Simulating two-level quantum systems via classical orbit motion: Adiabatic cyclic evolution and Berry phase

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We propose a method to simulate adiabatically driven two-level quantum systems and detect the Berry phase induced in the cyclic evolution by observing the classical orbit motion of a charged particle under a time-varying magnetic field. Analogy between the dynamics of these two species of systems is revealed and the correspondence relationship is established by virtue of their equations of motion, i.e., the von Neumman equation of the quantum binary system and the Newton-Lorentz equation of the charged particle. In particular, we show how the complex phase information of the quantum system, say, the Berry phase shift acquired by its wave function during the loop evolution, can be visually captured by examining the spatial motion of the classical particle.

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I. INTRODUCTION

The cyclic evolution of adiabatically driven quantum systems can lead to the emergence of the Berry phase [1], which is a geometric phase shift acquired by the eigenstates of the Hamiltonian in addition to the dynamical phase. The Berry phase only arises when the eigenfunctions of the Hamiltonian are complex, and it is defined by the contour integral along the loop evolution of the complex eigenfunction within the parameter space. As a measurable quantity, the Berry phase has significant implications in various physical domains [2–4]. The geometric or holonomic approach to quantum computation [5–10], which exploits the Berry phase shift and its non-Abelian counterpart, is believed to provide a fault-tolerant way to implement quantum information processing. In the field of condensed matter physics, the Berry phase offers valuable insights into the topological properties of materials and underlies the phenomena like Hall effects and charge pumping [11–14].

Direct observation of the Berry phase in quantum systems has been reported, e.g., in the spin qubit associated with the nitrogen-vacancy color center in diamond [15–17], in the superconducting charge qubits by means of the microwave radiation [18], and recently in the optical Möbius-strip microcavity [19]. These experiments surely require a fair amount of demanding conditions and technologies. Specifically, since decoherence has a long time to bite in the slow adiabatic evolution of the quantum system, it may seriously affect the visibility of the phase-dependent interference experiments. Meanwhile, there have been many studies upon the classical analogy and simulation for the dynamical behavior of quantum systems. For example, it has been shown that a classical system which obeys the Landau-Lifshitz equation is in the general ability to mimic a qubit's behavior [20]. Such classical analogues for strongly driven qubit systems have been manifested with various resonator systems, e.g., the nanomechanical resonator [21,22] and coupled electrical resonators [23].

In this paper we propose to simulate the dynamical evolution and detect the Berry phase generated by the adiabatically driven two-level quantum systems through the spatial motion of a classical charged particle subjected to a time-varying magnetic field. The usage of complex numbers is at the heart of quantum physics, especially the quantum coherence effects associated with the information of relative phase shifts. To simulate quantum states and their unitary evolution by real amplitudes and matrix entries is generally not trivial which should resort to a Hilbert space with higher dimensions [24]. For the two-level quantum system, the evolution of the wave function can be equally described by that of the three-parameter Bloch vector, which corresponds to the homomorphism between the unitary SU(2) group and the real orthogonal SO(3) group. We will demonstrate the analogy between the dynamical evolution of the Bloch vector of the driven binary quantum system and the orbit motion of the classical particle governed by the Newton-Lorentz equation, which allows for the classical simulation for this particular quantum system. Consequently, we aim at how the complex phase information, say, the Berry phase shift induced during the adiabatic cyclic evolution can be captured by the orbit motion of the classical charged particle in the real space.

The rest of the paper is organized as follows. In Sec. II, we will elucidate the analogy between the adiabatic evolution of the driven two-level quantum system and the spatial motion of a charged particle under a time-varying magnetic field. The correspondence relationship is established between the equations of motion of these two species of systems, i.e., the Bloch equation responsible for the density state of the quantum system and the Newton-Lorentz equation describing the orbit motion of the classical particle. Subsequently,

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we employ the driven spin-1/2 model and explore how the phase information can be captured by the orbit motion of the classical particle (Sec. III). In Sec. IV, we apply a double-loop strategy to achieve the pure Berry phase of the adiabatic cyclic evolution and show how its effect can be detected by the corresponding particle velocity. Finally, relevant discussions about the experimental realization and a brief summary are presented in Sec. V.

II. CORRESPONDENCE RELATIONSHIP BETWEEN EQUATIONS OF MOTION OF THE TWO SPECIES OF DRIVEN SYSTEMS

Let us consider the two-level quantum system described by a Hamiltonian

$$H(t) = \tfrac{1}{2}\boldsymbol{\Omega}(t)\cdot\boldsymbol{\sigma}, \tag{1}$$

where the driving field $\boldsymbol{\Omega}(t) \equiv [\Omega_x(t), \Omega_y(t), \Omega_z(t)]$ is generally time dependent and σ_i $(i = x, y, z)$ are Pauli matrices satisfying $[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k$. To describe the dynamics of the system, we look into the time evolution of the density state which is governed by the von Neumann equation

$$i\hbar\frac{\partial}{\partial t}\rho(t) = [H(t), \rho(t)]. \tag{2}$$

For a closed two-level system the density operator takes the form of $\rho(t) = \frac{1}{2}[I_2 + \boldsymbol{\alpha}(t)\cdot\boldsymbol{\sigma}]$, in which $\boldsymbol{\alpha}(t) \equiv \sum_i \alpha_i\hat{e}_i$ denotes the Bloch vector and its modulus remains constant over the time evolution. It is verified that the components $\alpha_i(t) = \mathrm{tr}[\rho(t)\sigma_i]$ and they satisfy the following Bloch equation:

$$\dot{\alpha}_i(t) = -\frac{1}{\hbar}[\boldsymbol{\alpha}(t)\times\boldsymbol{\Omega}(t)]_i. \tag{3}$$

The key point is to recognize the similarity between Eq. (3) and the Newton-Lorentz equation characterizing the spatial motion of a charged particle under a magnetic field, i.e.,

$$\dot{\boldsymbol{v}}(t) = \frac{q}{m}\boldsymbol{v}(t)\times\boldsymbol{B}(t). \tag{4}$$

Here, q/m is the charge-to-mass ratio of the particle and the time-varying $\boldsymbol{B}(t)$ is assumed to be spatially uniform. By comparing the above two equations (3) and (4), it is readily seen that, if we set $\boldsymbol{B}(t) \equiv -m\boldsymbol{\Omega}(t)/(\hbar q)$, the correspondence is established between the Bloch vector $\boldsymbol{\alpha}(t)$ of the quantum system and the velocity vector $\boldsymbol{v}(t)$ of the classical particle, differing only by a constant ratio with respect to their initial magnitudes.

It should be noted that the time-varying magnetic field may create an inductive electric field which, satisfying $\nabla\times\boldsymbol{E}(\boldsymbol{r}, t) = -\frac{\partial}{\partial t}\boldsymbol{B}(t)$, has not been included in the above Newton-Lorentz equation (4). Nonetheless, the above modest correspondence can be utilized as a preliminary tool to seek potential magnetic confinement protocols [25], which builds upon the methods and results already achieved in the field of quantum control for the driven quantum system. In what follows we shall consider that the driving field $\boldsymbol{\Omega}(t)$ varies adiabatically, allowing us to neglect the electric field generated by the corresponding magnetic field. The above correspondence relationship hence offers an effective way to simulate dynamical evolution for the quantum system by virtue of the spatial motion of the classical particle. In particular, we concentrate below on how the effect of the Berry phase shift induced in the cyclically driven quantum system can be detected by the particle's motion in the real space.

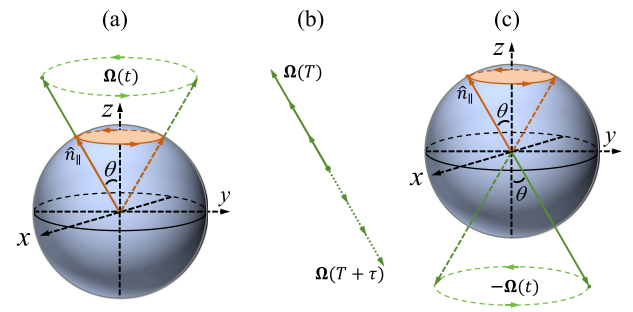

FIG. 1. Two times the loop evolution of the adiabatically driven spin-1/2 system. (a) The first cyclic evolution during which the wave function $|\psi^+_{\rm ad}(0)\rangle$ (corresponding to the Bloch vector $\hat{n}_{\parallel}$) acquires a total phase $\phi_{\rm tot}(t) = \phi^d(t) + \phi^g(t)$. (b) Adiabatic reversal process of the field orientation embedded in the two times loop evolution. (c) The second cyclic evolution during which $|\psi^+_{\rm ad}(0)\rangle$ acquires a total phase $\bar{\phi}_{\rm tot}(t) = -\phi^d(t) + \phi^g(t)$.

III. CAPTURING PHASE INFORMATION OF THE QUANTUM SYSTEM BY THE CLASSICAL ORBIT MOTION

Let us move to consider a paradigmatic quantum model, a spin subjected to a cyclic field which evolves adiabatically with a cone angle θ [see Fig. 1(a)]. The Hamiltonian reads

$$H_C(t) = \tfrac{1}{2}\Omega[\sin\theta(\cos\varphi\sigma_x + \sin\varphi\sigma_y) + \cos\theta\sigma_z], \tag{5}$$

where $\varphi(t) = \gamma t$ goes from 0 to 2π with a fixed frequency. Accordingly, the classical charged particle is subjected to a magnetic field $\boldsymbol{B}(t) = B_z\hat{e}_z + \boldsymbol{B}_\tau(t)$, in which $B_z = -m\Omega\cos\theta/(\hbar q)$ denotes a longitudinal field along the z axis and $\boldsymbol{B}_\tau(t) = B_\tau[\cos\varphi(t)\hat{e}_x + \sin\varphi(t)\hat{e}_y]$ represents a time-varying transverse field with $B_\tau = -m\Omega\sin\theta/(\hbar q)$. We assume that the adiabatic condition is fulfilled, i.e., $\hbar\gamma \ll \Omega$ or $\gamma \ll \frac{q}{m}|\boldsymbol{B}|$, equivalently. It indicates that the inductive electric field created by $\boldsymbol{B}_\tau(t)$, supposed to be of the form $\boldsymbol{E}(z, t) = \gamma z\boldsymbol{B}_\tau(t)$, can be neglected since the resulting electric force is much less than the magnetic force in view of the fact that the size of the particle orbit is of the order $v/(\frac{q}{m}|\boldsymbol{B}|)$.

During the adiabatic loop evolution, the energy eigenstates of the Hamiltonian $H_C(t)$ will remain to be in its instantaneous eigenstates $|\psi^\pm_{\rm ad}(t)\rangle = \cos\frac{\theta}{2}e^{\mp i\varphi/2}|\pm\rangle \pm \sin\frac{\theta}{2}e^{\pm i\varphi/2}|\mp\rangle$, but attaining a total phase during the time evolution $|\psi^\pm_{\rm ad}(0)\rangle \rightarrow e^{\mp i\phi_{\rm tot}(t)}|\psi^\pm_{\rm ad}(t)\rangle$, where $\phi_{\rm tot}(t) = \phi^d(t) + \phi^g(t)$ contains a dynamical part $\phi^d(t) = \frac{1}{2}\Omega t/\hbar$ and an extra Berry phase

$$\phi^g(t) = -i\int_0^t \langle\psi^+_{\rm ad}(\tau)|\partial_\tau|\psi^+_{\rm ad}(\tau)\rangle d\tau = -\frac{\gamma t}{2}\cos\theta. \tag{6}$$

The evolution of the density state of the system is then characterized by $\rho(t) = U_C(t)\rho(0)U_C^\dagger$, where the time evolution operator is shown to be

$$U_C(t) = \sum_{\pm} e^{\mp i\phi_{\rm tot}(t)} |\psi_{\rm ad}^{\pm}(t)\rangle\langle\psi_{\rm ad}^{\pm}(0)| = \begin{pmatrix} \cos\vartheta\, e^{-i(\chi+\frac{\varphi}{2})} & -i\sin\vartheta\, e^{-i\frac{\varphi}{2}} \\ -i\sin\vartheta\, e^{i\frac{\varphi}{2}} & \cos\vartheta\, e^{i(\chi+\frac{\varphi}{2})} \end{pmatrix}. \tag{7}$$

We chose "$|\pm\rangle$" as the computational basis for the representative matrix and the contained parameters $\vartheta(t)$ and $\chi(t)$ are specified by

$$\begin{aligned} \sin\vartheta(t) &= \sin\theta\sin\phi_{\rm tot}(t), \\ \tan\chi(t) &= \cos\theta\tan\phi_{\rm tot}(t). \end{aligned} \tag{8}$$

Following the group homomorphism between SU(2) and SO(3), the evolution of the Bloch vector of $\rho(t)$ is thus described by $\alpha_i(t) = \sum_j \mathcal{U}_C^{ij}(t)\alpha_j(0)$, in which $\mathcal{U}_C(t)$ is a real 3×3 orthogonal matrix given by

$$\mathcal{U}_C(t) = \begin{pmatrix} \cos\varphi\sin^2\vartheta + \cos(2\chi+\varphi)\cos^2\vartheta & \sin\varphi\sin^2\vartheta - \sin(2\chi+\varphi)\cos^2\vartheta & \sin2\vartheta\sin(\chi+\varphi) \\ \sin\varphi\sin^2\vartheta + \sin(2\chi+\varphi)\cos^2\vartheta & \cos(2\chi+\varphi)\cos^2\vartheta - \cos\varphi\sin^2\vartheta & -\sin2\vartheta\cos(\chi+\varphi) \\ \sin2\vartheta\sin\chi & \sin2\vartheta\cos\chi & \cos2\vartheta \end{pmatrix}. \tag{9}$$

Note that the time evolution of the density state $|\psi_{\rm ad}^{\pm}(t)\rangle\langle\psi_{\rm ad}^{\pm}(t)|$, with the Bloch vector expressed as $\hat{n}_{\shortparallel}(t) = \sin\theta(\cos\varphi\hat{e}_x + \sin\varphi\hat{e}_y) + \cos\theta\hat{e}_z$, does not reflect any of the above phase information. To detect the effect of the Berry phase shift, the initial state should be prepared in a superposition of the two basis states $|\psi_{\rm ad}^{\pm}(0)\rangle$. Furthermore, to ensure that the corresponding classical particle possesses a bounded orbit, the initial Bloch vector must be perpendicular to $\hat{n}_{\shortparallel}(0)$, i.e., within the plane determined by $\hat{n}_{\perp}(0) = \cos\theta\hat{e}_x - \sin\theta\hat{e}_z$ and $\hat{n}_y(0) = \hat{e}_y$. This is because the characteristic solution $\hat{n}_{\shortparallel}(t)$ has a constant z component, which would lead to unbounded motion of the classical particle in the z direction if the initial Bloch vector possesses any of its ingredients. To be specific, we consider that at $t=0$ the quantum system is in a state $|\psi(0)\rangle = [|\psi_{\rm ad}^{+}(0)\rangle + |\psi_{\rm ad}^{-}(0)\rangle]/\sqrt{2}$, that is, the initial Bloch vector reads $\boldsymbol{\alpha}_C(0) = \hat{n}_{\perp}(0)$. Under the described adiabatic evolution, the Bloch vector of the density operator $\rho(t) = |\psi(t)\rangle\langle\psi(t)|$ is obtained as

$$\boldsymbol{\alpha}_C(t) = \cos[2\phi_{\rm tot}(t)]\hat{n}_{\perp}(t) + \sin[2\phi_{\rm tot}(t)]\hat{n}_y(t), \tag{10}$$

in which

$$\begin{aligned} \hat{n}_{\perp}(t) &= \cos\theta(\cos\varphi\hat{e}_x + \sin\varphi\hat{e}_y) - \sin\theta\hat{e}_z, \\ \hat{n}_y(t) &= -\sin\varphi\hat{e}_x + \cos\varphi\hat{e}_y. \end{aligned} \tag{11}$$

It is seen that $\boldsymbol{\alpha}_C(t)$ evolves along a geodesic loop that is normal to the axis $\hat{n}_{\shortparallel}(t)$, while the latter processes adiabatically around the z axis.

Following the correspondence $\boldsymbol{\alpha}_C(t) \leftrightarrow \boldsymbol{v}(t)$, the velocity of the classical particle hence is described by a combination of the cyclotron around $\hat{n}_{\shortparallel}(t)$ with a frequency $\omega_n = \Omega/\hbar - \gamma\cos\theta$ and an adiabatic cyclic motion of its reference frame $\{\hat{n}_{\perp}(t), \hat{n}_y(t), \hat{n}_{\shortparallel}(t)\}$ around the z axis. In the laboratory frame, the velocity reads [Fig. 2(a)]

$$\begin{aligned} v_x(t) =& v(\cos\theta\cos\gamma t\cos\omega_n t - \sin\gamma t\sin\omega_n t), \\ v_y(t) =& v(\cos\theta\sin\gamma t\cos\omega_n t + \cos\gamma t\sin\omega_n t), \\ v_z(t) =& -v\sin\theta\cos\omega_n t. \end{aligned} \tag{12}$$

The bounded property of the orbit motion [see Fig. 2(b)] is easily understood since the displacement along all three directions, $r_i(t) = \int_0^t v_i(t)dt + r_i(0)$, is either periodic or quasi-periodic (depending on whether the ratio $\hbar\gamma/\Omega$ is a rational number or not). The Berry phase information of the quantum system can be extracted by observing the orbit motion of the classical particle. For example, it can be captured by the y component of the velocity in view of $v_y(0) = 0$ and $v_y(T) = v\sin[\Omega T/\hbar + 2\phi_g(T)]$, where $\phi_g(T) = -\pi\cos\theta$ accounts for the Berry phase generated during the cyclic evolution.

IV. DOUBLE-LOOP STRATEGY AND PURE GEOMETRIC PHASE SHIFT

In the cyclic evolution described above, the Berry phase is always accompanied by the dynamical phase. To enhance the scheme for detecting a pure geometric phase shift, one can employ a double-loop strategy which effectively cancels out the dynamical phase, leaving only a pure geometric phase at the ending of the evolution. Such a strategy has been exploited to achieve fault-tolerant gate operations for geometric quantum computation [6,26,27]. Specifically, the quantum system should undergo cyclic evolution twice and the evolution of the second loop is driven by a Hamiltonian

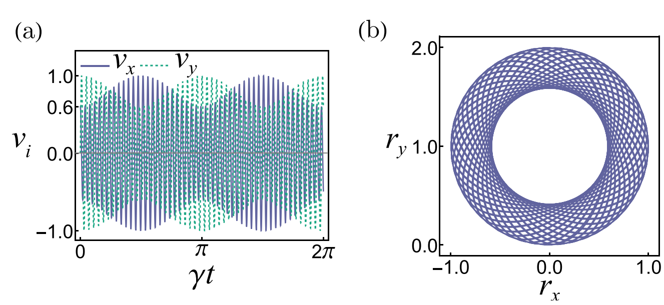

FIG. 2. The bounded orbit motion of the classical charged particle under the corresponding magnetic field, in which the initial $\boldsymbol{r}(0) = 0$, $\boldsymbol{v}(0) = \cos\theta\hat{e}_x - \sin\theta\hat{e}_z$ (being dimensionless by setting $v=1$), and the parameters are set as $\hbar\gamma/\Omega = 0.02$ and $\theta = \arccos\frac{3}{5}$. (a) Time evolution of the velocity vector specified by the components $v_x(t)$ (blue solid) and $v_y(t)$ (green dashed). (b) Planar projection of the particle's trajectory on the x-y plane, which is bounded within an annular range with the major and minor radii being approximately v/ω_n and $v\cos\theta/\omega_n$, respectively.

$H_{\bar{C}}(t) = -H_C(t)$ (see Fig. 1). Note that $H_{\bar{C}}(t)$ possesses the same instantaneous eigenstates with $H_C(t)$, $H_{\bar{C}}(t)|\psi^{\pm}_{\rm ad}(t)\rangle = \mp\frac{\Omega}{2}|\psi^{\pm}_{\rm ad}(t)\rangle$, but has opposite eigenvalues, i.e., with $+\frac{\Omega}{2}$ corresponding to $|\psi^{-}_{\rm ad}(t)\rangle$ and $-\frac{\Omega}{2}$ to $|\psi^{+}_{\rm ad}(t)\rangle$, respectively. Therefore, the dynamical phases induced during the two times loop evolution will cancel each other out. As a consequence, the entire double-loop process yields that

$$|\psi^{\pm}_{\rm ad}(0)\rangle \rightarrow |\psi^{\pm}_{\rm ad}(2T)\rangle = e^{\mp i2\phi^g(T)}|\psi^{\pm}_{\rm ad}(0)\rangle. \qquad (13)$$

In the previous proposal of the double-loop strategy for geometric quantum gate operations [26,27], the field direction is reversed suddenly at the ending of the first loop, i.e., $\mathbf{\Omega}(T) \rightarrow \mathbf{\Omega}(T+0^+) = -\mathbf{\Omega}(T)$, so that the wave function of the quantum system is kept unchanged at that moment. Nevertheless, for the present scheme, the reversal process of the field direction should be performed adiabatically, otherwise the sudden change of the counterpart magnetic field of the classical system will lead to the creation of inductive electric fields. Suppose that this embedded adiabatic process (in between the double loop evolution) is implemented by altering only the amplitude $\Omega(t)$ of the driving field [see Fig. 1(b)], e.g., from Ω to 0 during $(T, T+\tau/2)$ and from 0 to $-\Omega$ during $(T+\tau/2, T+\tau)$. It will just give rise to a null operation on the wave function if there is $\int_{T+\tau/2}^{T+\tau}\Omega(t)dt = -\int_{T}^{T+\tau/2}\Omega(t)dt$, that is, such an adiabatic change of the driving field will not affect the result of the double-loop evolution. We therefore omit the time duration $(T, T+\tau)$ in the discussion below.

Let us go on to consider the Bloch vector of the quantum system in the second loop evolution. The time evolution operator generated by $H_{\bar{C}}(t)$ is given by

$$U_{\bar{C}}(t) = \sum_{\pm} e^{\mp i\bar{\phi}_{\rm tot}(t)}|\psi^{\pm}_{\rm ad}(t)\rangle\langle\psi^{\pm}_{\rm ad}(0)|, \qquad (14)$$

in which $\bar{\phi}_{\rm tot}(t) = -\phi^d(t) + \phi^g(t)$. By applying the corresponding SO(3) group element $\mathcal{U}_{\bar{C}}(t)$ on the Bloch vector $\boldsymbol{\alpha}_{\bar{C}}(0) \equiv \boldsymbol{\alpha}_C(T)$ (the output state of the first loop evolution), one obtains the evolution over the second loop

$$\boldsymbol{\alpha}_{\bar{C}}(t) = \cos\Phi(T+t)\hat{n}_{\perp}(t) + \sin\Phi(T+t)\hat{n}_y(t), \qquad (15)$$

where $\Phi(T+t) = 2[\phi_{\rm tot}(T) + \bar{\phi}_{\rm tot}(t)]$. The final output of the velocity vector of the classical particle hence is obtained as

$$\boldsymbol{v}(2T) = v\cos[4\phi^g(T)]\hat{n}_{\perp}(0) + v\sin[4\phi^g(T)]\hat{n}_y(0). \qquad (16)$$

As a result, the Berry phase may be determined either by the relative orientation between the initial velocity vector $\boldsymbol{v}(0) = v\hat{n}_{\perp}(0)$ and the above $\boldsymbol{v}(2T)$, or by the ratio of any of the components $v_i(2T)$ to the initial velocity magnitude v.

V. DISCUSSION AND CONSLUSION

A critical issue regarding the experimental realization of the above simulation scheme is the requirement for a high charge-to-mass ratio of the dielectric particle. Assuming that the gyro frequency is measured in seconds and the magnetic induction intensity is approximately 1 Tesla, this implies that the ratio q/m of the classical particle must be around 1 C/kg, which is significantly higher than that of the Millikan oil droplet (typically within 10^{-4}–10^{-3} C/kg). Possible approaches to enhance the charge-to-mass ratio of the dielectric particles include, for example, using materials with an appropriately high dielectric constant and subjecting the particles to corona discharge at a high voltage. In addition, air damping and stochastic disturbances (resulting the Brownian motion) should be considered in a practical process. Such experimental setups may also offer a visual demonstration about the dissipative effect on the geometric phase caused by the classical damping and noise.

The Aharonov-Anandan phase [28] is known to be a nonadiabatic extension of the Berry phase. Thus it is of interest to ask whether the Aharonov-Anandan phase can be detected similarly by the above-described classical system. One possible approach to addressing this issue is to simulate the Aharonov-Anandan phase generated during parallel transport [29]. We note that a similar problem will be encountered about how to prevent or reduce the generation of an electric field since the time dependency of the classical magnetic field is also implicated in this process. Additionally, we would like to mention the Hannay angle [30], an anholonomy effect known as the classical analog of the Berry phase, and those geometric phases discovered in the non-Hamiltonian setting [31,32]. Exploring the Hannay angle of the described charged particle system and its relationship to the Berry phase detected by the currently proposed scheme would be another interesting topic.

In summary, we proposed a scheme to simulate the adiabatically driven two-level quantum system using a classical charged particle moving in a corresponding magnetic field. We revealed the analogy between the dynamics governed by the equations of motion of these two kinds of systems. The real-value simulation described in this paper hence can be understood as a physical realization of the group homomorphism SU(2)∼SO(3), not by the Bloch vector but by the velocity vector of the orbit motion of the classical particle. We demonstrated how the phase information accumulated during the cyclic adiabatic evolution of the quantum system can be captured by the motion of the classical particle with bounded orbits. Based on a double-loop scheme, we show that it is possible to detect a pure geometric phase shift through probing the velocity of the particle at the ending of the two times cyclic evolution.

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