

Symmetric multipartite Bell inequalities via Frank-Wolfe algorithms

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In multipartite Bell scenarios, we study the nonlocality robustness of the Greenberger-Horne-Zeilinger (GHZ) state. When each party performs planar measurements forming a regular polygon, we exploit the symmetry of the resulting correlation tensor to drastically accelerate the computation of (i) a Bell inequality via Frank-Wolfe algorithms and (ii) the corresponding local bound. The Bell inequalities obtained are facets of the symmetrized local polytope and they give the best-known upper bounds on the nonlocality robustness of the GHZ state for three to ten parties. Moreover, for four measurements per party, we generalize our facets and hence show, for any number of parties, an improvement on Mermin's inequality in terms of noise robustness. We also compute the detection efficiency of our inequalities and show that some give rise to the activation of nonlocality in star networks, a property that was only shown with an infinite number of measurements.

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I. INTRODUCTION

Bell nonlocality of multipartite quantum states is one of the most counterintuitive aspects of nature [1]. Measurements on entangled quantum states are indeed able to produce correlations that cannot be explained by any local classical mechanism. These nonlocal correlations can be witnessed in experiments by violating Bell inequalities [1,2]. In addition to its fundamental importance, Bell nonlocality also enables a technology based on the so-called device-independent paradigm of quantum information processing [3,4]. However, to certify the nonlocal nature of multipartite correlations and make it useful in real-life applications, one quickly runs into the problem of exponential scaling of the parameter space with the number of parties. Already for Bell inequalities with binary settings and outcomes, the number of extremal vertices and the dimension of the Bell polytope grows exponentially with the number of parties. When it comes to computing the nonlocality robustness of multipartite states, standard approaches relying on linear programming [5–7] or Frank-Wolfe algorithms [8,9] are then very quickly intractable, and hence limited to very low numbers of parties. However, symmetries, already leveraged in [10], can be advantageously used in this context to overcome these difficulties.

In this article, we exploit specific symmetries to reduce the space and iteration complexity of the Frank-Wolfe approach from [9]. We further use them to accelerate the computation of the local bounds of the resulting Bell inequalities. The symmetries considered are not ambient (they do not depend on the scenario itself as in [10]) but embedded in the specific instance we consider, namely, the multipartite GHZ state where each party performs measurements forming a regular polygon in the XY plane of the Bloch sphere. We obtain facets of the symmetrized local polytope for up to $N = 10$ parties and $m = 9$ measurements and give some consequences of these results. For instance, the case just mentioned allows us to show the activation of nonlocality in a star network, a fact that

was known only by using an infinite number of measurements [11]. Moreover, the inequalities obtained with our method for $m = 4$ appear to follow a pattern that we can generalize to all N , improving on the asymptotic results given by Mermin's inequality [12].

We introduce the relevant notions and illustrate the main symmetry reductions with a small example in Sec. II before generalizing to arbitrary scenarios in Sec. III. Then we give the main results in Sec. IV before discussing some consequences in Sec. V.

II. PRELIMINARIES

A. Notations

We consider a scenario in which N parties, labeled by $n \in [N]$, upon receiving inputs $x_n \in [m]$, give answers $a_n = \pm 1$ according to a strategy that they predefined. When repeating this for many rounds, one can construct the probabilities $p(a_1, \dots, a_N | x_1, \dots, x_N)$ corresponding to this strategy. In this work, marginals will always be zero and we work in the correlation notation [4]. If the parties have access to some shared quantum state, they may obtain correlations that cannot be explained with local means. Geometrically, in this finite scenario, this amounts to saying that these correlations are outside of the local polytope defined as the convex hull of local deterministic strategies.

Formally, for a choice of local deterministic strategies $\vec{a}^{(n)} := (a_1^{(n)} \dots a_m^{(n)}) \in \{\pm 1\}^m$ for all $n \in [N]$, the deterministic strategy $\mathbf{d}^{\vec{a}^{(1)} \dots \vec{a}^{(N)}}$ has elements

$$d_{x_1 \dots x_N}^{\vec{a}^{(1)} \dots \vec{a}^{(N)}} := \prod_{n=1}^N a_{x_n}^{(n)}, \quad (1)$$

and the local polytope is defined as

$$\mathcal{L}_N^{(m)} := \text{conv} \{ \mathbf{d}^{\vec{a}^{(1)} \dots \vec{a}^{(N)}} \mid \vec{a}^{(1)} \dots \vec{a}^{(N)} \in \{\pm 1\}^m \}. \quad (2)$$

A point inside this polytope hence admits a convex decomposition in terms of these deterministic strategies. Such a decomposition is called a local model. Conversely, points outside of the polytope may be detected by separating hyperplanes, named Bell inequalities.

Consider the following N -partite GHZ state [13]:

$$|\text{GHZ}_N\rangle := \frac{|0\dots 0\rangle + |1\dots 1\rangle}{\sqrt{2}}, \quad (3)$$

where $|0\rangle$ and $|1\rangle$ correspond to eigenvectors of σ_Z ; in other words, they are oriented vertically in the Bloch sphere. We are primarily interested in the *nonlocality robustness* of this state [14]. In general, for a density matrix ρ , this quantity is defined to be

$$v_c^\rho := \inf \{v \mid \exists m, \exists \vec{A}, v \mathbf{p}^{\rho, \vec{A}} \notin \mathcal{L}_N^{(m)}\}, \quad (4)$$

where, given the observables $\vec{A} := \{\{A_{x_n}^{(n)}\}_{x_n}\}_n$, the correlation tensor $\mathbf{p}^{\rho, \vec{A}}$ has elements

$$p_{x_1 \dots x_N}^{\rho, \vec{A}} := \text{Tr}[(A_{x_1}^{(1)} \otimes \dots \otimes A_{x_N}^{(N)})\rho], \quad (5)$$

according to Born's rule.

In this work, we will detect the nonlocality of the quantum state in Eq. (3) (so $\rho = |\text{GHZ}_N\rangle\langle\text{GHZ}_N|$) by having each party perform m measurements forming a regular polygon in the XY plane of the Bloch sphere [8], that is,

$$A_{x_n}^{(n)} := \cos\left(\frac{\pi}{m}x_n\right)\sigma_X + \sin\left(\frac{\pi}{m}x_n\right)\sigma_Y. \quad (6)$$

When doing so, it follows from Eq. (5) that the resulting correlation tensor, denoted $\mathbf{r}_N^{(m)}$, has elements

$$r_{x_1 \dots x_N}^{(m)} := \cos\left(\frac{\pi}{m}\sum_{n=1}^N x_n\right). \quad (7)$$

The main focus of this article is to compute the critical visibility

$$v_m^{\text{GHZ}_N} := \max \{v \mid v \mathbf{r}_N^{(m)} \in \mathcal{L}_N^{(m)}\}, \quad (8)$$

which is a well-defined quantity as $\mathbf{0} \in \mathcal{L}_N^{(m)}$ and $\mathcal{L}_N^{(m)}$ is compact. By definition of the nonlocality robustness in Eq. (4), one immediately has, for all m ,

$$v_c^{\text{GHZ}_N} \leq v_m^{\text{GHZ}_N}. \quad (9)$$

B. Simple example

We start with an illustration of the main symmetry reduction techniques when $N = 2$ and $m = 3$. In this case, the correlation tensor from Eq. (7) simply reads

$$\mathbf{r}_2^{(3)} = \begin{pmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -1 \\ -\frac{1}{2} & -1 & -\frac{1}{2} \end{pmatrix}. \quad (10)$$

The first step is to identify the symmetries of this matrix. They can be seen in Eq. (7) and boil down, in this simple case,

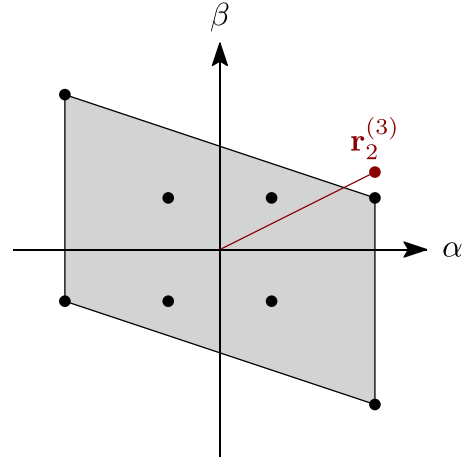


FIG. 1. The symmetrised local polytope for $N = 2$ and $m = 3$, that is, its projection on the subspace of matrices of the form of Eq. (13). The point $\mathbf{r}_2^{(3)}$ outside of the polytope is given in Eq. (10).

to the following equalities:

$$A \mathbf{r}_2^{(3)} A = \mathbf{r}_2^{(3)} \quad \text{and} \quad B \mathbf{r}_2^{(3)} B = \mathbf{r}_2^{(3)}, \quad (11)$$

where A corresponds to a cycling with a sign flip of the overflowing element and B acts as a mirror on the last two elements on top of flipping their signs, namely,

$$A := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}. \quad (12)$$

The Reynolds operator Γ associated to the action generated by A and B reads

$$\Gamma(\mathbf{p}) := \mathbf{p} + A\mathbf{p}A + A^2\mathbf{p}A^2 + B\mathbf{p}B + BA\mathbf{p}AB + AB\mathbf{p}BA$$

and projects any 3×3 matrix \mathbf{p} onto the subspace of matrices of the form

$$\begin{pmatrix} \alpha & \beta & -\beta \\ \beta & -\beta & -\alpha \\ -\beta & -\alpha & -\beta \end{pmatrix} =: \begin{bmatrix} \alpha \\ \beta \end{bmatrix}. \quad (13)$$

Now consider the local polytope projected onto this subspace; it is the convex hull of the following eight points:

$$\begin{bmatrix} -1 \\ -\frac{1}{3} \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix}, \quad (14)$$

$$\begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{3} \end{bmatrix}, \quad \begin{bmatrix} -\frac{1}{3} \\ \frac{1}{3} \end{bmatrix}, \quad \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \end{bmatrix}, \quad \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}, \quad (15)$$

and is depicted on Fig. 1. As can be seen there, only the four points of Eq. (14) are extreme points of the projected polytope. This symmetrised polytope has four facets given by the following inequalities:

$$\left\langle \pm \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \right\rangle \leq 3 \quad \text{and} \quad \left\langle \pm \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \right\rangle \leq 12. \quad (16)$$

TABLE I. First values of the sequence $\{u_m\}_m$ defined in Eq. (26).

m	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
u_m	1	2	2	4	6	10	16	30	52	94	172	316	586	1096	2048	3856	7286	13 798	26 216	25 482

Notice that the right-hand sides in Eq. (16) correspond to the scalar product in the initial nonsymmetrized space, not to the one represented in Fig. 1.

The noise robustness is the scalar v at which the point $v\mathbf{r}_2^{(3)}$ enters the local polytope. In this case it is exactly $\frac{4}{5}$ as shown by the following equality:

$$\frac{4}{5}\mathbf{r}_2^{(3)} = \frac{4}{5} \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} = \frac{9}{10} \begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix} + \frac{1}{10} \begin{bmatrix} -1 \\ 1 \end{bmatrix}. \quad (17)$$

The decomposition indeed lies precisely on the facet optimally detecting $\mathbf{r}_2^{(3)}$, that is, $\mathbf{f} := [2, 3]$.

When it comes to the computation of the local bound of this Bell inequality, namely, 12 as given in Eq. (16), the symmetry can also play a role. Formally, this bound $L_2^{(3)}$ is defined by

$$L_2^{(3)} := \max_{\mathbf{d}^{\bar{a}, \bar{b}}} \langle \mathbf{f}, \mathbf{d}^{\bar{a}, \bar{b}} \rangle = \max_{\bar{a}, \bar{b} \in \{\pm 1\}^m} \sum_{x, y \in [m]} f_{xy} a_x b_y. \quad (18)$$

Because \mathbf{f} is symmetric, i.e., it satisfies $\mathbf{f} = \Gamma(\mathbf{f})$, we have

$$\langle \mathbf{f}, \mathbf{d}^{\bar{a}, \bar{b}} \rangle = \langle \Gamma(\mathbf{f}), \mathbf{d}^{\bar{a}, \bar{b}} \rangle = \langle \mathbf{f}, \Gamma(\mathbf{d}^{\bar{a}, \bar{b}}) \rangle \quad (19)$$

since Γ is self-adjoint. This ensures that we can restrict the search to the symmetrized deterministic strategies. Actually, \bar{b} can be fixed once the strategy \bar{a} is chosen. This simplifies the computation of $L_2^{(3)}$ to

$$L_2^{(3)} = \max_{\bar{a} \in \{\pm 1\}^m} \sum_{y \in [m]} \left| \sum_{x \in [m]} f_{xy} a_x \right|, \quad (20)$$

which reduces the number of strategies to the number of orbits of a suitable action discussed below in the general case. For the simple example of this section, instead of running over the eight possible strategies \bar{a} , it suffices to consider the following two strategies:

$$\vec{a} = (-1, -1, -1) \quad \text{and} \quad \bar{a} = (-1, 1, -1), \quad (21)$$

giving rise, respectively, to

$$\vec{b} = (-1, 1, -1) \quad \text{and} \quad \bar{b} = (1, -1, 1), \quad (22)$$

which both attain the optimal value of 12 here.

Sections III A to III D generalize these symmetry arguments to any number of parties N and any number of measurements m , showing, in particular, the following reductions:

- (a) from m^N to $\lceil \frac{m}{2} \rceil$ for the dimension of the space,
- (b) from $2^{(N-1)(m-1)}$ to $\binom{u_m + N - 2}{N-1}$ for the number of strategies to enumerate, where the first terms of u_m are given in Table I.

III. METHOD

A. Group action

We define the action by how its generators transform the basis elements of the space of correlation tensors:

$$\begin{aligned} \mathbf{e} &:= e_{x_1} \otimes e_{x_2} \otimes e_{x_3} \otimes \dots \otimes e_{x_N}, \\ g_1 \cdot \mathbf{e} &:= e_{x_2} \otimes e_{x_1} \otimes e_{x_3} \otimes \dots \otimes e_{x_N}, \\ g_2 \cdot \mathbf{e} &:= e_{x_N} \otimes e_{x_1} \otimes e_{x_2} \otimes \dots \otimes e_{x_{N-1}}, \\ g_3 \cdot \mathbf{e} &:= e_{x_{1+1}} \otimes e_{x_{2-1}} \otimes e_{x_3} \otimes \dots \otimes e_{x_N}, \\ g_4 \cdot \mathbf{e} &:= e_{\bar{x}_1} \otimes e_{\bar{x}_2} \otimes e_{\bar{x}_3} \otimes \dots \otimes e_{\bar{x}_N}, \end{aligned} \quad (23)$$

where $e_m := -e_1$ and $e_{\bar{x}_n} := -e_{m-x_n}$. g_1 and g_2 generate the symmetric group permuting the parties, g_3 encodes the periodic structure of the tensor, and g_4 its reflection structure. We denote by G the group generated by g_1, g_2, g_3 , and g_4 . Note that we do not study the structure of G itself, as we are only interested in its action on tensors defined in Eq. (23).

The action defined by the generators preserves the deterministic strategies of Eq. (1). The orbits of this action on deterministic strategies will play an important role later. We can accelerate their enumeration by exploiting the invariance under permutation of parties (given by g_1 and g_2) and the fact that g_3 cycles only two parties at a time. This means that, for $N - 1$ parties, we can restrict the enumeration of the strategies to the orbits of the subgroup generated by g_3 , effectively using the last party to freely cycle the first $N - 1$ parties. This procedure gives a simple upper bound on the number of orbits

$$2^m u_m^{N-1}, \quad (24)$$

which can be refined by ordering the orbits of the first $N - 1$ parties to

$$2^m \binom{u_m + N - 2}{N - 1}, \quad (25)$$

where u_m is the number of orbits of the subgroup generated by g_3 , namely,

$$u_m := \frac{1}{2m} \sum_{2 \nmid d \mid m} \varphi(d) 2^{\frac{m}{d}}, \quad (26)$$

where this formula comes from the enumeration of binary necklaces, see the on-line encyclopedia of integer sequences [15] and references therein; see also Table I for the first terms of this sequence. Equation (25) can be slightly improved, for instance, by exploiting g_4 (which we did in practice), but it is already good enough for our needs.

Next we study the Reynolds operator associated to the generic action on tensors, that is,

$$\Gamma(\mathbf{p}) := \frac{1}{|G|} \sum_{g \in G} g \cdot \mathbf{p}. \quad (27)$$

By construction, this operator projects the correlation tensor \mathbf{p} onto a subspace of dimension $\lceil \frac{m}{2} \rceil$ such that the coordinates

TABLE II. Number of vertices of the symmetrized local polytope, that is, number of different correlation tensors of the form Eq. (28) obtained when applying the Reynolds operator Eq. (27) on all deterministic strategies defined in Eq. (1).

$m \backslash N$	3	4	5	6	7	8	9	10
2	2	3	2	3	2	3	2	3
3	10	12	14	16	18	20	22	24
4	10	21	14	29	18	37	22	45
5	60	90	126	168	216	270	330	396
6	100	249	336	657	816	1367		
7	640	1640	3740	7774	14 990			
8	1540	7889	22 008					
9	10 032	51 260						
10	30 494	340 349						
11	243 090							
12	799 980							

$\tilde{p}_{x_1 \dots x_N}$ of $\Gamma(\mathbf{p})$ satisfy

$$\tilde{p}_{x_1 \dots x_N} = \begin{cases} (-1)^{q_{\text{tot}}} \tilde{p}_{r_{\text{tot}}, 0 \dots 0} & \text{if } r_{\text{tot}} \leq \frac{m}{2}, \\ (-1)^{q_{\text{tot}}+1} \tilde{p}_{m-r_{\text{tot}}, 0 \dots 0} & \text{if } r_{\text{tot}} \geq \frac{m}{2}, \end{cases} \quad (28)$$

where q_{tot} and r_{tot} are the quotient and remainder of the Euclidean division of $\sum_n x_n$ by m . Note that, when m is even, the two conditions in Eq. (28) are simultaneously fulfilled for $r_{\text{tot}} = m/2$ so that the corresponding elements in the projected correlation tensor are 0. This justifies the dimension $\lceil \frac{m}{2} \rceil$ given above.

Note that [16], Appendix A considers a similar type of Bell inequalities in the case where $N = 2$ and the flip generator g_4 is omitted. The construction of inequalities there was done by enumeration of all matrices respecting the symmetry, which was very soon intractable as m grows. In the following, we give two methods to generate multipartite Bell inequalities satisfying the symmetry: Computing all facets of the symmetrized polytope (limited to small m) or deriving the one separating our point of interest given in Eq. (7) (via Frank-Wolfe algorithms).

B. Symmetrized local polytopes

Following [10] we first try to fully characterize the projected polytope, that is, $\Gamma(\mathcal{L})$. To do so, we can enumerate all symmetrized deterministic strategies and then compute the facets of the resulting polytope. The number of vertices found in the computationally tractable cases can be found in Table II. For all these examples, the number of facets is equal to $2^{\lceil m/2 \rceil}$. This seems to indicate that the symmetric polytope is affinely equivalent to the cross-polytope in dimension $\lceil m/2 \rceil$. Although we considered the orbits giving rise to the extreme points of this polytope and tried to infer a generalizable pattern, we could not establish this fact. We conjecture, however, that it holds in general, and hope that further research will identify these general extreme points. Note that proving this may be motivated by the observation that symmetric facets seem to be facets of the nonsymmetrized local polytope when N is odd and m even, so that obtaining them in general may directly give infinite families of facets for the local polytope.

C. Symmetric Frank-Wolfe

As m and N grow, the enumeration of all symmetrized vertices soon becomes intractable, and *a fortiori* the enumeration of all facets of the symmetrized polytope, see Table II. Since our main interest lies on a specific facet, namely, the one separating the correlation tensor in Eq. (7), we can use a different approach to get this exact facet. The most natural reformulation leads to a linear programming (LP) instance, but it becomes intractable very soon [6,7], essentially because enumerating all vertices of the (symmetrized) local polytope soon becomes impossible. This problem has been addressed using the Gilbert algorithm [17] adapted for quantum applications [8], and later reconnected to the corresponding field of constrained convex optimization working on Frank-Wolfe algorithms [9]. In this paper, we pursue this Frank-Wolfe approach, which allows us to move inside the local polytope without the need of enumerating its vertices, as only those which are relevant to our problem are explored.

More precisely, this class of first-order algorithms, also called conditional gradient methods [18], uses a linear minimization oracle (LMO) to find directions of progress within the feasible region and is quite memory efficient, which is suited for high-dimensional problems. Moreover, in our case where the feasible region is a polytope, the decomposition of the final point returned by the algorithm is usually very sparse, which makes this approach a good way of obtaining the points defining the facet of interest.

Formally, we are solving

$$\min_{\mathbf{x} \in \mathcal{L}_N^{(m)}} \frac{1}{2} \|\mathbf{x} - v_0 \mathbf{r}_N^{(m)}\|_2^2, \quad (29)$$

where the choice of the initial visibility v_0 will be discussed below (take $v_0 = 1$ to start with). The core idea of the original Frank-Wolfe algorithm [19,20] is to repeatedly move towards the minimizer of the so-called LMO, which is the linearization of our function at the current iterate \mathbf{x}_t :

$$\mathbf{v}_t := \arg \min_{\mathbf{x} \in \mathcal{L}_N^{(m)}} \langle \nabla f(\mathbf{x}_t), \mathbf{x}_t - v_0 \mathbf{r}_N^{(m)}, \mathbf{x} \rangle. \quad (30)$$

In our case, this oracle amounts to finding a deterministic strategy reaching the local bound of the Bell inequality defined by the gradient at the current iterate. We explain in more details how this is done in Sec. III D. Since this LMO can be costly, variants of this algorithm have been developed that feature a memory (active set) used to recycle previously found vertices and hence reduce the number of calls to the LMO. Moreover, for high m , we employ a heuristic LMO. We refer to [9] and references therein for more details on the method and only comment here on the choice of v_0 .

As already mentioned, we start with $v_0 = 1$, which gives a first separating hyperplane, and then use the corresponding visibility (that is, v such that $v \mathbf{r}_N^{(m)}$ lies on this hyperplane) to start again the algorithm, keeping the active set, i.e., the deterministic strategies already found. Since we are interested in facets of the symmetrized polytope, we repeat this procedure until the number of points in the active set reaches the dimension of the search space, which is conveniently quite small here.

TABLE III. Critical visibility $v_m^{\text{GHZ}_N}$ for the nonlocality of the N -partite GHZ state with each party performing m measurements forming a regular polygon on the XY plane. We also include robustnesses obtained with large number of measurements for which computing the local bound is out of reach, even with our symmetric approach, see Table VI for more values. The lower bound on $v_{XY}^{\text{GHZ}_N}$ (see Sec. VB) is, however, rigorously proven as it only requires a valid local model.

N	3	4	5	6	7	8	9	10	Comments
$2^{(1-N)/2}$	0.5	0.35355	0.25	0.17678	0.1250	0.08839	0.0625	0.04419	Mermin [12]
$m = 4$	0.5	0.35355	0.21875	0.15468	0.09375	0.06629	0.04004	0.02748	Sec. VA
m	22	20	18	16	12	11	10	9	
$v_m^{\text{GHZ}_N}$	0.49143	0.32493	0.20872	0.13422	0.08569	0.05544	0.03521	0.02301	
m	224	128	64	32	16				Putative
$v_m^{\text{GHZ}_N}$	0.49132	0.32384	0.20824	0.13327	0.08526				
$v_{XY}^{\text{GHZ}_N} \geq$	0.49129	0.32374	0.20793	0.13231	0.08243	0.05108	0.03149	0.01974	Sec. VB
v_{low}^N	0.32489	0.22335	0.15354	0.10555	0.07256	0.04989	0.03429	0.02358	Sec. VD

In our symmetric case, we can indeed navigate in the reduced space of dimension $\lceil \frac{m}{2} \rceil$, keeping in mind that the scalar product has to be properly weighted to reflect the actual geometry of the initial tensors. What allows us to perform this dimension reduction is the fact, mentioned above in Sec. III B, that symmetrized vertices can be viewed both geometrically (as the projection on the symmetric subspace) and algebraically [as the sum of the orbit of the atom found by the LMO, see Eq. (27)]. Phrased differently, this means that every atom found is virtually added to the active set together with its entire orbit, placing the same weight on all these symmetrically equivalent atoms.

Although the dimension of the space is independent of N , the complexity of the LMO depends on N so that it becomes more and more expensive as this number increases. More precisely, the heuristic approach used in [9] (alternating minimization) acts on full tensors (of size m^N) and requires more and more iterations to converge when N grows. One can naturally avoid to store these full tensors in memory, but the complexity still scales quite poorly. We nonetheless underline that this LMO is called very infrequently in the version of Frank-Wolfe that we use here (with active set [21]). Typically the total number of calls is a few times the dimension $\lceil \frac{m}{2} \rceil$, hence very reasonable; for example, in the instance with $N = 3$ and $m = 224$ given in Table III, it was called 539 times.

Moreover, as N grows, the enumeration of orbits discussed below becomes more and more competitive with respect to the alternating heuristic just mentioned. For $N \geq 8$ we systematically use this enumeration as it is faster on top of being exact.

D. Local bounds

The question of computing local bounds is notoriously complex. Already for $N = 2$, it is a quadratic unconstrained binary optimization (QUBO) instance, and the degree of the problem increases with N . One could reformulate these problems into linear ones at the expense of increasing the number of variables, but such a reformulation is out of question in practice as the resulting problems are far too big, already for $N = 2$. In this section, we explain how symmetry can significantly reduce the naive enumeration of cases.

Formally, for an N -partite tensor \mathbf{f} defining a Bell inequality, the problem reads

$$\max_{\vec{a}^{(1)} \dots \vec{a}^{(N)}} \sum_{\vec{x} \in [m]^N} f_{\vec{x}} \prod_{n=1}^N a_{x_n}^{(n)}, \tag{31}$$

where the maximization is performed for $\vec{a}^{(n)} \in \{\pm 1\}^m$, that is, over deterministic strategies.

In general, without symmetries, it suffices to enumerate $2^{(N-1)(m-1)}$ strategies. This is because the optimal strategy for the last party is fixed when the other $N - 1$ parties have chosen theirs, as follows:

$$a^{(N)} = \text{sign} \left(\sum_{\vec{x} \in [m]^{N-1}} f_{\vec{x}} \prod_{n=1}^{N-1} a_{x_n}^{(n)} \right), \tag{32}$$

where we set $\text{sign}(0) = 1$. Moreover, since we do not consider marginals here, we can fix $a_0^{(n)} = -1$ for $n \in [N - 1]$, up to flipping the signs of both $\vec{a}^{(n)}$ and $\vec{a}^{(N)}$.

With symmetries, we can further restrict to the orbits of the deterministic strategies. Since the last strategy is fixed as in the nonsymmetric case, see Eq. (32), the factor 2^m in Eq. (25) can be dropped to obtain an upper bound on the number of cases to be considered

$$\binom{u_m + N - 2}{N - 1}, \tag{33}$$

where we recall that u_m is defined in Eq. (26) and its first terms given in Table I. In practice, we used GAP [22] to obtain the orbits of interest (up to $m = 26$).

IV. RESULTS

We are now ready to present the robustnesses obtained by using the method presented in the previous sections. The evolution of the value of $v_m^{\text{GHZ}_N}$ is shown in Fig. 2 and we summarize the best values in Table III together with some values relevant for the discussion in Sec. V. All Bell inequalities and closed-form expressions of the values given here can be found in the Supplemental Material accompanying this article [23].

Let us show an elegant example to give a flavor of the kind of inequalities found by our method. We pick $N = 5$ and

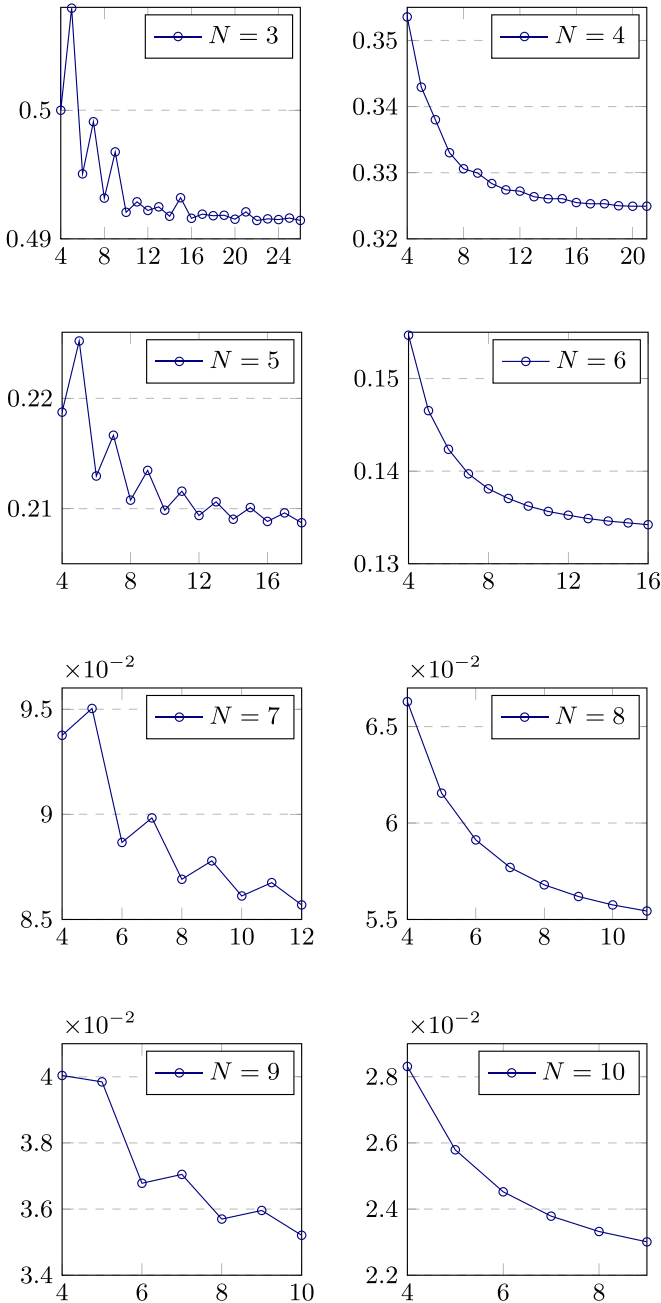


FIG. 2. Critical visibility $v_m^{\text{GHZ}_N}$ (y axis) for the nonlocality of the N -partite GHZ state with each party performing m (x axis) measurements forming a regular polygon on the XY plane. These visibilities naturally give lower bounds on the nonlocality robustness of the N -partite GHZ state, see Eqs. (4) and (9). We refer to the supplemental file for the analytical values and corresponding Bell inequalities and to Table VI for approximate values. The nonmonotonic behavior is related to the nonoptimality of the quantum strategy chosen to witness the nonlocality of the N -partite GHZ state for m measurements per party. We expect optimal strategies to enjoy similar symmetries than the ones exploited here but do not study this point in this work.

$m = 10$, so that the facet \mathbf{f} derived with our procedure and given in the Supplemental Material has its first m elements

$f_{x_1 0000}$ for $x_1 \in [m]$ being

$$[988, 0, 575, 0, -575, 0, 575, 0, -575, 0],$$

all the others being deduced by symmetry, see Eq. (28). With a slight improvement on Eq. (33) obtained by using g_4 from Eq. (23), we obtain the upper bound 242 873 on the number of orbits. By enumerating only this many cases instead of all $2^{36} \approx 7 \times 10^{10}$ strategies, we get the local bound $L_5^{(10)} = 3\,280\,000$. The quantum value reached by the GHZ state is $Q_5^{(10)} = 15\,630\,000$, which finally gives $v_{10}^{\text{GHZ}_5} = L_5^{(10)}/Q_5^{(10)} \approx 0.20985$.

One immediate comment when considering Fig. 2 is the nonmonotonicity of the curves when N is odd. This happens because the measurements used are the same on all parties, which is not optimal in terms of robustness. For instance, for $N = 3$, we could use the following asymmetric setting: the first and third parties perform the same measurements given by a regular polygon in the XY plane, and the second uses the same polygon, but rotated by an angle $\pi/(2m)$. The resulting correlation tensor also enjoys symmetries, but they are slightly different, although the very same method as the one presented in this article can be equally successfully applied.

Interestingly, the integer coefficients of the inequalities found become quite large when m increases (for instance, 14 digits for $N = 3$ and $m = 24$). Although this property is not surprising from an algebraic point of view (they are hyperplanes going through integer vertices in an increasing dimension), most inequalities studied in the literature contain small integers, and the authors of [16] also restricted its search of symmetric inequalities to such small values. Maybe a close examination of the inequalities constructed in this work (or similar ones that the authors can compute on demand) could reveal an interesting method to build general inequalities of interest.

As far as the computation time is concerned, we ran all instances on a 64-core Intel® Xeon® Gold 6338 machine and the longest one was for $N = 10$ and $m = 9$, which took ten days. For $N \leq 6$, however, the running time is more reasonable: From a few seconds for $m \leq 10$ to a few hours beyond. Typically, the case $N = 3$ and $m = 224$ took 3 hours.

V. CONSEQUENCES

A. Extension of the inequality for $m = 4$ to all N

For $m = 4$, the inequalities found with our method have a very simple form: $[1, 0, 0, 0]$ for odd N and $[0, 1, 0, -1]$ for even N , straightforwardly reaching the quantum value $Q_N^{(4)}$ of 4^{N-1} and $4^{N-1}\sqrt{2}$, respectively. However, the computation of the local bound $L_N^{(4)}$, whose first values are given in Table IV, is not as simple. In Appendix A, we show that $L_{N+2}^{(4)} = 8(L_{N+1}^{(4)} - L_N^{(4)})$ so that this local bound forms a Lucas sequence. From there its value follows:

$$L_{2n-1}^{(4)} = \frac{4^{n-1}l_n}{\sqrt{2}} \quad \text{and} \quad L_{2n}^{(4)} = \frac{4^n l_n}{\sqrt{2}}, \quad (34)$$

TABLE IV. Local bounds of the inequalities defined by [1, 0, 0, 0] for odd N and [0, 1, 0, -1] for even N , see Eq. (34).

N	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$L_N^{(4)}$	8	32	56	224	384	1536	2624	10 496	17 920	71 680	122 368	489 472	835 584	3 342 336	5 705 728

where

$$l_n := \left(1 + \frac{1}{\sqrt{2}}\right)^n - \left(1 - \frac{1}{\sqrt{2}}\right)^n. \tag{35}$$

The gist of the proof of the validity of this local bound is the following: There are only $u_4 = 2$ orbits per party, so that the total number of strategies to check is N , see Eq. (33). In this example, the best two strategies reaching the local bound are the ones in which all $N - 1$ first parties use the same deterministic strategy (out of the two orbits available). Note that, in this case, as the dimension of the space is also $\lceil \frac{m}{2} \rceil = 2$, we can extract the explicit local model of $v_4^{\text{GHZ}_N} \mathbf{r}_N^{(4)}$ as in Eq. (17).

Similar inequalities were already mentioned in [24], but the computation of the local bound, which did not use any symmetry reduction, was only done up to $N = 5$. Although the proof in Appendix A is admittedly tedious, we expect further research to simplify the argumentation and to generalize it to more measurements. A motivation to investigate in this direction is the observation that the trivial strategy is saturating all inequalities given in this work, see Sec. VB below.

B. Lower bounds on the nonlocality robustness with all projective measurements in the XY plane

In the qubit case, the visibility of $1/\sqrt{2}$ does not only correspond to the noise threshold reached by Clauser-Horne-Shimony-Holt (CHSH), but also to $1/K_G(2)$, that is, the best robustness achievable with measurements lying on a great circle of the Bloch sphere [25]. With our method we can obtain bounds on generalizations of this number: $v_m^{\text{GHZ}_N}$ can indeed be turned into a lower bound on the nonlocality threshold $v_{XY}^{\text{GHZ}_N}$ of the N -partite GHZ state under projective measurements in the XY plane. More precisely,

$$v_{XY}^{\text{GHZ}_N} \geq \left[\cos\left(\frac{\pi}{2m}\right) \right]^N v_m^{\text{GHZ}_N}, \tag{36}$$

where the trigonometric factor is the shrinking factor of the regular polygon with $2m$ vertices.

Note that valid lower bounds on $v_m^{\text{GHZ}_N}$ can safely be plugged in Eq. (36) and that our algorithm can easily produce such lower bounds. The main bottleneck to obtain $v_m^{\text{GHZ}_N}$ is the computation of the local bound, which soon becomes intractable, although we have considerably pruned the number of strategies to consider. But, obtain a lower bound v on $v_m^{\text{GHZ}_N}$, we only need an explicit local model of $v \mathbf{r}_N^{(m)}$, that is, a convex decomposition of this point in terms of deterministic strategies; this is precisely what the Frank-Wolfe algorithm produces. Moreover, given the small number of symmetrized vertices involved in the final decomposition, we can retrieve the *putative* facet of the symmetrized local polytope. This wording underlines the fact that we are unable to rigorously establish the corresponding local bound and hence the

property of indeed being a facet (and not simply a separating hyperplane containing $\lceil \frac{m}{2} \rceil$ deterministic strategies). Interestingly, as mentioned above, the strategy consisting of always answering -1 on all $N - 1$ first parties is always reaching the value given by our heuristic, which seems to indicate that the geometry of the problem could be leveraged to prove the validity of the heuristic local bound. We could not use it to our advantage and leave this question open for further work.

With the local model obtained above for $m = 4$, we can also derive bounds on $v_{XY}^{\text{GHZ}_N}$ for all N by combining Eqs. (34) and (36). Contrary to those derived with Mermin’s inequality, these bounds are higher than the entanglement threshold [26].

C. Detection efficiency

We consider the case where the parties share an N -partite GHZ state (3) and detect the particles with the same efficiency η . When the particle is not detected, the parties agree to output $+1$; this ensures that the measurements remain dichotomic. Note, however, that the parties could also use a third outcome in the case of no detection [27,28]. If all detectors fire, which happens with a probability of η^N , then the N -partite full correlation Bell inequality I can be maximally violated, that is, we have $I = Q_N^{(m)}$, the quantum value. If only some of the detectors fire, we have $I = 0$ since the inequality I contains no marginal terms. On the other hand, if no detector fires, which happens with a probability of $(1 - \eta)^N$, the local bound can be reached, i.e., we have $I = L_N^{(m)}$. Consequently, the entire data violate the inequality I if and only if

$$\eta^N Q_N^{(m)} + (1 - \eta)^N L_N^{(m)} > L_N^{(m)}, \tag{37}$$

and then dividing by $L_N^{(m)}$ to make $v_m^{\text{GHZ}_N} = L_N^{(m)} / Q_N^{(m)}$ appear, we arrive at

$$\frac{\eta_{\text{crit}}^N}{v_m^{\text{GHZ}_N}} + (1 - \eta_{\text{crit}})^N = 1, \tag{38}$$

which defines the critical detection efficiency threshold η_{crit} for N parties and m measurements.

In [29], the value of $\eta_{\text{crit}} = 0.7706$ was calculated (based on the WWWKZB inequalities, see references therein) for the special case of $N = 4$ and $m = 2$. This value is already reproduced by our inequality for $N = 4$ and $m = 4$, and we steadily improve on this critical efficiency when m increases, reaching, for $N = 4$ and $m = 19$, the value $\eta_{\text{crit}} = 0.7544$. More generally, the bank of inequalities provided in this work should be of interest as they all feature good detection efficiency with few measurements. Naturally, these critical efficiencies get better as the number of parties increases [30].

D. Activation of nonlocality in star networks

Consider a star network in which a central party shares two-qubit isotropic states with visibility v with N surrounding parties. Upon projection onto the GHZ state by this central

party, the state shared by the surrounding parties reads

$$\rho_N = v^N |\text{GHZ}_N\rangle\langle\text{GHZ}_N| + (1 - v^{N-1}) \frac{\mathbb{1}_{2^N}}{2^N} + \rho_{\text{rest}}, \quad (39)$$

where ρ_{rest} vanishes when each party performs measurements on the XY plane, so that we are effectively left with the correlations of an N -partite GHZ state with noise v^N .

In [31] this entanglement swapping procedure was used to show that nonlocality can be activated for $N \geq 21$. This means that there is a visibility v for which the initial two-qubit states are local, but where nonlocality can be demonstrated in the star network using the method described above. As a first remark, the proof used in [31] relied on the lower bound 0.6595 on the nonlocality threshold of the two-qubit isotropic state. As this bound has been improved since then to $v_{\text{low}} \approx 0.6875$ [9], the number of parties for which nonlocality activation occurs is $N = 10$. Indeed, for this number of parties we have

$$\left(\frac{2}{\pi}\right) 2^{\frac{1}{N}} < v_{\text{low}}, \quad (40)$$

which shows that the inequality from [11], with an infinite number of measurements, can be used to witness the activation of nonlocality in networks.

Our results strengthen this result though. For $N = 10$ and $m = 8$ or $m = 9$, we indeed have

$$\left(v_m^{\text{GHZ}_N}\right)^{\frac{1}{N}} < v_{\text{low}}, \quad (41)$$

which shows that the activation of nonlocality in networks can be demonstrated with a finite (and fairly small) number of measurements.

VI. CONCLUSION

In this article we studied the nonlocality robustness of the N -partite GHZ state through a specific quantum strategy where all parties perform measurements forming a regular polygon in the XY plane of the Bloch sphere. The specificity of this setting is motivated by the symmetry of the resulting correlation tensor and the quality of the corresponding critical visibilities, giving bounds on the nonlocality robustness. Studying these symmetries allows us to devise efficient ways of finding tight Bell inequalities for which the local bound can also be computed thanks to symmetries. The largest instance we solve is $N = 10$ and $m = 9$, which would feature, without symmetrization, correlation tensors with 3.5×10^9 elements and would require enumerating 4.7×10^{21} deterministic strategies; instead, we only need five elements to represent the correlation tensors and iterate over 1.4×10^8 orbits to obtain the local bound. These critical visibilities have some immediate consequences in terms of detection efficiency and activation of nonlocality in star networks, but they should essentially be seen as a proof of concept that symmetries can be leveraged in the main algorithmic ingredient of this work: Frank-Wolfe algorithms.

As the inequalities derived in this article naturally fall in the framework of [32], they can be extended to any number of outcomes following the technique therein. Studying the resulting inequalities would be interesting as it could potentially reveal experimentally relevant inequalities.

Finding other relevant symmetric cases would be a natural next step; in particular, identifying a bipartite correlation tensor featuring nice symmetries and a good robustness would be an excellent way to improve on the upper bound on the Grothendieck constant of order three. But more generally, showcasing the use of symmetrization in this context may be taken as an inspiration for all contexts in which Frank-Wolfe algorithms may play a role, e.g., entanglement detection, inflation in networks, and large-scale semi-definite programming.

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APPENDIX A: COMPUTATION OF THE LOCAL BOUND FOR $m = 4$

Here we outline the proof of the local bound in the case of an odd number of parties, that is, for the facet $[1, 0, 0, 0]$. The other case, namely, $[0, 1, 0, -1]$ for an even number of parties, can be treated similarly.

We start by defining the matrices

$$R := \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{pmatrix} = V \begin{pmatrix} \alpha_{-+} & & & \\ & \alpha_{++} & & \\ & & \alpha_{--} & \\ & & & \alpha_{+-} \end{pmatrix} V^\top \quad (A1)$$

and

$$S := \begin{pmatrix} 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \end{pmatrix} = V \begin{pmatrix} \alpha_{--} & & & \\ & \alpha_{+-} & & \\ & & \alpha_{-+} & \\ & & & \alpha_{++} \end{pmatrix} V^\top, \quad (A2)$$

which we directly diagonalized with the following eigenvalues and eigenvectors:

$$\alpha_{\pm\pm} = 1 \pm (\sqrt{2} \pm 1)i \text{ and } V = \frac{1}{2} \begin{pmatrix} \omega^{-3} & \omega^3 & \omega^{-1} & \omega \\ -i & i & i & -i \\ \omega^{-1} & \omega & \omega^{-3} & \omega^3 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad (A3)$$

where $\omega = e^{i\pi/4}$. Note that V is, up to trivial operations, a Vandermonde matrix.

The reason to introduce these matrices is that

$$L_{i,j} := \|R^i S^j\|_1 \quad (A4)$$

TABLE V. Values of L_{ij} for small i and j . The antidiagonal in bold corresponds to $N = i + j + 1 = 9$ and reaches its highest value when $i = N - 1$ or $j = N - 1$, that is, when the deterministic strategies chosen by all parties coincide.

$i \backslash j$	0	1	2	3	4	5	6	7	8	9	10
0	1	4	8	24	56	160	384	1088	2624	7424	17920
1	4	4	8	24	64	160	448	1088	3072	7424	20992
2	8	8	8	32	64	192	448	1280	3072	8704	20992
3	24	24	32	32	64	192	512	1280	3584	8704	24576
4	56	64	64	64	64	256	512	1536	3584	10240	24576
5	160	160	192	192	256	256	512	1536	4096	10240	28672
6	384	448	448	512	512	512	512	2048	4096	12288	28672
7	1088	1088	1280	1280	1536	1536	2048	2048	4096	12288	32768
8	2624	3072	3072	3584	3584	4096	4096	4096	4096	16384	32768
9	7424	7424	8704	8704	10240	10240	12288	12288	16384	16384	32768
10	17920	20992	20992	24576	24576	28672	28672	32768	32768	32768	32768

is exactly the value attained in a scenario with $i + j + 1$ parties, when i of them choose the trivial strategy $(1, 1, 1, 1)$ and j of them the strategy $(1, 1, -1, 1)$. Because $m = 4$ measurements, these two possibilities are the only two orbits. Note that the order of the parties does not matter thanks to the symmetries, which can be seen in the commutation of R and S . Actually, only the first column of $R^i S^j$ truly matters in the definition of $L_{i,j}$ since the Bell inequality considered here is $[1, 0, 0, 0]$, but, normalizing the 1-norm (so that it is submultiplicative) and noting that all columns have the same 1-norm by symmetry, Eq. (A4) is equivalent to this.

Having diagonalized the matrices R and S allows us to explicitly derive the expression of $L_{i,j}$:

$$L_{i,j} = \frac{1}{4}(|a_{ij} + b_{ij} + c_{ij} + d_{ij}| + |a_{ij} - b_{ij} - c_{ij} + d_{ij}| + |a_{ij} + ib_{ij} - ic_{ij} - d_{ij}| + |a_{ij} - ib_{ij} + ic_{ij} - d_{ij}|) \tag{A5}$$

where we defined

$$\begin{aligned} a_{ij} &:= \alpha_{-+}^i \alpha_{--}^j, & b_{ij} &:= \alpha_{++}^i \alpha_{+-}^j, \\ c_{ij} &:= \alpha_{--}^i \alpha_{-+}^j, & d_{ij} &:= \alpha_{+-}^i \alpha_{++}^j. \end{aligned} \tag{A6}$$

We give some values of $L_{i,j}$ in Table V.

To prove the validity of the local bound, we have to show that the pattern emphasized in Table V for $N = 9$ generalizes, that is, that all antidiagonals of this array grow from the diagonal to the edges. As the form in Eq. (A5) is not very practical for this purpose, we derive inductive formulas from there.

First, it is clear that

$$R^2 S^2 = 8 \cdot \mathbb{1} \quad \text{gives} \quad L_{i+2,j+2} = 8L_{i,j}. \tag{A7}$$

Second, we can derive the explicit phases involved in the moduli in Eq. (A5) by manually showing that the pattern has a period of eight (in i or j), which, in turn, allows us, thanks to the newly linearized equation, to derive the following relation:

$$L_{i,j+4} = 8(L_{i,j+2} - L_{i,j}), \tag{A8}$$

which we have to prove separately for odd and even j and for $i = 0$ and $i = 1$, the other values following from Eq. (A7). With Eqs. (A7) and (A8) at hand, a simple induction suffices

to prove the desired result, namely, that the antidiagonals of Table V reach their maximum on the edges.

Given the tedium of this proof, it would be interesting for further work to simplify it by connecting it with the abundant literature on generalized Fibonacci numbers and identities involving multinomial coefficients, from which we consulted the authors of [33,34], just to mention a few. We were unsuccessful in our attempts to do so but we expect elegant methods to easily solve this small case and to generalize to more measurements. To this end, note that the matrix from Eq. (A1) can be written

$$R = \mathbb{1} + A + A^2 + A^3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ A \\ A^2 \\ A^3 \end{pmatrix}$$

and the one from Eq. (A2)

$$S = \mathbb{1} + A - A^2 + A^3 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ A \\ A^2 \\ A^3 \end{pmatrix}$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

The sign pattern is precisely the one of the underlying orbit. These observations allow to write, for instance, R^i as a combinatorial sum instead of the analytic expression from Eq. (A5), namely,

$$\begin{aligned} R^i &= \sum_{k_0+k_1+k_2+k_3=i} \binom{i}{k_0, k_1, k_2, k_3} \mathbb{1}^{k_0} A^{k_1} A^{2k_2} A^{3k_3} \\ &= \sum_{k_0+k_1+k_2+k_3=i} \frac{i!}{k_0!k_1!k_2!k_3!} A^{k_1+2k_2+3k_3}, \end{aligned} \tag{A9}$$

which can be simplified further given that $A^4 = -\mathbb{1}$.

APPENDIX B: FULL TABLE OF RESULTS

Here is a full table of all results obtained in this work.

TABLE VI. Critical visibility $v_m^{\text{GHZ}_N}$ for the nonlocality of the N -partite GHZ state with each party performing m measurements forming a regular polygon on the XY plane. Each value is obtained analytically by finding the facet of the symmetrized local polytope that optimally detects $\mathbf{r}_N^{(m)}$ from Eq. (7). The local bound is rigorously established only for the bold values; the other ones rely on a heuristic, although the trivial deterministic strategy always saturates it so that there might be some symmetry arguments enabling to generalize the proof from Appendix A. Note that, for $m = 12$, the computation of the local bound by enumeration is barely possible for $N = 4$ (2^{44} cases) whereas we can reach $N = 7$ with symmetries ($25\,879\,574\,945 < 2^{35}$ cases).

$m \backslash N$	3	4	5	6	7	8	9	10
2	0.5	0.5	0.25	0.25	0.125	0.125	0.0625	0.0625
3	0.57143	0.39024	0.26230	0.17534	0.11700	0.07802	0.05202	0.03426
4	0.5	0.35355	0.21875	0.15468	0.09375	0.06629	0.04004	0.02748
5	0.50794	0.34293	0.22521	0.14653	0.09503	0.06155	0.03985	0.02521
6	0.49505	0.33800	0.21296	0.14236	0.08866	0.05913	0.03678	0.02452
7	0.49911	0.33302	0.21666	0.13971	0.08983	0.05770	0.03705	0.02379
8	0.49317	0.33059	0.21077	0.13809	0.08691	0.05680	0.03570	0.02332
9	0.49676	0.32994	0.21347	0.13705	0.08779	0.05619	0.03596	0.02301
10	0.49206	0.32836	0.20985	0.13623	0.08612	0.05575	0.03521	
11	0.49288	0.32740	0.21158	0.13565	0.08675	0.05544		
12	0.49221	0.32721	0.20938	0.13524	0.08569			
13	0.49249	0.32637	0.21062	0.13488				
14	0.49176	0.32606	0.20904	0.13461				
15	0.49320	0.32607	0.21010	0.13441				
16	0.49160	0.32548	0.20884	0.13422	0.08526			
17	0.49192	0.32530	0.20961					
18	0.49180	0.32531	0.20872					
19	0.49184	0.32500	0.20933					
20	0.49153	0.32493	0.20861					
21	0.49210	0.32494	0.20916					
22	0.49143	0.32471	0.20854					
23	0.49155	0.32463	0.20897					
24	0.49150	0.32465	0.20849	0.13352				
25	0.49162							
26	0.49143							
27	0.49162							
28	0.49141							
29	0.49147							
30	0.49145							
31	0.49146							
32	0.49138	0.32423	0.20836	0.13327				
64	0.49134	0.32391	0.20824					
96	0.49133	0.32386						
128	0.49133	0.32384						
224	0.49132							

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