Quantum coherence between subspaces: State transformation, cohering power, *k* coherence, and other properties

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(Received 24 November 2023; accepted 9 January 2024; published 30 January 2024)

The concept of block coherence encompasses the case where experimental capabilities are not so delicate to perform arbitrary refined measurements on individual atoms. We develop a framework which facilitates further investigation of this resource theory in several respects. Using this framework, we investigate the problem of state conversion by incoherent operations and show that a majorization condition is the necessary and sufficient condition for state transformation by block-incoherent operations. We also determine the form of the maximally coherent state from which all other states and all unitary gates can be constructed by incoherent operations. Thereafter, we define the concept of block-cohering and block-decohering powers of quantum channels and determine these powers for several types of channels. Finally, we explore the relation between block coherence and a previous extension of coherence, known as *k* coherence.

DOI: 10.1103/PhysRevA.109.012435

I. INTRODUCTION

From the very beginning of quantum theory, coherence of states, as a property that radically distinguishes the quantum superposition from classical mixtures, has been the subject of much discussion. While coherence is not entirely unfamiliar to physicists, as it has been present in all forms of wave phenomena, only in quantum mechanics has it revealed its most exciting properties. It is here that one encounters intriguing concepts like superposition of spatial degrees of freedom as in a double-slit experiment [1,2], superposition of macroscopic states of many-body systems [3,4] as in a quantum phase transition, and entanglement which itself underlies the unique features of quantum computation and quantum information processing [5]. A plethora of theoretical and experimental techniques are known for manipulating coherence in optical experiments [6–8], and theoretical limitations for manipulation of superposition have been studied in various works [9,10]. Nevertheless, attempts to quantify the superposition of orthogonal states are rather recent. General measures of coherence were first introduced by Aberg in [11] and then formulated in a quantitative resource-theory-based form in [12,13], which was further developed in various directions in [14–35].

The resource theory was itself inspired by the understanding that entanglement can be considered as a resource that is used in an efficient and useful way and consumed at the end of most quantum communication tasks [36–39]. Likewise, superposition and coherence can also be thought of as kinds of resources that are used in a quantum process and consumed at the end. The core concept of any resource theory is the operational restrictions that we have for manipulating quantum states in our laboratory [40–46]. In entanglement these restrictions derive from locality and in superposition and coherence they derive from our insufficient means for accessing any kind of basis for quantum states, either in our measurements and filtering operations or in other kinds of operations. For example, a laboratory may easily put spin-$\frac{1}{2}$ particles in states $|\uparrow, z\rangle$ or $|\downarrow, z\rangle$, but not in their arbitrary superposition. To summarize the notions introduced in [12], a basis $\{|i\rangle, i = 1, \dots, d\}$ for a Hilbert space is chosen as the preferred basis. A state is called incoherent if it is diagonal in this preferred basis, i.e., if

$$\rho_{\text{inc}} = \sum_{i=0}^{d-1} p_i |i\rangle\langle i|, \tag{1}$$

where $\{p_i\}$ is a probability distribution. In fact, the incoherent states are the ones that can be freely generated by the measurements of the experimenter in the preferred basis. The totality of such states form a convex set $\mathcal{I}_{\text{inc}}$ in the space of all conceivable quantum states. The incoherent operations are then defined to be the ones that do not generate any coherence from an incoherent state, i.e., they are trace-preserving and completely positive operations which map the set of incoherent states into itself, i.e., a quantum operation $\mathcal{E}$ is incoherent if $\mathcal{E} : \mathcal{I}_{\text{inc}} \to \mathcal{I}_{\text{inc}}$.

Finally, a maximally coherent state is a state from which all other states and all unitary operators (quantum gates) can be constructed purely by incoherent operations, i.e., by operations that are at the disposal of the experimenter in the laboratory. It was shown in [12] that a state like

$$|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle \tag{2}$$

is a maximally coherent state of a d-dimensional Hilbert space, in the above sense. Therefore, it is a resource state in the context of coherence theory. Once the free states and free operations were recognized, quantitative measures of coherence could be defined in the spirit of resource theory.

This primary resource theory of coherence is based on an orthogonal basis for the description of the density matrix and rank-1 measurement of the experimenter and it leads to measures of coherence which should be expressed in terms of the individual matrix elements of the density matrix, the determination of which may not be experimentally feasible. The removal of the constraints of this standard resource theory of coherence has led to other generalized theories. For example, the requirement of orthogonality of the basis vectors is relaxed to their linear independence in [22], and the authors of [23] write the density matrix in terms of expectation values of Hermitian operators and express the known measures of coherence in terms of what they call the observable matrix, all the elements of which are directly measurable in the laboratory.

In a different development, the author of [11] introduced the notion of block coherence, and different block coherence measures were defined in [11,20,21]. In the resource theory of block coherence, the block-incoherent states have a block-diagonal structure which is determined by a projective (not necessarily rank-1) measurement. The resource theory of coherence based on positive-operator-valued measurements (POVMs) was also introduced in [14], where the authors use the Naimark theorem to define the POVM-based coherence. This generalization was also quantified in [15,20,21].

The theory of block coherence is of special importance in cases where the experimenter does not have an ability to measure a complete set of observables and prepare a complete basis of states, which is often the case. Mathematically, this means that the projectors of measurement are not rank-1 projectors. For example, one may only be interested in measuring a property of a group of particles, in which case the projectors will be $\Pi_j = \mathbb{I}\otimes\mathbb{I}\otimes\cdots\otimes\pi_j\otimes\cdots\otimes\mathbb{I}\otimes\mathbb{I}$, where π_j are projectors on that specified group. This is the case where the projectors Π_j are no longer of unit rank. Even for one particle, one may only be able to measure its total spin and not the z component of its spin, or one may only be able to determine whether the spins of two particles are parallel or antiparallel, e.g., in a communication task where these pairs of particles are sent between two parties with no shared reference frame [47–54]. In other quantum protocols, one may need to determine whether the majority of spins are up or down in a given precision [55]. All these refer to realistic situations where a preferred and complete basis and the refined operations induced by that measurement are not accessible to us. Under such circumstances, we should adapt our notions and measures of coherence to these new limitations. For example, in a situation where we can only do projective measurements with $\pi_1 = |0\rangle\langle 0| + |1\rangle\langle 1|$ and $\pi_1 = |2\rangle\langle 2|$, in a three-level system, it is meaningless to assign nonzero coherence to a state like $a|0\rangle + b|1\rangle$ and zero coherence to a state like $a^2|0\rangle\langle 0| + b^2|1\rangle\langle 1|$.

Besides its theoretical interest, the resource theory of block coherence may have significant practical consequences. It is important to know that the availability of less-refined measurements in a laboratory affects our definition of incoherent states and operations, and the resourceful states. Do we need more or less coherence, according to the initial definition of [12], in order to produce a certain state? Do we need more complicated incoherent operations to construct arbitrary states from our resource states? Which states can be transformed to each other freely? Can one define block-cohering and block-decohering powers of quantum channels as in [24,25]? What is the relation of block coherence and the notion of k coherence developed in [16–19]. These questions have been left unanswered in previous studies and, as we will see, these questions guide us to a rich and comprehensive structure of the resource theory of block coherence.

In the present work we provide answers to these questions. To this end, we first introduce a mathematical framework, which facilitates many of the subsequent calculations. Then we prove a majorizationlike sufficient and necessary condition for pure state transformation and find the explicit form of the incoherent operations which perform state conversion. We also explicitly show that one can use the action of incoherent operations on the maximally coherent state to construct any arbitrary gate. We will see that the more course grained our measurements are, the more complicated incoherent operations are necessary to convert this state to an arbitrary state and construct an arbitrary unitary operation. We also define the block-cohering and block-decohering powers of quantum channels and derive closed formulas of these quantities for several families of quantum channels. Finally, we elaborate on the relation between block coherence and an interesting notion called k coherence [16–19]. The latter concept, which is different from block coherence, is based on the number of basis states which are in a superposition in a given general state. We find a curious and interesting relation between the two notions, which we will clarify by an explicit and yet general example.

The structure of the paper is as follows. In Sec. II we state our notation and conventions. In Sec. III we recapitulate the previous results in a simple mathematical form. We then briefly review two of the previously defined block-coherence measures in Sec. IV. In Sec. V we investigate the pure state conversion and show that in the context of block coherence, majorization is still a sufficient condition for state transformation by incoherent operations. We show in Sec. VI how, by having access to a maximally incoherent state and by using only incoherent operations, one can implement any arbitrary unitary gate. To this end, we obtain the explicit form of the appropriate Kraus operators. We also define the block-cohering and -decohering powers of a quantum map in Sec. VII and we calculate these powers for a few families of channels. The relation between block coherence and k coherence is investigated in Sec. VIII. We summarize in Sec. IX. In the Appendix we show that the majorization condition of Sec. V is also necessary for state transformation by block-incoherent operations.

II. NOTATION

Let $\mathcal{H}$ be a Hilbert space which can be decomposed into subspaces such that

$$\mathcal{H} = \bigoplus_{\mu=1}^{M} H_\mu, \tag{3}$$

with $\dim(H_\mu) = d_\mu$ and let π_μ be the projection operator on the subspace H_μ:

$$\sum_{\mu=1}^{M} \pi_\mu = \mathbb{I}_{\mathcal{H}}.$$

These projectors define the only measurements that are at our disposal in our laboratory. Let the subspace H_μ be spanned by an orthonormal basis $\{|e_{i_\mu}\rangle, i_\mu = 1, \ldots, d_\mu\}$. Here we have abbreviated the more detailed denotation $e^{(\mu)}_{i_\mu}$, where (μ) indicates the subspace and i_μ the basis state in the subspace, simply to e_{i_μ}, hoping that this will not lead to confusion. Obviously, we have $\langle e_{i_\mu}|e_{j_\mu}\rangle = \delta_{i_\mu,j_\mu}$ and $\langle e_{i_\mu}|e_{j_\nu}\rangle = 0$, with $\mu \neq \nu$. We also define an auxiliary space $Q = \text{Span}\{|1\rangle, |2\rangle, \ldots, |\mu\rangle, \ldots, |M\rangle\}$ to specify different subspaces in the following way. A block-diagonal operator is then defined as

$$A = \sum_{\mu=1}^{M} |\mu\rangle\langle\mu| \otimes A_\mu \tag{4}$$

and an operator that is nonzero only on the block $\mu\nu$ is written as $B = |\mu\rangle\langle\nu| \otimes B_{\mu\nu}$. Such an operator maps H_ν to H_μ and acts as a zero operator on all other subspaces.

III. PRELIMINARIES

Consider the d-dimensional Hilbert space $\mathcal{H}$ in (3) and let $\mathcal{M} = \{\pi_\mu \mid \mu = 1, \ldots, M\}$ describe a measurement with projective operators π_μ, not necessarily of rank 1. A quantum state is defined to be block incoherent if it is in block-diagonal form, that is, if

$$\rho_{\text{inc}} = \sum_{\mu=1}^{M} p_\mu |\mu\rangle\langle\mu| \otimes \rho_\mu, \quad \sum_\mu x_\mu = 1, \tag{5}$$

in which each ρ_μ is a density matrix in H_μ. The outcome of any measurement $\mathcal{M}$ on any state is of the above form. Incoherent states, when being measured, remain intact. In a more explicit form, an incoherent state has the matrix form

$$\rho_{\text{inc}} = \begin{pmatrix} p_1\rho_1 & & & & \\ & p_2\rho_2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & p_M\rho_M \end{pmatrix}, \tag{6}$$

where ρ_μ is a $d_\mu \times d_\mu$ density matrix. When $d_\mu = 1$ for all $\mu = 1, \ldots, M$, this definition coincides with the usual definition of incoherent states. Obviously, the set of all block incoherent states is a convex set, which is denoted by $\mathcal{I}_{\text{inc}}$. Note that, in any subspace, no preferred basis is assigned.

Incoherent operations are the ones that do not create coherence out of incoherent states. Different approaches are used to define these operations [31]. The largest class of incoherent operations are the so-called maximal incoherent operations and consist of all operations that map $\mathcal{I}_{\text{inc}}$ to itself. A quantum operation is said to be an incoherent operation if it has a Kraus representation such that each Kraus operator maps $\mathcal{I}_{\text{inc}}$ to itself.

An operation $\mathcal{E} = \sum_a K_a \rho K_a^\dagger$ is a block-incoherent operation if and only if its Kraus operators have the form [15]

$$K_a = \sum_\mu |a(\mu)\rangle\langle\mu| \otimes K^a_\mu, \tag{7}$$

in which $a : \{1, 2, \ldots, M\} \to \{1, 2, \ldots, M\}$ is an arbitrary function, not necessarily a permutation, and K^a_μ is any arbitrary operator. The proof is presented in [15], but can also be shown straightforwardly by using the auxiliary space introduced in Sec. II. We simply note that a Kraus operator of the form $K_a = \sum_\mu |a(\mu)\rangle\langle\mu| \otimes K^a_\mu$, when acting on $\rho = \sum_\mu p_\mu |\mu\rangle\langle\mu| \otimes \rho_\mu$, produces another incoherent state of the same form.

The explicit form of these Kraus operators are such that in each column only one block should be nonzero. For example, when $M = 2$, regardless of the dimensions of blocks, the admissible forms of Kraus operators are

$$K_1 = \begin{pmatrix} A_1 & \\ & A_2 \end{pmatrix}, \quad K_2 = \begin{pmatrix} B_1 & B_2 \\ & \end{pmatrix},$$
$$K_3 = \begin{pmatrix} & C_1 \\ C_2 & \end{pmatrix}, \quad K_4 = \begin{pmatrix} & \\ D_1 & D_2 \end{pmatrix}. \tag{8}$$

IV. MEASURES OF BLOCK COHERENCE

A. Measure based on relative entropy

Given an arbitrary state ρ, one can define its measure of block coherence as its minimum distance from the set of block-incoherent states. This approach has been followed in [11,14,15,20,21], which leads to the block-coherent content of the state ρ,

$$C(\rho) = \min_{\delta \in \mathcal{I}_{\text{inc}}} D(\rho, \delta), \tag{9}$$

where D is any distance. Although relative entropy does not have all the properties of distance, it is usually used in measures like (9) to quantify various resources. If one takes $D(\rho, \delta)$ to be the relative entropy between the two states, then the closest block-incoherent state to a given state ρ is obtained by simply removing all the off-diagonal blocks in the density matrix [11]. Hence a closed and easily calculable formula for the entropy-based measure of block coherence of a given state ρ is

$$C^M_s(\rho) = S(\rho^*) - S(\rho), \tag{10}$$

where

$$\rho^* := \sum_{\mu=1}^{M} \pi_\mu \rho \pi_\mu = \sum_{\mu=1}^{M} |\mu\rangle\langle\mu| \otimes \rho_\mu. \tag{11}$$

We now ask what kind of pure state has the largest value of block coherence, when we fix the measurement or the block structure. Consider an arbitrary pure state

$$|\Psi\rangle = \begin{pmatrix} x_1|\psi_1\rangle \\ x_2|\psi_2\rangle \\ \vdots \\ \vdots \\ x_M|\psi_M\rangle \end{pmatrix}, \tag{12}$$

subject to $\sum_{\mu=1}^{M} |x_\mu|^2 = 1$, where the $|\psi_\mu\rangle$ are arbitrary normalized pure states of dimension d_μ. Using (10), we find that

$$C^M_s(|\Psi\rangle) = S(\rho^*) = -\sum_\mu |x_\mu|^2 \ln |x_\mu|^2, \tag{13}$$

which means that the highest coherence belongs to states of the form

$$|\Psi\rangle_{\mathrm{MC}} = \frac{1}{\sqrt{M}}\begin{pmatrix} |\psi_1\rangle \\ |\psi_2\rangle \\ \vdots \\ \vdots \\ |\psi_M\rangle \end{pmatrix}, \tag{14}$$

i.e., the highest coherence corresponds to the states with equal probabilities of the subspaces. It is important to note that the states $|\psi_\mu\rangle$ are arbitrary as long as they are not zero. They need not have any coherence in their own subspace at all.

B. Measure based on the l_1-norm

Starting from ρ^*, another measure of block coherence based on the l_1-norm, which is a natural generalization of this measure for the coherence introduced in [12], is defined as [20]

$$C_1^M(\rho) = \|\rho - \rho^*\|_{l_1} = \sum_{\mu\neq\nu} \|\rho_{\mu\nu}\|_1, \tag{15}$$

where $\rho_{\mu\nu}$ is the matrix in the block $\mu\nu$ of ρ and

$$\|A\|_1 = \mathrm{Tr}(\sqrt{A^\dagger A}) \tag{16}$$

is the trace norm of A. For this measure, the block coherence of a general pure state (12) is found to be

$$C_1^M(|\Psi\rangle) = \sum_{\mu\neq\nu} |x_\mu x_\nu| \, \||\psi_\mu\rangle\langle\psi_\nu|\|, \tag{17}$$

and since $\||\psi\rangle\langle\phi|\| = \sqrt{\langle\psi|\psi\rangle\langle\phi|\phi\rangle}$,

$$C_1^M(|\Psi\rangle) = \sum_{\mu\neq\nu} |x_\mu x_\nu|. \tag{18}$$

According to this measure, the maximally coherent state (14) has a coherence given by

$$C_1^M(|\Psi\rangle_{\mathrm{MC}}) = M - 1. \tag{19}$$

Obviously, when there is only one block, there is no coherence and when all blocks are one dimensional ($M = d$), this measure coincides with the usual measure of coherence [12].

It should be noted that, by using both measures (9) and (15), the arbitrary state (12) has the same coherence as the state

$$|\Phi\rangle = \begin{pmatrix} x_1|\phi_1\rangle \\ x_2|\phi_2\rangle \\ \vdots \\ \vdots \\ x_M|\phi_M\rangle \end{pmatrix}, \tag{20}$$

where

$$|\phi_\mu\rangle = \frac{1}{\sqrt{d_\mu}} \sum_{i_\mu=1}^{d_\mu} |e_{i_\mu}\rangle \tag{21}$$

is the maximally coherent state of the subspace H_μ. In fact, the states (12) and (20) are equivalent since they can be converted to each other by applying block-diagonal incoherent unitary operators of the form $U = \bigoplus_{\mu=1}^M u_\mu$. For future use, we also state that the maximally coherent state (14) is equivalent to the state

$$|\Phi\rangle_{\mathrm{MC}} = \frac{1}{\sqrt{M}}\begin{pmatrix} |\phi_1\rangle \\ |\phi_2\rangle \\ \vdots \\ \vdots \\ |\phi_M\rangle \end{pmatrix} \tag{22}$$

with regard to their coherence.

For the maximally coherent states, by suppressing the notation for states, we have

$$0 = C_{l_1}^1 \leqslant C_{l_1}^2 \leqslant \cdots \leqslant C_{l_1}^d = d - 1, \tag{23}$$

which shows that the value of coherence increases as M increases from 1 to d, a result which is expected on physical grounds, and for $M = d$ we find $C_{l_1}^d(|\Phi\rangle) = d - 1$, which understandably coincides with the l_1 value of standard definition of coherence. In Sec. V we will show that for any value of M and for both types of coherence measures (9) and (15), it is indeed possible to start from the maximally coherent state for that partition and obtain any other arbitrary state by simply using the incoherent operations allowable for that type of partition. This verifies that these two measures are indeed correct measures of block coherence in terms of resource theory.

Remark 1. Both measures of coherence are invariant under multiplication of each state $|e_{i_\mu}\rangle$ by a local phase $e^{i\phi_{i_\mu}}|e_{i_\mu}\rangle$. Therefore, we take all the coefficients in maximally coherent states to be real.

Remark 2. Throughout the text, we use $|\Psi\rangle$ and $|\psi\rangle$ to show arbitrary states of the form (12) and we use the terms $|\Phi\rangle$ and $|\phi\rangle$ to denote the states of the forms (20) and (21).

V. PURE STATE CONVERSION BY BLOCK-INCOHERENT OPERATIONS

In this section, by explicit analytical derivation of Kraus operators, we show that majorization is a sufficient condition for pure state conversion by block-incoherent operations. In the Appendix we will show that it is also the necessary condition. This is the counterpart of the theorem for state transformation under the standard incoherent operations derived in [46]. Before proceeding, let us recall the definition of majorization. Consider two probability distributions $\mathbf{p} = (p_1 \geqslant p_2 \geqslant \cdots \geqslant p_M)$ and $\mathbf{q} = (q_1 \geqslant q_2 \geqslant \cdots \geqslant q_M)$. We say that $\mathbf{p}$ majorizes $\mathbf{q}$ and write $\mathbf{p} \succ \mathbf{q}$ if for all $k = 1, 2, \ldots, M$ it holds that $\sum_{i=1}^k p_i \geqslant \sum_{i=1}^k q_i$. To avoid cluttering of notation, we first explain the idea by considering the case where there are two subspaces of arbitrary dimensions and then we will then extend the argument to the general case, where there is an arbitrary number of subspaces.

A. Case of two subspaces of arbitrary dimensions

Consider the case where we have only two subspaces of dimensions d_1 and d_2, i.e., $\mathcal{H} = H_1 \oplus H_2$, with corresponding projectors

$$\pi_1 = \sum_{i=1}^{d_1} |e_i\rangle\langle e_i|, \quad \pi_2 = \sum_{j=1}^{d_2} |f_j\rangle\langle f_j|.$$

Our task is to show that the initial state

$$|\Psi_{\mathbf{x}}\rangle = \begin{pmatrix} x_1|\psi_1\rangle \\ x_2|\psi_2\rangle \end{pmatrix}, \quad x_1^2 + x_2^2 = 1 \tag{24}$$

can be converted by an incoherent operation to the final state

$$|\Psi_{\mathbf{y}}\rangle = \begin{pmatrix} y_1|\psi_1'\rangle \\ y_2|\psi_2'\rangle \end{pmatrix}, \quad y_1^2 + y_2^2 = 1, \tag{25}$$

where $|\psi_i\rangle$ and $|\psi_i'\rangle$ are arbitrary normalized states in their own subspaces if the majorization condition is valid for the probability vectors $\mathbf{x} = (x_1^2, x_2^2)$ and $\mathbf{y} = (y_1^2, y_2^2)$, i.e., $\mathbf{y} \succ \mathbf{x}$. Note that we have taken the coefficients x_1, x_2, y_1, and y_2 to be real, since block-diagonal unitary operators can always remove any phases from these numbers.

First, we use the fact that in the context of block coherence, the block unitaries of the form $U_1 \oplus U_2$ are regarded as free incoherent operations; hence the states (24) and (25) can freely be converted to

$$|\Phi_{\mathbf{x}}\rangle = \begin{pmatrix} x_1|\phi_1\rangle \\ x_2|\phi_2\rangle \end{pmatrix}, \quad x_1^2 + x_2^2 = 1 \tag{26}$$

and

$$|\Phi_{\mathbf{y}}\rangle = \begin{pmatrix} y_1|\phi_1\rangle \\ y_2|\phi_2\rangle \end{pmatrix}, \quad y_1^2 + y_2^2 = 1, \tag{27}$$

respectively, where

$$|\phi_1\rangle = \frac{1}{\sqrt{d_1}} \sum_{i=1}^{d_1} |e_i\rangle, \quad |\phi_2\rangle = \frac{1}{\sqrt{d_2}} \sum_{i=1}^{d_2} |f_i\rangle \tag{28}$$

are the maximally coherent states of their own subspaces, and to study the state conversion problem it will be enough to investigate the conversion from (26) to (27), without loss of generality. Now consider the generalized incoherent Kraus operators in block-diagonal and anti-block-diagonal forms

$$A_0 = \gamma_0 \begin{pmatrix} \frac{y_1}{x_1}\mathbb{I}_{d_1} & \\ & \frac{y_2}{x_2}\mathbb{I}_{d_2} \end{pmatrix} \tag{29}$$

and

$$A_{ij} = \gamma \begin{pmatrix} & \frac{y_1}{x_2\sqrt{d_1}}|\phi_1\rangle\langle f_j| \\ \frac{y_2}{x_1\sqrt{d_2}}|\phi_2\rangle\langle e_i| & \end{pmatrix}, \tag{30}$$

where the coefficients γ_0 and γ are considered to be real without loss of generality. The completely positive and trace-preserving map $\mathcal{E}_{\text{inc}}$ can then be defined as

$$\mathcal{E}_{\text{inc}}(\rho) = A_0 \rho A_0^\dagger + \sum_{ij} A_{ij} \rho A_{ij}^\dagger. \tag{31}$$

It is now straightforward to check that

$$A_0|\Phi_{\mathbf{x}}\rangle = \gamma_0|\Phi_{\mathbf{y}}\rangle, \quad A_{ij}|\Phi_{\mathbf{x}}\rangle = \frac{\gamma}{\sqrt{d_1 d_2}}|\Phi_{\mathbf{y}}\rangle, \tag{32}$$

which leads to

$$\mathcal{E}_{\text{inc}}(|\Phi_{\mathbf{x}}\rangle\langle\Phi_{\mathbf{x}}) = |\Phi_{\mathbf{y}}\rangle\langle\Phi_{\mathbf{y}}|. \tag{33}$$

Note that we have used the trace-preserving property

$$A_0^\dagger A_0 + \sum_{ij} A_{ij}^\dagger A_{ij} = \mathbb{I}_{d_1+d_2}, \tag{34}$$

which leaves the following constrains on the coefficients γ_0 and γ:

$$\gamma_0^2\left(\frac{y_1}{x_1}\right)^2 + \gamma^2\left(\frac{y_2}{x_1}\right)^2 = 1, \quad \gamma_0^2\left(\frac{y_2}{x_2}\right)^2 + \gamma^2\left(\frac{y_1}{x_2}\right)^2 = 1. \tag{35}$$

One can now easily check that the above conditions, i.e., positivity of γ_0^2 and γ^2, can be satisfied if and only if $\mathbf{x} \prec \mathbf{y}$. To see this, multiply both equations of (35) to $(x_1x_2)^2$ and then subtract them from each other, which after simplification leads to

$$\gamma_0^2\left(x_2^2 y_1^2 - x_1^2 y_2^2\right) + \gamma^2\left(x_2^2 y_2^2 - x_1^2 y_1^2\right) = 0. \tag{36}$$

Now suppose that $x_2 < x_1$ and $y_2 < y_1$. Then positivity of γ_0^2 and γ^2 together with the normalization of probability vectors $\mathbf{x} = (x_1^2, x_2^2)$ and $\mathbf{y} = (y_1^2, y_2^2)$ implies that $x_1 < y_1$. Having the same arguments for the other possible orderings of $\{x_1, x_2\}$ and $\{y_1, y_2\}$, we see that the conditions (35) are equivalent to the majorization condition $\mathbf{x} \prec \mathbf{y}$. Here it should be emphasized that the dimensions d_1 and d_2 of the subspaces are not necessarily equal and the derived majorization condition is solely based on the coefficients x_i and y_i, regardless of the dimensions of subspaces. Finally, note that the role of off-diagonal blocks in A_{ij} is crucial; otherwise one cannot satisfy the trace-preserving condition necessary for the quantum operation.

B. Case of an arbitrary number of subspaces of arbitrary dimensions

The method of the preceding section can be generalized to this case in a straightforward manner. We only need to use a compact notation, as remarked in the beginning of the paper. With the notation introduced in Sec. II and following the discussion presented after Eqs. (24) and (25), it will be enough to investigate the convertibility of the initial general state

$$|\Phi_{\mathbf{x}}\rangle = \sum_{\mu=1}^{M} x_\mu|\mu\rangle \otimes |\phi_\mu\rangle, \quad \sum_{\mu=1}^{M} x_\mu^2 = 1 \tag{37}$$

to the state

$$|\Phi_{\mathbf{y}}\rangle = \sum_{\mu=1}^{M} y_\mu|\mu\rangle \otimes |\phi_\mu\rangle, \quad \sum_{\mu=1}^{M} y_\mu^2 = 1, \tag{38}$$

where

$$|\phi_\mu\rangle = \frac{1}{\sqrt{d_\mu}} \sum_{i_\mu=1}^{d_\mu} |e_{i_\mu}\rangle$$

was defined in (21). Without loss of generality, we assume that all the coefficients are positive. This assumption is justified because block unitary operators can remove any phase from these coefficients.

Let us define the block diagonal operator A_0 as

$$A_0 := \gamma_0 \sum_{\mu=1}^{M} \frac{y_\mu}{x_\mu}\, |\mu\rangle\langle\mu| \otimes \mathbb{I}_{d_\mu}, \tag{39}$$

which in matrix form looks like

$$A_0 = \gamma_0 \begin{pmatrix} \frac{y_1}{x_1}\mathbb{I}_{d_1} & & & & \\ & \frac{y_2}{x_2}\mathbb{I}_{d_2} & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \frac{y_M}{x_M}\mathbb{I}_{d_M} \end{pmatrix}. \tag{40}$$

Let $I = (i_1, i_2, \dots, i_M)$, where $i_\mu \in \{1, 2, \dots, d_\mu\}$, and set $\not{d} := d_1 d_2 \cdots d_M$. Then for any $s \in \{1, 2, \dots, M-1\}$ we define the incoherent Kraus operators

$$A^s{}_I = \frac{\gamma_s}{\sqrt{\not{d}}} \sum_{\mu=1}^{M} \frac{y_\mu}{x_{\mu+s}} \sqrt{d_{\mu+s}} |\mu\rangle\langle\mu+s| \otimes |\phi_\mu\rangle\langle e_{i_{\mu+s}}|. \tag{41}$$

Direct calculation now shows that
(i)

$$A_0|\Phi_{\mathbf{x}}\rangle = \gamma_0|\Phi_{\mathbf{y}}\rangle, \tag{42}$$

(ii)

$$A^s_I|\Phi_{\mathbf{x}}\rangle = \frac{\gamma_s}{\sqrt{\not{d}}}|\Phi_{\mathbf{y}}\rangle\ \forall\, s, I, \tag{43}$$

and (iii) the quantum operation

$$\mathcal{E}(\rho) = A_0\rho A_0^\dagger + \sum_{s,I} A^s_I \rho A^{s\dagger}_I \tag{44}$$

is trace preserving if and only if

$$\gamma_0^2 y_\mu^2 + \sum_{s=1}^{M-1} \gamma_s^2 y_{\mu-s}^2 = x_\mu^2\ \forall\, \mu. \tag{45}$$

The above statements show that the incoherent quantum operation $\mathcal{E}$ can convert the state $|\Phi_{\mathbf{x}}\rangle$ to $|\Phi_{\mathbf{y}}\rangle$ provided the equality (45) holds. This equality is nothing but the condition that the vector $\mathbf{y} = (y_1^2, y_2^2, \dots, y_M^2)$ majorizes $\mathbf{x} = (x_1^2, x_2^2, \dots, x_M^2)$, denoted by $\mathbf{x} \prec \mathbf{y}$ [5].[1] Note that any probability vector $\mathbf{y}$ majorizes the normalized coefficient vector $\frac{1}{M}(1, 1, \dots, 1)$ and hence any quantum state can be obtained by applying a suitable incoherent operation on the maximally coherent states (14), as it is expected from the resource theory of block coherence.

It is instructive to explicitly show this last conversion by another explicit example which conveys the basic idea in a simple and yet general way. Let the Hilbert space be partitioned into three parts $\mathcal{H} = H_1 \oplus H_2 \oplus H_3$ with dimensions d_1, d_2, and d_3, respectively. The orthonormal bases of these Hilbert spaces are given by $\{|e^1_i\rangle, i = 1, \dots, d_1\}$, $\{|e^2_j\rangle, j = 1, \dots, d_2\}$, and $\{|e^3_k\rangle, k = 1, \dots, d_3\}$, respectively. Then the Kraus operators (39) and (41) take the following matrix forms:

$$A_0 = \gamma_0 \begin{pmatrix} \frac{y_1}{x_1}\mathbb{I}_{d_1} & & \\ & \frac{y_2}{x_2}\mathbb{I}_{d_2} & \\ & & \frac{y_3}{x_3}\mathbb{I}_{d_3} \end{pmatrix}, \tag{46}$$

$$A^1_{ijk} = \frac{\gamma_1}{\sqrt{\not{d}}} \begin{pmatrix} 0 & \frac{y_1}{x_2}\sqrt{d_2}|\phi_1\rangle\langle e^2_j| & 0 \\ 0 & 0 & \frac{y_2}{x_3}\sqrt{d_3}|\phi_2\rangle\langle e^3_k| \\ \frac{y_3}{x_1}\sqrt{d_1}|\phi_3\rangle\langle e^1_i| & 0 & 0 \end{pmatrix},$$

$$A^2_{ijk} = \frac{\gamma_2}{\sqrt{\not{d}}} \begin{pmatrix} 0 & 0 & \frac{y_1}{x_3}\sqrt{d_3}|\phi_1\rangle\langle e^3_k| \\ \frac{y_2}{x_1}\sqrt{d_1}|\phi_2\rangle\langle e^1_i| & 0 & 0 \\ 0 & \frac{y_3}{x_2}\sqrt{d_2}|\phi_3\rangle\langle e^2_j| & 0 \end{pmatrix}. \tag{47}$$

It is now easy to check that

$$A_0^\dagger A_0 = \gamma_0^2 \begin{pmatrix} \left(\frac{y_1}{x_1}\right)^2 & & \\ & \left(\frac{y_2}{x_2}\right)^2 & \\ & & \left(\frac{y_3}{x_3}\right)^2 \end{pmatrix} \tag{48}$$

and

$$\sum_{i,j,k} A^{1\dagger}_{ijk}A^1_{ijk} = \gamma_1^2 \begin{pmatrix} \left(\frac{y_3}{x_1}\right)^2 & & \\ & \left(\frac{y_1}{x_2}\right)^2 & \\ & & \left(\frac{y_2}{x_3}\right)^2 \end{pmatrix}, \quad \sum_{i,j,k} A^{2\dagger}_{ijk}A^2_{ijk} = \gamma_2^2 \begin{pmatrix} \left(\frac{y_2}{x_1}\right)^2 & & \\ & \left(\frac{y_3}{x_2}\right)^2 & \\ & & \left(\frac{y_1}{x_3}\right)^2 \end{pmatrix}, \tag{49}$$

which when added together prove the trace-preserving condition (45) for the channel $\mathcal{E}$ defined in (44).

VI. CONSTRUCTING ARBITRARY GATES

In the preceding section, as a result of the majorization condition, we saw that maximally coherent states (14) and (22) are the most resourceful states in the context of state

[1]Proposition (12.11) of [5] states that for two probability vectors $\mathbf{x}$ and $\mathbf{y}$, $\mathbf{y} \succ \mathbf{x}$ if and only if $\mathbf{x} = \sum_j p_j P_j \mathbf{y}$ for some probability distribution p_j and permutation matrices P_j.

conversion. Now we also prove that, starting from these states and only by using incoherent operations, one can implement any arbitrary quantum gate U.

The goal is to perform the unitary operation $U = \sum_{\mu,\nu=1}^{M} |\mu\rangle\langle\nu| \otimes A_{\mu\nu}$ on the arbitrary quantum state $|\Psi\rangle = \sum_{\alpha=1}^{M} x_\alpha |\alpha\rangle \otimes |\psi_\alpha\rangle$. Following the same idea as in [12] and without loss of generality, we use an ancillary system with the maximally coherent state (22) and we define the joint state $|\xi\rangle$,

$$|\xi\rangle = |\Psi\rangle \bigotimes |\Phi\rangle = |\Psi\rangle \bigotimes \frac{1}{\sqrt{M}} \begin{pmatrix} |\phi_1\rangle \\ |\phi_2\rangle \\ \vdots \\ \vdots \\ |\phi_M\rangle \end{pmatrix}. \tag{50}$$

Now consider the incoherent Kraus operators $\mathcal{K}_s$, $s = 1, \dots, M$, defined as

$$\mathcal{K}_s = \sum_{\mu,\nu=1}^{M} |\mu\rangle\langle\nu| \otimes A_{\mu\nu} \bigotimes |s\rangle\langle\mu+s| \otimes |\phi_s\rangle\langle\phi_{\mu+s}|. \tag{51}$$

It is easy to show that $\sum_{s=1}^{M} \mathcal{K}_s^\dagger \mathcal{K}_s = I$ and $\mathcal{K}_s \mathcal{I}_{\text{inc}} \mathcal{K}_s^\dagger \subset \mathcal{I}_{\text{inc}}$. By straightforward calculations, we find that

$$\mathcal{K}_s|\xi\rangle = \frac{1}{\sqrt{M}} U|\Psi\rangle \bigotimes (|s\rangle \otimes |\phi_s\rangle), \tag{52}$$

which leads to the quantum channel

$$\sum_s \mathcal{K}_s(|\xi\rangle\langle\xi|)\mathcal{K}_s^\dagger = U|\Psi\rangle\langle\Psi|U^\dagger \bigotimes \rho_{\text{inc}}, \tag{53}$$

where

$$\rho_{\text{inc}} = \frac{1}{M} \sum_\mu |\mu\rangle\langle\mu| \otimes |\phi_\mu\rangle\langle\phi_\mu| \tag{54}$$

is the completely decohered form of the maximally coherent state $|\Phi\rangle$ which we started with. Thus, by consuming a maximally coherent state we can implement any unitary operator on any arbitrary state.

VII. BLOCK-COHERING AND -DECOHERING POWER

Using the definition of block coherence, one can also define the block-cohering and block-decohering powers of a quantum channel $\mathcal{E}$, just like the definitions of [24] or [25] for cohering and decohering powers. Following the definitions presented in [24], the block-cohering power (BCP) and the block-decohering power (BDP) of a channel will be defined as

$$\text{BCP}(\mathcal{E}) = \max_{\rho_{\text{inc}} \in \mathcal{I}_{\text{inc}}} C(\mathcal{E}(\rho_{\text{inc}})), \tag{55}$$

$$\text{BDP}(\mathcal{E}) = \max_{|\Psi\rangle_{\text{MC}}} [C(|\Psi\rangle_{\text{MC}}\langle\Psi|) - C(\mathcal{E}(|\Psi\rangle_{\text{MC}}\langle\Psi|))], \tag{56}$$

respectively, where C is any well-defined block-coherence measure, ρ_{inc} is chosen from the set of block-incoherent states (6), and $|\Psi\rangle_{\text{MC}}$ stands for maximally block-coherent states of the form (14). Equations (55) and (56) mean that the BCP of a channel is equal to the maximum amount of block coherence that can be generated for an initial block-incoherent state, and the BDP of a channel is the maximum amount of block coherence of a maximally block-coherent state that is destroyed by the quantum channel. Using the above definitions, one can now calculate the BCP and BDP of any quantum channel. Below we will study some channels that are of practical importance.

A. Examples of block-cohering power

We first follow the same argument as in [24] to show that for any quantum channel $\mathcal{E}$, linearity of the channel $\mathcal{E}$ and convexity of the coherence measure allow us to write

$$\text{BCP}(\mathcal{E}) = \max_{|\Psi_{\text{inc}}\rangle} C(\mathcal{E}(|\Psi_{\text{inc}}\rangle\langle\Psi_{\text{inc}}|)), \tag{57}$$

where $|\Psi_{\text{inc}}\rangle$ is an incoherent pure state. Note that an incoherent pure state $|\Psi_{\text{inc}}\rangle$ has only one nonzero state in a given subspace, i.e.,

$$|\Psi_\nu\rangle = |\nu\rangle \otimes |\psi_\nu\rangle = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ |\psi_\nu\rangle \\ \cdot \\ 0 \end{pmatrix}. \tag{58}$$

We now proceed to show Eq. (57) and then we will prove a theorem and study a few examples.

Lemma. The BCP of a channel that is defined in (55) is equal to (57).

Proof. Consider an incoherent state $\rho_{\text{inc}} = \sum_{\mu=1}^{M} p_\mu |\mu\rangle\langle\mu| \otimes \rho_\mu$. Then by considering the pure state decomposition of each ρ_μ, the above incoherent state can be written as

$$\rho_{\text{inc}} = \sum_{\mu,j} p_\mu q_\mu^{(j)} |\mu\rangle\langle\mu| \otimes |\psi_\mu^j\rangle\langle\psi_\mu^j|,$$

where $\{q_\mu^{(j)}\}$ is a probability distribution for each μ. From there,

$$\begin{aligned} C(\mathcal{E}(\rho_{\text{inc}})) &= C\left(\sum_{\mu,j} p_\mu q_\mu^{(j)} \mathcal{E}\big(|\mu\rangle\langle\mu| \otimes |\psi_\mu^j\rangle\langle\psi_\mu^j|\big)\right) \\ &\leqslant \sum_{\mu,j} p_\mu q_\mu^{(j)} C\big(\mathcal{E}\big(|\mu\rangle\langle\mu| \otimes |\psi_\mu^j\rangle\langle\psi_\mu^j|\big)\big) \\ &\leqslant C\big(\mathcal{E}\big(|\alpha\rangle\langle\alpha| \otimes |\psi_\alpha^i\rangle\langle\psi_\alpha^i|\big)\big), \end{aligned} \tag{59}$$

where α and i are the block and state numbers that have the largest value of $C(\mathcal{E}(|\mu\rangle\langle\mu| \otimes |\psi_\mu^j\rangle\langle\psi_\mu^j|))$ among all possible values of μ and j. Equation (58) proves the theorem, which states that the maximization of (55) can only be performed over pure incoherent states.

This lemma not only simplifies the calculation of cohering power, but also gives us an alternative method for characterization of incoherent Kraus operators. In Sec. III we indicated that any quantum channel whose Kraus operators are of the form (7) [exemplified in (8)] cannot produce any coherence. We now present an alternative proof of this fact. This proof provides us with tools which enable us to calculate in a direct way the BCP of many other channels.

Theorem. Based on the definition (55), the block-cohering power of any quantum channel whose Kraus operators are of the form (7) is zero.

Proof. For definiteness, consider a pure incoherent state of the form $|\Psi_1\rangle$ (a similar analysis applies to other states $|\Psi_\mu\rangle$). Consider a quantum channel $\mathcal{E}$, with Kraus operators of the form $K^i = |\mu\rangle\langle\nu| \otimes K^i_{\mu\nu}$. Note that in each block of K^i we have an operator $K^i_{\mu\nu} : L(H_\nu) \to L(H_\mu)$. The action of this Kraus operator on the state $|\Psi_1\rangle$ leads to a non-normalized vector of the form

$$|\Phi^i\rangle := \begin{pmatrix} K^i_{11}|\psi_1\rangle \\ K^i_{21}|\psi_1\rangle \\ K^i_{31}|\psi_1\rangle \\ \vdots \\ \vdots \\ K^i_{M1}|\psi_1\rangle \end{pmatrix}, \tag{60}$$

and from there we can write

$$\mathcal{E}(|\Psi_1\rangle\langle\Psi_1|) = \sum_i |\Phi^i\rangle\langle\Phi^i|. \tag{61}$$

The convexity of coherence measure again leads to

$$C(\mathcal{E}(|\Psi_1\rangle\langle\Psi_1|)) = C\left(\sum_i (|\Phi^i\rangle\langle\Phi^i|)\right) \leqslant \sum_i C(|\Phi^i\rangle\langle\Phi^i|), \tag{62}$$

and by using (18) for the coherence measure of pure states we find

$$C_1(\mathcal{E}(|\Psi_1\rangle\langle\Psi_1|)) \leqslant \sum_i \sum_{\alpha\neq\beta} \sqrt{\langle\psi_1|{K^i_{\beta 1}}^\dagger K^i_{\beta 1}|\psi_1\rangle\langle\psi_1|{K^i_{\alpha 1}}^\dagger K^i_{\alpha 1}|\psi_1\rangle}. \tag{63}$$

This means that if a quantum channel is such that all its Kraus operators have only one nonzero element in each column block, then the cohering power of that channel is zero. This is in accord with our previous statement in (7).

Example 1: The BCP of a unitary operator. Let $\mathcal{E}_u(\rho) = U\rho U^\dagger$ be a unitary channel acting on a pure incoherent state $|\Psi_\nu\rangle = |\nu\rangle \otimes |\psi_\nu\rangle$, where the nonzero state exists in the νth block. The block structure of U is revealed when we write it as $U = \sum_{\alpha\beta} |\alpha\rangle\langle\beta| \otimes A_{\alpha\beta}$. We then find

$$U|\Psi_\nu\rangle = \begin{pmatrix} A_{1\nu}|\psi_\nu\rangle \\ A_{2\nu}|\psi_\nu\rangle \\ \vdots \\ \vdots \\ A_{M\nu}|\psi_\nu\rangle \end{pmatrix}.$$

We then find, from (18) and (57),

$$\begin{aligned} C_1^M(U|\Psi_\nu\rangle) &= \sum_{\mu\neq\mu'} \sqrt{\langle\psi_\nu|A^\dagger_{\mu\nu}A_{\mu\nu}|\psi_\nu\rangle\langle\psi_\nu|A^\dagger_{\mu'\nu}A_{\mu'\nu}|\psi_\nu\rangle} \\ &= \sum_{\mu,\mu'} \sqrt{\langle\xi_{\mu,\nu}|\xi_{\mu,\nu}\rangle\langle\xi_{\mu',\nu}|\xi_{\mu'\nu}\rangle} - \sum_\mu \langle\xi_{\mu,\nu}|\xi_{\mu,\nu}\rangle \\ &= \left(\sum_\mu \sqrt{\langle\xi_{\mu,\nu}|\xi_{\mu,\nu}\rangle}\right)^2 - 1, \end{aligned} \tag{64}$$

where $|\xi_{\mu,\nu}\rangle = A_{\mu\nu}|\psi_\nu\rangle$ and in the last line we have used unitarity of U to set $\sum_\mu \langle\xi_{\mu,\nu}|\xi_{\mu,\nu}\rangle = 1$. Therefore, by taking the initial incoherent state to be a state where $|\psi\rangle$ can be in any of the rows, we find the BCP of a general unitary operator

$$\mathrm{BCP}(U) = \max_{\{\nu,|\psi_\nu\rangle\}} \left(\sum_\mu \sqrt{\langle\xi_{\mu\nu}|\xi_{\mu\nu}\rangle}\right)^2 - 1, \tag{65}$$

where

$$|\xi_{\mu\nu}\rangle = A_{\mu\nu}|\psi_\nu\rangle. \tag{66}$$

As the simplest case, let $M = 2$ and consider the cohering power of a unitary operator $U = \begin{pmatrix} A & B \\ C & D\end{pmatrix}$, acting on $\mathcal{H} = H_{d_1} \oplus H_{d_2}$, where A, B, C, and D are $d_1 \times d_1, d_1 \times d_2, d_2 \times d_1$, and $d_2 \times d_2$ dimensional, respectively. Following (65), we find for this unitary operator

$$\begin{aligned} \mathrm{BCP}(U) = \max_{|\psi_1\rangle,|\psi_2\rangle} [&(\sqrt{\langle\psi_1|A^\dagger A|\psi_1\rangle} + \sqrt{\langle\psi_1|C^\dagger C|\psi_1\rangle})^2, \\ &(\sqrt{\langle\psi_2|B^\dagger B|\psi_2\rangle} + \sqrt{\langle\psi_2|D^\dagger D|\psi_2\rangle})^2] - 1. \end{aligned} \tag{67}$$

As an explicit example, let $M = 2$ and U be a unitary operator acting on $\mathcal{H} = H_2 \otimes H_N = H_N \oplus H_N$, of the form $U = \begin{pmatrix} a\mathbb{I}_N & bV \\ -b^* V^\dagger & a^*\mathbb{I}_N\end{pmatrix}$, where a and b are complex numbers subject to $|a|^2 + |b|^2 = 1$, $\mathbb{I}_N$ is the identity operator, and V is an arbitrary unitary operator acting on H_N. For this operator, a simple calculation shows that

$$\mathrm{BCP}(U) = (|a| + |b|)^2 - 1 = 2|ab|, \tag{68}$$

which shows that the Hadamard-like block operator $U = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{I}_N & V \\ -V^\dagger & \mathbb{I}_N\end{pmatrix}$ has maximum BCP, as it should.

Example 2: The BCP of the tensor product of two operators. Consider the tensor product of two unitary operators $W = U \otimes V$, where U is M dimensional and V is N dimensional. This operator acts on $\mathcal{H} = \oplus_{\mu=1}^M H_\mu$ where all the subspaces H_μ are N dimensional. Each block $\mu\nu$ of the unitary matrix W is of the form $W_{\mu\nu} = u_{\mu\nu}V$, where $u_{\mu\nu}$ is a complex number and is the $(\mu\nu)$th entry of the unitary matrix U. Inserting this in Eq. (66), we see that $|\xi_{\mu\nu}\rangle = u_{\mu\nu}V|\psi_\nu\rangle$, and following the result (65), we find

$$\mathrm{BCP}(U \otimes V) = \max_\nu \left(\sum_\mu \sqrt{u_{\mu\nu}u^*_{\mu\nu}}\right)^2 - 1. \tag{69}$$

However, this is nothing but the ordinary cohering power of a unitary matrix U as defined in [24]. Thus we have shown that understandably the BCP of a unitary operator $U \otimes V$ is nothing but the ordinary cohering power of the unitary matrix U, as the matrix V acts within each block and it is the matrix U which acts between blocks.

Example 3: The BCP of a random unitary channel. Consider now a random unitary operator of the form

$$\mathcal{E}(\rho) = \sum_i p_i(U^i \otimes V^i)\rho(U^i \otimes V^i)^\dagger,$$

acting on $\mathcal{H} = (H_M \otimes H_N)$. By considering the block structure $\mathcal{H} = \oplus_{\mu=1}^M H_\mu$, where all the subspaces H_μ are N

dimensional, from (57) we have

$$\mathrm{BCP}(\mathcal{E}) = \max_{|\Psi_{\mathrm{inc}}\rangle} C_1^M\left(\sum_i p_i(U^i\otimes V^i)|\Psi_{\mathrm{inc}}\rangle\langle\Psi_{\mathrm{inc}}|(U^i)^\dagger\otimes(V^i)^\dagger\right). \tag{70}$$

As in previous examples, consider the pure incoherent input state $|\Psi_\nu\rangle$, with the only nonzero entity $|\psi_\nu\rangle$ in the νth block. We find after straightforward calculations

$$\mathcal{E}(|\Psi_\nu\rangle\langle\Psi_\nu|) = \begin{pmatrix} B_{11}^\nu & B_{12}^\nu & \cdots & B_{1M}^\nu \\ B_{21}^\nu & B_{22}^\nu & \cdots & B_{2M}^\nu \\ \vdots & \vdots & \vdots & \vdots \\ B_{M1}^\nu & B_{M2}^\nu & \cdots & B_{MM}^\nu \end{pmatrix}, \tag{71}$$

where $B_{\mu\mu'}^\nu$ are the N-dimensional matrices

$$B_{\mu\mu'}^\nu = \sum_i p_i(U^i)_{\mu\nu}\overline{(U^i)_{\mu'\nu}}\, V_i|\psi_\nu\rangle\langle\psi_\nu|V_i^\dagger. \tag{72}$$

In view of the relations (15) and (57), we find

$$\mathrm{BCP}(\mathcal{E}) = \max_{\nu,|\psi_\nu\rangle}\sum_{\mu\neq\mu'}\left\|B_{\mu\mu'}^\nu\right\|_1. \tag{73}$$

As a very simple example, we find, after some simple calculations, that for the channel $\mathcal{E}(\rho) = (1-p)\rho + (U\otimes V)\rho(U\otimes V)^\dagger$ acting on two qubits, with $U = \begin{pmatrix} a & b \\ -b^* & a\end{pmatrix}$, $\mathrm{BCP}(\mathcal{E}) = 2p|ab|$.

B. Examples of block-decohering power

In this section we use (56) to calculate the block-decohering power of a few channels.

Example: The BDP of a unitary channel. Let $\mathcal{E}_u(\rho) = U\rho U^\dagger$ be a unitary channel acting on a maximally block-coherent state $|\Psi_{\mathrm{MC}}\rangle = \frac{1}{\sqrt{M}}\sum_\mu|\mu\rangle\otimes|\psi_\mu\rangle$. The block coherence of this state is, from (19), equal to $C_1^M(|\Psi_{\mathrm{MC}}\rangle) = M-1$. The block structure of U is revealed when we write it as $U = \sum_{\alpha\beta}|\alpha\rangle\langle\beta|\otimes A_{\alpha\beta}$. We then find $U|\Psi_{\mathrm{MC}}\rangle = \frac{1}{\sqrt{M}}\sum_{\alpha\mu}|\alpha\rangle\otimes A_{\alpha\mu}|\psi_\mu\rangle$. The block coherence of this state is determined from (18) to be

$$C_1^M(U|\Psi_{\mathrm{MC}}\rangle) = \frac{1}{M}\sum_{\alpha\neq\alpha'}\sqrt{\langle\chi_\alpha|\chi_\alpha\rangle\langle\chi_{\alpha'}|\chi_{\alpha'}\rangle}, \tag{74}$$

where $|\chi_\alpha\rangle = \sum_\mu A_{\alpha\mu}|\psi_\mu\rangle$. This can be rewritten as

$$C_1^M(U|\Psi_{\mathrm{MC}}\rangle) = \frac{1}{M}\left[\left(\sum_\alpha\sqrt{\langle\chi_\alpha|\chi_\alpha\rangle}\right)^2 - \sum_\alpha\langle\chi_\alpha|\chi_\alpha\rangle\right]. \tag{75}$$

Using unitarity of U, we note that $\sum_\alpha\langle\chi_\alpha|\chi_\alpha\rangle = \sum_\alpha\sum_{\mu,\nu}\langle\psi_\mu|\psi_\mu\rangle = M$. The block-decohering power will then be

$$\mathrm{BDP}(\mathcal{E}_u) = M - \frac{1}{M}\min_{\{\psi_\mu\}}\left(\sum_\alpha\sqrt{\langle\chi_\alpha|\chi_\alpha\rangle}\right)^2. \tag{76}$$

As an explicit example, let $M=2$ and U be a unitary operator acting on $\mathcal{H} = H_2\otimes H_N$ of the form $U = \begin{pmatrix} aI_N & bV \\ -b^*V^\dagger & a^*I_N\end{pmatrix}$, where a and b are complex numbers subject to $|a|^2+|b|^2=1$, I_N is the identity operator, and V is an arbitrary unitary operator acting on H_N. For this unitary operator we have

$$|\chi_1\rangle = a|\psi_1\rangle + bV|\psi_2\rangle, \quad |\chi_2\rangle = -b^*V^\dagger|\psi_1\rangle + a^*|\psi_2\rangle, \tag{77}$$

leading to

$$\mathrm{BDP}(U) = 2 - \tfrac{1}{2}\min_{\{|\psi_1\rangle,|\psi_2\rangle\}}(\sqrt{1+x}+\sqrt{1-x})^2, \tag{78}$$

where $x = 2\,\mathrm{Re}(a^*b\langle\psi_1|V|\psi_2\rangle)$. The minimum value of the function $f(x) = \sqrt{1+x}+\sqrt{1-x}$ is obtained at $x=\pm1$. This demands that the maximally coherent state in (56) which defines the decohering power of the above unitary operator U should be chosen such that

$$|\psi_2\rangle = e^{-i\arg(a^*b)}V^\dagger|\psi_1\rangle, \tag{79}$$

which leads to the following BDP for the operator U:

$$\mathrm{BDP}(U) = 1 - \sqrt{1-4|ab|^2}. \tag{80}$$

Understandably, for any block-diagonal or block-anti-diagonal operator, this will give zero BDP and for the block Hadamard operator $H = \frac{1}{\sqrt{2}}\begin{pmatrix}\mathbb{I}_N & \mathbb{I}_N \\ \mathbb{I}_N & -\mathbb{I}_N\end{pmatrix}$ it will give $\mathrm{BDP}(\mathcal{E}_H) = 1$. This last example is in fact a manifestation of a more general pattern which can be proved by a simple and similar equation for any block structure.

Proposition. For any unitary operator $U\otimes V$ acting on $\mathcal{H} = H_M\otimes H_N = \oplus_{\mu=1}^M H_N$, we have $\mathrm{BDP}(U\otimes V) = \mathrm{BDP}(U)$, which is intuitively plausible.

VIII. RELATION BETWEEN BLOCK COHERENCE AND k COHERENCE

The original notion of incoherence [11,12], which defines incoherent states as diagonal density matrices in a specific basis, has been aptly generalized to multilevel or k coherence [16–19]. In this generalized setting, a state

$$|\psi\rangle = \sum_{i=1}^d c_i|i\rangle \tag{81}$$

is said to have coherence at level k if exactly k of the coefficients c_i are nonzero. Thus an incoherent state has coherence at level 1, a state like $|\psi\rangle = a|0\rangle + b|1\rangle$ in $\mathcal{H}_d$ has coherence at level 2, and so on. A state with coherence at level k is said to have coherence rank equal to k: $r_C(|\psi\rangle) = k$. The generalization to mixed states is done by defining the states with coherence level k to be the convex combination of all pure states whose coherence level is less than or equal to k, i.e.,

$$\mathcal{C}_k := \mathrm{conv}\{|\psi\rangle\langle\psi|,\,|\,r_C(|\psi\rangle)\leqslant k\}. \tag{82}$$

Obviously these sets obey the following inclusion relation:

$$\mathcal{C}_1\subset\mathcal{C}_2\subset\mathcal{C}_3\subset\cdots\subset\mathcal{C}_d. \tag{83}$$

The relation between k coherence and block coherence is interesting, and we explore it in this section. For the

$$\rho^{(1)} = \begin{pmatrix} * & * & & \\ * & * & & \\ & & \circ & \circ \\ & & \circ & \circ \end{pmatrix} \qquad \rho^{(2)} = \begin{pmatrix} * & & * & \\ & \circ & & \circ \\ * & & * & \\ & \circ & & \circ \end{pmatrix} \qquad \rho^{(3)} = \begin{pmatrix} * & & & * \\ & \circ & \circ & \\ & \circ & \circ & \\ * & & & * \end{pmatrix}$$

(a) (b) (c)

FIG. 1. Three different block structures for a density matrix of two particles: (a) $\rho^{(1)}$, (b) $\rho^{(2)}$, and (c) $\rho^{(3)}$. The basis states are ordered as $|00\rangle$, $|01\rangle$, $|10\rangle$, and $|11\rangle$. These correspond to three different block structures $\mathcal{H}_4 = H_2 \oplus H_2$. All these states belong to $\mathcal{C}_2 \subset \mathcal{H}_4$.

sake of simplicity, we describe this relation by presenting an explicit simple example. The basic idea can then be understood in the general setting. Consider a density matrix $\rho \in L(\mathcal{H}_4)$. It is physically more interesting to consider the example of two particles (ions in an ion trap), although this restriction is not necessary. Thus $\mathcal{H}_4$ is the four-dimensional space of two qubits and the preferred basis is taken to be $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. Figure 1 shows three different block structures of this matrix.

In Fig. 1(a) the density matrix is given by

$$\rho^{(1)} = |0\rangle\langle 0| \otimes \rho_0 + |1\rangle\langle 1| \otimes \rho_1, \tag{84}$$

which indicates that the first particle has no coherence at all, due to a measurement of the first particle in the computational basis. Decomposition of the states ρ_0 and ρ_1 casts this state into the form

$$\rho^{(1)} = \sum_i |\psi_i\rangle\langle\psi_i| + \sum_i |\phi_i\rangle\langle\phi_i|, \tag{85}$$

where

$$|\psi_i\rangle = \alpha_i|0,0\rangle + \beta_i|0,1\rangle, \quad |\phi_i\rangle = \gamma_i|1,0\rangle + \delta_i|1,1\rangle. \tag{86}$$

(Note that in the above equations and in the ones that follow in this section, we use a minimal notation in order not to clutter the notation. Thus we use non-normalized states and density matrices and we also use repetitive symbols.) This shows that $\rho^{(1)}$ is the convex combination of coherent states of level 2 and thus $\rho^{(1)} \in \mathcal{C}_2$. However, not all states of $\mathcal{C}_2$ are of this form, since not all 2-coherent pure states are involved in this decomposition. Consider now another block structure shown in Fig. 1(b), induced by measurements on the second particle, again in the computational basis. Following the same argument as before, the state is now given by

$$\rho^{(2)} = \rho_0 \otimes |0\rangle\langle 0| + \rho_1 \otimes |1\rangle\langle 1|, \tag{87}$$

or after decomposition of the states ρ_0 and ρ_1,

$$\rho^{(2)} = \sum_i |\psi_i\rangle\langle\psi_i| + \sum_i |\phi_i\rangle\langle\phi_i|, \tag{88}$$

where

$$|\psi_i\rangle = \alpha_i|0,0\rangle + \beta_i|1,0\rangle, \quad |\phi_i\rangle = \gamma_i|0,1\rangle + \delta_i|1,1\rangle, \tag{89}$$

and thus again we find $\rho^{(2)} \in \mathcal{C}_2$. The states of the form $\rho^{(1)}$ and $\rho^{(2)}$ do not still comprise all the states of $\mathcal{C}_2$. This is due to the fact that we have not exhausted all the block structures, i.e., measurements. The last block structure is shown in Fig. 1(c) and is induced by a measurement with projectors $\pi_0 = |00\rangle\langle 00| + |11\rangle\langle 11|$ and $\pi_1 = |01\rangle\langle 01| + |10\rangle\langle 10|$, i.e., a measurement which determines the equality or difference of the two qubits. The state is then written as

$$\rho^{(3)} = \sum_i |\psi_i\rangle\langle\psi_i| + \sum_i |\phi_i\rangle\langle\phi_i|, \tag{90}$$

where

$$|\psi_i\rangle = \alpha_i|0,0\rangle + \beta_i|1,1\rangle, \quad |\phi_i\rangle = \gamma_i|0,1\rangle + \delta_i|1,0\rangle. \tag{91}$$

This shows that $\rho^{(3)} \in \mathcal{C}_2$ too and any state in $\mathcal{C}_2$ is of the form $\rho^{(1)}$, $\rho^{(2)}$, or $\rho^{(3)}$. After seeing this simple example, we are ready to state the relation between block coherence and k coherence.

Suppose that we have a block structure B_k based on the decomposition of the Hilbert space $\mathcal{H}_d = \oplus_{\mu=1}^{M} H_\mu$, subject to the constraint

$$\dim(H_\mu) \leqslant k \,\forall\, \mu. \tag{92}$$

Then, according to Eq. (6), the set of incoherent states with regard to block structure B_k is

$$\mathcal{I}_{\text{inc}}^{(B_k)} = \left\{ \rho \mid \rho = \sum_\mu \pi_\mu \rho \pi_\mu,\ \text{rank}(\pi_\mu) \leqslant k \right\}, \tag{93}$$

where π_μ is the projection operator on the subspace H_μ. We now conjecture the relation between block coherence and k coherence,

$$\bigcup_B \mathcal{I}_{\text{inc}}^{(B_k)} = \mathcal{C}_k, \tag{94}$$

where $\bigcup_B$ means a union over all block structures of the form (92).

In passing, one may be tempted to ask why an alternative definition of k coherence has not been adopted from the very beginning for mixed states, i.e., one in which diagonal density matrices are 1-coherent states, three-diagonal density matrices are 2-coherent states, five-diagonal states are 2-coherent states, etc. We think that while this categorization is in principle possible, it is not motivated by physical measurements, even on adjacent particles in a many-body system. The simple two-particle system that we have analyzed in this section may not show this clearly, but it is easily seen in a three-particle system with basis states $\{|000\rangle, |001\rangle, |010\rangle, |011\rangle, |100\rangle, |101\rangle, |110\rangle, |111\rangle\}$ that measurement of the second particle in the basis $\{|0\rangle, 1\rangle\}$ entails a block structure which contains nonzero elements far from the diagonal. To our understanding, this explains why the definition of k coherence as adopted in [16–19] is the natural one.

IX. CONCLUSION

The concept of block coherence, based on projective measurement, was first introduced in [11] and then generalized via Naimark extension in [14,15] to include POVMs. In these works certain general properties of the resource theory of block coherence were proved. In the present work we restricted ourselves to projective measurements and adopted a notational framework which facilitated many explicit calculations. In particular, this enabled us to prove that a majorization condition is sufficient and necessary for state transformation using block-incoherent operations (Sec. V and the Appendix). Moreover, we were able to define the block-cohering power and block-decohering power of quantum operations (Sec. VII), as an extension of the works in [24,25].

This framework makes it also possible to connect block coherence, in a transparent way, with other generalized notions of coherence. An example is the connection with k coherence, which is discussed via a simple example in Sec. VIII. Within this framework it is also possible to extend other classes of resource theories to their block form. An example is the dephasing covariant incoherent operations. In ordinary resource theory of coherence, a quantum operation $\mathcal{E}$ is a dephasing covariant incoherent operation if it commutes with the dephasing operation $\Delta:\rho\rightarrow\sum_\mu|\mu\rangle\langle\mu|\rho|\mu\rangle\langle\mu|$. Many of the results in [26,30,31] on this kind of resource theory can be readily extended after proper modifications by defining block-dephasing covariant incoherent operations as those which commute with the block-dephasing operator Δ^B : $\rho\rightarrow\sum_\mu|\mu\rangle\langle\mu|\otimes\rho_{\mu\nu}$, where ρ_μ is now the operator on a block. Actually, such an extension seems to be present also in [26], where a large system is partitioned into subsystems, each carrying out a different representation of the translation symmetry group.

ACKNOWLEDGMENT

We would like to thank one of the anonymous referees for valuable comments and suggestions, which led to an extensive revision of this paper in many respects.

APPENDIX: PROOF OF THE NECESSARY CONDITION FOR STATE TRANSFORMATION

In the main text we constructed a block-incoherent operation with specific forms for the Kraus operators which, provided that $\mathbf{x}\prec\mathbf{y}$, transforms the state $|\Phi_\mathbf{x}\rangle$ to $|\Phi_\mathbf{y}\rangle$. We now prove the converse statement: If there is any block-incoherent operation (with any type of incoherent Kraus operators) which transforms $|\Phi_\mathbf{x}\rangle$ into $|\Phi_\mathbf{y}\rangle$, then necessarily the majorization condition holds, that is, $\mathbf{x}\prec\mathbf{y}$. Thus majorization is both a necessary and sufficient condition for this transformation. The basic idea of the proof of necessity can be conveyed in the simple case where we have two subspaces, i.e., $M=2$. This saves us and the reader from cluttered formulas and notation. The argument for the general case of arbitrary number of subspaces is a straightforward generalization.

So let $\mathcal{H}=H_1\oplus H_2$ and suppose that there is an incoherent operation $\mathcal{E}(\rho)=\sum_n K_n\rho K_n^\dagger$, which converts the initial state

$$|\Phi_\mathbf{x}\rangle=\begin{pmatrix}x_1|\phi_1\rangle\\ x_2|\phi_2\rangle\end{pmatrix},\quad x_1^2+x_2^2=1 \tag{A1}$$

to the final state

$$|\Phi_\mathbf{y}\rangle=\begin{pmatrix}y_1|\phi_1\rangle\\ y_2|\phi_2\rangle\end{pmatrix},\quad y_1^2+y_2^2=1, \tag{A2}$$

where without loss of generality we have taken the coefficients x_μ and y_μ to be real. We will now prove that if

$$\sum_a K_a|\Phi_\mathbf{x}\rangle\langle\Phi_\mathbf{x}|K_a^\dagger=|\Phi_\mathbf{y}\rangle\langle\Phi_\mathbf{y}|, \tag{A3}$$

then $\mathbf{x}\prec\mathbf{y}$, where $\mathbf{x}=(x_1^2,x_2^2)$ and $\mathbf{y}=(y_1^2,y_2^2)$.

According to (7), the general form of an incoherent Kraus operator is such that it has exactly only one nonzero block in each column and can be written in the form

$$K_a=\sum_\mu|a(\mu)\rangle\langle\mu|\otimes K_\mu^a, \tag{A4}$$

in which $a:\{1,2\}\rightarrow\{1,2\}$ is an arbitrary function. By using a suitable permutation P_a on the blocks, the above Kraus operator can be cast into the form

$$K_a=P_a\sum_\mu|a(\mu)\rangle\langle\mu|\otimes K_\mu^a, \tag{A5}$$

where $a(\mu)$ is now restricted such that $1\leqslant a(\mu)\leqslant\mu$. Hence, without loss of generality, we can write the following form for the incoherent Kraus operator K_a;

$$K_a=P_a\begin{pmatrix}K_1^a & \delta_{1,a(2)}K_2^a\\ 0 & \delta_{2,a(2)}K_2^a\end{pmatrix}. \tag{A6}$$

The permutation matrix P_a preceding the upper triangular Kraus operator effectively covers all the possible forms of the incoherent Kraus operators K_a. (For a higher number of subspaces, e.g., when $M=3$, the above form of the Kraus operators is replaced with

$$K_a=P_a\begin{pmatrix}K_1^a & \delta_{1,a(2)}K_2^a & \delta_{1,a(3)}K_3^a\\ 0 & \delta_{2,a(2)}K_2^a & \delta_{2,a(3)}K_3^a\\ 0 & 0 & \delta_{3,a(3)}K_3^a\end{pmatrix}$$

and all the arguments which follow are repeated.) From the condition $\sum_a K_a^\dagger K_a=I$ we get

$$\sum_a {K_\mu^a}^\dagger K_\mu^a=I_\mu,\quad \mu=1,2$$
$$\sum_a\delta_{1,a(2)}{K_1^a}^\dagger K_2^a=\mathbf{0}_{d_1\times d_2}. \tag{A7}$$

On the other hand, according to Eq. (A3), for each a there exists a complex number α_a such that $K_a|\Phi_\mathbf{x}\rangle=\alpha_a|\Phi_\mathbf{y}\rangle$ and hence

$$P_a\begin{pmatrix}x_1K_1^a|\phi_1\rangle+x_2\delta_{1,a(2)}K_2^a|\phi_2\rangle\\ x_2\delta_{2,a(2)}K_2^a|\phi_2\rangle\end{pmatrix}=\alpha_a\begin{pmatrix}y_1|\phi_1\rangle\\ y_2|\phi_1\rangle\end{pmatrix}, \tag{A8}$$

or equivalently

$$\begin{pmatrix}x_1K_1^a|\phi_1\rangle+x_2\delta_{1,a(2)}K_2^a|\phi_2\rangle\\ x_2\delta_{2,a(2)}K_2^a|\phi_2\rangle\end{pmatrix}=\alpha_a\begin{pmatrix}y_{P_a^{-1}(1)}|\phi_{P_a^{-1}(1)}\rangle\\ y_{P_a^{-1}(2)}|\phi_{P_a^{-1}(2)}\rangle\end{pmatrix}, \tag{A9}$$

where P_a^{-1} is the inverse of the permutation operator P_a.

Equating the norms of vectors in each block on both sides of (A9) and summing over a and using (A7), we find

$$x_1^2 + x_2^2 \sum_a \delta_{1,a(2)} = \sum_a |\alpha_a|^2 y^2_{P_a^{-1}(1)}, \quad \text{(A10a)}$$

$$x_2^2 \sum_a \delta_{2,a(2)} = \sum_a |\alpha_a|^2 y^2_{P_a^{-1}(2)}. \quad \text{(A10b)}$$

From Eq. (A10a) it is evident that

$$x_1^2 \leqslant \sum_a |\alpha_a|^2 y^2_{P_a^{-1}(1)}. \quad \text{(A11)}$$

By adding Eqs. (A10a) and (A10b) we will also find that

$$x_1^2 + x_2^2 = \sum_a |\alpha_a|^2 y^2_{P_a^{-1}(1)} + \sum_a |\alpha_a|^2 y^2_{P_a^{-1}(2)}. \quad \text{(A12)}$$

From Eqs. (A10) it is evident that

$$\left(x_1^2, x_2^2\right) \prec \left(\sum_a |\alpha_a|^2 y^2_{P_a^{-1}(1)}, \sum_a |\alpha_a|^2 y^2_{P_a^{-1}(2)}\right). \quad \text{(A13)}$$

Now note that for $\mu = 1, 2$,

$$\sum_a |\alpha_a|^2 y^2_{P_a^{-1}(\mu)} = \sum_{a, P_a^{-1}(\mu)=1} |\alpha_a|^2\, y_1^2 + \sum_{a, P_a^{-1}(\mu)=2} |\alpha_a|^2\, y_2^2. \quad \text{(A14)}$$

Let $b_{\mu\nu} := \sum_{a, P_a^{-1}(\mu)=\nu} |\alpha_a|^2$ for $\mu, \nu \in \{1, 2\}$. Then, in view of the relation $\sum_a |\alpha_a|^2 = 1$, the matrix $B = (b_{\mu\nu})$ is a doubly stochastic matrix and

$$B\left(y_1^2, y_2^2\right)^t = \left(\sum_a |\alpha_a|^2 y^2_{P_a^{-1}(1)}, \sum_a |\alpha_a|^2 y^2_{P_a^{-1}(2)}\right)^t, \quad \text{(A15)}$$

which implies that [5]

$$\left(\sum_a |\alpha_a|^2 y^2_{P_a^{-1}(1)}, \sum_a |\alpha_a|^2 y^2_{P_a^{-1}(2)}\right) \prec \left(y_1^2, y_2^2\right). \quad \text{(A16)}$$

From Eqs. (A13) and (A16) we infer that

$$\left(x_1^2, x_2^2\right) \prec \left(y_1^2, y_2^2\right). \quad \text{(A17)}$$

This proves the theorem.

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