Noisy Demkov-Kunike model

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The Demkov-Kunike (DK) model, characterized by a time-dependent Rabi coupling $J\,\mathrm{sech}(t/T)$ and on-site detuning $\Delta_0 + \Delta_1 \tanh(t/T)$, has one of the most general forms of an exactly solvable two-state quantum system and therefore it provides a paradigm for coherent manipulations of a qubit's quantum state. Despite its extensive application in noise-free cases, the exploration of the noisy DK model remains limited. Here we extend the coherent DK model to take into account a noisy coupling term $J \rightarrow J_{\mathrm{noisy}}(t)$. We consider colored Markovian noise sources represented by telegraph noise and Gaussian noise. We present exact solutions for the survival probability $Q_{\mathrm{DK}}^{\mathrm{noisy}}$ of the noisy DK model, namely, the probability of the system to remain in its initial state. For slow telegraph noise, we identify parameter regimes where the survival probability $Q_{\mathrm{DK}}^{\mathrm{noisy}}$ is suppressed rather than enhanced by noise. In contrast, for slow Gaussian noise, the noise always enhances the survival probability $Q_{\mathrm{DK}}^{\mathrm{noisy}}$, due to the absorption of noise quanta across the energy gap. This study not only complements the existing research on the noisy Landau-Zener model but also provides valuable insights into the control of two-level quantum systems.

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I. INTRODUCTION

The two-state quantum system not only serves as a building block for quantum information and quantum computation, but also underpins our understanding of various phenomena, such as atomic collisions [1], molecular magnets [2], and chemical reactions [3]. In the study of two-state quantum systems, those models that can be solved analytically are particularly important as they provide benchmarks. Notable examples include the Landau-Zener (LZ) model [4–8], the Rosen-Zener (RZ) model [9], the Allen-Eberly (AE) model [10,11], and the Bambini-Berman (BB) [12] model. In this context, the Demkov-Kunike (DK) model, originally proposed in Ref. [13], represents a rather universal model [14–16]: It can reduce to the RZ, AE, BB, and LZ models under appropriate parameter choices while avoiding some of their intrinsic drawbacks [17,18]. Indeed, the DK model, in which the Rabi coupling and the on-site detuning depend on time as $J\,\mathrm{sech}(t/T)$ and $\Delta_0 + \Delta_1 \tanh(t/T)$, respectively, provides one of the most general forms of a two-state quantum model that can be analytically solved.

In recent years, the exploration of two-state systems coupled to an environment has attracted considerable attention. Apart from the fundamental interest in open systems, understanding and controlling noise is also crucial in practical applications [19,20] such as in noisy intermediate-scale quantum computers [21] and cloud service of quantum computers [22]. However, the investigation of the impact of noisy environments remains a significant challenge. In general, the noise mainly affects a qubit in two ways [23], namely, by destroying the superpositions through the randomization of the phase coherence between the two eigenstates and by generating excitations and altering the state occupations. Therefore, the aforementioned analytically solvable models need to be revisited to account for the presence of noise. Similarly, the noisy LZ problems have been intensively studied both theoretically [24–32] and experimentally [33–36]. However, the study of the noisy DK model still remains elusive. In comparison with the noisy LZ model, the noisy DK model has the distinctive advantage that it may provide a general form of an exactly solvable two-state quantum model coupled to an environment. Hence, a timely and fundamental question is how the noise influences the quantum transitions in the DK model.

In this work we investigate the DK tunneling rate in the presence of colored Markovian noises, as exemplified by telegraph noise and colored Gaussian noise. Specifically, we focus on the slow noise case as in Ref. [31], i.e., when the noise correlation time is long compared to typical transition time, where we can derive exact analytical solutions for the survival probability of the system remaining in the initial state. While coupling to classical noise typically results in an enhanced survival probability, we observe the suppression in certain parameter regimes for slow telegraph noise. In contrast, for slow Gaussian noise, we always observe an enhancement of the survival probability, due to the absorption of the noise quanta across the gap. This observation provides valuable insights into the intricate interplay between the noise and the transition dynamics. Our findings not only contribute to the understanding of the noisy DK model but also offer a complementary perspective to the existing studies on the noisy Landau-Zener model. Furthermore, our work introduces new possibilities for the control of two-level quantum systems.

This paper is organized as follows. In Sec. II we describe the noisy DK model. In Secs. III and IV we systematically investigate how telegraph noise and Gaussian noise affect

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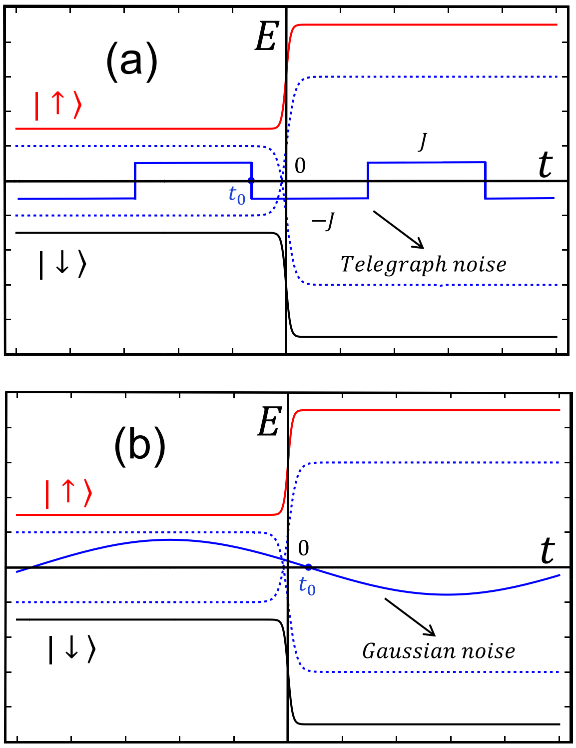

FIG. 1. (a) Telegraph noisy Demkov-Kunike model. In two-state telegraph noise, the stochastic parameter $J_{\text{noisy}}(t)$ in the Hamiltonian (1) switches from J and $-J$ at time t_0. (b) Gaussian noisy DK model with $J_{\text{noisy}}(t) = J[\tanh(t/T) - \tanh(t_0/T)]$. The time $t = t_0$ where $J_{\text{noisy}}(t) = 0$ is close to the time when the level crossing (blue dotted curves) of the diabatic levels closes.

the tunneling rate of the noisy DK model, respectively. In Sec. V we discuss the experimental conditions for observing the described phenomena. We briefly summarize our work in Sec. VI.

II. NOISY DEMKOV-KUNIKE MODEL

The standard DK model, as described in Refs. [14–16], is characterized by the quasilinear level crossing of a bell-shaped pulse with finite detuning. When noise is considered (see Fig. 1), the Hamiltonian takes the form

$$H = [\Delta_0 + \Delta_1 \tanh(t/T)]\sigma_z + J_{\text{noisy}}(t)\text{sech}(t/T)\sigma_x, \quad (1)$$

where σ_i $(i = x, z)$ are the Pauli matrices. Here the Δ_0 and Δ_1 are referred to as the static and chirp detuning parameters, respectively, and T is the scanning period of the external pulse field. The second term in the Hamiltonian (1) represents the intrinsic interactions between the two diabatic states $|\uparrow\rangle$ and $|\downarrow\rangle$, which induces the transitions. The external noise appears as the stochastic parameter [37,38] $J_{\text{noisy}}(t)$, which fluctuates over time according to the colored Markovian noise sources [37,38], as exemplified by the telegraph noise and Gaussian noise. For both types of noise, we have the mean value $\langle J_{\text{noisy}}(t)\rangle = 0$ and the first-order correlation $\langle J_{\text{noisy}}(t + \tau), J_{\text{noisy}}(t)\rangle = \sigma^2 \exp(-|\tau|/\tau_c)$, where τ_c is the correlation time, σ^2 is the variance, angular brackets denote the stochastic average, and $\langle X, Y\rangle \equiv \langle XY\rangle - \langle X\rangle\langle Y\rangle$. The noisy DK model contains three important noisy models as the special cases, namely, the noisy RZ model [9] for $\Delta_1 = 0$, the noisy AE model [10,11] for $\Delta_0 = 0$, and the noisy BB model [12] for $\Delta_0 = \Delta_1$.

As a reference, let us first briefly recapitulate the properties of the noise-free DK model with $J_{\text{noisy}}(t) = J$. There the time evolution of the wave function $\psi(t) = [C_1(t), C_2(t)]^T$ is governed by $i\partial\psi/\partial t = H\psi$. By introducing the new variable $z = [1 + \tanh(t/T)]/2$, with $z \in [0, 1]$ corresponding to $t \in (-\infty, +\infty)$, the solutions of $C_1(t)$ can be transformed into the solutions of the Gauss hypergeometric equation

$$z(1-z)\frac{d^2C_1}{dz^2} + [\nu - (\lambda + \mu + 1)z]\frac{dC_1}{dz} - \lambda\mu C_1 = 0, \quad (2)$$

where $\nu = 1/2 - iT(\Delta_0 - \Delta_1)$, $\lambda = iT(\sqrt{\Delta_1^2 - J^2} + \Delta_1)$, and $\mu = iT(\Delta_1 - \sqrt{\Delta_1^2 - J^2})$. Equation (2) has two linearly independent solutions expressed by the hypergeometric function, i.e., ${}_2F_1(\lambda, \mu, \nu, z)$ and $z^{1-\nu}{}_2F_1(\lambda + 1 - \nu, \mu + 1 - \nu, 2 - \nu, z)$.

The key quantity of interest is the survival probability $Q_{\text{DK}} = |C_1(t \to \infty)|^2$ under the initial condition $|C_1(t \to -\infty)| = 1$. For the noise-free DK model, the exact expression of Q_{DK} was obtained in Refs. [14–16] (or the detailed derivation can be found in Appendix A of Ref. [16]):

$$Q_{\text{DK}} = \frac{\cosh(2\pi T\Delta_1) + \cosh\left(2\pi T\sqrt{\Delta_1^2 - J^2}\right)}{\cosh(2\pi T\Delta_0) + \cosh(2\pi T\Delta_1)}. \quad (3)$$

Setting $\Delta_0 = 0$ yields the transition probability for the AE model, $Q_{\text{AE}} = 1 - \sinh^2(\pi T\sqrt{\Delta_1^2 - J^2})/\cosh^2(\pi T\Delta_1)$ [10,11], whereas if $\Delta_1 = 0$ one obtains $Q_{\text{RZ}} = \cos^2(\pi TJ)/\cosh^2(\pi T\Delta_0)$ for the RZ model [9].

The presence of the noisy component $J_{\text{noisy}}(t)$ in the Hamiltonian parameter leads to random shaking of the system. We denote the survival probability in the presence of noise by $Q_{\text{DK}}^{\text{noisy}}$. Depending on the ratio between the noise correlation time τ_c and the typical transition time $\tau_{\text{DK}} \propto 1/J$ of the DK model, there are two limits. (i) In the limit of fast noise $\tau_c/\tau_{\text{DK}} \to 0$, the stochastic parameter $J_{\text{noisy}}(t)$ is expected to undergo many oscillations within the transition time, so the occupations in each of the two diabatic levels are nearly identical. (ii) In the limit of slow noise $\tau_c/\tau_{\text{DK}} \to \infty$, the $J_{\text{noisy}}(t)$ can be treated as a constant on the timescale of the transition. This implies that the resulting $Q_{\text{DK}}^{\text{noisy}}$ can be roughly averaged over the distribution $P(J)$ of the stochastic parameter $J_{\text{noisy}}(t)$ as $\langle Q_{\text{DK}}^{\text{noisy}}\rangle = \int dJ\, P(J) Q_{\text{DK}}^{\text{noisy}}$. In our work we focus on the slow noise case in the sense of (ii), but taking into account the finite $1/\tau_c$ correction associated with the level crossing regime, along the lines of Ref. [31].

We will explore two typical kinds of colored Markovian noises [37,38]. In Sec. III we consider a random telegraph process as illustrated in Fig. 1(a) and analytically study the survival probability in the DK model in the fast noise limit. In Sec. IV we consider Gaussian noise where $J_{\text{noisy}}(t)$ changes continuously [cf. Fig. 1(b)] and investigate the transition behavior in the slow noise limit.

III. TELEGRAPH NOISY DK MODEL

In the two-state telegraph noise model, the stochastic parameter $J_{\text{noisy}}(t)$ in the Hamiltonian (1) randomly switches between two discrete values $-J$ and J. The telegraph noise property is characterized by $\langle J_{\text{noisy}}(t)\rangle = 0$ and $\langle J_{\text{noisy}}(t+\tau), J_{\text{noisy}}(t)\rangle = J^2\exp(-|\tau|/\tau_c)$. We follow Ref. [31] and consider a sufficiently slow noise but with finite $1/\tau_c$ correction. During the course of the transition, the noise jump typically occurs at time $t_0 \sim \tau_c \gg \tau_{\text{DK}} \sim 1/J$, in agreement with the slow noise assumption. However, as illustrated in Fig. 1(a), there is some (small) chance that the random switch occurs near the level crossing point that may significantly affect the tunneling probability.

Below we exactly solve the dynamics governed by the Hamiltonian (1) for the survival probability $Q_{\text{DK}}^{\text{noisy}}$, under the initial conditions $C_1(-\infty) = 1$ and $C_2(-\infty) = 0$. We proceed in two steps. First, we consider $J_{\text{noisy}}(t)$ flipping its sign once at some random time t_0 during the transition and calculate the corresponding $Q_{\text{DK}}^{\text{noisy}}$. Then, since t_0 is random, we average $Q_{\text{DK}}^{\text{noisy}}$ over t_0 to get the average $\overline{Q_{\text{DK}}^{\text{noisy}}}$.

We begin by calculating the transition dynamics when one switch occurs between the two discrete values $-J$ and J at some t_0. For times $t < t_0$, the system dynamics is governed by Eq. (2) with $J_{\text{noisy}}(t) = J$. Therefore, the instantaneous state can be expressed in terms of Gauss hypergeometric functions as

$$\begin{pmatrix} C_1(z) \\ C_2(z) \end{pmatrix} = \begin{pmatrix} {}_2F_1(\lambda, \mu, \nu, z) \\ \frac{\sqrt{\lambda\mu z(1-z)}}{\nu}\, {}_2F_1(\lambda+1, \mu+1, \nu+1, z) \end{pmatrix}, \tag{4}$$

where the parameters μ, ν, and λ are defined in Eq. (2).

For times $t > t_0$, the stochastic parameter $J_{\text{noisy}}(t)$ is switched to $J_{\text{noisy}}(t) = -J$. In this case, the general solution of Eq. (2) involves a linear superposition of two hypergeometric functions, i.e.,

$$\begin{pmatrix} C_1(z) \\ C_2(z) \end{pmatrix} = A\begin{pmatrix} {}_2F_1(\lambda, \mu, \nu, z) \\ -\frac{\sqrt{\lambda\mu z(1-z)}}{\nu}\, {}_2F_1(\lambda+1, \mu+1, \nu+1, z) \end{pmatrix} + B\begin{pmatrix} z^{1-\nu}\, {}_2F_1(\lambda+1-\nu, \mu+1-\nu, 2-\nu, z) \\ -\sqrt{\frac{z(1-z)}{\lambda\mu}}(1-\nu)z^{-\nu}\, {}_2F_1(\lambda+1-\mu, \mu+1-\nu, 1-\nu, z) \end{pmatrix}. \tag{5}$$

Here the coefficients A and B are determined by the continuity condition of C_1 and C_2 in Eqs. (4) and (5) at t_0 as

$$A(t_0) = \frac{\frac{{}_2F_1(\lambda+1-\nu,\mu+1-\nu,1-\nu,z_0)}{{}_2F_1(\lambda+1,\mu+1,\nu+1,z_0)} + \frac{\lambda\mu z_0}{\nu(1-\nu)}\frac{{}_2F_1(\lambda+1-\nu,\mu+1-\nu,2-\nu,z_0)}{{}_2F_1(\lambda,\mu,\nu,z_0)}}{\frac{{}_2F_1(\lambda+1-\nu,\mu+1-\nu,1-\nu,z_0)}{{}_2F_1(\lambda+1,\mu+1,\nu+1,z_0)} - \frac{\lambda\mu z_0}{\nu(1-\nu)}\frac{{}_2F_1(\lambda+1-\nu,\mu+1-\nu,2-\nu,z_0)}{{}_2F_1(\lambda,\mu,\nu,z_0)}}, \tag{6}$$

$$B(t_0) = \frac{2}{-\frac{\nu(1-\nu)}{\lambda\mu}z_0^{-\nu}\frac{{}_2F_1(\lambda+1-\nu,\mu+1-\nu,1-\nu,z_0)}{{}_2F_1(\lambda+1,\mu+1,\nu+1,z_0)} + z_0^{1-\nu}\frac{{}_2F_1(\lambda+1-\nu,\mu+1-\nu,2-\nu,z_0)}{{}_2F_1(\lambda,\mu,\nu,z_0)}}. \tag{7}$$

For $t_0 \to \infty$, corresponding to the absence of parameter switching, we obtain $A = 1$ and $B = 0$ as expected. The introduction of parameter switching leads to $B \neq 0$, causing a significant impact on the survival probability, as demonstrated below.

Using Eq. (5), we obtain an exact expression for the probability to remain at the same adiabatic level as

$$\begin{aligned} Q_{\text{DK}}^{\text{noisy}} = {} & |A(t_0)|^2\frac{\cosh(2\pi T\Delta_0) + \cosh\left(2\pi T\sqrt{\Delta_1^2 - J^2}\right)}{\cosh(2\pi T\Delta_0) + \cosh(2\pi T\Delta_1)} \\ & + |B(t_0)|^2\frac{[1-(\Delta_0-\Delta_1)^2T^2]\left[\cosh(2\pi T\Delta_1) - \cosh\left(2\pi T\sqrt{\Delta_1^2 - J^2}\right)\right]}{J^2T^2[\cosh(2\pi T\Delta_0) + \cosh(2\pi T\Delta_1)]} \\ & + A^*B\frac{\Gamma\left(\frac{1}{2} - \frac{i}{2}(\Delta_0+\Delta_1)T\right)\Gamma\left(12 - \frac{i}{2}(\Delta_0-\Delta_1)T\right)\Gamma\left(\frac{3}{2} - \frac{i}{2}(\Delta_0-\Delta_1)T\right)\Gamma\left(\frac{1}{2} + \frac{i}{2}(\Delta_0+\Delta_1)T\right)}{\Gamma\left(\frac{1-T\sqrt{J^2+\Delta_1^2}+i\Delta_0T}{2}\right)\Gamma\left(\frac{1+T\sqrt{J^2+\Delta_1^2}+i\Delta_0T}{2}\right)\Gamma\left(\frac{2-T\sqrt{J^2-\Delta_1^2}-\Delta_1T)}{2}\right)\Gamma\left(\frac{2+T\sqrt{J^2-\Delta_1^2}-\Delta_1T)}{2}\right)} + \text{H.c.}, \end{aligned} \tag{8}$$

where Γ is the Gamma function. Equation (8) constitutes the first key result of this study, which describes the impact of one random switch on the tunneling rate in the DK model. By setting $A = 1$ and $B = 0$, Eq. (8) precisely reproduces the noise-free results reported in Refs. [14–16].

Since in Eq. (8) the noise is fully encoded in the coefficients $A(t_0)$ and $B(t_0)$, we now analyze how the telegraph switching time t_0 affects $A(t_0)$ and $B(t_0)$, as shown in Fig. 2(a). In the asymptotic limit $t_0 \to -\infty$, corresponding to the case with $J_{\text{noisy}} = -J$, it is clear that $A \to 1$ and $B \to 0$. This can be understood as follows: (i) The energy gap of the DK model is approximately $2\sqrt{(\Delta_0 - \Delta_1)^2 + J^2}$ at $t_0 \to -\infty$ and (ii) the system is initially prepared in a state with $A = 1$ and $B = 0$, and J_{noisy} is switched from J to $-J$, so the telegraph noise is not strong enough to excite the B state. In contrast, when $t_0 \to 0$, there exists a level crossing in the parameter regime $\Delta_0 < \Delta_1$ [see the dotted curves in Fig. 1(a)]. In this case, an arbitrarily small J can excite the system, changing B from zero to nonzero [see the black curves in Fig. 2(a)]. In Fig. 2(c) we further study how the gap closing affects the coefficients $A(0)$ and $B(0)$ at $t_0 = 0$ by varying Δ_1, when $\Delta_0 = 4$. We see that the optimal enhancement of B occurs at $\Delta_1 \approx 4$, corresponding to where the energy gap almost closes.

Next, based on the behaviors of $A(t_0)$ and $B(t_0)$ for various t_0 and Δ_1, we study how the onset of one random jump affects the tunneling rate. In Fig. 2(b) we fix Δ_1 and show Q_{DK} as a

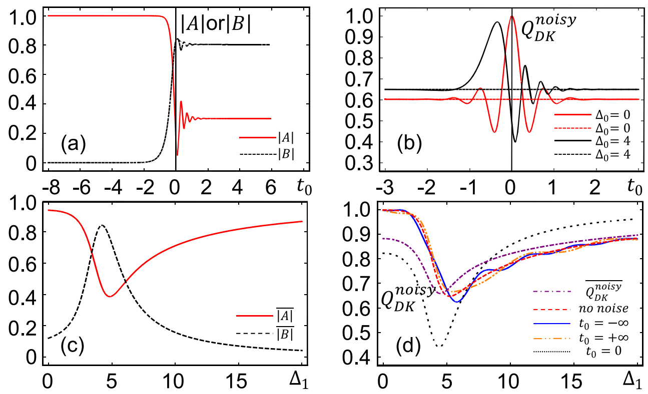

FIG. 2. Effects of fast telegraph noise on the tunneling rate $Q_{\rm DK}^{\rm noisy}$ of the DK model. The magnitude of the coefficients A in Eq. (6) and B in Eq. (7) is plotted as a function of (a) the switching time t_0 and (c) the chirp detuning parameter Δ_1. The tunneling rate $Q_{\rm DK}^{\rm noisy}$ in Eq. (8) is plotted as a function of (b) t_0 and (d) Δ_1. In (b) the dashed curves represent the tunneling rate in the absence of noise. The parameters are (a) $J=\pi/2$, $\Delta_0=4$, and $\Delta_1=5$; (b) $J=\pi/2$ and $\Delta_1=5$; (c) $J=\pi/2$, $\Delta_0=4$, and $t_0=0$; and (d) $J=\pi/2$ and $\Delta_0=4$.

function of t_0. There, when $\Delta_0=0$ (red curves), the results are similar to those of the telegraph noisy LZ model studied in Ref. [31]. Moreover, $Q_{\rm DK}^{\rm noisy}$ is symmetric with respect to t_0 and exactly recovers the noise-free counterpart in the limit $t_0\to\pm\infty$. When $\Delta_0\neq 0$ (black curves), however, the $Q_{\rm DK}^{\rm noisy}$ becomes asymmetric with respect to t_0. In addition, it exactly recovers the noise-free counterpart in the limit of $t_0\to\pm\infty$. The symmetry can be understood as arising from the symmetry of the energy levels with respect to $t=0$. Surprisingly, we see that $Q_{\rm DK}^{\rm noisy}$ decreases with t_0 in the regime where $\Delta_0+\Delta_1\tanh(t_0/T)\to 0$ in the Hamiltonian (1).

To further understand the noise-suppressed tunneling in the regime $\Delta_0+\Delta_1\tanh(t_0/T)\to 0$, we set $t_0=0$ and analyze how the tunneling rate depends on Δ_1. In Fig. 2(d) we show $Q_{\rm DK}^{\rm noisy}$ as a function of Δ_1 for different t_0 and compare it with the noise-free case (red dashed curves). As expected, both asymptotic results in the limit $t_0\to-\infty$ (blue solid curves) and the limit $t_0\to+\infty$ (orange dash–double-dotted curves) almost coincide with the noise-free case. In contrast, the result of the noisy DK model for $t_0\to 0$ (black dotted curves) differs significantly from the noise-free case. In particular, there is a dip of $Q_{\rm DK}^{\rm noisy}$ where Δ_1 corresponds to the level crossing closing. Thus we conclude that the smaller the energy gap of the telegraph noisy DK model is, the stronger the suppression of $Q_{\rm DK}^{\rm noisy}$ is.

Finally, we account for the random nature of t_0 and average Eq. (8) over t_0 [31] to obtain the corresponding results. The average result $\overline{Q_{\rm DK}^{\rm noisy}}$ is plotted as the purple dash-dotted curve in Fig. 2(d). Note that $\overline{Q_{\rm DK}^{\rm noisy}}$ is qualitatively similar to the result for a single $t_0\sim 0$. This suggests that the transition can be particularly strongly affected by a random occurrence of switching near the level crossing.

IV. GAUSSIAN NOISY DK MODEL

So far, we have systematically studied how the fast telegraph noise affects the DK tunneling rate $Q_{\rm DK}^{\rm noisy}$ based on Eq. (8). In this section we consider slow Gaussian noise characterized by $\langle J_{\rm noisy}(t)\rangle=0$ and $\langle J_{\rm noisy}(t+\tau),J_{\rm noisy}(t)\rangle=J^2\exp(-|\tau|/\tau_c)$ and investigate its effect on $Q_{\rm DK}^{\rm noisy}$. Note that Ref. [39] first investigated the effect of the slow Gaussian noise on the tunneling rate in the context of the LZ model. Here, extending the approach developed in Refs. [31,39] for the study of the LZ model with slow Gaussian noise, we seek to exactly solve the Gaussian noisy DK model.

Specifically, we assume the following form for the off-diagonal term of the Hamiltonian (1):

$$J_{\rm noisy}(t){\rm sech}(t/T)=J\left[\tanh\left(\frac{t}{T}\right)-\tanh\left(\frac{t_0}{T}\right)\right]. \tag{9}$$

Here t_0 is a random number. Similar to the case of telegraph noise, the relevant situation is expected to be when t_0 is near the level crossing closing.

We start by considering a single choice of t_0. Using Eq. (9), the dynamics of the Gaussian noisy DK model is governed by two coupled equations

$$i\frac{dC_1}{dt}=\left[\Delta_0+\Delta_1\tanh\left(\frac{t}{T}\right)\right]C_1+J\left[\tanh\left(\frac{t}{T}\right)-\tanh\left(\frac{t_0}{T}\right)\right]C_2, \tag{10}$$

$$i\frac{dC_2}{dt}=J\left[\tanh\left(\frac{t}{T}\right)-\tanh\left(\frac{t_0}{T}\right)\right]C_1-\left[\Delta_0+\Delta_1\tanh\left(\frac{t}{T}\right)\right]C_2. \tag{11}$$

Equation (11) can be exactly solved by introducing the new variables

$$\begin{pmatrix}C_1'\\ C_2'\end{pmatrix}=\begin{pmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{pmatrix}\begin{pmatrix}C_1\\ C_2\end{pmatrix}, \tag{12}$$

with $\tan(2\theta)=J/\Delta_1$. Using Eq. (12) transforms Eq. (11) into the dynamical equations for C_1' and C_2', i.e.,

$$i\frac{dC_1'}{dt}=\left[\Delta_0'+\Delta_1'\tanh\left(\frac{t}{T}\right)\right]C_1'+J'C_2', \tag{13}$$

$$i\frac{dC_2'}{dt}=J'C_1'-\left[\Delta_0'+\Delta_1'\tanh\left(\frac{t}{T}\right)\right]C_2', \tag{14}$$

with $\Delta_0'=[\Delta_0\cos^2 2\theta-\Delta_1\sin^2 2\theta\tanh(t_0/T)]/\cos 2\theta$, $\Delta_1'=\Delta_1/\cos 2\theta$, and $J'=[\Delta_0+\Delta_1\tanh(t_0/T)]\sin 2\theta$.

It turns out that Eqs. (13) and (14) are the second Demkov-Kunike model [31] with the renormalized parameters Δ_0', Δ_1', and J'. Therefore, after straightforward yet tedious calculations we analytically obtain the exact result

$$Q_{\rm DK}^{\rm noisy}(t_0, J) = \frac{\sinh[\pi T(E_e - E_a + 2\Delta_1')/2]\sinh[\pi T(E_a - E_e + 2\Delta_1')/2]}{\sinh(\pi T E_a)\sinh(\pi T E_e)}, \tag{15}$$

with $E_a = \sqrt{(\Delta_0' - \Delta_1')^2 + J'^2}$ and $E_e = \sqrt{(\Delta_0' + \Delta_1')^2 + J'^2}$. Equation (15) is another key result of this study, which describes the survival probability of the system at the initial level for the Gaussian noisy DK model.

Since t_0 is random, next we average $Q_{\rm DK}^{\rm noisy}(t_0, J)$ in Eq. (15) over t_0 as $\overline{Q_{\rm DK}^{\rm noisy}} = \int_{-\infty}^{+\infty} dt_0/\tau_c Q_{\rm DK}^{\rm noisy}$. Consequently, relevant for the tunneling rate are three free parameters Δ_0, Δ_1, and J. In Fig. 3(a) we show $\overline{Q_{\rm DK}^{\rm noisy}}$ as a function of Δ_1 for various J. We see that there is a steep increase of $\overline{Q_{\rm DK}^{\rm noisy}}$ as Δ_1 approaches $\Delta_1 \sim 4$, where the level crossing occurs [cf. the blue dotted curves in Fig. 1(b)]. Moreover, we observe an increase in $\overline{Q_{\rm DK}^{\rm noisy}}$ with J before the level crossing. This can be understood as the stronger J is, the easier the transition is. In contrast, in the regime $\Delta_1 > 4$ after the level crossing, we see that $\overline{Q_{\rm DK}^{\rm noisy}}$ decreases with J.

Finally, for the slow Gaussian noise, $\overline{Q_{\rm DK}^{\rm noisy}}$ should be further averaged over the Gaussian-type distribution $P(J)$. We have $\langle\overline{Q_{\rm DK}^{\rm noisy}}\rangle = \int dJ\, P(J)\overline{Q_{\rm DK}^{\rm noisy}}$. The resulting noise-averaged DK tunneling rate $\langle\overline{Q_{\rm DK}^{\rm noisy}}\rangle$ only depends on Δ_0 and Δ_1. As shown in Fig. 3(b), increasing Δ_1 always leads to an enhanced $\langle\overline{Q_{\rm DK}^{\rm noisy}}\rangle$.

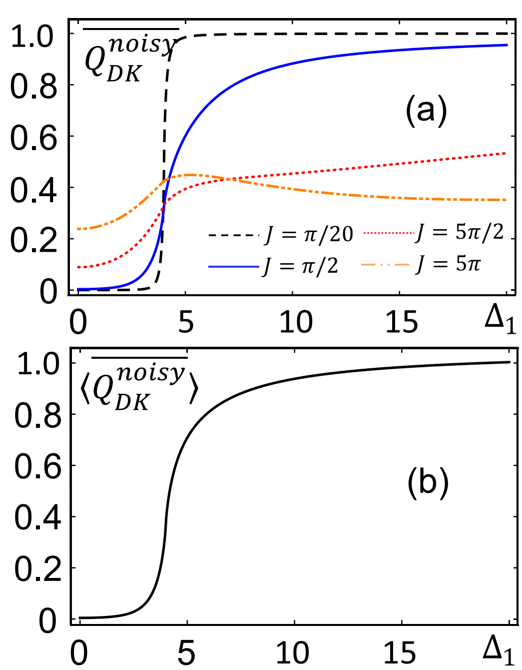

FIG. 3. Superadiabaticity of the Gaussian noisy DK model. (a) Time-averaged tunneling rate $\overline{Q_{\rm DK}^{\rm noisy}} = \int_{-\infty}^{+\infty} dt_0/\tau_c Q_{\rm DK}^{\rm noisy}$ as a function of Δ_1. (b) Noise-averaged tunneling rate $\langle\overline{Q_{\rm DK}^{\rm noisy}}\rangle = \int dJ\, P(J)\overline{Q_{\rm DK}^{\rm noisy}}$ as a function of Δ_1. Here $P(J)$ is the Gaussian-type distribution. In both plots we use $\Delta_0 = 4$ and $\tau_c = 1$.

V. DISCUSSION

The emphasis as well as value of this study is a general and exactly solvable model that is capable of describing a noisy two-level quantum system. The unavoidable presence of impurities in most real-world physical systems has provided a strong motivation to study noisy two-level models. While usually the noisy model is considered as phenomenological in the context of condensed matter, the noisy DK model studied here is of relevance in cold-atom experiments, where the noise can be engineered [40,41]. An optically trapped atomic realization of the DK model may thus serve as an ideal platform to study the effect of time-dependent disorder in a controlled setting by appropriate modulation of the laser beams to mimic various noise sources [42], thus providing an experimental counterpart of the present theoretical study. At the same time, there also exist several other quantum simulation platforms, such as trapped ions [43], Rydberg atoms [44,45], and cavity quantum electrodynamics [46], which have displayed the capability to implement controlled disorder in otherwise clean many-body systems. With these state-of-the-art experimental technologies, we hope the predicted results can be observed in future experiments.

We should also bear in mind the assumptions that underlie our results. Our study is based on the two-level model and primarily focuses on transition probabilities. In other words, our theoretical framework only considers coherent noise and has ignored the decoherence or purity of the state after the transition has passed. To study the effects of noise on both transition probabilities and decoherence, one needs to use the generalized master equation for the marginal system density operator [37,38,47–49], which is beyond the scope of the present study.

VI. CONCLUSION

In summary, we have explored the dynamics of the DK model in a noisy environment, where the influence of the (classical) environment was modeled by telegraph noise and Gaussian noise characterized by $J \to J_{\rm noisy}(t)$. We analytical obtained exact expressions for the survival probability $Q_{\rm DK}^{\rm noisy}$ of finding the system to remain in the initial state. For the slow telegraph noise, we found parameter regimes where $Q_{\rm DK}^{\rm noisy}$ is suppressed, rather than enhanced. For slow Gaussian noise, we found that the noise always leads to an enhanced $Q_{\rm DK}^{\rm noisy}$, which originates from the absorption of the noise quanta across the gap. Our study introduces a different perspective for quantum control of two-level quantum systems.

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