Strong quantum nonlocality with genuine entanglement in an N-qutrit system

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In this paper, we construct orthogonal genuinely multipartite entangled bases in $(\mathbb{C}^3)^{\otimes N}$ for $N \geqslant 3$, where every state is a one-uniform state. By modifying this construction, we successfully obtain strongly nonlocal orthogonal genuinely entangled sets and strongly nonlocal orthogonal genuinely entangled bases, which provide an answer to the problem raised by Halder *et al.* [Phys. Rev. Lett. **122**, 040403 (2019)]. The strongly nonlocal orthogonal genuinely entangled set we constructed in $(\mathbb{C}^3)^{\otimes N}$ contains much fewer quantum states than all known ones. Meanwhile, our results also answer the question given by Wang *et al.* [Phys. Rev. A **104**, 012424 (2021)].

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I. INTRODUCTION

Quantum nonlocality is a fundamental property of quantum mechanics, manifesting the nonclassical aspects of quantum phenomena. The most well-known manifestation of quantum nonlocality is Bell nonlocality [1], which arises from entangled states [2,3]. Entangled states show nonlocality by violating Bell-type inequalities [4–12]. It is well known that entanglement is an important resource in areas such as quantum teleportation [13–15], quantum key distribution [16–18], and quantum networks [19]. On the other hand, the local indistinguishability of quantum states exhibits nonlocal properties in a way fundamentally different from Bell nonlocality. Local indistinguishability means that a known set of orthogonal quantum states distributed among spatially separated parties is not possible to be exactly distinguished by local operations and classical communication (LOCC) [20]. In 1999, Bennett *et al.* [21] presented a locally indistinguishable orthogonal product basis (OPB) in the Hilbert space $\mathbb{C}^3 \otimes \mathbb{C}^3$, which shows the phenomenon of nonlocality without entanglement. Then, locally indistinguishable orthogonal product sets (OPSs) and orthogonal entangled sets (OESs) aroused much research interest [22–33] and found useful applications in data hiding [34,35] and quantum secret sharing [36,37].

In 2019, Halder *et al.* [38] introduced a stronger form of nonlocality, i.e., strong nonlocality, by the notion of local irreducibility of multipartite quantum states under every bipartition. A set of multiparty orthogonal quantum states is defined to be locally irreducible if it is not possible to eliminate one or more states from the set by orthogonal-preserving local measurements (OPLMs) [38]. Such a set, by definition, is locally indistinguishable, but the converse does not hold in general. As an OPS in $\mathbb{C}^2 \otimes \mathbb{C}^d$, $d \geqslant 2$ is locally distinguishable and it is also locally reducible; the locally irreducible phenomenon of OPS does not exist in the systems where one of the subsystems has a two-dimensional complex space. So Halder *et al.* claimed that the multiparty OPS with strong nonlocality can only exist, if at all, on $\mathcal{H} = \otimes_{i=1}^{N} \mathcal{H}_i, N \geqslant 3$, where $\dim H_i \geqslant 3$ for every i. Then, they provided two strongly nonlocal OPBs [38] in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ and $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$, respectively. Zhang *et al.* [39] presented a general definition of strong nonlocality for multipartite quantum systems, and distinguished the nonlocality of two sets of orthogonal quantum states. Later, strong quantum nonlocality without entanglement was widely studied and many results have been obtained [40–46].

For genuinely entangled orthogonal bases (in which each element is entangled in every bipartition), intuition suggests that they are easier to exhibit strong nonlocality. However, Halder *et al.* [38] found that the three-qubit Greenberger-Horne-Zeilinger (GHZ) basis (unnormalized) $\{|000\rangle \pm |111\rangle, |011\rangle \pm |100\rangle, |001\rangle \pm |110\rangle, |010\rangle \pm |101\rangle\}$, which is genuinely entangled and locally irreducible (when all parts are separated), is locally reducible in all bipartitions. Then they asked whether one can find entangled bases that are locally irreducible in all bipartitions, that is, whether one can find entangled bases that possess strong nonlocality. References [47–50] answered this open question. In Ref. [47], the authors showed strongly nonlocal OESs and strongly nonlocal orthogonal entangled bases (OEBs) in $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$ ($d \geqslant 3$). However, these states are not genuinely entangled. Wang *et al.* [48] presented strongly nonlocal orthogonal genuinely entangled sets (OGESs) in $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$ by using graph connectivity. For a multipartite quantum system, the authors of Ref. [49] provided strongly nonlocal OESs in $(\mathbb{C}^d)^{\otimes N}$ for all $N \geqslant 3$ and $d \geqslant 2$, and strongly nonlocal

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OGESs when $N = 3$ and 4, but they did not present strongly nonlocal OGESs for $N \geqslant 5$. Li *et al.* [50] constructed a strongly nonlocal OGES of size $\prod_{n=1}^{N} d_n - \prod_{n=1}^{N}(d_n - 1) + 1$ in multipartite quantum systems $\mathcal{H} = \otimes_{n=1}^{N}\mathcal{H}_n$ with $N \geqslant 3$, where d_n is the dimension of the nth local subsystem $\mathcal{H}_n$.

In this paper, we construct strongly nonlocal OGESs of size $2 \times 3^{N-1}$ and a strongly nonlocal orthogonal genuinely entangled basis (OGEB) in $(\mathbb{C}^3)^{\otimes N}$ ($N \geqslant 3$). As a consequence, our constructions once again answer the open question of whether one can find entangled bases that are locally irreducible in all bipartitions, given by Halder *et al.* [38], that is, OEBs that are strongly nonlocal do exist. Not only that, we also show that both strongly nonlocal OGESs and strongly nonlocal OGEBs do indeed exist. Our constructions also partially answer an open question given in [49], "How do we construct a strongly nonlocal orthogonal genuinely entangled set in $(\mathbb{C}^d)^{\otimes N}$ for any $d \geqslant 2$ and $N \geqslant 5$?"

In an N-qutrit system, the OGES in our construction has $3^{N-1} - 2^N + 1$ states fewer than that constructed in Ref. [50]. In $\otimes_{i=1}^{N}\mathbb{C}^{d_i}$ ($N, d_i \geqslant 3$), the authors of Refs. [45,46] constructed strongly nonlocal OPSs with odd N and even N, respectively, and both of them have size $3^N - 1$ when $d_i = 3$. Reference [45] also provided unextendible product bases (UPBs) in N-partite systems for all odd $N \geqslant 3$. An UPB is a set of orthogonal product states which span a subspace of a given Hilbert space, while the complementary subspace contains no product state [51]. In the same system, compared with these OPSs, the size $2 \times 3^{N-1}$ of OGESs in our construction is much smaller. Thus, our results provide an answer to the problem in Ref. [48], "Can we construct some smaller set that has the property of the strongest nonlocality via the OGES than the OPS." Note that in a $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ system, Shi *et al.* [49] showed a strongly nonlocal OGES with 18 elements, which is consistent with our size. Che *et al.* [43] also constructed a strongly nonlocal UPB of size 12.

The rest of this paper is organized as follows. In Sec. II, we introduce some necessary notations and definitions used in the sequel. In Sec. III, we construct an OGEB in $(\mathbb{C}^3)^{\otimes N}$ ($N \geqslant 3$). In Sec. IV, we exhibit strongly nonlocal OGESs and strongly nonlocal OGEBs in space $(\mathbb{C}^3)^{\otimes N}$ ($N \geqslant 3$). We end with conclusions in Sec. V.

II. PRELIMINARIES

Throughout this paper, we consider only a pure state and do not normalize the states for simplicity. For a d-dimensional Hilbert space $\mathbb{C}^d$ ($d \geqslant 2$), we assume that $\mathcal{B} := \{|0\rangle, |1\rangle, \dots, |d-1\rangle\}$ is the computational basis of $\mathbb{C}^d$, and $\mathbb{Z}_d := \{0, 1, \dots, d-1\}$, $\mathbb{Z}_d^N := (\mathbb{Z}_d)^{\times N}$. Given a $d \times d$ matrix $E := \sum_{i=0}^{d-1}\sum_{j=0}^{d-1} a_{i,j}|i\rangle\langle j|$, for $\mathcal{S}, \mathcal{T} \subseteq \{|0\rangle, |1\rangle, \dots, |d-1\rangle\}$, we define

$$_{\mathcal{S}}E_{\mathcal{T}} := \sum_{|s\rangle\in\mathcal{S}}\sum_{|t\rangle\in\mathcal{T}} a_{s,t}|s\rangle\langle t|,$$

which is a submatrix of E with row coordinates $\mathcal{S}$ and column coordinates $\mathcal{T}$. In particular, $_{\mathcal{S}}E_{\mathcal{S}}$ is represented by $E_{\mathcal{S}}$. A positive operator-valued measure (POVM) is a set of semidefinite operators $\{E_m = M_m^{\dagger}M_m\}$ such that $\sum_m E_m = \mathbb{I}$, where $\mathbb{I}$ is the identity operation. A measurement is trivial if all the POVM elements are proportional to the identity operator; otherwise, the measurement is nontrivial [52]. Clearly, the trivial measurement means that no information about the state can be yielded.

In a multipartite system $\mathcal{H}_{A_1} \otimes \cdots \otimes \mathcal{H}_{A_N}$ with a local dimension d each, we say that $|\Psi\rangle_{A_1A_2\cdots A_N}$ is a one-uniform state [53] if its reduced density matrices for each subsystem are maximally mixed, i.e., $\rho_{A_i} = \mathrm{Tr}_{\bar{A}_i}(|\psi\rangle\langle\psi|) = \mathrm{I}/d$. A well-known example is the N-qudit GHZ state $|\mathrm{GHZ}_N^d\rangle = \sum_{i=0}^{d-1}|i\rangle^{\otimes N}$.

Now we restate the definition of a locally irreducible set and strong nonlocality [38].

Definition 1 (Locally irreducible set). A set $\{|\psi\rangle\}$ of orthogonal quantum states in $\mathcal{H} = \otimes_{i=1}^{N}\mathbb{C}^{d_i}$ with $N \geqslant 2$ and $d_i \geqslant 2$, $i = 1, 2, \dots, N$, is locally irreducible if it is not possible to locally eliminate one or more states from the set while preserving orthogonality of the postmeasurement states.

Definition 2 (Strong nonlocality). A set $\{|\psi\rangle\}$ of orthogonal quantum states in multipartite systems $\mathcal{H} = \otimes_{i=1}^{N}\mathbb{C}^{d_i}$ with $N \geqslant 3$ and $d_i \geqslant 2$, $i = 1, 2, \dots, N$, has the property of strong nonlocality if it is locally irreducible in every bipartition.

There is a sufficient condition for local irreducibility: if any parties can only perform a trivial OPLM, then the set of states must be locally irreducible. Therefore, one can show that a set $\{|\psi\rangle\}$ of orthogonal states is strongly nonlocal by proving that each subsystem of any bipartition can only perform a trivial OPLM.

Next, we state three lemmas of Shi *et al.* [42] as follows.

Lemma 1 (Block zeros lemma). Let an $n \times n$ matrix $E = (a_{i,j})_{i,j\in\mathbb{Z}_n}$ be the matrix representation of the operator $E = M^{\dagger}M$ under the bases $\mathcal{B} := \{|0\rangle, |1\rangle, \dots, |n-1\rangle\}$. Given two nonempty disjoint subsets $\mathcal{S}$ and $\mathcal{T}$ of $\mathcal{B}$, assume that $\{|\psi_i\rangle\}_{i=0}^{s-1}$, $\{|\phi_j\rangle\}_{j=0}^{t-1}$ are two orthogonal sets spanned by $\mathcal{S}$ and $\mathcal{T}$, respectively, where $s = |\mathcal{S}|$, $t = |\mathcal{T}|$. For $_{\mathcal{S}}E_{\mathcal{T}} := \sum_{|i\rangle\in\mathcal{S}}\sum_{|j\rangle\in\mathcal{T}} a_{i,j}|i\rangle\langle j|$, if $\langle\psi_i|E|\phi_j\rangle = 0$ for any $i \in \mathbb{Z}_s$, $j \in \mathbb{Z}_t$, then $_{\mathcal{S}}E_{\mathcal{T}} = \mathbf{0}$ and $_{\mathcal{T}}E_{\mathcal{S}} = \mathbf{0}$.

Lemma 2 (Block trivial lemma). Let an $n \times n$ matrix $E = (a_{i,j})_{i,j\in\mathbb{Z}_n}$ be the matrix representation of the operator $E = M^{\dagger}M$ under the basis $\mathcal{B} := \{|0\rangle, |1\rangle, \dots, |n-1\rangle\}$. Given a nonempty subset $\mathcal{S}$ of $\mathcal{B}$, assume that $\{|\psi_i\rangle\}_{i=0}^{s-1}$ is an orthogonal set spanned by $\mathcal{S}$. Suppose that $\langle\psi_i|E|\psi_j\rangle = 0$ for any $i \neq j \in \mathbb{Z}_s$. If there exists a state $|u_0\rangle$, such that $_{\{|u_0\rangle\}}E_{\mathcal{S}\setminus\{|u_0\rangle\}} = 0$ and $\langle u_0|\psi_j\rangle \neq 0$ for any $j \in \mathbb{Z}_s$, then $E_{\mathcal{S}} \propto \mathbb{I}_{\mathcal{S}}$.

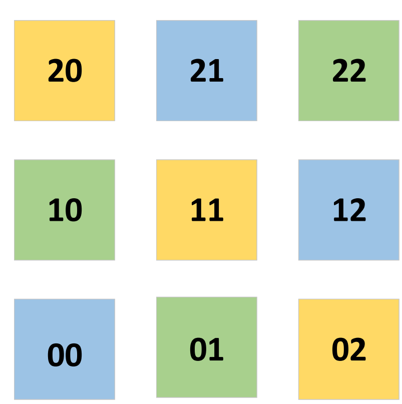

FIG. 1. The set $\mathbb{Z}_3 \times \mathbb{Z}_3$ is depicted by a 3×3 grid, where the blue, green, and yellow regions correspond to the set $\mathcal{G}_0^2$, $\mathcal{G}_1^2$, and $\mathcal{G}_2^2$ in Eq. (3), respectively.

Lemma 3. Let $\{|\psi_j\rangle\}$ be a set of orthogonal states in multipartite system $\otimes_{i=1}^{N}\mathbb{C}^{d_i}$. For each $i = 1, 2, \ldots, N$, define $\bar{A}_i = \{A_1A_2\cdots A_N\}\backslash\{A_i\}$ is the joint party of all but the ith party. If any OPLM on $\bar{A}_i$ is trivial, then the set $\{|\psi_j\rangle\}$ is of the strongest nonlocality.

III. THE CONSTRUCTION OF OGEBs in $(\mathbb{C}^3)^{\otimes N}$

In this section, we construct three sets $\mathcal{G}_i^N$ $(i \in \mathbb{Z}_3)$ of strings on $\mathbb{Z}_3^N$ and give two propositions to characterize them. Then we exhibit OGEBs in $(\mathbb{C}^3)^{\otimes N}$ for $N \geqslant 3$ in Theorem 1.

Given a set $\mathcal{G}$ of n-tuples, we construct $(n+1)$-tuples by the following method:

$$\begin{aligned}\{j_0\} \times \mathcal{G} &= \{j_0\} \times \{(j_1^1, \ldots, j_{n-1}^1, j_n^1), (j_1^2, \ldots, j_{n-1}^2, j_n^2), \ldots, (j_1^t, \ldots, j_{n-1}^t, j_n^t)\}\\ &= \{(j_0, j_1^1, \ldots, j_{n-1}^1, j_n^1), (j_0, j_1^2, \ldots, j_{n-1}^2, j_n^2), \ldots, (j_0, j_1^t, \ldots, j_{n-1}^t, j_n^t)\},\end{aligned} \tag{1}$$

where $t = |\mathcal{G}|$.

A. THREE SETS on $\mathbb{Z}_3^N$

First, we consider $\mathbb{Z}_3$, and denote

$$\mathcal{G}_0^1 = \{0\},\ \ \mathcal{G}_1^1 = \{1\},\ \ \mathcal{G}_2^1 = \{2\}. \tag{2}$$

For $N = 2$, we give three subsets of $\mathbb{Z}_3 \times \mathbb{Z}_3$,

$$\begin{aligned}\mathcal{G}_0^2 &= (\{0\} \times \mathcal{G}_0^1) \bigcup (\{2\} \times \mathcal{G}_1^1) \bigcup (\{1\} \times \mathcal{G}_2^1) = \{(0,0), (2,1), (1,2)\},\\ \mathcal{G}_1^2 &= (\{1\} \times \mathcal{G}_0^1) \bigcup (\{0\} \times \mathcal{G}_1^1) \bigcup (\{2\} \times \mathcal{G}_2^1) = \{(1,0), (0,1), (2,2)\},\\ \mathcal{G}_2^2 &= (\{2\} \times \mathcal{G}_0^1) \bigcup (\{1\} \times \mathcal{G}_1^1) \bigcup (\{0\} \times \mathcal{G}_2^1) = \{(2,0), (1,1), (0,2)\}.\end{aligned} \tag{3}$$

Obviously, the sets $\mathcal{G}_i^2$ are pairwise disjoint and the union of all sets is $\mathbb{Z}_3 \times \mathbb{Z}_3$, as shown in Fig. 1.

Now, we construct three subsets of $\mathbb{Z}_3^N$ for $N \geqslant 2$,

$$\mathcal{G}_i^N = (\{i\} \times \mathcal{G}_0^{N-1}) \bigcup (\{i \oplus_3 2\} \times \mathcal{G}_1^{N-1}) \bigcup (\{i \oplus_3 1\} \times \mathcal{G}_2^{N-1}), \tag{4}$$

where $i \in \mathbb{Z}_3$, $i \oplus_3 t = (i+t) \bmod 3$. For each i, one can also exhibit the exact description

$$\begin{aligned}\mathcal{G}_0^N &= (\{0\} \times \mathcal{G}_0^{N-1}) \bigcup (\{2\} \times \mathcal{G}_1^{N-1}) \bigcup (\{1\} \times \mathcal{G}_2^{N-1}),\\ \mathcal{G}_1^N &= (\{1\} \times \mathcal{G}_0^{N-1}) \bigcup (\{0\} \times \mathcal{G}_1^{N-1}) \bigcup (\{2\} \times \mathcal{G}_2^{N-1}),\\ \mathcal{G}_2^N &= (\{2\} \times \mathcal{G}_0^{N-1}) \bigcup (\{1\} \times \mathcal{G}_1^{N-1}) \bigcup (\{0\} \times \mathcal{G}_2^{N-1}).\end{aligned} \tag{5}$$

Then, we have two propositions.

Proposition 1. The sets given by Eq. (4) are pairwise disjoint and the union of all sets is $\mathbb{Z}_3^N$, that is,

$$\mathcal{G}_0^N \cup \mathcal{G}_1^N \cup \mathcal{G}_2^N = \mathbb{Z}_3^N \ \text{ and } \ \mathcal{G}_i^N \cap \mathcal{G}_j^N = \emptyset, \ \text{ where } \ i \neq j \in \mathbb{Z}_3.$$

Proof. According to Eq. (3), it is clear that the claim is true for $N = 2$.

We proceed by induction. Assume that the result has been proved for $N = k$, i.e., $\mathcal{G}_0^k \cup \mathcal{G}_1^k \cup \mathcal{G}_2^k = \mathbb{Z}_3^k$ and $\mathcal{G}_i^k \cap \mathcal{G}_j^k = \emptyset$, $i \neq j \in \mathbb{Z}_3$. Let $l = k+1$, and one gets

$$\begin{aligned}\mathcal{G}_0^l \cup \mathcal{G}_1^l \cup \mathcal{G}_2^l &= [(\{0\} \times \mathcal{G}_0^k) \cup (\{2\} \times \mathcal{G}_1^k) \cup (\{1\} \times \mathcal{G}_2^k)] \bigcup [(\{1\} \times \mathcal{G}_0^k) \cup (\{0\} \times \mathcal{G}_1^k) \cup (\{2\} \times \mathcal{G}_2^k)]\\ &\quad \bigcup [(\{2\} \times \mathcal{G}_0^k) \cup (\{1\} \times \mathcal{G}_1^k) \cup (\{0\} \times \mathcal{G}_2^k)]\\ &= (\{0,1,2\} \times \mathcal{G}_0^k) \bigcup (\{0,1,2\} \times \mathcal{G}_1^k) \bigcup \left(\{0,1,2\} \times \mathcal{G}_2^k\right)\\ &= \{0,1,2\} \times (\mathcal{G}_0^k \cup \mathcal{G}_1^k \cup \mathcal{G}_2^k)\\ &= \{0,1,2\} \times \mathbb{Z}_3^k\\ &= \mathbb{Z}_3^l.\end{aligned}$$

By the induction hypothesis, $\mathcal{G}_i^k \cap \mathcal{G}_j^k = \emptyset$ is true for k. Note that

$$(\{0\} \times \mathcal{G}_0^k) \bigcap (\{1\} \times \mathcal{G}_0^k) = \emptyset,$$

$$\left(\{0\}\times\mathcal{G}_0^k\right)\bigcap\left(\{0\}\times\mathcal{G}_1^k\right)=\emptyset,$$
$$\left(\{0\}\times\mathcal{G}_0^k\right)\bigcap\left(\{2\}\times\mathcal{G}_2^k\right)=\emptyset,$$

and it follows from Eq. (5) that

$$\left(\{0\}\times\mathcal{G}_0^k\right)\bigcap\mathcal{G}_1^l=\emptyset. \tag{6}$$

Similarly, there are

$$\left(\{2\}\times\mathcal{G}_1^k\right)\bigcap\mathcal{G}_1^l=\emptyset,$$
$$\left(\{1\}\times\mathcal{G}_2^k\right)\bigcap\mathcal{G}_1^l=\emptyset. \tag{7}$$

Combining Eqs. (6) and (7) with Eq. (5) (when $N=l$, $i=1$), we get $\mathcal{G}_0^l\cap\mathcal{G}_1^l=\emptyset$. Similarly, $\mathcal{G}_0^l\cap\mathcal{G}_2^l=\emptyset$ and $\mathcal{G}_1^l\cap\mathcal{G}_2^l=\emptyset$ can be deduced. ■

Proposition 2. The set $\mathcal{G}_i^N$ given by Eq. (4) is invariant under arbitrary permutation of the positions of the N components.

Proof. First, we rewrite Eq. (5) as the following form:

$$\begin{bmatrix}\mathcal{G}_0^N\\ \mathcal{G}_1^N\\ \mathcal{G}_2^N\end{bmatrix}=\begin{bmatrix}\{0\}&\{2\}&\{1\}\\ \{1\}&\{0\}&\{2\}\\ \{2\}&\{1\}&\{0\}\end{bmatrix}\times\begin{bmatrix}\mathcal{G}_0^{N-1}\\ \mathcal{G}_1^{N-1}\\ \mathcal{G}_2^{N-1}\end{bmatrix}=\begin{bmatrix}(\{0\}\times\mathcal{G}_0^{N-1})\bigcup(\{2\}\times\mathcal{G}_1^{N-1})\bigcup(\{1\}\times\mathcal{G}_2^{N-1})\\ (\{1\}\times\mathcal{G}_0^{N-1})\bigcup(\{0\}\times\mathcal{G}_1^{N-1})\bigcup(\{2\}\times\mathcal{G}_2^{N-1})\\ (\{2\}\times\mathcal{G}_0^{N-1})\bigcup(\{1\}\times\mathcal{G}_1^{N-1})\bigcup(\{0\}\times\mathcal{G}_2^{N-1})\end{bmatrix}. \tag{8}$$

Similar to matrix multiplication, we get

$$\begin{bmatrix}\mathcal{G}_0^N&\mathcal{G}_2^N&\mathcal{G}_1^N\\ \mathcal{G}_1^N&\mathcal{G}_0^N&\mathcal{G}_2^N\\ \mathcal{G}_2^N&\mathcal{G}_1^N&\mathcal{G}_0^N\end{bmatrix}=\begin{bmatrix}\{0\}&\{2\}&\{1\}\\ \{1\}&\{0\}&\{2\}\\ \{2\}&\{1\}&\{0\}\end{bmatrix}\times\begin{bmatrix}\mathcal{G}_0^{N-1}&\mathcal{G}_2^{N-1}&\mathcal{G}_1^{N-1}\\ \mathcal{G}_1^{N-1}&\mathcal{G}_0^{N-1}&\mathcal{G}_2^{N-1}\\ \mathcal{G}_2^{N-1}&\mathcal{G}_1^{N-1}&\mathcal{G}_0^{N-1}\end{bmatrix}, \tag{9}$$

where the result of the right-hand side of Eq. (9) is

$$\begin{bmatrix}(\{0\}\times\mathcal{G}_0^{N-1})\bigcup(\{2\}\times\mathcal{G}_1^{N-1})\bigcup(\{1\}\times\mathcal{G}_2^{N-1}) & (\{0\}\times\mathcal{G}_2^{N-1})\bigcup(\{2\}\times\mathcal{G}_0^{N-1})\bigcup(\{1\}\times\mathcal{G}_1^{N-1}) & (\{0\}\times\mathcal{G}_1^{N-1})\bigcup(\{2\}\times\mathcal{G}_2^{N-1})\bigcup(\{1\}\times\mathcal{G}_0^{N-1})\\ (\{1\}\times\mathcal{G}_0^{N-1})\bigcup(\{0\}\times\mathcal{G}_1^{N-1})\bigcup(\{2\}\times\mathcal{G}_2^{N-1}) & (\{1\}\times\mathcal{G}_2^{N-1})\bigcup(\{0\}\times\mathcal{G}_0^{N-1})\bigcup(\{2\}\times\mathcal{G}_1^{N-1}) & (\{1\}\times\mathcal{G}_1^{N-1})\bigcup(\{0\}\times\mathcal{G}_2^{N-1})\bigcup(\{2\}\times\mathcal{G}_0^{N-1})\\ (\{2\}\times\mathcal{G}_0^{N-1})\bigcup(\{1\}\times\mathcal{G}_1^{N-1})\bigcup(\{0\}\times\mathcal{G}_2^{N-1}) & (\{2\}\times\mathcal{G}_2^{N-1})\bigcup(\{1\}\times\mathcal{G}_0^{N-1})\bigcup(\{0\}\times\mathcal{G}_1^{N-1}) & (\{2\}\times\mathcal{G}_1^{N-1})\bigcup(\{1\}\times\mathcal{G}_2^{N-1})\bigcup(\{0\}\times\mathcal{G}_0^{N-1})\end{bmatrix}.$$

Repeating this argument reveals

$$\begin{bmatrix}\mathcal{G}_0^N&\mathcal{G}_2^N&\mathcal{G}_1^N\\ \mathcal{G}_1^N&\mathcal{G}_0^N&\mathcal{G}_2^N\\ \mathcal{G}_2^N&\mathcal{G}_1^N&\mathcal{G}_0^N\end{bmatrix}_{[N,\dots,2,1]}=\begin{bmatrix}\{0\}&\{2\}&\{1\}\\ \{1\}&\{0\}&\{2\}\\ \{2\}&\{1\}&\{0\}\end{bmatrix}_N\times\cdots\times\begin{bmatrix}\{0\}&\{2\}&\{1\}\\ \{1\}&\{0\}&\{2\}\\ \{2\}&\{1\}&\{0\}\end{bmatrix}_2\times\begin{bmatrix}\mathcal{G}_0^1&\mathcal{G}_2^1&\mathcal{G}_1^1\\ \mathcal{G}_1^1&\mathcal{G}_0^1&\mathcal{G}_2^1\\ \mathcal{G}_2^1&\mathcal{G}_1^1&\mathcal{G}_0^1\end{bmatrix}_1. \tag{10}$$

Suppose that the elements in $\mathcal{G}_i^N$ that we constructed are ordered strings. For example, consider any string $(c_N,\dots,b_x,\dots,a_1)_{[N,\dots,x,\dots,1]}$ belongs to $\mathcal{G}_i^N$, where $[N,\dots,x,\dots,1]$ indicates the position order of each element in this string, and the index x means that the element b_x comes from the xth square matrix on the right-hand side.

To prove $\mathcal{G}_i^N$ is invariant under arbitrary permutation, we only need to show that $\mathcal{G}_i^N$ will not change when we perform arbitrary permutation on the N component positions in the N-tuples. Substituting Eq. (2) into Eq. (10), we get

$$\begin{bmatrix}\mathcal{G}_0^N&\mathcal{G}_2^N&\mathcal{G}_1^N\\ \mathcal{G}_1^N&\mathcal{G}_0^N&\mathcal{G}_2^N\\ \mathcal{G}_2^N&\mathcal{G}_1^N&\mathcal{G}_0^N\end{bmatrix}_{[N,\dots,2,1]}=\begin{bmatrix}\{0\}&\{2\}&\{1\}\\ \{1\}&\{0\}&\{2\}\\ \{2\}&\{1\}&\{0\}\end{bmatrix}_N\times\cdots\times\begin{bmatrix}\{0\}&\{2\}&\{1\}\\ \{1\}&\{0\}&\{2\}\\ \{2\}&\{1\}&\{0\}\end{bmatrix}_2\times\begin{bmatrix}\{0\}&\{2\}&\{1\}\\ \{1\}&\{0\}&\{2\}\\ \{2\}&\{1\}&\{0\}\end{bmatrix}_1. \tag{11}$$

Because the right square matrices are the same, let $[i_N,\dots,i_2,i_1]$ be an arbitrary permutation of $[N,\dots,2,1]$; then we have

$$\begin{bmatrix}\mathcal{G}_0^N&\mathcal{G}_2^N&\mathcal{G}_1^N\\ \mathcal{G}_1^N&\mathcal{G}_0^N&\mathcal{G}_2^N\\ \mathcal{G}_2^N&\mathcal{G}_1^N&\mathcal{G}_0^N\end{bmatrix}_{[i_N,\dots,i_2,i_1]}=\begin{bmatrix}\{0\}&\{2\}&\{1\}\\ \{1\}&\{0\}&\{2\}\\ \{2\}&\{1\}&\{0\}\end{bmatrix}_{i_N}\times\cdots\times\begin{bmatrix}\{0\}&\{2\}&\{1\}\\ \{1\}&\{0\}&\{2\}\\ \{2\}&\{1\}&\{0\}\end{bmatrix}_{i_2}\times\begin{bmatrix}\{0\}&\{2\}&\{1\}\\ \{1\}&\{0\}&\{2\}\\ \{2\}&\{1\}&\{0\}\end{bmatrix}_{i_1}. \tag{12}$$

Therefore, the proof is now complete. ■

B. OGEBs IN $(\mathbb{C}^3)^{\otimes N}$

Let $\mathcal{H}:=(\mathbb{C}^3)^{\otimes N}$, s_i be the cardinality of the set $\mathcal{G}_i^N$ given by Eq. (4), $i\in\mathbb{Z}_3$. Define

$$\mathcal{S}_i:=\{|\Psi_{i,k}\rangle\in\mathcal{H}\,\big|\,k\in\mathbb{Z}_{s_i},|\Psi_{i,k}\rangle:=\sum_{j\in\mathcal{G}_i^N}\omega_{s_i}^{kf_i(j)}|\boldsymbol{j}\rangle\}. \tag{13}$$

Here, $f_i:\mathcal{G}_i^N\longrightarrow\mathbb{Z}_{s_i}$ is any fixed bijection and $\omega_{s_i}:=e^{\frac{2\pi\sqrt{-1}}{s_i}}$, $s_i=|\mathcal{G}_i^N|$. For example, consider the set of states $\mathcal{S}_0:=\{|\Psi_{0,k}\rangle=\sum_{j\in\mathcal{G}_0^N}\omega_{s_0}^{kf_0(j)}|\boldsymbol{j}\rangle|k\in\mathbb{Z}_{s_0}\}$ when $N=3$. Here, f_0 is a bijection from $\mathcal{G}_0^3$ to $\mathbb{Z}_9$, which is shown in Table I. Then we

TABLE I. $f_0 : \mathcal{G}_0^3 \longrightarrow \mathbb{Z}_9$.

$\boldsymbol{j}$	(000)	(021)	(012)	(210)	(201)	(222)	(120)	(111)	(102)
$f_0(\boldsymbol{j})$	0	1	2	3	4	5	6	7	8

obtain that

$$\begin{aligned}|\Psi_{0,k}\rangle := \{&|000\rangle + \omega_9^k|021\rangle + \omega_9^{2k}|012\rangle \\ &+ \omega_9^{3k}|210\rangle + \omega_9^{4k}|201\rangle + \omega_9^{5k}|222\rangle + \omega_9^{6k}|120\rangle \\ &+ \omega_9^{7k}|111\rangle + \omega_9^{8k}|102\rangle\}_{k\in\mathbb{Z}_9}.\end{aligned} \tag{14}$$

Evidently, the set $\{\mathcal{S}_i\}$ of states is an orthogonal basis in $(\mathbb{C}^3)^{\otimes N}$. Furthermore, it forms a genuinely entangled orthogonal basis.

Theorem 1. The set $\bigcup_{i=0}^2 \mathcal{S}_i$ of states given by Eq. (13) is an OGEB in $(\mathbb{C}^3)^{\otimes N}$.

Proof. We only need to show that $|\Psi_{i,k}\rangle$ is entangled for each bipartition of the system $\{A_1, A_2, \dots, A_N\}$. Let $\{A_{x_1}A_{x_2}\cdots A_{x_s}\}|\{A_{x_{s+1}}A_{x_{s+2}}\cdots A_{x_N}\}$ $(1 \leqslant s \leqslant N-1)$ be a bipartition of the subsystem, where $\{x_1, x_2, \dots, x_N\}$ is an arbitrary permutation of $\{1, 2, \dots, N\}$. We denote $\mathcal{A}$ and $\mathcal{B}$ as the computational bases of the systems $\{A_{x_1}A_{x_2}\cdots A_{x_s}\}$ and $\{A_{x_{s+1}}A_{x_{s+2}}\cdots A_{x_N}\}$, respectively, and express state $|\Psi_{i,k}\rangle$ as

$$|\Psi_{i,k}\rangle = \Sigma_{|a\rangle\in\mathcal{A}}\Sigma_{|b\rangle\in\mathcal{B}}\psi_{a,b}|a\rangle|b\rangle.$$

Then, $|\Psi_{i,k}\rangle$ is entangled if the rank of the matrix $(\psi_{a,b})$ is greater than one.

Now, we state a fact obtained by Eq. (5): for arbitrary $(i_1, i_2, \dots, i_{N-1}) \in \mathbb{Z}_3^{N-1}$, the strings $(0, i_1, i_2, \dots, i_{N-1})$, $(1, i_1, i_2, \dots, i_{N-1})$, and $(2, i_1, i_2, \dots, i_{N-1})$ are distributed in different sets. As a consequence, given $(j_1, j_2, \dots, j_{N-2}) \in \mathcal{G}_0^{N-2}$, we get

$$\begin{aligned}&(0, j_1, j_2, \dots, j_{N-2}) \in \mathcal{G}_0^{N-1},\\ &(1, j_1, j_2, \dots, j_{N-2}) \in \mathcal{G}_1^{N-1},\\ &(2, j_1, j_2, \dots, j_{N-2}) \in \mathcal{G}_2^{N-1}.\end{aligned} \tag{15}$$

By Proposition 2, we have

$$\begin{aligned}&(j_1, j_2, \dots, j_{N-2}, 0) \in \mathcal{G}_0^{N-1},\\ &(j_1, j_2, \dots, j_{N-2}, 1) \in \mathcal{G}_1^{N-1},\\ &(j_1, j_2, \dots, j_{N-2}, 2) \in \mathcal{G}_2^{N-1}.\end{aligned}$$

Then, it follows immediately from Eq. (4) that for any $i \in \mathbb{Z}_3$,

$$\begin{aligned}&(i, j_1, j_2, \dots, j_{N-2}, 0) \in \mathcal{G}_i^N,\\ &(i \oplus_3 2, j_1, j_2, \dots, j_{N-2}, 1) \in \mathcal{G}_i^N,\\ &(i \oplus_3 1, j_1, j_2, \dots, j_{N-2}, 2) \in \mathcal{G}_i^N,\end{aligned}$$

and

$$\begin{aligned}&(i, j_1, j_2, \dots, j_{N-2}, 1) \notin \mathcal{G}_i^N,\\ &(i, j_1, j_2, \dots, j_{N-2}, 2) \notin \mathcal{G}_i^N,\\ &(i \oplus_3 2, j_1, j_2, \dots, j_{N-2}, 0) \notin \mathcal{G}_i^N,\\ &(i \oplus_3 2, j_1, j_2, \dots, j_{N-2}, 2) \notin \mathcal{G}_i^N,\\ &(i \oplus_3 1, j_1, j_2, \dots, j_{N-2}, 0) \notin \mathcal{G}_i^N,\\ &(i \oplus_3 1, j_1, j_2, \dots, j_{N-2}, 1) \notin \mathcal{G}_i^N.\end{aligned} \tag{16}$$

If the statement about Eq. (16) is not true, then at least one of the above strings belongs to $\mathcal{G}_i^N$. Assume that $(i, j_1, j_2, \dots, j_{N-2}, 1) \in \mathcal{G}_i^N$. Proposition 2 ensures that $(1, i, j_1, j_2, \dots, j_{N-2})$ and $(0, i, j_1, j_2, \dots, j_{N-2})$ belong to the same set $\mathcal{G}_i^N$. Evidently, this contradicts the fact we originally stated, and thus $(i, j_1, \dots, j_{N-2}, 1) \notin \mathcal{G}_i^N$. The other cases of Eq. (16) can be proved similarly.

Based on the above argument, we can conclude that matrix $(\psi_{a,b})$ has one of the following two 2×2 submatrices:

	$\lvert j_s\cdots j_{(N-2)}0\rangle$	$\lvert j_s\cdots j_{(N-2)}1\rangle$
$\lvert i\, j_1 j_2\cdots j_{s-1}\rangle$	α_1	0
$\lvert (i\oplus_3 2) j_1 j_2\cdots j_{s-1}\rangle$	0	β_1

	$\lvert j_s\cdots j_{(N-2)}0\rangle$	$\lvert j_s\cdots j_{(N-2)}1\rangle$
$\lvert (i\oplus_3 2) j_1 j_2\cdots j_{s-1}\rangle$	0	β_2
$\lvert i\, j_1 j_2\cdots j_{s-1}\rangle$	α_2	0,

where $\alpha_m\beta_m \neq 0\, (m = 1, 2)$. So the Schmidt rank of $|\Psi_{i,k}\rangle$ under each bipartition is greater than one. Hence, $|\Psi_{i,k}\rangle$ is a genuinely entangled state.

Similarly, if one chooses the string $(j_1, j_2, \dots, j_{N-2})$ from $\mathcal{G}_1^{N-2}$ or $\mathcal{G}_2^{N-2}$, the same conclusion can be obtained. ■

Furthermore, each state in $\bigcup_{i=0}^2 \mathcal{S}_i$ is one-uniform, which means that all their reductions to one qutrit are $I/3$. Now, we give the reduced density matrix of subsystem A_1. By the construction of Eq. (5), for any string $(j_1, \dots, j_{N-1}) \in \mathbb{Z}_3^{N-1}$, the N-tuples $(0, j_1, \dots, j_{N-1})$, $(1, j_1, \dots, j_{N-1})$, and $(2, j_1, \dots, j_{N-1})$ must not belong to the same set $\mathcal{G}_i^N$. Correspondingly, the vectors $|0, j_1, \dots, j_{N-1}\rangle$, $|1, j_1, \dots, j_{N-1}\rangle$, and $|2, j_1, \dots, j_{N-1}\rangle$ are not in the same state $|\Psi_{i,k}\rangle$. Thus, we derive

$$\begin{aligned}\rho_{A_1} &= \mathrm{Tr}_{\bar{A}_1}(|\Psi'_{i,k}\rangle\langle\Psi'_{i,k}|)\\ &= \frac{1}{3^{N-1}}\mathrm{Tr}_{\bar{A}_1}\left(\sum_{\boldsymbol{j}\in\mathcal{G}_i^N}\omega_{s_i}^{kf_i(\boldsymbol{j})}|\boldsymbol{j}\rangle\sum_{\boldsymbol{g}\in\mathcal{G}_i^N}\bar{\omega}_{s_i}^{kf_i(\boldsymbol{g})}\langle\boldsymbol{g}|\right)\\ &= \frac{1}{3^{N-1}}\sum_{\boldsymbol{j}\in\mathcal{G}_i^N}\sum_{\boldsymbol{g}\in\mathcal{G}_i^N}\omega_{s_i}^{kf_i(\boldsymbol{j})}\bar{\omega}_{s_i}^{kf_i(\boldsymbol{g})}\mathrm{Tr}_{\bar{A}_1}(|\boldsymbol{j}\rangle\langle\boldsymbol{g}|)\\ &= \frac{1}{3^{N-1}}\left(\sum_{\boldsymbol{j}\in\{i\}\times\mathcal{G}_0^{N-1}}\omega_{s_i}^{kf_i(\boldsymbol{j})}\bar{\omega}_{s_i}^{kf_i(\boldsymbol{j})}|i\rangle\langle i|\right.\end{aligned}$$

TABLE II. The distribution of $(0)^{\times N}$, $(1)^{\times N}$, and $(2)^{\times N}$, where $n \geqslant 1$.

System N	$(0)^{\times N}$	$(1)^{\times N}$	$(2)^{\times N}$
$N=2$	$(0,0)\in\mathcal{G}_0^2$	$(1,1)\in\mathcal{G}_2^2$	$(2,2)\in\mathcal{G}_1^2$
$N=3$	$(0,0,0)\in(\{0\}\times\mathcal{G}_0^2)\subset\mathcal{G}_0^3$	$(1,1,1)\in(\{1\}\times\mathcal{G}_2^2)\subset\mathcal{G}_0^3$	$(2,2,2)\in(\{2\}\times\mathcal{G}_1^2)\subset\mathcal{G}_0^3$
$N=4$	$(0)^{\times 4}\in(\{0\}\times\mathcal{G}_0^3)\subset\mathcal{G}_0^4$	$(1)^{\times 4}\in(\{1\}\times\mathcal{G}_0^3)\subset\mathcal{G}_1^4$	$(2)^{\times 4}\in(\{2\}\times\mathcal{G}_0^3)\subset\mathcal{G}_2^4$
$N=5$	$(0)^{\times 5}\in(\{0\}\times\mathcal{G}_0^4)\subset\mathcal{G}_0^5$	$(1)^{\times 5}\in(\{1\}\times\mathcal{G}_1^4)\subset\mathcal{G}_2^5$	$(2)^{\times 5}\in(\{2\}\times\mathcal{G}_2^4)\subset\mathcal{G}_1^5$
$N=6$	$(0)^{\times 6}\in(\{0\}\times\mathcal{G}_0^5)\subset\mathcal{G}_0^6$	$(1)^{\times 6}\in(\{1\}\times\mathcal{G}_2^5)\subset\mathcal{G}_0^6$	$(2)^{\times 6}\in(\{2\}\times\mathcal{G}_1^5)\subset\mathcal{G}_0^6$
$N=7$	$(0)^{\times 7}\in(\{0\}\times\mathcal{G}_0^6)\subset\mathcal{G}_0^7$	$(1)^{\times 7}\in(\{1\}\times\mathcal{G}_0^6)\subset\mathcal{G}_1^7$	$(2)^{\times 7}\in(\{2\}\times\mathcal{G}_0^6)\subset\mathcal{G}_2^7$
$N=8$	$(0)^{\times 8}\in(\{0\}\times\mathcal{G}_0^7)\subset\mathcal{G}_0^8$	$(1)^{\times 8}\in(\{1\}\times\mathcal{G}_1^7)\subset\mathcal{G}_2^8$	$(2)^{\times 8}\in(\{2\}\times\mathcal{G}_2^7)\subset\mathcal{G}_1^8$
$N=9$	$(0)^{\times 9}\in(\{0\}\times\mathcal{G}_0^8)\subset\mathcal{G}_0^9$	$(1)^{\times 9}\in(\{1\}\times\mathcal{G}_2^8)\subset\mathcal{G}_0^9$	$(2)^{\times 9}\in(\{2\}\times\mathcal{G}_1^8)\subset\mathcal{G}_0^9$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$N=3n$	$(0)^{\times 3n}\in(\{0\}\times\mathcal{G}_0^{(3n-1)})\subset\mathcal{G}_0^N$	$(1)^{\times 3n}\in(\{1\}\times\mathcal{G}_2^{(3n-1)})\subset\mathcal{G}_0^N$	$(2)^{\times 3n}\in(\{2\}\times\mathcal{G}_1^{(3n-1)})\subset\mathcal{G}_0^N$
$N=3n+1$	$(0)^{\times(3n+1)}\in(\{0\}\times\mathcal{G}_0^{3n})\subset\mathcal{G}_0^N$	$(1)^{\times(3n+1)}\in(\{1\}\times\mathcal{G}_0^{3n})\subset\mathcal{G}_1^N$	$(2)^{\times(3n+1)}\in(\{2\}\times\mathcal{G}_0^{3n})\subset\mathcal{G}_2^N$
$N=3n+2$	$(0)^{\times(3n+2)}\in(\{0\}\times\mathcal{G}_0^{(3n+1)})\subset\mathcal{G}_0^N$	$(1)^{\times(3n+2)}\in(\{1\}\times\mathcal{G}_1^{(3n+1)})\subset\mathcal{G}_2^N$	$(2)^{\times(3n+2)}\in(\{2\}\times\mathcal{G}_2^{(3n+1)})\subset\mathcal{G}_1^N$

$$
\begin{aligned}
&+ \sum_{\boldsymbol{j}\in\{i\oplus_3 2\}\times\mathcal{G}_1^{N-1}} \omega_{s_i}^{kf_i(\boldsymbol{j})}\bar{\omega}_{s_i}^{kf_i(\boldsymbol{j})}|i\oplus_3 2\rangle\langle i\oplus_3 2| \\
&+ \left.\sum_{\boldsymbol{j}\in\{i\oplus_3 1\}\times\mathcal{G}_2^{N-1}} \omega_{s_i}^{kf_i(\boldsymbol{j})}\bar{\omega}_{s_i}^{kf_i(\boldsymbol{j})}|i\oplus_3 1\rangle\langle i\oplus_3 1|\right) \\
&= \frac{1}{3^{N-1}}[3^{N-2}(|0\rangle\langle 0|+|1\rangle\langle 1|+|2\rangle\langle 2|)] \\
&= I/3,
\end{aligned} \tag{17}
$$

where $|\Psi'_{i,k}\rangle = \frac{1}{\sqrt{3^{N-1}}}|\Psi_{i,k}\rangle$ is the normalized form of $|\Psi_{i,k}\rangle$ and $\bar{\omega}_{s_i}^{kf_i(\boldsymbol{g})}$ is the complex conjugate of $\omega_{s_i}^{kf_i(\boldsymbol{g})}$. Similarly, there are $\rho_{A_2}=\rho_{A_3}=\cdots=\rho_{A_N}=\mathrm{I}/3$. That is, all of the one-qutrit reductions are maximally mixed.

The basis in an N-qudit system is called a "maximum entangled basis" (MEB) [54] if each element is a one-uniform state. MEB has been found to be useful in applications in quantum information masking. For example, the authors of Ref. [54] used MEB and showed that it is possible to mask arbitrary unknown quantum states into multipartite lower-dimensional systems.

IV. STRONGLY NONLOCAL OGEs AND STRONGLY NONLOCAL OGEBs IN $(\mathbb{C}^3)^{\otimes N}$

In this section, by modifying the previous construction, we successfully show strongly nonlocal OGESs of size $2\times 3^{N-1}$ and strongly nonlocal OGEBs in Hilbert space $\mathcal{H}=(\mathbb{C}^3)^{\otimes N}$. Our OGESs are strictly fewer, $3^{N-1}-2^N+1$ fewer to be precise, than the size 3^N-2^N+1 of the strongly nonlocal OGES in Ref. [50]. We prove that only $2\times 3^{N-1}$ entangled states can also exhibit strong nonlocality in an N-qutrit system.

Let $(0)^{\times N} := (\underbrace{0,0,\dots,0}_{N})$, $(1)^{\times N} := (\underbrace{1,1,\dots,1}_{N})$, $(2)^{\times N} := (\underbrace{2,2,\dots,2}_{N})$ and denote $\mathbf{0}=(0)^{\times(N-1)}$, $\mathbf{1}=(1)^{\times(N-1)}$, $\mathbf{2}=(2)^{\times(N-1)}$. Based on the distribution of $(0)^{\times N}$, $(1)^{\times N}$, and $(2)^{\times N}$ in sets $\mathcal{G}_i^N$ $(i\in\mathbb{Z}_3)$, our construction includes three cases. For details of the distribution, please see Table II.

Case I. When $N=3n\,(n\geqslant 1)$, we redefine

$$
\begin{aligned}
\widetilde{\mathcal{G}_0^N} &= [\{0\}\times(\mathcal{G}_0^{N-1}\backslash\{\mathbf{0}\})]\bigcup[\{2\}\times(\mathcal{G}_1^{N-1}\backslash\{\mathbf{2}\})] \\
&\quad \bigcup[\{1\}\times(\mathcal{G}_2^{N-1}\backslash\{\mathbf{1}\})], \\
\widetilde{\mathcal{G}_1^N} &= (\{1\}\times\mathcal{G}_0^{N-1})\bigcup(\{0\}\times\mathcal{G}_1^{N-1})\bigcup(\{2\}\times\mathcal{G}_2^{N-1}), \\
\widetilde{\mathcal{G}_2^N} &= (\{2\}\times\mathcal{G}_0^{N-1})\bigcup(\{1\}\times\mathcal{G}_1^{N-1})\bigcup(\{0\}\times\mathcal{G}_2^{N-1}), \\
\widetilde{\mathcal{G}_3^N} &= \left\{(0,\underbrace{0,\dots,0}_{N-1}),(1,\underbrace{1,\dots,1}_{N-1}),(2,\underbrace{2,\dots,2}_{N-1})\right\}.
\end{aligned} \tag{18}
$$

Here, $\mathcal{G}_i^{N-1}$ $(i=0,1,2)$ is given by Eq. (4). Clearly, we just rename the sets $\mathcal{G}_1^N$ and $\mathcal{G}_2^N$ that are given by Eq. (5) as $\widetilde{\mathcal{G}_1^N}$ and $\widetilde{\mathcal{G}_2^N}$ without changing their structure.

Let $\widetilde{s_i}$ be the cardinality of the set $\widetilde{\mathcal{G}_i^N}$, $i\in\mathbb{Z}_4$, and we define

$$
\widetilde{\mathcal{S}_i} := \left\{|\widetilde{\Psi}_{i,k}\rangle\in\mathcal{H}\,\middle|\,k\in\mathbb{Z}_{\widetilde{s_i}},\,|\widetilde{\Psi}_{i,k}\rangle := \sum_{\boldsymbol{j}\in\widetilde{\mathcal{G}_i^N}}\omega_{\widetilde{s_i}}^{kf_i(\boldsymbol{j})}|\boldsymbol{j}\rangle\right\}. \tag{19}
$$

Here, $f_i:\widetilde{\mathcal{G}_i^N}\longrightarrow\mathbb{Z}_{\widetilde{s_i}}$ is any fixed bijection and $\omega_{\widetilde{s_i}} := \mathrm{e}^{\frac{2\pi\sqrt{-1}}{\widetilde{s_i}}}$.

Comparing Eq. (18) with Eq. (5), we only change the position of $(0)^{\times N}$, $(2)^{\times N}$, and $(1)^{\times N}$. Then, according to the proof of Proposition 2 and Theorem 1, the change of these elements does not change the permutation invariance of sets $\widetilde{\mathcal{G}_i}$ in Eq. (18), nor does it change the genuine entanglement of sets $\widetilde{\mathcal{S}_i}$ in Eq. (19).

Theorem 2. In $(\mathbb{C}^3)^{\otimes N}$ $(N=3n,\ n\geqslant 1)$, the set $\widetilde{\mathcal{S}} := \cup_{i=0}^3\widetilde{\mathcal{S}_i}$ given by Eq. (19) is a strongly nonlocal OGEB. The set $\widetilde{\mathcal{S}}\backslash\widetilde{\mathcal{S}_2}=\widetilde{\mathcal{S}_0}\cup\widetilde{\mathcal{S}_1}\cup\widetilde{\mathcal{S}_3}$ is a strongly nonlocal OGES of size $2\times 3^{N-1}$.

Proof. Evidently, $\widetilde{\mathcal{S}}$ is an OGEB in $(\mathbb{C}^3)^{\otimes N}$. Thus, we only need to prove that $\widetilde{\mathcal{S}}\backslash\widetilde{\mathcal{S}_2}$ has the property of strong nonlocality. First, we show that $A_2A_3\cdots A_N$ can only perform a trivial OPLM $\{\Pi_\alpha\}$, $\alpha=1,2,\dots$. Let $\Pi_\alpha=M_\alpha^\dagger M_\alpha$; since the measurement is orthogonality preserving, for every α, the

TABLE III. Some off-diagonal elements of Π_α for $N = 3n$. Here we apply the block zeros lemma to the sets $\widetilde{\mathcal{S}}_i$, $\widetilde{\mathcal{S}}_j$ $(i \neq j \in \{0, 1, 3\})$ given by Eq. (19).

Sets	Elements
$\widetilde{\mathcal{S}}_0, \widetilde{\mathcal{S}}_1$	${}_{\mathcal{G}_0^{N-1}\backslash\{\mathbf{0}\}}(\Pi_\alpha)_{\mathcal{G}_1^{N-1}} = 0$, ${}_{\mathcal{G}_0^{N-1}}(\Pi_\alpha)_{\mathcal{G}_2^{N-1}\backslash\{\mathbf{1}\}} = 0$, ${}_{\mathcal{G}_1^{N-1}\backslash\{\mathbf{2}\}}(\Pi_\alpha)_{\mathcal{G}_2^{N-1}} = 0$
$\widetilde{\mathcal{S}}_1, \widetilde{\mathcal{S}}_3$	${}_{\{\mathbf{0}\}}(\Pi_\alpha)_{\mathcal{G}_1^{N-1}} = 0$, ${}_{\mathcal{G}_0^{N-1}}(\Pi_\alpha)_{\{\mathbf{1}\}} = 0$, ${}_{\{\mathbf{2}\}}(\Pi_\alpha)_{\mathcal{G}_2^{N-1}} = 0$
$\widetilde{\mathcal{S}}_0, \widetilde{\mathcal{S}}_3$	${}_{\{\mathbf{0}\}}(\Pi_\alpha)_{\mathcal{G}_0^{N-1}\backslash\{\mathbf{0}\}} = 0$, ${}_{\{\mathbf{1}\}}(\Pi_\alpha)_{\mathcal{G}_2^{N-1}\backslash\{\mathbf{1}\}} = 0$, ${}_{\{\mathbf{2}\}}(\Pi_\alpha)_{\mathcal{G}_1^{N-1}\backslash\{\mathbf{2}\}} = 0$

postmeasurement states must be pairwise orthogonal,

$$\langle\Psi|\mathbb{I}_{A_1}\otimes M_\alpha^\dagger M_\alpha|\Phi\rangle = \langle\Psi|\mathbb{I}_{A_1}\otimes\Pi_\alpha|\Phi\rangle = 0,$$

for any two different states $|\Psi\rangle, |\Phi\rangle \in \widetilde{\mathcal{S}}\backslash\widetilde{\mathcal{S}}_2$. According to Proposition 1, the sets of basis vectors corresponding to $\widetilde{\mathcal{G}_0^N}$, $\widetilde{\mathcal{G}_1^N}$, and $\widetilde{\mathcal{G}_3^N}$ are disjoint subsets of the computational basis $\mathcal{B} = \{\otimes_{k=1}^N|\eta_k\rangle|\eta_k = 0, 1, 2\}$ of $(\mathbb{C}^3)^{\otimes N}$.

Applying the block zeros lemma to any two different sets $\widetilde{\mathcal{S}}_i$ and $\widetilde{\mathcal{S}}_j$ $(i \neq j \in \{0, 1, 3\})$, we obtain

$$\langle i_1, i_2, \dots, i_N|\mathbb{I}_{A_1}\otimes\Pi_\alpha|j_1, j_2, \dots, j_N\rangle = \langle\boldsymbol{i}|E|\boldsymbol{j}\rangle = 0,$$

for any $\boldsymbol{i} = (i_1, i_2, \dots, i_N) \in \widetilde{\mathcal{G}_i^N}$ and $\boldsymbol{j} = (j_1, j_2, \dots, j_N) \in \widetilde{\mathcal{G}_j^N}$. Here, $E := \mathbb{I}_{A_1}\otimes\Pi_\alpha$.

When $i_1 = j_1$, one gets

$$\langle i_2, \dots, i_N|\Pi_\alpha|j_2, \dots, j_N\rangle = 0.$$

Based on the above argument, we deduce that some off-diagonal elements of Π_α, shown in Table III, are zero.

From Table II, we know that $\mathbf{0} \in \mathcal{G}_0^{N-1}$, which means $\boldsymbol{i_0} = (1, 0, 0, \dots, 0) \in \widetilde{\mathcal{G}_1^N}$.

For any $\boldsymbol{j} = (j_1, j_2, \dots, j_N) \in \widetilde{\mathcal{G}_1^N}$ and $\boldsymbol{j} \neq \boldsymbol{i_0}$, $j_1 \neq 1$, there is

$$\langle\boldsymbol{i_0}|E|\boldsymbol{j}\rangle = \langle\boldsymbol{i_0}|\mathbb{I}_{A_1}\otimes\Pi_\alpha|\boldsymbol{j}\rangle = 0.$$

If $j_1 = 1$, then we get $\boldsymbol{j} \in \{1\}\times\mathcal{G}_0^{N-1}$, and therefore $(j_2, \dots, j_N) \in \mathcal{G}_0^{N-1}$ and $(j_2, \dots, j_N) \neq \mathbf{0}$. Noticing that ${}_{\{\mathbf{0}\}}\Pi_{\mathcal{G}_0^{N-1}\backslash\{\mathbf{0}\}} = 0$ in Table III, one obtains

$$\begin{aligned}\langle\boldsymbol{i_0}|E|\boldsymbol{j}\rangle &= \langle 1, 0, \dots, 0|\mathbb{I}_{A_1}\otimes\Pi_\alpha|1, j_2 \dots, j_N\rangle \\ &= \langle 0, \dots, 0|\Pi_\alpha|j_2 \dots, j_N\rangle = 0.\end{aligned}$$

Applying the block trivial lemma to the set of basis vectors corresponding to $\widetilde{\mathcal{G}_1^N}$, the set $\widetilde{\mathcal{S}}_1$ of states, and the vector $|1, 0, \dots, 0\rangle$, for any different strings $\boldsymbol{i}' = (i_1', i_2', \dots, i_N')$ and $\boldsymbol{j}' = (j_1', j_2', \dots, j_N')$ belonging to $\widetilde{\mathcal{G}_1^N}$, we have

$$\langle\boldsymbol{i}'|E|\boldsymbol{j}'\rangle = \langle\boldsymbol{j}'|E|\boldsymbol{i}'\rangle = 0, \quad \langle\boldsymbol{i}'|E|\boldsymbol{i}'\rangle = \langle\boldsymbol{j}'|E|\boldsymbol{j}'\rangle.$$

This implies that

$$\langle i_2', \dots, i_N'|\Pi_\alpha|i_2', \dots, i_N'\rangle = \langle j_2', \dots, j_N'|\Pi_\alpha|j_2', \dots, j_N'\rangle.$$

If $i_1' = j_1'$, one has

$$\begin{aligned}&\langle i_1', i_2', \dots, i_N'|E|j_1', j_2', \dots, j_N'\rangle \\ &= \langle i_2', \dots, i_N'|\Pi_\alpha|j_2', \dots, j_N'\rangle = 0.\end{aligned}$$

The diagonal elements and some off-diagonal elements of Π_α for $N = 3n$ are illustrated in Table IV.

Observe the first and second rows of Table III, where the results ${}_{\mathcal{G}_0^{N-1}\backslash\{\mathbf{0}\}}(\Pi_\alpha)_{\mathcal{G}_1^{N-1}} = 0$ and ${}_{\{\mathbf{0}\}}(\Pi_\alpha)_{\mathcal{G}_1^{N-1}} = 0$ in the first column yield ${}_{\mathcal{G}_0^{N-1}}(\Pi_\alpha)_{\mathcal{G}_1^{N-1}} = 0$. Similarly, we can obtain ${}_{\mathcal{G}_0^{N-1}}(\Pi_\alpha)_{\mathcal{G}_2^{N-1}} = 0$ and ${}_{\mathcal{G}_1^{N-1}}(\Pi_\alpha)_{\mathcal{G}_2^{N-1}} = 0$ by the second and third columns. Combining the results $\langle\boldsymbol{i}|\Pi_\alpha|\boldsymbol{j}\rangle = 0$ for $\boldsymbol{i} \neq \boldsymbol{j} \in \mathcal{G}_i^{N-1} (i \in \{0, 1, 2\})$ in Table IV ensures that the off-diagonal elements of Π_α are all zeros. By the results in last row of Table IV, we obtain that the diagonal elements of Π_α are all equal. Thus, Π_α is proportional to the identity for $\alpha = 1, 2, \dots$. Because of the symmetrical structure, we can also show that any $(N-1)$ parties could only perform a trivial OPLM. ■

It is worth noting that the set $\widetilde{\mathcal{S}}\backslash\widetilde{\mathcal{S}}_1 = \widetilde{\mathcal{S}}_0 \cup \widetilde{\mathcal{S}}_2 \cup \widetilde{\mathcal{S}}_3$ has the same effect with $\widetilde{\mathcal{S}}\backslash\widetilde{\mathcal{S}}_2 = \widetilde{\mathcal{S}}_0 \cup \widetilde{\mathcal{S}}_1 \cup \widetilde{\mathcal{S}}_3$, that is, $\widetilde{\mathcal{S}}\backslash\widetilde{\mathcal{S}}_1$ also exhibits strong quantum nonlocality. The detailed analysis is shown in Tables V and VI.

For the case of $N = 3n + 1$ and $N = 3n + 2$, we give two theorems similar to Theorem 2. What we need to do is to prove that the set $\widetilde{\mathcal{S}}\backslash\widetilde{\mathcal{S}}_2$ has the property of strong nonlocality, which means that any $(N-1)$ parties could only perform a trivial OPLM. We omit the detailed proof, but give four tables for the complete analysis, because it is similar to that of Theorem 2.

Case II. $N = 3n + 1$ $(n \geqslant 1)$.

We redefine

$$\widetilde{\mathcal{G}_0^N} = \left[\{0\}\times\left(\mathcal{G}_0^{N-1}\backslash\{\mathbf{0}\}\right)\right]\bigcup\left(\{2\}\times\mathcal{G}_1^{N-1}\right)\bigcup\left(\{1\}\times\mathcal{G}_2^{N-1}\right),$$

$$\widetilde{\mathcal{G}_1^N} = \left[\{1\}\times\left(\mathcal{G}_0^{N-1}\backslash\{\mathbf{1}\}\right)\right]\bigcup\left(\{0\}\times\mathcal{G}_1^{N-1}\right)\bigcup\left(\{2\}\times\mathcal{G}_2^{N-1}\right),$$

$$\widetilde{\mathcal{G}_2^N} = \left(\{2\}\times\mathcal{G}_0^{N-1}\right)\bigcup\left(\{1\}\times\mathcal{G}_1^{N-1}\right)\bigcup\left(\{0\}\times\mathcal{G}_2^{N-1}\right),$$

$$\widetilde{\mathcal{G}_3^N} = \left\{(0, \underbrace{0, \dots, 0}_{N-1}), (1, \underbrace{1, \dots, 1}_{N-1})\right\}. \tag{20}$$

Here, $\mathcal{G}_i^{N-1}$ $(i = 0, 1, 2)$ is given by Eq. (4). The set $\mathcal{G}_2^N$ given by Eq. (5) is renamed $\widetilde{\mathcal{G}_2^N}$. Let $\widetilde{s}_i$ be the cardinality of the set

TABLE IV. Diagonal elements and some off-diagonal elements of Π_α when $N = 3n$. Here we apply the block trivial lemma to the set $\widetilde{\mathcal{S}}_1$, the set of basis vectors corresponding to $\widetilde{\mathcal{G}_1^N}$, and the vector $|1, 0, \dots, 0\rangle$.

Subsets of $\widetilde{\mathcal{G}_1^N}$	Elements
$\{1\}\times\mathcal{G}_0^{N-1}$	$\langle\boldsymbol{i}|\Pi_\alpha|\boldsymbol{j}\rangle = 0$ for $\boldsymbol{i} \neq \boldsymbol{j} \in \mathcal{G}_0^{N-1}$
$\{0\}\times\mathcal{G}_1^{N-1}$	$\langle\boldsymbol{i}|\Pi_\alpha|\boldsymbol{j}\rangle = 0$ for $\boldsymbol{i} \neq \boldsymbol{j} \in \mathcal{G}_1^{N-1}$
$\{2\}\times\mathcal{G}_2^{N-1}$	$\langle\boldsymbol{i}|\Pi_\alpha|\boldsymbol{j}\rangle = 0$ for $\boldsymbol{i} \neq \boldsymbol{j} \in \mathcal{G}_2^{N-1}$
$\widetilde{\mathcal{G}_1^N}$	$\langle\boldsymbol{i}|\Pi_\alpha|\boldsymbol{i}\rangle = \langle\boldsymbol{j}|\Pi_\alpha|\boldsymbol{j}\rangle$ for $\boldsymbol{i} \neq \boldsymbol{j} \in \mathbb{Z}_3^{N-1}$

TABLE V. Some off-diagonal elements of Π_α for $N = 3n$. Here we apply the block zeros lemma to the sets $\widetilde{\mathcal{S}}_i$, $\widetilde{\mathcal{S}}_j$ ($i \neq j \in \{0, 2, 3\}$) given by Eq. (19).

Sets	Elements
$\widetilde{\mathcal{S}}_0, \widetilde{\mathcal{S}}_2$	${}_{\mathcal{G}_0^{N-1}\backslash\{\mathbf{0}\}}(\Pi_\alpha)_{\mathcal{G}_2^{N-1}} = 0,\ {}_{\mathcal{G}_1^{N-1}\backslash\{\mathbf{2}\}}(\Pi_\alpha)_{\mathcal{G}_0^{N-1}} = 0,\ {}_{\mathcal{G}_2^{N-1}\backslash\{\mathbf{1}\}}(\Pi_\alpha)_{\mathcal{G}_1^{N-1}} = 0$
$\widetilde{\mathcal{S}}_2, \widetilde{\mathcal{S}}_3$	${}_{\{\mathbf{2}\}}(\Pi_\alpha)_{\mathcal{G}_0^{N-1}} = 0,\ {}_{\{\mathbf{1}\}}(\Pi_\alpha)_{\mathcal{G}_1^{N-1}} = 0,\ {}_{\{\mathbf{0}\}}(\Pi_\alpha)_{\mathcal{G}_2^{N-1}} = 0$
$\widetilde{\mathcal{S}}_0, \widetilde{\mathcal{S}}_3$	${}_{\{\mathbf{0}\}}(\Pi_\alpha)_{\mathcal{G}_0^{N-1}\backslash\{\mathbf{0}\}} = 0,\ {}_{\{\mathbf{1}\}}(\Pi_\alpha)_{\mathcal{G}_2^{N-1}\backslash\{\mathbf{1}\}} = 0,\ {}_{\{\mathbf{2}\}}(\Pi_\alpha)_{\mathcal{G}_1^{N-1}\backslash\{\mathbf{2}\}} = 0$

TABLE VI. Diagonal elements and some off-diagonal elements of Π_α when $N = 3n$. Here we apply the block trivial lemma to the set $\widetilde{\mathcal{S}}_2$, the set of basis vectors corresponding to $\widetilde{\mathcal{G}_2^N}$, and the vector $|2, 0, \ldots, 0\rangle$.

Subsets of $\widetilde{\mathcal{G}_2^N}$	Elements
$\{2\} \times \mathcal{G}_0^{N-1}$	$\langle \boldsymbol{i}\vert\Pi_\alpha\vert\boldsymbol{j}\rangle = 0$ for $\boldsymbol{i} \neq \boldsymbol{j} \in \mathcal{G}_0^{N-1}$
$\{1\} \times \mathcal{G}_1^{N-1}$	$\langle \boldsymbol{i}\vert\Pi_\alpha\vert\boldsymbol{j}\rangle = 0$ for $\boldsymbol{i} \neq \boldsymbol{j} \in \mathcal{G}_1^{N-1}$
$\{0\} \times \mathcal{G}_2^{N-1}$	$\langle \boldsymbol{i}\vert\Pi_\alpha\vert\boldsymbol{j}\rangle = 0$ for $\boldsymbol{i} \neq \boldsymbol{j} \in \mathcal{G}_2^{N-1}$
$\widetilde{\mathcal{G}_2^N}$	$\langle \boldsymbol{i}\vert\Pi_\alpha\vert\boldsymbol{i}\rangle = \langle \boldsymbol{j}\vert\Pi_\alpha\vert\boldsymbol{j}\rangle$ for $\boldsymbol{i} \neq \boldsymbol{j} \in \mathbb{Z}_3^{N-1}$

TABLE VII. Some off-diagonal elements of Π_α when $N = 3n + 1$. Here we apply the block zeros lemma to any two different sets $\widetilde{\mathcal{S}}_i$ ($i = 0, 1, 3$) given by Eq. (21).

Sets	Elements
$\widetilde{\mathcal{S}}_0, \widetilde{\mathcal{S}}_1$	${}_{\mathcal{G}_0^{N-1}\backslash\{\mathbf{0}\}}(\Pi_\alpha)_{\mathcal{G}_1^{N-1}} = 0,\ {}_{\mathcal{G}_0^{N-1}\backslash\{\mathbf{1}\}}(\Pi_\alpha)_{\mathcal{G}_2^{N-1}} = 0,\ {}_{\mathcal{G}_1^{N-1}}(\Pi_\alpha)_{\mathcal{G}_2^{N-1}} = 0$
$\widetilde{\mathcal{S}}_1, \widetilde{\mathcal{S}}_3$	${}_{\{\mathbf{0}\}}(\Pi_\alpha)_{\mathcal{G}_1^{N-1}} = 0,\ {}_{\{\mathbf{1}\}}(\Pi_\alpha)_{\mathcal{G}_0^{N-1}\backslash\{\mathbf{1}\}} = 0$
$\widetilde{\mathcal{S}}_0, \widetilde{\mathcal{S}}_3$	${}_{\{\mathbf{0}\}}(\Pi_\alpha)_{\mathcal{G}_0^{N-1}\backslash\{\mathbf{0}\}} = 0,\ {}_{\{\mathbf{1}\}}(\Pi_\alpha)_{\mathcal{G}_2^{N-1}} = 0$

TABLE VIII. When $N = 3n + 1$, diagonal elements and some off-diagonal elements of Π_α are shown. Here we apply the block trivial lemma to the sets $\widetilde{\mathcal{S}}_1$ and $\widetilde{\mathcal{S}}_3$ given by Eq. (21), the set of base vectors corresponding to $\widetilde{\mathcal{G}_1^N}$ and $\widetilde{\mathcal{G}_3^N}$, and the vector $|1, 0, \ldots, 0\rangle$.

Subsets of $\widetilde{\mathcal{G}_1^N}$	Elements	Set	Elements
$\{1\} \times \mathcal{G}_0^{N-1}\backslash\{\mathbf{1}\}$	$\langle \boldsymbol{i}\vert\Pi_\alpha\vert\boldsymbol{j}\rangle = 0$ for $\boldsymbol{i} \neq \boldsymbol{j} \in \mathcal{G}_0^{N-1}\backslash\{\mathbf{1}\}$	$\widetilde{\mathcal{G}_1^N}$	$\langle \boldsymbol{i}\vert\Pi_\alpha\vert\boldsymbol{i}\rangle = \langle \boldsymbol{j}\vert\Pi_\alpha\vert\boldsymbol{j}\rangle$ for $\boldsymbol{i} \neq \boldsymbol{j} \in \mathbb{Z}_3^{N-1}\backslash\{\mathbf{1}\}$
$\{0\} \times \mathcal{G}_1^{N-1}$	$\langle \boldsymbol{i}\vert\Pi_\alpha\vert\boldsymbol{j}\rangle = 0$ for $\boldsymbol{i} \neq \boldsymbol{j} \in \mathcal{G}_1^{N-1}$	$\widetilde{\mathcal{G}_3^N}$	$\langle \mathbf{0}\vert\Pi_\alpha\vert\mathbf{0}\rangle = \langle \mathbf{1}\vert\Pi_\alpha\vert\mathbf{1}\rangle$
$\{2\} \times \mathcal{G}_2^{N-1}$	$\langle \boldsymbol{i}\vert\Pi_\alpha\vert\boldsymbol{j}\rangle = 0$ for $\boldsymbol{i} \neq \boldsymbol{j} \in \mathcal{G}_2^{N-1}$		

TABLE IX. Some off-diagonal elements of Π_α when $N = 3n + 2$. Here we apply the block zeros lemma to the sets $\widetilde{\mathcal{S}}_i$ ($i = 0, 1, 3$) given by Eq. (23).

Sets	Elements
$\widetilde{\mathcal{S}}_0, \widetilde{\mathcal{S}}_1$	${}_{\mathcal{G}_0^{N-1}\backslash\{\mathbf{0}\}}(\Pi_\alpha)_{\mathcal{G}_1^{N-1}} = 0,\ {}_{\mathcal{G}_0^{N-1}}(\Pi_\alpha)_{\mathcal{G}_2^{N-1}} = 0,\ {}_{\mathcal{G}_1^{N-1}}(\Pi_\alpha)_{\mathcal{G}_2^{N-1}\backslash\{\mathbf{2}\}} = 0$
$\widetilde{\mathcal{S}}_1, \widetilde{\mathcal{S}}_3$	${}_{\{\mathbf{0}\}}(\Pi_\alpha)_{\mathcal{G}_1^{N-1}} = 0,\ {}_{\{\mathbf{2}\}}(\Pi_\alpha)_{\mathcal{G}_2^{N-1}\backslash\{\mathbf{2}\}} = 0$
$\widetilde{\mathcal{S}}_0, \widetilde{\mathcal{S}}_3$	${}_{\{\mathbf{0}\}}(\Pi_\alpha)_{\mathcal{G}_0^{N-1}\backslash\{\mathbf{0}\}} = 0,\ {}_{\mathcal{G}_1^{N-1}}(\Pi_\alpha)_{\{\mathbf{2}\}} = 0$

TABLE X. Diagonal elements and some off-diagonal elements of Π_α when $N = 3n + 2$. Here, one applies the block trivial lemma to the sets $\widetilde{\mathcal{S}}_1$ and $\widetilde{\mathcal{S}}_3$ given by Eq. (23), the sets of base vectors corresponding to $\widetilde{\mathcal{G}_1^N}$ and $\widetilde{\mathcal{G}_3^N}$, and the vector $|1, 0, \ldots, 0\rangle$.

Subsets of $\widetilde{\mathcal{G}_1^N}$	Elements	Set	Elements
$\{1\} \times \mathcal{G}_0^{N-1}$	$\langle \boldsymbol{i}\vert\Pi_\alpha\vert\boldsymbol{j}\rangle = 0$ for $\boldsymbol{i} \neq \boldsymbol{j} \in \mathcal{G}_0^{N-1}$	$\widetilde{\mathcal{G}_1^N}$	$\langle \boldsymbol{i}\vert\Pi_\alpha\vert\boldsymbol{i}\rangle = \langle \boldsymbol{j}\vert\Pi_\alpha\vert\boldsymbol{j}\rangle$ for $\boldsymbol{i} \neq \boldsymbol{j} \in \mathbb{Z}_3^{N-1}\backslash\{\mathbf{2}\}$
$\{0\} \times \mathcal{G}_1^{N-1}$	$\langle \boldsymbol{i}\vert\Pi_\alpha\vert\boldsymbol{j}\rangle = 0$ for $\boldsymbol{i} \neq \boldsymbol{j} \in \mathcal{G}_1^{N-1}$	$\widetilde{\mathcal{G}_3^N}$	$\langle \mathbf{0}\vert\Pi_\alpha\vert\mathbf{0}\rangle = \langle \mathbf{2}\vert\Pi_\alpha\vert\mathbf{2}\rangle$
$\{2\} \times \mathcal{G}_2^{N-1}\backslash\{\mathbf{2}\}$	$\langle \boldsymbol{i}\vert\Pi_\alpha\vert\boldsymbol{j}\rangle = 0$ for $\boldsymbol{i} \neq \boldsymbol{j} \in \mathcal{G}_2^{N-1}\backslash\{\mathbf{2}\}$		

$\widetilde{\mathcal{G}_i^N}$, $i \in \mathbb{Z}_4$, and we define

$$\widetilde{\mathcal{S}_i} := \left\{ |\widetilde{\Psi}_{i,k}\rangle \in \mathcal{H} \,\middle|\, k \in \mathbb{Z}_{\widetilde{s_i}}, |\widetilde{\Psi}_{i,k}\rangle := \sum_{\boldsymbol{j}\in\widetilde{\mathcal{G}_i^N}} \omega_{\widetilde{s_i}}^{kf_i(\boldsymbol{j})} |\boldsymbol{j}\rangle \right\}. \tag{21}$$

Here, $f_i : \widetilde{\mathcal{G}_i^N} \longrightarrow \mathbb{Z}_{\widetilde{s_i}}$ is any fixed bijection and $\omega_{\widetilde{s_i}} := e^{\frac{2\pi\sqrt{-1}}{\widetilde{s_i}}}$.

Theorem 3. In $(\mathbb{C}^3)^{\otimes N}$ ($N = 3n + 1$, $n \geqslant 1$), the set $\widetilde{\mathcal{S}} := \cup_{i=0}^3 \widetilde{\mathcal{S}_i}$ given by Eq. (21) is a strongly nonlocal OGEB. The set $\widetilde{\mathcal{S}} \backslash \widetilde{\mathcal{S}_2} = \widetilde{\mathcal{S}_0} \cup \widetilde{\mathcal{S}_1} \cup \widetilde{\mathcal{S}_3}$ is a strongly nonlocal OGES of size $2 \times 3^{N-1}$.

Proof. See Table VII and Table VIII for the complete analysis. Thus we obtain $\Pi_\alpha \propto \mathbb{I}$. ∎

Case III. $N = 3n + 2$ ($n \geqslant 1$).

We redefine

$$\widetilde{\mathcal{G}_0^N} = [\{0\}\times(\mathcal{G}_0^{N-1}\backslash\{\mathbf{0}\})] \bigcup (\{2\}\times\mathcal{G}_1^{N-1}) \bigcup (\{1\}\times\mathcal{G}_2^{N-1}),$$

$$\widetilde{\mathcal{G}_1^N} = (\{1\}\times\mathcal{G}_0^{N-1}) \bigcup (\{0\}\times\mathcal{G}_1^{N-1}) \bigcup [\{2\}\times(\mathcal{G}_2^{N-1}\backslash\{\mathbf{2}\})],$$

$$\widetilde{\mathcal{G}_2^N} = (\{2\}\times\mathcal{G}_0^{N-1}) \bigcup (\{1\}\times\mathcal{G}_1^{N-1}) \bigcup (\{0\}\times\mathcal{G}_2^{N-1}),$$

$$\widetilde{\mathcal{G}_3^N} = \left\{ (0, \underbrace{0, \ldots, 0}_{N-1}), (2, \underbrace{2, \ldots, 2}_{N-1}) \right\}. \tag{22}$$

Here, $\mathcal{G}_i^{N-1}$ ($i = 0, 1, 2$) is given by Eq. (4). The set $\mathcal{G}_2^N$ given by Eq. (5) is renamed $\widetilde{\mathcal{G}_2^N}$. Let $\widetilde{s_i}$ be the cardinality of the set $\widetilde{\mathcal{G}_i^N}$, $i \in \mathbb{Z}_4$, and we define

$$\widetilde{\mathcal{S}_i} := \left\{ |\widetilde{\Psi}_{i,k}\rangle \in \mathcal{H} \,\middle|\, k \in \mathbb{Z}_{\widetilde{s_i}}, |\widetilde{\Psi}_{i,k}\rangle := \sum_{\boldsymbol{j}\in\widetilde{\mathcal{G}_i^N}} \omega_{\widetilde{s_i}}^{kf_i(\boldsymbol{j})} |\boldsymbol{j}\rangle \right\}. \tag{23}$$

Here, $f_i : \widetilde{\mathcal{G}_i^N} \longrightarrow \mathbb{Z}_{\widetilde{s_i}}$ is any fixed bijection and $\omega_{\widetilde{s_i}} := e^{\frac{2\pi\sqrt{-1}}{\widetilde{s_i}}}$.

Theorem 4. In $(\mathbb{C}^3)^{\otimes N}$ ($N = 3n + 2$, $n \geqslant 1$), the set $\widetilde{\mathcal{S}} := \cup_{i=0}^3 \widetilde{\mathcal{S}_i}$ given by Eq. (23) is a strongly nonlocal OGEB. The set $\widetilde{\mathcal{S}} \backslash \widetilde{\mathcal{S}_2} = \widetilde{\mathcal{S}_0} \cup \widetilde{\mathcal{S}_1} \cup \widetilde{\mathcal{S}_3}$ is a strongly nonlocal OGES of size $2 \times 3^{N-1}$.

Proof. The complete analysis is given in Table IX and Table X. Thereby, we obtain $\Pi_\alpha \propto \mathbb{I}$. ∎

We construct strongly nonlocal OGESs containing $2 \times 3^{N-1}$ states in $(\mathbb{C}^3)^{\otimes N}$, which is $3^{N-1} - 2^N + 1$ fewer than the OGESs presented in Ref. [50] and $3^{N-1} - 1$ fewer than the OPSs in Refs. [45,46]. It should be pointed out that our OGES is also of the strongest nonlocality. A set of orthogonal states is said to have the property of the strongest nonlocality [48] if only a trivial orthogonality-preserving POVM can be performed for each bipartition of the subsystems. As a consequence, we successfully show that there does exist a smaller size of the strongest nonlocal OGESs in an N-qutrit system.

V. CONCLUSION

In this work, we constructed OGESs and OGEBs with strong nonlocality in $(\mathbb{C}^3)^{\otimes N}$ ($N \geqslant 3$), which positively answer the question in Ref. [38] of "whether one can find orthogonal entangled bases that are locally irreducible in all bipartitions" and partially answer an open question given in [49], "How do we construct a strongly nonlocal orthogonal genuinely entangled set in $(\mathbb{C}^d)^{\otimes N}$ for any $d \geqslant 2$ and $N \geqslant 5$?" Furthermore, in an N-qutrit system, the strongly nonlocal OGESs in our construction have a much smaller size than that of the strongly nonlocal OGESs in Ref. [50] and strongly nonlocal OPSs in Refs. [45,46]. Thus, this work is also an answer to the question in Ref. [48], "can we construct some smaller set that has the property of the strongest nonlocality via the OGES than the OPS?" Our result could also be helpful in better understanding the structure of "local irreducibility in all bipartitions " of entangled states.

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