# Landau-Zener formula for the adiabatic gauge potential

Gabriel Cardoso [*]
*Tsung-Dao Lee Institute, Shanghai Jiao Tong University, Shanghai 201210, China*

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We study the stability of counterdiabatic drive by computing the joint dependence of the probability of nonadiabatic transitions $P_\eta(\delta)$ on the adiabatic parameter $\delta$ and on the normalized amplitude of the adiabatic gauge potential (AGP) $\eta$ in the Landau-Zener-Stückelberg-Majorana model. We show that the Dykhne-Davis-Pechukas formula cannot be readily applied since the AGP introduces a singularity in the Hamiltonian which makes the wave function multivalued in the complex-time plane. This can be understood as the non-Abelian Aharonov-Bohm phase introduced by the AGP and leads to the counterdiabatic correction of the Landau-Zener formula. In particular, it shows that, unlike the nonperturbative suppression of transitions in the adiabatic limit $\delta \to 0$, the probability is only perturbatively suppressed in the counterdiabatic limit $\eta \to 1$. We then consider the extension of our results to integrable time-dependent quantum Hamiltonians. We prove that the AGP satisfies the flatness constraint which characterizes integrability in these models, which allows us to derive simple expressions for the AGP and the probability of transitions $P_\eta(\delta)$ near adiabatic or counterdiabatic evolution in three- and four-state integrable examples.

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## I. INTRODUCTION

Motivated by a variety of applications in quantum technology, interest in control protocols which effectively realize adiabatic evolution in time-dependent quantum systems has increased recently [1–10]. In counterdiabatic driving, one modifies the Hamiltonian by an extra term which is engineered to minimize the probability of nonadiabatic transitions [11–14]. This term is known as the adiabatic gauge potential (AGP) since it is the gauge generator of unitary rotations onto the time-dependent eigenbasis [15–17]. The stability of this method against perturbations has been considered both theoretically and experimentally in different contexts [18–25], but naturally, it is a hard question to answer with any degree of generality due to the wealth of possible Hamiltonian perturbations.

We approach this problem in a manner inspired by the famous Landau-Zener (LZ) formula. In the Landau-Zener-Stückelberg-Majorana (LZSM) model [26–30], the deviations of the Hamiltonian from the adiabatic regime are parametrized by a single adiabatic parameter $\delta$, and the probability of nonadiabatic transitions is found to vanish nonperturbatively fast in the adiabatic limit $\delta \to 0$ as $P(\delta) = e^{-\pi/\delta}$, the LZ formula. In this sense, the adiabatic regime is nonperturbatively stable against Hamiltonian perturbations parametrized by $\delta$. Subsequent generalizations led to the Dykhne-Davis-Pechukas (DDP) formula [31–33], which considers the analytic continuation of the Hamiltonian to complex time and computes the exponent in $P(\delta)$ from the singularities of the eigenstates which lie closest to the real axis, with corrections given by a power series in $\delta$ [34–36].

We show that one can similarly quantify the stability of the counterdiabatic regime in terms of the dependence of the probability of nonadiabatic transitions $P_\eta(\delta)$ on a suitably defined counterdiabatic parameter $\eta$. Here, $\eta$ is the amplitude of the AGP, normalized so that $\eta = 0$ is the original drive and $\eta = 1$ is the counterdiabatic drive. For small but fixed $\delta$, $P_\eta(\delta)$ vanishes perturbatively as $(\eta - 1)^2$ in the counterdiabatic limit $\eta \to 1$, so that the counterdiabatic regime is only perturbatively stable against such perturbations. In terms of the small $\delta$ expansion of $\log[P_\eta(\delta)]$, we find that the LZ term, of order $\delta^{-1}$, is not modified. Instead, the presence of the AGP changes the terms at the next order $\delta^0$, leading to a modification of the LZ formula by a universal prefactor. We also confirm these findings by comparing our formula with the numerical solution of the time-dependent Schrödinger equation (TDSE).

Our calculation is based on a generalization of the DDP formula. While application of the DDP formula assumes that the Hamiltonian extends analytically to a strip of the complex-time plane which includes the leading singularities of the eigenvectors, we find that singularities of the Hamiltonian itself are intrinsic to the AGP and to the mechanism behind counterdiabatic driving. Our analysis shows that the singularity of the AGP leads to relative phases between different paths in the complex-time plane, which suppresses nonadiabatic transitions by destructive Aharonov-Bohm interference [37].

In an independent recent development, the exact solvability of a number of models was understood to be due to an underlying integrability structure that allows them to be factorized into a sequence of LZSM problems [38–43]. In particular, in these models, the condition for adiabaticity can be quantified in terms of a few $\delta$ parameters corresponding to the different pairwise crossings. We show that counterdiabatic driving of these models is also special. Namely, we prove that the

[*]gabriel.jg.cardoso@outlook.com

 

integrability condition which leads to the factorizability extends to the AGP. This leads to simplifications in computing the AGP and in evaluating the probability of transitions $P_\eta(\delta)$ close to the adiabatic or counterdiabatic regime, as we illustrate with three- and four-level examples.

This paper is organized as follows: Sec. II reviews the LZ formula and the method of counterdiabatic driving. Section III reviews the derivation of the DDP formula from a more general result: the adiabatic theorem in the complex plane. In Sec. IV, this method is applied to the LZSM model with AGP, leading to the AGP phase and the modified Landau-Zener formula $P_\eta(\delta)$. Section V discusses the extension of the integrability condition to the AGP, with the three-state bow-tie model as an explicit example. In Sec. VI the results from Secs. IV and V are combined so that we can evaluate the transition probability $P_\eta(\delta)$ in an integrable four-state model. Section VII includes a discussion of the results and interesting future directions. Three Appendixes are included for additional computational details.

## II. TRANSITION PROBABILITY

The problem we consider is as follows. Let $H(t)$ be a nondegenerate, time-dependent, two-level Hamiltonian with a gap of order $\mathcal{E}$ which changes over a timescale $T$. It is then interesting to rewrite the TDSE in the dimensionless form

$$i\delta\partial_\tau|\psi\rangle = h|\psi\rangle, \tag{1}$$

where $\tau = t/T$ is the dimensionless "slow time," $h(\tau) = \frac{1}{\mathcal{E}}H$, and

$$\delta = \frac{\hbar}{\mathcal{E}T} \tag{2}$$

is the adiabatic parameter. The instantaneous eigenvalues $e_n$ and eigenstates $|n\rangle$ are given by

$$h(\tau)|n(\tau)\rangle = e_n(\tau)|n(\tau)\rangle, \;\; n = 0, 1. \tag{3}$$

We consider the solution of (1) in the form

$$|\psi(\tau)\rangle = \sum_n c_n(\tau)e^{-\frac{i}{\delta}\int_0^\tau e_n - i\int_0^\tau a_n}|n(\tau)\rangle, \tag{4}$$

which starts from the ground state,

$$\lim_{\tau\to-\infty} c_n(\tau) = \delta_{n,0}. \tag{5}$$

Here, we also defined the Berry connection,

$$a_n = -i\langle n|\partial_\tau|n\rangle. \tag{6}$$

Then the probability of nonadiabatic transitions is

$$P \equiv \lim_{\tau\to+\infty} |c_1(\tau)|^2. \tag{7}$$

We can compute $P$ for Hamiltonians with different parameters. Then the adiabatic theorem says that its dependence on the adiabatic parameter $P(\delta)$ is such that

$$P \xrightarrow[\delta\to 0]{} 0, \tag{8}$$

and the LZ formula gives the leading behavior of $P(\delta)$ as we deviate away from this limit,

$$P \sim e^{-\pi/\delta}. \tag{9}$$

This formula is quite generic and has been applied to many different systems. It is often interpreted in terms of the robustness of the adiabatic limit: for small but nonzero $\delta$, the probability is nonperturbatively small. For the LZSM model we discuss below it is actually exact, although in this paper we are concerned with the small-$\delta$ regime. We also note that the $\pi$ value of the coefficient in the exponent corresponds to a particular choice of the $\delta$ parameter in the LZSM Hamiltonian [Eqs. (16) and (17)]. The probability is, of course, independent of this choice, which will become clear from the DDP formula discussed in the next section.

Importantly, various exceptions, to the adiabatic theorem in general and to the LZ formula in particular, exist. In many cases, the physics of these exceptions can be traced back to the existence of additional Hamiltonian parameters that introduce time and energy scales not accounted for by the adiabatic parameter $\delta$ [44]. For example, by repeatedly driving a system through LZSM transitions with certain periods, one can resonantly populate the excited state [45]. The resulting deviations from the intuitive picture above can range from an effective shift of the adiabatic parameter [46] to essential deviations from the DDP formula, for example, when the drive is strongly nonlinear [47]. In this sense, the LZ formula should be thought of as (1) a paradigm of adiabatic stability, on the basis of which exceptions can be analyzed, and (2) a building block which one can use to construct toy models with richer dynamics.

On the other hand, suppose the eigenstates (3) depend on time through the parameter $\theta$, and let $U(\theta)$ be the corresponding unitary operator which diagonalizes the Hamiltonian. Then the AGP is

$$\mathcal{A}_\theta = -i\delta U\partial_\theta U^\dagger, \tag{10}$$

and it is easy to check by direct computation that the original eigenstates (3),

$$e^{-\frac{i}{\delta}\int_0^\tau e_n - i\int_0^\tau \dot{\theta} a_n}|n(\tau)\rangle, \tag{11}$$

where $\theta = \theta(\tau)$, are exact solutions of the TDSE with the new Hamiltonian

$$h(\theta) + \dot{\theta}\mathcal{A}_\theta. \tag{12}$$

In other words, adding the AGP to the Hamiltonian replaces (9) by

$$P = 0. \tag{13}$$

We investigate the stability of this exact suppression of nonadiabatic transitions by calculating $P_\eta(\delta)$, the probability of transitions for the one-parameter family of Hamiltonians

$$h + \eta\dot{\theta}\mathcal{A}_\theta, \tag{14}$$

where the strength $\eta$ of the AGP term tunes the Hamiltonian from the original drive at $\eta = 0$ to the counterdiabatic drive at $\eta = 1$. We find that, for small $\delta$,

$$P_\eta \sim \cos^2\frac{\eta\pi}{2}e^{-2\eta/3}e^{-\pi/\delta}. \tag{15}$$

Interestingly, the usual LZ exponent of order $1/\delta$ is unchanged. Instead, the AGP suppresses this estimate by a $\delta$-independent prefactor which vanishes perturbatively as $(\eta - 1)^2$ in the counterdiabatic limit $\eta \to 1$.

### III. THE DDP METHOD

Note that formulas (9) and (15) vanish in all orders of perturbation theory in the parameter $\delta$. To capture the nonperturbative result, we proceed using the method of Dykhne, Davis, and Pechukas, which makes use of the analytic properties of the Hamiltonian. Since, as we will see, the singularity of the AGP does not allow us to immediately apply the DDP formula, we first review how to derive it from a more elementary result in the familiar LZSM case (for more details, see [33]). The LZSM Hamiltonian is

$$H = \begin{pmatrix} at & b \\ b & -at \end{pmatrix}. \tag{16}$$

The eigenvalues $\pm\sqrt{a^2t^2+b^2}$ have an avoided crossing at $t = 0$, with minimal gap $2b$. Thus, it is natural to take $\mathcal{E} = b$ and $T = b/a$, so that the TDSE takes the form (1) with

$$h = e(\tau)[\sin\theta\sigma^x + \cos\theta\sigma^z], \quad \delta = \frac{\hbar a}{b^2}, \tag{17}$$

where

$$e(\tau) = \sqrt{1+\tau^2}, \;\tan\theta = \frac{1}{\tau}. \tag{18}$$

The instantaneous eigenstates are

$$|0\rangle = \begin{pmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}, \; |1\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}, \; e_{0,1} = \mp e(\tau), \tag{19}$$

and we consider the decomposition of the solution in terms of the eigenstates as in (4). Then the coefficients $c_n(\tau)$ satisfy the equations

$$\dot{c}_0 = p_{01} e^{\frac{i}{\delta}\int_0^\tau (e_0 - e_1) + i\int_0^\tau (a_0 - a_1)} c_1(\tau), \tag{20}$$

$$\dot{c}_1 = p_{10} e^{\frac{i}{\delta}\int_0^\tau (e_1 - e_0) + i\int_0^\tau (a_1 - a_0)} c_0(\tau), \tag{21}$$

with the anti-Hermitian coefficients $p_{nm} = -\langle n|\partial_\tau|m\rangle$. We consider the solution with initial condition (5).

The essence of the method lies in extending these equations to complex $\tau$. Since the Hamiltonian (17) is analytic in the complex plane, the solution of the TDSE (1) is well defined everywhere [48]. However, a subtlety appears when writing the equations for the amplitudes $c_n$ (20,21): although the solution $|\psi(\tau)\rangle$ is single valued, the eigenstates are not. Indeed, the eigenvalues $\pm e(\tau) = \pm\sqrt{1+\tau^2}$ have square-root branch points at the complex degeneracies $\pm\tau^*$, where

$$\tau^* = i, \tag{22}$$

so that going around one of these points amounts to swapping the two eigenstates. Therefore, the decomposition (4) of the solution of the TDSE only makes sense with respect to a specific curve, which we will keep track of by using superscripts: $c_n^{A,B,\dots}$ will denote the amplitudes of $|\psi(\tau)\rangle$ in terms of the eigenstates defined continuously along the curve $\gamma^{A,B,\dots}$. Additionally, since we are ultimately interested in the amplitudes $c_n(\tau)$ for real $\tau$, we introduce branch cuts from $\pm\tau^*$ to infinity for when we extend the eigenstates continuously from the real axis as in Fig. 1. With this definition, the labeling of adiabatic amplitudes $c_n^B$ on a given curve $\gamma^B$ coincides with the one on the real axis $\gamma^A$ if the curve does not cross the branch cut and gets relatively swapped if the curve crosses it. The LZ formula follows from comparing the amplitudes corresponding to different curves in the adiabatic limit.

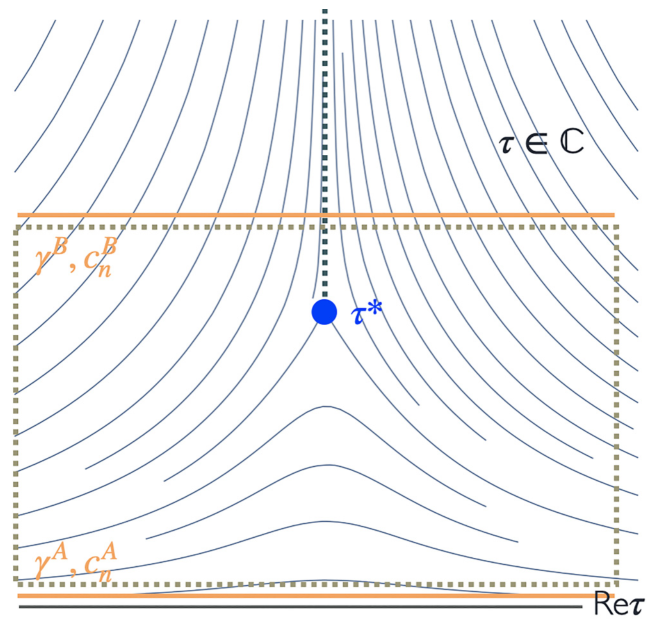


FIG. 1. Level lines of the function $\Delta^A(\tau)$ (23) for the eigenvalues of the LZSM model. There is a square-root branch point $\tau^*$, from which an extra level line extends vertically. We place the branch cut on top of this extra level line. Then we can see that both the real axis $\gamma^A$ and the curve $\gamma^B$ satisfy the condition of nonincreasing $\Delta(\tau)$ for the adiabatic theorem.

In the limit $\delta \to 0$, Eqs. (20) and (21) are dominated by the dynamical phase factors, and this leads to an important result. Let $\gamma^A(s)$ be a curve on the complex plane such that $\mathrm{Re}[\gamma^A(s)] \xrightarrow[s\to\pm\infty]{} \pm\infty$. If, on $\gamma^A$, the function

$$\Delta^A(\tau) = \mathrm{Im}\left[\int_0^{\tau=\gamma^A(s)} \left(e_0^A - e_1^A\right) ds'\right] \tag{23}$$

is nonincreasing, then it can be shown that the adiabatic theorem is valid, that is,

$$c_n(\gamma^A(s)) \xrightarrow[s\to+\infty]{\delta\to 0} \delta_{n0}. \tag{24}$$

This result includes the adiabatic theorem since $\Delta^A(\tau)$ vanishes when the real axis is taken as the path $\gamma^A$, but the point is that there are other curves to which it also applies, thus earning it the name "adiabatic theorem in the complex plane." In Fig. 1, we plot the level lines of the function $\Delta(\tau)$, which is defined by analytic continuation of $\Delta^A(\tau)$ evaluated on the real axis, with the branch cut as shown. It increases away from the real axis, and the level line splits into three at the branch point $\tau^*$. Consider now the curve $\gamma^B$ in Fig. 1. To the left of the branch cut, it traverses the decreasing level lines, while to the right it traverses the increasing level lines. But since the labeling of eigenstates is swapped as the curve crosses the branch cut, $\Delta^B(\tau)$, which is defined analogously to (23), is also nonincreasing on all of $\gamma^B$.

We compare the eigenstate decompositions of $|\psi(\tau)\rangle$ on the real axis $\gamma^A$ and on the curve $\gamma^B$. Since, to the left of the cut, the labels of the eigenstates agree, the initial condition becomes

$$c_n^A(\tau) \xrightarrow[\tau\to-\infty]{} \delta_{n0}, \tag{25}$$

$$c_n^B(\tau) \xrightarrow[\tau\to-\infty]{} \delta_{n0}. \tag{26}$$

And since in both $\gamma^A$ and $\gamma^B$ the adiabatic theorem is satisfied, we have that

$$c_n^A(\tau) \xrightarrow[\tau\to+\infty]{\delta\to 0} \delta_{n0}, \tag{27}$$

$$c_n^B(\tau) \xrightarrow[\tau\to+\infty]{\delta\to 0} \delta_{n0}. \tag{28}$$

However, due to the relative crossing of the branch point, the adiabatic-limit answer (28) gives the leading correction to the adiabatic limit in (27) by matching the coefficients of the wave function. With a slight abuse of notation,

$$\begin{aligned}|\psi(+\infty)\rangle &= e^{-i\int_0^{\gamma^B(\infty)}\left(\frac{e_0}{\delta}-a_0\right)}|0^B(+\infty)\rangle \\ &= \sum_n c_n^A(+\infty)e^{-i\int_0^{\gamma^A(\infty)}\left(\frac{e_n}{\delta}-a_n\right)}|n^A(+\infty)\rangle. \end{aligned} \tag{29}$$

Since $|0^B(+\infty)\rangle = |1^A(+\infty)\rangle$, matching this coefficient gives

$$c_1^A(+\infty) = e^{i\int_0^{\gamma^A(\infty)}\left(\frac{e_1^A}{\delta}-a_1^A\right)}e^{-i\int_0^{\gamma^B(\infty)}\left(\frac{e_0^B}{\delta}-a_0^B\right)} \tag{30}$$

$$= e^{-\frac{i}{\delta}\int_0^{\tau^*}(e_0-e_1)}, \tag{31}$$

where in the second line we deformed the integration contour, represented by the dotted line in Fig. 1 [49]. Because the LZSM Hamiltonian (17) is purely real, the Berry phase factor vanishes. The result is Dykhne's formula: the probability of transitions is fixed by the phase integral of the gap between the eigenstates up to the closest branch point,

$$P = \left|c_1^A(+\infty)\right|^2 \sim e^{-\frac{2}{\delta}\mathrm{Im}\left[\int_0^{\tau^*}(e_1-e_0)\right]}. \tag{32}$$

Using the explicit eigenvalues of the LZSM model, we find

$$P \sim e^{-\frac{4}{\delta}\mathrm{Im}\left[\int_0^{\tau^*}\sqrt{1+\tau^2}\right]} = e^{-\pi/\delta}, \tag{33}$$

which is the LZ formula. The more general contour integral (30), known as the DDP formula, can account for a general distribution of branch-point singularities, and it is covariant with respect to different choices of the $\delta$ parameter. Different applications, extensions, and limitations of the DDP formula have been studied in the literature [47,50–53].

## IV. THE AGP PHASE

Let us now see how this calculation is changed by the AGP. The LZSM Hamiltonian (17) is diagonalized by

$$U(\theta) = \begin{pmatrix} -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \\ \cos\frac{\theta}{2} & \sin\frac{\theta}{2}\end{pmatrix} = e^{-i\frac{\theta}{2}\sigma^y}\sigma^x, \tag{34}$$

with time dependence $\theta = \theta(\tau)$ from (18), so that the AGP takes on the form [11]

$$\dot{\theta}\mathcal{A}_\theta = -\frac{\delta}{2(1+\tau^2)}\sigma^y. \tag{35}$$

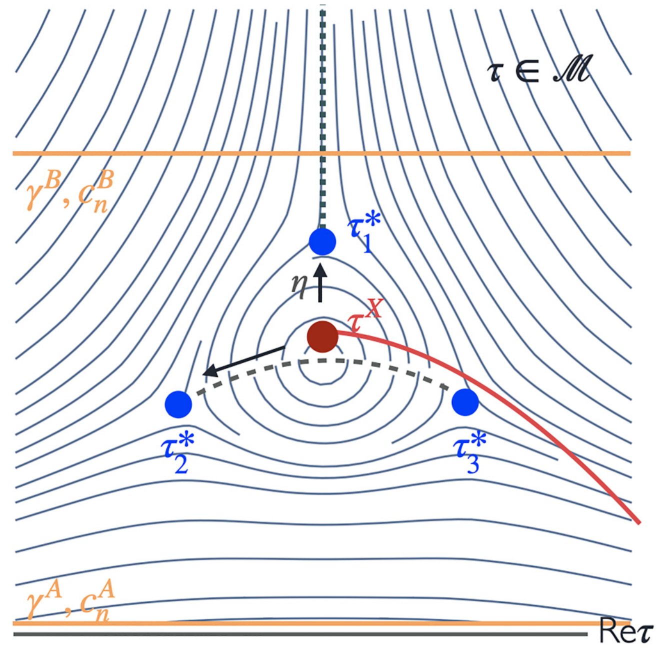


FIG. 2. Level lines of the function $\Delta^A(\tau)$ (23) for the eigenvalues of the counterdiabatic LZSM model (14). We introduce a Dirac string extending from the singularity of the Hamiltonian at $\tau^X$ (red). In the upper half of the complement of the Dirac string $\mathcal{M}^+$, there are three branch points, from which we define the branch cuts as above: one extending up from $\tau_1^*$ along the vertical level line and one connecting $\tau_2^*$ to $\tau_3^*$. Then we can see that both the real axis $\gamma^A$ and the curve $\gamma^B$ satisfy the condition of nonincreasing $\Delta(\tau)$ for the adiabatic theorem.

Thus, the Hamiltonian (14), parametrized by $\eta \in [0, 1]$, is given by

$$h(\eta) = e(\tau)[\sin\theta\sigma^x + \cos\theta\sigma^z] + \eta\dot{\theta}\mathcal{A}_\theta \tag{36}$$

$$= \sigma^x + \tau\sigma^z - \frac{\eta\delta}{2(1+\tau^2)}\sigma^y. \tag{37}$$

We see that, while the LZSM Hamiltonian is analytic everywhere, the presence of the AGP leads to poles at $\pm\tau^X$, where

$$\tau^X = i. \tag{38}$$

In particular, this means that the solution of the TDSE (1) is not well defined everywhere. In fact, in the punctured plane $\mathbb{C}\setminus\{\tau^X, -\tau^X\}$, it is multivalued, and the DDP formula cannot be applied.

A simple work-around for dealing with singular gauge fields that have nontrivial holonomy is known from the theory of Dirac's magnetic monopole [54]: we make the domain of the TDSE simply connected by introducing a Dirac string extending from each pole to infinity (represented by the red line in Fig. 2). Then, on the complement of the Dirac string $\mathcal{M}$, the solution is well defined and single valued. For simplicity, we take the strings to become parallel to and very close to the real axis as $\mathrm{Re}[\tau] \to +\infty$.

On the other hand, the eigenvalues of the new Hamiltonian (37) are

$$e_{0,1} = \mp \frac{1}{1+\tau^2}\sqrt{(1+\tau^2)^3 + \frac{\eta^2\delta^2}{4}}, \tag{39}$$

which vanish at the six square-root branch points $\pm\tau^*_{1,2,3}$ at

$$\tau^*_{1,2,3} = i\left[1 + \left(\frac{\eta\delta}{2}\right)^{\frac{2}{3}} e^{i\frac{2\pi}{3}k}\right]^{\frac{1}{2}}, \tag{40}$$

with $k = 0, 1, 2$. Compare Figs. 1 and 2. For $\eta > 0$, there are three branch points $\tau^*_{1,2,3}$ surrounding each pole of the Hamiltonian $\tau^X$, while for $\eta \to 0$ all of these collapse to the single branch point of the LZSM model.

Again, we find that the expansion (4) of the wave function is not well defined in the whole $\mathcal{M}$. Thus, we extend the labeling of eigenstates from the real time axis to the upper half-plane sector $\mathcal{M}^+$ by introducing branch cuts as in Fig. 2: one cut extending from $\tau^*_1$ to infinity along the imaginary axis and one connecting $\tau^*_2$ to $\tau^*_3$. The level lines of $\Delta(\tau)$ are plotted in Fig. 2. Outside the region occupied by the singular points, they look similar to the LZSM case, so we can easily check that the adiabatic theorem applies to the curve $\gamma^B$, and we have Eqs. (25)–(28). However, since the curves $\gamma^A$ and $\gamma^B$ are separated by the Dirac string, the two expansions cannot be immediately matched at $\mathrm{Re}[\tau] \to +\infty$. Indeed, the wave function itself is not continuous upon jumping across the string, and the discontinuity is found by integrating the TDSE on a path going around the Dirac string. Recalling that we take the string to lie arbitrarily close to the real axis, we find the matching condition

$$|\psi(+\infty)\rangle = e^{-\eta\frac{i}{\delta}\oint_{\tau^X} d\tau\dot{\theta}\mathcal{A}_\theta}|\psi(+\infty + i\epsilon)\rangle, \tag{41}$$

where the exponent corresponds to a residue integral of the AGP around the pole at $\tau^X$,

$$e^{-\frac{i}{\delta}\eta\oint\mathcal{A}_\theta d\theta} = e^{i\eta\frac{\pi}{2}\sigma^y}, \tag{42}$$

so that

$$e^{-\frac{i}{\delta}\oint_{\tau^X} d\theta\mathcal{A}_\theta}|0^B(+\infty)\rangle = \cos\left(\frac{\eta\pi}{2}\right)|1^A(+\infty)\rangle - \sin\left(\frac{\eta\pi}{2}\right)|0^A(+\infty)\rangle. \tag{43}$$

Therefore, comparing the $|1^A(+\infty)\rangle$ coefficients of (41), we find

$$c_1^A(+\infty) = e^{i\int_0^{\gamma^A(\infty)}\left(\frac{e_1^A}{\delta} - a_1^A\right)} e^{-i\int_0^{\gamma^B(\infty)}\left(\frac{e_0^B}{\delta} - a_0^B\right)} \cos\frac{\eta\pi}{2} = \cos\frac{\eta\pi}{2} e^{-\frac{i}{\delta}\int_0^{\tau_1^*}(e_0 - e_1)} e^{i\int_0^{\tau_1^*}(a_0 - a_1)}, \tag{44}$$

where the integrations from zero to the branch point $\tau^*_1$ in these expressions are along paths circumventing the branch cuts. Thus, the transition probability is

$$P_\eta \sim \cos^2\frac{\eta\pi}{2} e^{2\mathrm{Im}[\int_0^{\tau_1^*}(a_1 - a_0)]} e^{-\frac{2}{\delta}\mathrm{Im}[\int_0^{\tau_1^*}(e_1 - e_0)]}. \tag{45}$$

We see that there are two modifications from the Dykhne formula (32): the Berry phase factor, which in this case is nonzero, and the first prefactor, which appears due to the topological phase introduced by the AGP.

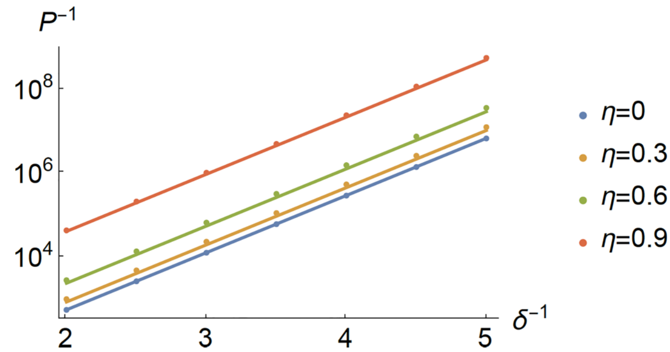


FIG. 3. Inverse probability of nonadiabatic transitions $P^{-1}$ as a function of inverse adiabatic parameter $\delta^{-1}$ for Hamiltonians (14) with different values of $\eta$. The solid lines are formula (45), and the dots are the results of numerically solving the TDSE (1) for the transition probability $P = |c_1(200)|^2$ with initial condition $c_n(-200) = \delta_{n0}$.

Again, the integrals can be evaluated by deforming the contour, although one should be careful with the branch cuts. In Appendix A, we find that the $O(1/\delta)$ and $O(1)$ terms in the dynamical phase are

$$e^{-\frac{2}{\delta}\mathrm{Im}[\int_0^{\tau_1^*}(e_1 - e_0)]} = e^{-\pi/\delta + 2\eta/3}, \tag{46}$$

while the geometric phase factor, although not identically vanishing, does not contribute at this order. Finally, we find the modified LZ formula

$$P_\eta \sim \cos^2\frac{\eta\pi}{2} e^{-2\eta/3} e^{-\pi/\delta}. \tag{47}$$

As shown in Fig. 3, this formula agrees with the results found by numerically solving the TDSE for different values of $\eta$.

We find that, for small but fixed $\delta$, $P_\eta(\delta)$ vanishes perturbatively as $(\eta - 1)^2$ in the counterdiabatic limit $\eta \to 1$. In terms of the small $\delta$ expansion of the exponent of $P_\eta(\delta)$, the LZ term, of order $\delta^{-1}$, is not modified. Instead, the presence of the AGP changes the terms at the next order $\delta^0$ and leads to a universal prefactor, independent of the adiabatic parameter $\delta$, which is not accounted for in the DDP formula. This prefactor is due to the non-Abelian holonomy (42) of the adiabatic gauge potential, which generates relative phases between the different paths. Thus, we find that the complex-time picture of counterdiabatic drive is the Aharonov-Bohm interference due to the topology of the AGP.

## V. ADIABATIC GAUGE POTENTIAL AND INTEGRABILITY

Formula (47) is particular to the LZSM model. However, it can be used to estimate deviations from counterdiabatic dynamics in Hamiltonians which are related to the LZSM model by integrability. The main ingredient is an integrability constraint which is satisfied by the AGP, as we now show. In the next section, we apply this result to compute $P_\eta(\delta)$ in an integrable four-state problem.

Consider again the TDSE,

$$i\partial_t|\psi\rangle = H_0|\psi\rangle, \tag{48}$$

in the $(-\infty, \infty)$ time interval, where we restored units to $\hbar = 1$. Integrable time-dependent quantum Hamiltonians (ITQHs) have an extra structure in which one can find Hamiltonian parameters $x^j$ (such as matrix elements in $H_0$) and corresponding auxiliary operators $H_j$ such that the collection $(H_\mu) = (H_0, H_j)$ satisfies the compatibility condition

$$\partial_\mu H_\nu - \partial_\nu H_\mu + i[H_\mu, H_\nu] = 0, \tag{49}$$

where $(x^\mu) = (t, x^j)$. Then we can extend the TDSE to an equation for parallel transport on the whole $(x^\mu)$ space with connection form $H_\mu dx^\mu$,

$$i\partial_\mu|\psi\rangle = H_\mu|\psi\rangle. \tag{50}$$

Since Eq. (49) is the zero-curvature condition for this connection, it implies that the solution of (50) is independent of the path of integration.

For a given initial condition at $t \to -\infty$ and fixed $x^j$, we can use this freedom to find the solution for $t \to +\infty$ by solving the TDSE on a convenient path in the $(x^\mu)$ space. It is interesting to choose a path for which $|x|$ is always large because, generally, this guarantees that the eigenvalues are far apart and the solution only gains the appropriate phases multiplying each instantaneous eigenstate. The exception is when the path intersects the hypersurfaces where an avoided crossing happens. Crossing one of these hypersurfaces is equivalent to a Landau-Zener problem for the corresponding states which come close together. This reduces the matrix of transition probabilities to a product of Landau-Zener factors

$$\begin{pmatrix} 1-P & P \\ P & 1-P \end{pmatrix}, \tag{51}$$

where the LZ transition probabilities are given by (9) with the appropriate adiabatic parameter for each crossing hypersurface. In this way, we can see the LZSM model as a building block of integrable time-dependent quantum Hamiltonians. Examples that fall under this paradigm include multilevel Landau-Zener models [39–42], Gaudin magnets, and driven BCS Hamiltonians [43].

In these examples, the Hamiltonians $H_\mu$ are real and symmetric. We can then separate the real and imaginary parts of (49),

$$\partial_\mu H_\nu - \partial_\nu H_\mu = 0, \tag{52}$$

$$[H_\mu, H_\nu] = 0. \tag{53}$$

We find that the AGP

$$\mathcal{A}_0 = -iU\partial_\tau U^\dagger, \tag{54}$$

which suppresses nonadiabatic transitions in $H_0$, inherits the integrable structure. More precisely, there are partner operators $\mathcal{A}_j$ such that $\mathcal{A}_\mu = (\mathcal{A}_0, \mathcal{A}_j)$ satisfy the flatness condition

$$\partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + i[\mathcal{A}_\mu, \mathcal{A}_\nu] = 0. \tag{55}$$

The partner operators $\mathcal{A}_j$ are the adiabatic gauge potentials for the extended TDSE (50),

$$\mathcal{A}_\mu = -iU\partial_\mu U^\dagger. \tag{56}$$

We can also show that

$$\partial_\mu(H_\nu + \mathcal{A}_\nu) - \partial_\nu(H_\mu + \mathcal{A}_\mu) + i[H_\mu + \mathcal{A}_\mu, H_\nu + \mathcal{A}_\nu] = 0. \tag{57}$$

If, additionally, $[\mathcal{A}_\mu, \mathcal{A}_\nu] = 0$, then, for any $\eta$,

$$\partial_\mu(H_\nu + \eta\mathcal{A}_\nu) - \partial_\nu(H_\mu + \eta\mathcal{A}_\mu) + i[H_\mu + \eta\mathcal{A}_\mu, H_\nu + \eta\mathcal{A}_\nu] = 0. \tag{58}$$

We include proofs of these facts in Appendix B.

Equation (57) is deeply connected to the multidimensional WKB approximation in $(x^\mu)$ space used to solve ITQHs [38], which relies on the property that nonadiabatic transitions are restricted to the same regions of parameter space for all the Hamiltonians $H_\mu$ and is also related to the fact that the auxiliary Hamiltonians $H_j$ can themselves be interpreted as counterdiabatic potentials [39]. Equation (58) shows that the integrable structure allows one to continuously suppress transitions in all directions of the extended parameter space by tuning a single parameter, $\eta$. In geometric terms, the AGP gives a continuous deformation of $H_\mu dx^\mu$ into the fully counterdiabatic regime through a path of flat connections parametrized by $\eta$.

As an explicit illustration of the flatness of the AGP, consider the three-state bow-tie model [55,56],

$$H_t = \begin{pmatrix} 0 & ag & b\gamma \\ ag & a^2t & 0 \\ b\gamma & 0 & b^2t \end{pmatrix}. \tag{59}$$

Corresponding to parameters $a$ and $b$, the partner Hamiltonians are

$$H_a = \begin{pmatrix} 0 & gt & 0 \\ gt & at^2 - \frac{b^2\gamma^2}{a(a^2-b^2)} & \frac{bg\gamma}{a^2-b^2} \\ 0 & \frac{bg\gamma}{a^2-b^2} & -\frac{ag^2}{a^2-b^2} \end{pmatrix}, \tag{60}$$

$$H_b = \begin{pmatrix} 0 & 0 & \gamma t \\ 0 & \frac{b\gamma^2}{a^2-b^2} & -\frac{ag\gamma}{a^2-b^2} \\ \gamma t & -\frac{ag\gamma}{a^2-b^2} & bt^2 + \frac{a^2g^2}{b(a^2-b^2)} \end{pmatrix}, \tag{61}$$

and we can directly check Eqs. (52) and (53). This integrable structure was used to rederive the exact solution of this model in [43]. As one can expect from the fact that the AGP $(\mathcal{A}_t, \mathcal{A}_a, \mathcal{A}_b)$ are defined as derivatives of the instantaneous eigenvectors, they are given by complicated expressions in terms of the Hamiltonian parameters (see Appendix C for details). The formulas simplify considerably in the $b \to a$ limit. Although $H_a$ and $H_b$ are singular in this limit, the combination $H_a + H_b$ is well defined,

$$\tilde{H}_a = \lim_{b\to a}(H_a + H_b) = \begin{pmatrix} 0 & gt & \gamma t \\ gt & at^2 + \frac{\gamma^2}{2a} & -\frac{g\gamma}{2a} \\ \gamma t & -\frac{g\gamma}{2a} & at^2 + \frac{g^2}{2a} \end{pmatrix}, \tag{62}$$

and together $\tilde{H}_t = H_t|_{b\to a}$ and $\tilde{H}_a$ satisfy (52) and (53). The corresponding AGP are

$$\mathcal{A}_{t,a} = iA_{t,a}\begin{pmatrix} 0 & -g & -\gamma \\ g & 0 & 0 \\ \gamma & 0 & 0 \end{pmatrix}, \tag{63}$$

with the coefficients $\vec{A} = (A_t, A_a)$ given by

$$\vec{A} = \frac{1}{2\sqrt{g^2+\gamma^2}} \nabla \left[ \tan^{-1}\left( \frac{at}{2\sqrt{g^2+\gamma^2}} \right) \right]. \qquad (64)$$

Note that (63) implies that $[\mathcal{A}_t, \mathcal{A}_a] = 0$, and (64) implies that $\partial_a \mathcal{A}_t - \partial_t \mathcal{A}_a = 0$. Thus, for the bow-tie model,

$$\partial_a \mathcal{A}_t - \partial_t \mathcal{A}_a + i[\mathcal{A}_a, \mathcal{A}_t] = 0. \qquad (65)$$

We can also directly check that $[\tilde{H}_t, \mathcal{A}_a] = [\tilde{H}_a, \mathcal{A}_t]$, which implies (57) and (58).

## VI. A MULTILEVEL EXAMPLE

We illustrate how formula (47) can be combined with (58) to estimate deviations from counterdiabatic evolution in an integrable four-level system. Let $(H_\mu) = (H_t, H_x)$, where

$$H_t = \begin{pmatrix} at+x & 0 & g & -\gamma \\ 0 & -at+x & \gamma & g \\ g & \gamma & bt & 0 \\ -\gamma & g & 0 & -bt \end{pmatrix}, \qquad (66)$$

$$H_x = \begin{pmatrix} t+\frac{ax}{a^2-b^2} & 0 & \frac{g}{a-b} & -\frac{\gamma}{a+b} \\ 0 & t-\frac{ax}{a^2-b^2} & -\frac{\gamma}{a+b} & -\frac{g}{a-b} \\ \frac{g}{a-b} & -\frac{\gamma}{a+b} & -\frac{bx}{a^2-b^2} & 0 \\ -\frac{\gamma}{a+b} & -\frac{g}{a-b} & 0 & \frac{bx}{a^2-b^2} \end{pmatrix}, \qquad (67)$$

which satisfy (52) and (53) in the $(x^\mu) = (t, x)$ space [38,57,58]. The integrable structure allows us to find new solvable models by considering evolution along different paths in this space, and in particular on the line $(t, x) = (\tau, x_0 + v\tau)$ we have the Hamiltonian

$$H_\tau = H_t + vH_x = \begin{pmatrix} a_+\tau + x_+ & 0 & g_+ & -\gamma_+ \\ 0 & -a_-\tau + x_- & \gamma_- & g_- \\ g_+ & \gamma_- & b'\tau + y & 0 \\ -\gamma_+ & g_- & 0 & -b'\tau - y \end{pmatrix}, \qquad (68)$$

where

$$\lambda_\pm = v/(a \pm b), \qquad a_\pm = a(1 + \lambda_+\lambda_-) \pm 2v,$$
$$x_\pm = x_0(1 \pm a\lambda_+\lambda_-/v), \qquad b' = b(1 - \lambda_+\lambda_-),$$
$$y = -x_0 b\lambda_+\lambda_-/v, \qquad g_\pm = g(1 \pm \lambda_-),\ \gamma_\pm = \gamma(1 \pm \lambda_+),$$

and we assume $a > b > 0$. Note that $H_\tau$ is similar to $H_t$, but a little more general. Indeed, it was shown in [38] that the solution is different from that of $H_t$ if $v > a - b$.

We wish to compute the probability of nonadiabatic transitions $P_\eta(\delta)$ as a function of the AGP amplitude $\eta$ in the model (68). Thus, we consider the dynamics of the Hamiltonian

$$H_\tau + \eta\mathcal{A}_\tau \qquad (69)$$

modified by the AGP $\mathcal{A}_\tau$. Note that, like in the LZSM case, being in the small-$\delta$ regime will be important for our calculation of $P_\eta(\delta)$. However, although adiabaticity certainly means that the coefficients $a_\pm$ and $b'$ are small compared to $g_\pm$ and $\gamma_\pm$, at this point the exact adiabaticity criterion is not yet clear since there are many ways to construct an admissible dimensionless quantity $\delta$ from the coefficients in (68). This point will become clear below, where we see that integrability narrows down the adiabaticity condition to the smallness of two combinations of the coefficients.

From our result (58), we know that the modified Hamiltonian $H_\tau + \eta H$ corresponds to a special direction of the integrable system defined by $H_t + \eta\mathcal{A}_t$ and $H_x + \eta\mathcal{A}_x$ in the $(t, x)$ plane. Thus, the same technique used to solve $H_\tau$ can be applied to the modified Hamiltonian (69). Instead of solving the TDSE for $H_\tau$ from $\tau = -T$ to $\tau = +T$, with $T \to \infty$, we can go from point $(t_i, x_i) = (-T, x_0 - vT)$ to point $(t_f, x_f) = (T, x_0 + vT)$ using a more convenient path. A good choice is to integrate first

$$i\partial_x|\psi\rangle = (H_x + \eta\mathcal{A}_x)|_{t=t_i}|\psi\rangle \qquad (70)$$

with $x$ from $x_i$ to $x_f$, followed by

$$i\partial_t|\psi\rangle = (H_t + \eta\mathcal{A}_t)|_{x=x_f}|\psi\rangle \qquad (71)$$

with $t$ from $t_i$ to $t_f$. The advantage is that, for large $T$, the Hamiltonians $H_t$ and $H_x$ have large diagonal entries. In the $T \to \infty$ limit, there are no transitions between the instantaneous eigenstates, and in fact, the adiabatic levels coincide with the diabatic levels (the diagonal entries). Moreover, as discussed in Appendix C, in this basis the AGP is off diagonal and is given by

$$\langle m|\mathcal{A}_\mu|n\rangle = i\frac{\langle m|\partial_\mu H_\mu|n\rangle}{E_n - E_m}, \qquad (72)$$

which therefore vanishes in the $T \to \infty$ limit.

The only exception to this argument happens around points of the path $(t^*, x^*)$ where two of the diabatic levels cross, which leads to an avoided crossing of the instantaneous eigenstates. For example, let us consider the case $a - b < v < a + b$. Then at the point $(t^*, x^*) = (-\frac{x_f}{a+b}, x_f)$ we can easily check that $H_t^{11} = H_t^{44}$. Let us expand $|\psi\rangle = \sum_n c_n|\tilde{n}\rangle$, where we introduced the notation $|\tilde{n}\rangle$ to distinguish the $n$th diabatic level from the $n$th instantaneous eigenstate $|n\rangle$. Ignoring the amplitudes on levels $|\tilde{2}, \tilde{3}\rangle$, which are separated by a gap proportional to $T$, and expanding $H_t$, we find that the TDSE at $(t^* + t', x^*)$ reads

$$i\frac{dc_1}{dt'} = at'c_1 - \gamma c_4 - i\frac{\eta(a+b)\gamma}{4\gamma^2 + (a+b)^2 t'^2}c_4,$$
$$i\frac{dc_4}{dt'} = -bt'c_4 - \gamma c_1 + i\frac{\eta(a+b)\gamma}{4\gamma^2 + (a+b)^2 t'^2}c_1, \qquad (73)$$

where we computed the AGP using the method in Appendix C. In particular, only the components of the AGP between states $|\tilde{1}\rangle$ and $|\tilde{4}\rangle$ survive the $T \to \infty$ limit. For example, we find that

$$\langle\tilde{1}|\mathcal{A}_t|\tilde{2}\rangle = i\frac{(a+b)^2 g\gamma}{2av[4\gamma^2 + (a+b)^2 t'^2]}\frac{1}{T} + O\left(\frac{1}{T^2}\right). \qquad (74)$$

By rescaling $t'$ and $H_t$, (73) reduces to the LZSM model modified by the AGP (37) with the adiabatic parameter

$$\delta_\gamma = \frac{a+b}{2\gamma^2} = \frac{a_+ + a_- + 2b' + 2\sqrt{(a_+ + b')(a_- + b')}}{2(\gamma_+ + \gamma_-)^2}$$

and counterdiabatic parameter $\eta$. It follows that, for small $\delta_\gamma$, integration across $(t^*, x^*)$ is equivalent to multiplication of the amplitudes $c_1$ and $c_4$ by the matrix of transition amplitudes in the modified LZSM model $M_\eta^{14}(\delta_\gamma)$, Eqs. (41)–(44). There are four pairwise crossings $(t_{1,2,3,4}, x_{1,2,3,4})$ in total, and the resulting matrix of transition amplitudes is

$$S = U^t(t_f, t_4)M_\eta^{23}(\delta_4)U^t(t_4, t_3)M_\eta^{14}(\delta_3)U^t(t_3, t_i) \times U^x(x_f, x_2)M_\eta^{13}(\delta_2)U^x(x_2, x_1)M_\eta^{24}(\delta_1)U^x(x_1, x_i),$$

where $U^\mu(x_i^\mu, x_j^\mu)$ is the diagonal matrix of adiabatic phases computed by integrating the diagonal entries of $H_\mu$ from $x_j^\mu$ to $x_i^\mu$, and we find that $\delta_{1,2} = \delta_g$ and $\delta_{3,4} = \delta_\gamma$, where

$$\delta_g = \frac{a-b}{2g^2} = \frac{a_+ + a_- - 2b' + 2\sqrt{(a_+ - b')(a_- - b')}}{2(g_+ + g_-)^2}.$$

To compute the probability of transitions between instantaneous eigenstates, we have to remember that on each LZSM avoided crossing the labels of instantaneous eigenstates are swapped with respect to the diabatic levels. Thus, for a system prepared in the instantaneous eigenstate $|1\rangle = |\tilde{1}\rangle$ at $\tau \to -\infty$, the probability of transitions to the instantaneous eigenstate $|2\rangle = |\tilde{1}\rangle$ at $\tau \to +\infty$ is

$$P_\eta^{1\to 2} = |S^{11}|^2 = \left|\left(M_\eta^{14}(\delta_3)\right)^{11}\right|^2 \left|\left(M_\eta^{13}(\delta_2)^{11}\right)\right|^2 \tag{75}$$

$$= \cos^4\left(\frac{\eta\pi}{2}\right)e^{-4\eta/3}e^{-\pi/\delta_\gamma}e^{-\pi/\delta_g}, \tag{76}$$

where we denote by $(A)^{ij}$ the $ij$ entry of matrix $A$, and in the last step we used Eq. (47). This result is valid in the adiabatic regime, which is now seen to correspond to small $\delta_{\gamma,g}$. When comparing this result to the LZSM model, we find that in the case of the integrable four-state Hamiltonian (68), the attenuation of transitions by the AGP is doubly strong, vanishing as $(\eta - 1)^4$ in the $\eta \to 1$ limit.

## VII. DISCUSSION

The adiabatic gauge potential completely suppresses nonadiabatic transitions. However, it does so only if one can exactly realize the time-dependent AGP Hamiltonian. In real experimental settings, perturbations to the original Hamiltonian and to the AGP are inevitable, so quantifying the stability of this method of counterdiabatic control is of both conceptual and practical importance. In this work we found that, in terms of deviations of the AGP amplitude $\eta$ from the counterdiabatic regime $\eta = 1$, the probability of transitions $P_\eta(\delta)$ has the form of a modified LZ formula [Eq. (47)]. The modification is given by a prefactor which depends on $\eta$ but not on the adiabatic parameter $\delta$. This prefactor was derived by extending the derivation of the DDP formula to account for a singularity of the AGP in complex time. This singularity leads to nontrivial holonomy of the AGP field, and the resulting Aharonov-Bohm phase interference between different complex-time paths gives the counterdiabatic suppression of the transition probability. In this way, our calculation gives geometric insight into the mechanism behind the counterdiabatic drive. Moreover, the modified LZ formula (47) implies that, unlike the nonperturbative suppression of transitions in the adiabatic limit $\delta \to 0$, they are only perturbatively suppressed in the counterdiabatic limit $\eta \to 1$.

Although intuitive, the choice of the amplitude of the AGP as the counterdiabatic parameter $\eta$ corresponds to a single type of deformation of the Hamiltonian. In a general experimental setting, the Hamiltonian can be perturbed in many other ways, and the counterdiabatic suppression of transitions might be even less stable under different types of perturbation. This is similar to the many deviations from the Landau-Zener formula which appear when the LZSM Hamiltonian is perturbed or enriched in interesting ways. Whether effects like resonance and interference can strongly destroy the control of eigenstates by the AGP is an interesting question for future work. In particular, it is possible that our method could be extended to more general models with analytic (or meromorphic) Hamiltonians which could probe into these questions. It would also be interesting to understand how the cosine prefactor derived here is related to the generic formulas derived in [59] by the method of superadiabatic renormalization.

Complementarily, we also studied the properties of the AGP in integrable time-dependent quantum Hamiltonians. We showed that, in a precise sense, the AGP is compatible with integrability. Namely, the AGP satisfies the flatness condition and gives an integrable deformation of these models. This leads to a number of simplifications, as illustrated with three- and four-level examples, which showed that ITQHs provide good models on which to study counterdiabatic control in a multilevel setting. As an example, we computed the dependence of the probability of nonadiabatic transitions $P_\eta$ on the counterdiabatic parameter in a four-state model. A bonus of our results is that the modified Hamiltonians $H_\mu + \eta\mathcal{A}_\mu$ might lead to new examples of ITQHs, as hinted at by the fact that the expressions for the AGP are, in general, quite complicated.

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## APPENDIX A: DYNAMICAL AND GEOMETRIC EXPONENTS

In evaluating the integrals in the dynamical and geometric exponents appearing in (45), we can again deform the contour to lie on the imaginary axis. However, this means crossing the branch cut at $\tau^X = i$ (see Fig. 2). For example, the dynamical phase becomes

$$\mathrm{Im}\left[\int_0^{\tau_1^*}(e_1 - e_0)\right] = \mathrm{Re}\left[2\lim_{\varepsilon\to 0}\left(\int_0^{1-\varepsilon} - \int_{1+\varepsilon}^{\sqrt{1+a^{\frac{2}{3}}}}\right)\sqrt{1 - y^2 + \frac{a^2}{(1-y^2)^2}}dy\right], \tag{A1}$$

where

$$a = \frac{\eta\delta}{2} \tag{A2}$$

is small. One important consequence of the change in sign at the cut is that the integral is finite. The leading term can easily be found from the first integral at $a = 0$,

$$\int_0^1 \sqrt{1-y^2}dy = \frac{\pi}{4}, \tag{A3}$$

and just gives the LZ exponent. To get the next term in $a$, we have to carefully expand the second integral. One way is to write the integrand as

$$\int_{1+\varepsilon}^{\sqrt{1+a^{\frac{2}{3}}}} \frac{\sqrt{a^{\frac{4}{3}} - a^{\frac{2}{3}}(1-y^2) + (1-y^2)^2}\sqrt{1-y^2+a^{\frac{2}{3}}}}{1-y^2}dy, \tag{A4}$$

which correctly separates out the pole at the lower limit and the zero at the upper limit of the integral, and expand the first square root in $a^{\frac{2}{3}}$. We find

$$\int_{1+\varepsilon}^{\sqrt{1+a^{\frac{2}{3}}}} \sqrt{1-y^2+a^{\frac{2}{3}}} = \frac{a}{3} + \cdots, \tag{A5}$$

which gives the leading value of the dynamical phase for small $\delta$ as

$$e^{-\frac{2}{\delta}\mathrm{Im}[\int_0^{\tau_1^*}(e_1-e_0)]} = e^{-\pi/\delta + 2\eta/3}. \tag{A6}$$

## APPENDIX B: INTEGRABILITY CONDITION FOR THE AGP

We consider real symmetric Hamiltonians $H_\mu$ satisfying

$$\partial_\mu H_\nu - \partial_\nu H_\mu = 0, \tag{B1}$$

$$[H_\mu, H_\nu] = 0. \tag{B2}$$

From (B2), we can find a unitary $U$ such that

$$\tilde{H}_\mu = U^\dagger H_\mu U \tag{B3}$$

are all diagonal. Then the AGPs are

$$\mathcal{A}_\mu = -iU\partial_\mu U^\dagger, \tag{B4}$$

and using the fact that $(\partial_\mu U)U^\dagger = -U\partial_\mu U^\dagger$, we find

$$[\mathcal{A}_\mu, \mathcal{A}_\nu] = -(U\partial_\mu U^\dagger U\partial_\nu U^\dagger - U\partial_\nu U^\dagger U\partial_\mu U^\dagger) \tag{B5}$$

$$= \partial_\mu U\partial_\nu U^\dagger - \partial_\nu U\partial_\mu U^\dagger \tag{B6}$$

$$= \partial_\mu(U\partial_\nu U^\dagger) - \partial_\nu(U\partial_\mu U^\dagger), \tag{B7}$$

so that the $\mathcal{A}_\mu$ satisfy the flatness condition

$$\partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + i[\mathcal{A}_\mu, \mathcal{A}_\nu] = 0. \tag{B8}$$

Now, from (B1),

$$0 = \partial_\mu(U\tilde{H}_\nu U^\dagger) - \partial_\nu(U\tilde{H}_\mu U^\dagger) \tag{B9}$$

$$= U(\partial_\mu \tilde{H}_\nu - \partial_\nu \tilde{H}_\mu)U^\dagger + [H_\nu, U\partial_\mu U^\dagger] - [H_\mu, U\partial_\nu U^\dagger]. \tag{B10}$$

Conjugating by $U$, we get

$$(\partial_\mu \tilde{H}_\nu - \partial_\nu \tilde{H}_\mu) + \{[\tilde{H}_\nu, (\partial_\mu U^\dagger)U] - [\tilde{H}_\mu, (\partial_\nu U^\dagger)U]\} = 0. \tag{B11}$$

Since the $\tilde{H}_\mu$ are diagonal, the contribution from the first term (in parentheses) is diagonal and that from the second term (in curly brackets) is off diagonal. They separately give

$$\partial_\mu E_{\nu,n} - \partial_\nu E_{\mu,n} = 0, \tag{B12}$$

$$[H_\mu, \mathcal{A}_\nu] - [H_\nu, \mathcal{A}_\mu] = 0. \tag{B13}$$

Finally, using (B1),(B2), (B8), and (B13), we find that

$$\partial_\mu(H_\nu + \mathcal{A}_\nu) - \partial_\nu(H_\mu + \mathcal{A}_\mu) + i[H_\mu + \mathcal{A}_\mu, H_\nu + \mathcal{A}_\nu] = 0, \tag{B14}$$

so that the Hamiltonians corrected by the AGP terms still satisfy the flatness condition. In other words, the adiabatic gauge potential is consistent with time-dependent quantum integrability. Additionally, if (B8) separates into

$$\partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu = 0, \tag{B15}$$

$$[\mathcal{A}_\mu, \mathcal{A}_\nu] = 0, \tag{B16}$$

then

$$\partial_\mu(H_\nu + \eta\mathcal{A}_\nu) - \partial_\nu(H_\mu + \eta\mathcal{A}_\mu) + i[H_\mu + \eta\mathcal{A}_\mu, H_\nu + \eta\mathcal{A}_\nu] = 0 \tag{B17}$$

for all $\eta$.

## APPENDIX C: EVALUATING THE AGP AND INTEGRABLE EXAMPLES

In this Appendix, we collect useful properties of the AGP and discuss how they apply to integrable Hamiltonians. First, note that we defined the AGP suppressing transitions in the $x^\mu$ direction as

$$\mathcal{A}_\mu = -iU\partial_\mu U^\dagger, \tag{C1}$$

where $U$ is a unitary such that

$$\tilde{H}_\mu = U^\dagger H_\mu U \tag{C2}$$

are all simultaneously diagonal. In general, calculating $\mathcal{A}_\mu$ through (C1) is difficult since $U$ has a complicated expression in terms of the Hamiltonian parameters. A more direct alternative is the following. First note that

$$\partial_\mu H_\mu = \partial_\mu U\tilde{H}_\mu U^\dagger + U\partial_\mu \tilde{H}_\mu U^\dagger + U\tilde{H}_\mu \partial_\mu U^\dagger \tag{C3}$$

$$= U\partial_\mu \tilde{H}_\mu U^\dagger - i[\mathcal{A}_\mu, H_\mu], \tag{C4}$$

where repeated indices are not summed over. $\partial_\mu \tilde{H}_\mu$ is the diagonal matrix of derivatives of the eigenvalues of $H_\mu$, so $U\partial_\mu \tilde{H}_\mu U^\dagger$ commutes with $H_\mu = U\tilde{H}_\mu U^\dagger$. Therefore, $\mathcal{A}_\mu$ satisfies the equation [15]

$$[H_\mu, i\partial_\mu H_\mu - [\mathcal{A}_\mu, H_\mu]] = 0. \tag{C5}$$

This gives a linear equation for $\mathcal{A}_\mu$ which we can solve by expanding $\mathcal{A}_\mu$ on a basis of Hermitian operators.

As an example, we consider the bow-tie model, which has an integrable structure in terms of the space of parameters $(x^\mu) = (t, a, b)$ and the Hamiltonians

$$H_t = \begin{pmatrix} 0 & ag & b\gamma \\ ag & a^2 t & 0 \\ b\gamma & 0 & b^2 t \end{pmatrix}, \tag{C6}$$

$$H_a = \begin{pmatrix} 0 & gt & 0 \\ gt & at^2 - \frac{b^2\gamma^2}{a(a^2-b^2)} & \frac{bg\gamma}{a^2-b^2} \\ 0 & \frac{bg\gamma}{a^2-b^2} & -\frac{ag^2}{a^2-b^2} \end{pmatrix}, \tag{C7}$$

$$H_b = \begin{pmatrix} 0 & 0 & \gamma t \\ 0 & \frac{b\gamma^2}{a^2-b^2} & -\frac{ag\gamma}{a^2-b^2} \\ \gamma t & -\frac{ag\gamma}{a^2-b^2} & bt^2 + \frac{a^2g^2}{b(a^2-b^2)} \end{pmatrix}. \tag{C8}$$

The eigenvectors can be made real so that the $\mathcal{A}_\mu$ are imaginary and antisymmetric. Solving (C5) for the three independent coefficients, we find

$$\mathcal{A}_t = \frac{i}{f_4}\begin{pmatrix} 0 & f_3 & -f_2 \\ -f_3 & 0 & f_1 \\ f_2 & -f_1 & 0 \end{pmatrix}, \tag{C9}$$

with the polynomial expressions

$$f_1(a,b,g,\gamma,t) = abtg\gamma(a^2-b^2)\{a^4(b^2t^2-g^2) - b^4\gamma^2 + a^2b^2[b^2t^2 + g(g^2+\gamma^2]\}, \tag{C10}$$

$$\begin{aligned} f_2(a,b,g,\gamma,t) &= b\gamma\{a^8t^2(b^2t^2-g^2) \\ &\quad + a^6(6b^2g^2t^2 - 2b^4t^4 - 3g^4) + b^6\gamma^4 \\ &\quad + a^2b^4[5g^2\gamma^2 + 2b^2t^2(2g^2+\gamma^2)] \\ &\quad + a^4b^2[4g^4 + b^4t^4 - 2b^2t^2\gamma^2 - 3g^2(3b^2t^2+\gamma^2)]\}, \end{aligned} \tag{C11}$$

$$f_3(a,b,g,\gamma,t) = -f_2(b,a,\gamma,g,t), \tag{C12}$$

$$\begin{aligned} f_4&(a,b,g,\gamma,t) \\ &= (a^2t^2+4g^2)[a^3g^2 + ab^2t^2(a^2-b^2)]^2 \\ &\quad + 2a^2b^2\gamma^2[6a^2g^4 + (10a^4 - 19a^2b^2 + 10b^4)g^2t^2 \\ &\quad + (2a^6 - 4a^4b^2 + a^2b^4 + b^6)t^4]. \end{aligned} \tag{C13}$$

Similarly, we can find $\mathcal{A}_{a,b}$. We find that $f_{1,2,3}$ simplify greatly in the $b \to a$ limit, so that

$$\mathcal{A}_t \to \frac{ia}{a^2t^2 + 4(g^2+\gamma^2)}\begin{pmatrix} 0 & -g & -\gamma \\ g & 0 & 0 \\ \gamma & 0 & 0 \end{pmatrix}. \tag{C14}$$

In this limit, we can easily solve (C5) for $\mathcal{A}_a$ as well, and the results are shown in (63) and (64), which agree with (C14).

Equation (C5) also leads to an important expression for the AGP in the basis of instantaneous eigenstates $|n\rangle$. From (C4),

$$i\langle m|(\partial_\mu H_\mu - U\partial_\mu \tilde{H}_\mu U^\dagger)|n\rangle = \langle m|[\mathcal{A}_\mu, H_\mu]|n\rangle \tag{C15}$$

$$= (E_n - E_m)\langle m|\mathcal{A}_\mu|n\rangle. \tag{C16}$$

Thus, in the basis of instantaneous eigenstates, where $\mathcal{A}_\mu$ is off-diagonal, its matrix elements are given by [11]

$$\langle m|\mathcal{A}_\mu|n\rangle = i\frac{\langle m|\partial_\mu H_\mu|n\rangle}{E_n - E_m}. \tag{C17}$$

In particular, in regions of the parameter space $x^\mu$ where the levels are far apart this formula shows that the AGP is suppressed.


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