Suppose that the operationally accessible resources *a priori* are those below the dashed line, i.e., those available before the communication starts. Then the following equivalence is impossible:

$$(36)$$
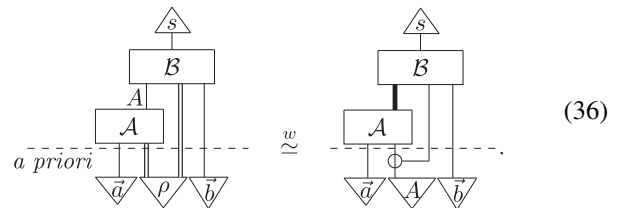
Proof. It is enough to observe that (36) implies that Bob's strategy (the values of k_i's) is fixed for all values of z; therefore, the only way to satisfy the equivalence $\mathcal{H} \simeq \mathcal{Q}$ formally written in Eq. (32) is via Eq. (34). But this is true only for $(x, y, z) \in \mathcal{D}$. Since there are still points $(x, y, z) \in \mathcal{C}/\mathcal{D}$, this proves that, without *a priori* Bob's knowledge of z, in *reductio ad absurdum* $\mathcal{H} \not\simeq \mathcal{Q}$. ∎

We thus infer that, in the absence of *a priori* knowledge of z, there is no z-invariant postprocessing strategy for Bob that

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[6]Remarkably, the parallelepiped $\mathcal{D}$ is the largest region in $\mathcal{C}$ shape invariant in the xy plane along the z axis and PW protocol is obtained from Eq. (25) with

$$q = \frac{1}{2}\left(1 - \frac{k_3 - k_4}{1 - |k_1 - k_2|}\right). \quad (35)$$

yields an output $\langle B(\vec{a}, \vec{b}, r, \vec{x}_{\vec{a}})\rangle_{\mathcal{H}}$ capable of reproducing all possible Bob responses $\langle B(\vec{a}, \vec{b}, A, \rho)\rangle_{\mathcal{Q}}$.

In the context of quantum theory, a physical interpretation suggests that, if Bob had access to the information of z, that is, if he knew Alice's preparation *a priori*, the communication tasks would be rendered trivial as Alice's laboratory would essentially become accessible to Bob. Observe that, in such a case, Bob would have three *a priori* inputs:

$$\tag{37}$$
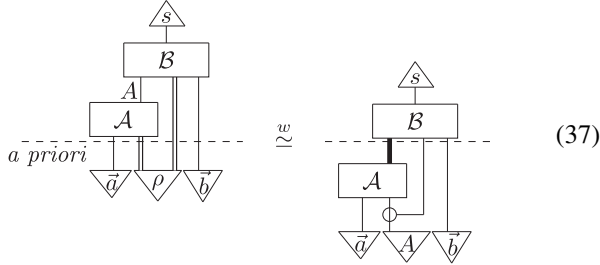
which contrasts the scenario of Eq. (36), where Bob has only two *a priori* inputs. In this situation, the hyperbit conveys the information of $\vec{x}_{\vec{a}}$, enabling Bob to adapt his strategy according to his z-dependent software program. Consequently, we have $\langle B(\vec{a}, \vec{b}, A, \rho)\rangle_{\mathcal{Q}} = \langle B(\vec{a}, \vec{b}, r, \vec{x}_{\vec{a}})\rangle_{\mathcal{H}}$. Note that Eq. (37) diagrammatically reformulates Proposition IV.

The resources available to Bob within quantum theory and hyperbit theory enhance our comprehension of quantum theory. Hyperbit theory, originally developed to simplify calculations, still remains categorized as a foil theory [26]. Foil theories are purposefully designed to highlight the distinctive features within quantum mechanics. This categorization helps in delineating and deepening our appreciation of the specific and unique aspects of quantum theory.

A. Related papers

In Ref. [2], a finite gap was identified that exposed a specific type of inequivalence between the entanglement-assisted (EA) bit value and the hyperbit value, primarily due to the constraints of the PW protocol. This means that only a counterexample was found that disproves the equivalence $\mathcal{Q} \simeq \mathcal{H}$ without any explanation at the level of the interpretation. Our results, however, suggest that even when the most general strategy of Eq. (25) is employed, under the assumption that Bob cannot access Alice's laboratory (i.e., Bob does not know the expectation value z), the equivalence is limited to the $\mathcal{D}$ region, as observed in the PW protocol (see Fig. 2). If the focus is solely on the violation of an inequality based on the upper bound of a linear functional, such as the witness used in [2], one can only observe the discrepancy between quantum theory and hyperbit theory without understanding why the equivalence cannot be achieved or the underlying assumption that leads to such a *no go* result. In our analysis, we pinpoint the no go result in diagram (36) and the equivalence of diagram (37) in cases where the communication is trivial. Both these diagrams are more informative than diagram (4).

V. CONCLUSION

In this paper, we explored the Tsirelson isomorphism with the potential to simplify computations and offer a physical

interpretation via the hyperbit concept. Yet, given the hypotheses that ground this isomorphism, such an interpretation could be perceived as a general probabilistic theory. Consequently, we demonstrated that, for communication tasks covered by the theorem in Ref. [9], postprocessing operations on hyperbits and their associated dual effects exhibit equivalence in a confined region given an input-independent protocol. This finding served to clarify the previously discussed equivalence $\mathcal{H} \simeq \mathcal{Q}$ [2,9]. Further, we examined the conditions leading to disrupting this equivalence based on certain characteristics of postprocessing and accessibility within the hyperbit theory.

Upon the hypotheses of Tsirelson's theorem, such as the commutativity among settings, further research might be directed toward formulating a CHSH-like correlator in hyperbit terms as it implies a form of "quantumness" certification that may only involve a single correlation term similar to those defining the CHSH functional [27,28]. In addition to these insights, our geometrical construction lends support to the idea that quantum mechanics may carry more information about Alice's laboratory, even when Bob has no direct access, as in the case of hyperbits. Our proof shows that if Bob already knows the value of z, a protocol satisfying $g = t$ in Eq. (32) always exists. When identifying (x, y) as expectation values of some operational procedure, the linear trade-off of Eq. (31) is known to be noncontextual [29,30] and the quadratic one of Eq. (22) is quantum. Such quantum superiority might be linked with contextuality akin to a random access code [31], opening new avenues for understanding and exploring the complex world of quantum communication.

APPENDIX: OPERATIONAL FEATURES

In this Appendix, we explain the details of Eq. (4) that summarize the whole paper. Let us start from the quantum protocol to compute the probability that Bob's outcome is s:

$$\tag{A1}$$
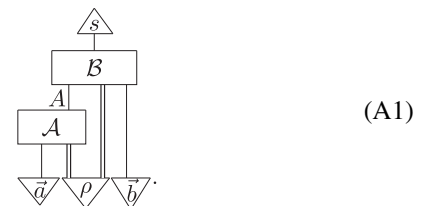
Alice's lab is labeled as $\mathcal{A}$ and depending on the input bitstring $\vec{a}$ she selects one of the possible dicothomic measurements

$A_{\vec{a}'}$,

$$\boxed{\mathcal{A}} = \sum_{\vec{a}'} \triangle{\vec{a}'} \boxed{A_{\vec{a}'}}. \tag{A2}$$

In the sum only one term is connected to the input so that the cardinality of the input and output *legs* on the LHS is the same as the one on the RHS by the action of the selector $\vec{a}'$, indeed

$$\delta_{\vec{a}'\vec{a}} = \frac{\triangle{\vec{a}'}}{\triangledown{\vec{a}}}. \tag{A3}$$

The spectral representation of Alice's dicothomic measurement $A_{\vec{a}'}$ with PVM elements $\{P_{\vec{a}'}^+, P_{\vec{a}'}^-\}$ is

$$\boxed{A_{\vec{a}'}} = \underset{P_{\vec{a}'}^+}{\bowtie^+} + \underset{P_{\vec{a}'}^-}{\bowtie^-}, \tag{A4}$$

where only one channel is activated so that the number of legs on both sides remains the same. After the measurement, Alice outputs the outcome $A \in \{-1, +1\}$ and sends it to Bob's lab labeled as $\mathcal{B}$. Here, Bob will receive two *a priori* inputs, i.e., before the communication start: the bitstring $\vec{b}$ and the shared entangled state ρ represented with a double wire. The third Bob's input is received during the communication, namely, Alice's outcome that selects one of his dichotomic observables $B_{A',\vec{b}'}$. Specifically,

$$\boxed{\mathcal{B}} = \sum_{\vec{b}'} \sum_{A'=\pm 1} \triangle{A'} \boxed{B_{A',\vec{b}'}} \triangle{\vec{b}'} \tag{A5}$$

where the possible dichotomic measurements output $s \in \{-1, +1\}$,

$$\boxed{B_{\vec{b}',A'}} = \underset{P_{A',\vec{b}'}^+}{\bowtie^+} + \underset{P_{A',\vec{b}'}^-}{\bowtie^-}. \tag{A6}$$

Notice that the spectral decomposition of the measurement $B_{\vec{b}',A'}$ leads to a different decomposition of the vector $\vec{y}_{\vec{b}',A'}$ than the decomposition in Fig. 1, which is more similar to the one of the vector $\vec{x}_{\vec{a}}$.

As regards the RHS of Eq. (A1) about the fragment of hyperbit theory, we have

$$\frac{\triangle{s}}{\underset{\triangledown{\vec{a}}\,\triangledown{r}\,\triangledown{\vec{b}'}}{\boxed{\mathcal{A}}}} \tag{A7}$$

It is clear from this scheme that Alice and Bob shared only a classical random variable r with a vanishing expectation value
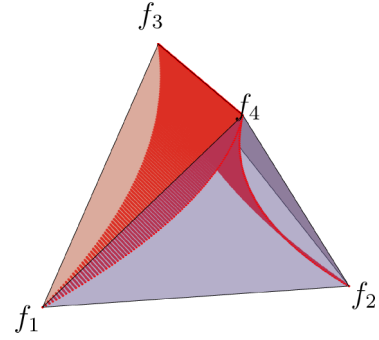

FIG. 4. Each point in the tetrahedron identifies a PW protocol at given $y \in [-1, 1]$ and $q \in [0, 1]$ that respectively characterize the probabilistic discarding and the probabilistic flip operations obtained as convex combinations of the function f_i of Eq. (24).

and the communication is powered by the hyperbit that Alice sends to Bob represented by a thicker output line. With a slight abuse of notation, we refer to the Alice operational procedure again with the label $\mathcal{A}$, where now she has two classical inputs and the hyperbit output:

$$\boxed{\mathcal{A}} = \sum_{\vec{a}'} \sum_{r'=\pm 1} \triangle{\vec{a}'} \underset{r'}{\bigvee^{r'\vec{x}_{\vec{a}}}} \tag{A8}$$

where we select as a shared copy the unbiased Alice's output such that $r' = A$. Once Bob receives the hyperbit multiplied by the random copied variable $A\vec{x}_{\vec{a}}$ his strategy is fixed: the expectation value associated with the effect $\vec{y}_{\perp,\vec{b},A}$ inputs into a preestablished postprocess. All the protocols can be visualized as a point in the tetrahedron with vertices f_i of Eq. (24) as in Fig 4. As an example, we specialize in the PW protocol. Then Bob's strategy is the one presented in the PW protocol of Sec. III,

$$\boxed{\mathcal{B}} = \underset{c}{\triangledown} + (1-|c|) \sum_{A',\vec{b}'} \underset{\tilde{y}_{\perp\vec{b}',A'} \quad A' \quad \vec{b}'}{\overset{\boxed{PS}}{y_{\perp,\vec{b}',A'}}} \tag{A9}$$

where PS is the postprocessing of the outcomes once the probabilistic discarding operation is applied. In the PW protocol PS $= f_q$, which is the probabilistic flip

$$\boxed{f_q} = q\left(\bowtie + \bowtie^+ \right) + (1-q)\left(\bowtie^+ + \bowtie \right) \tag{A10}$$

where the last term is the identity operation that can be equivalently represented by a solid vertical line.

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