Detection of multipartite correlation transfer via discrete Rényi entropy

Carlos Muñoz [1], Luis Roa [2], and Andrei B. Klimov [1,*]

[1] *Departamento de Física, Universidad de Guadalajara, Avenida Revolución 1500, 44420 Guadalajara, Jalisco, México*

[2] *Departamento de Física, Universidad de Concepción, Casilla 160-C, Concepción, Chile*

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It is shown that the total amount of correlation stored in N-qubit systems, as characterized by the discrete Rényi entropy, can be effectively employed to detect the presence of N-partite correlations within the framework of deterministic measurements. An associated optimization procedure can be analytically performed for a broad range of N-qubit states, encompassing both symmetric and nonsymmetric ones. This analytical approach enables the analysis of the asymptotic limit $N \gg 1$. It is proved that the appropriately normalized quadratic discrete Rényi entropy always decreases in the process of deterministic measurements. This allows us to introduce a robustness parameter for assessing the stability of pure multipartite states under the protocol of measurement-induced optimal correlation transfer.

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I. INTRODUCTION

Phase-space quasidistributions have been widely used to characterize quantum states of various types, whether described by compact or noncompact dynamic symmetry groups (see Ref. [1] and references therein). For instance, properties like positivity and Gaussianity of the Wigner function (and the Glauber P function) are commonly employed [2] as indicators of the quantumness of states [3]. The Husimi Q function allows us to explore other aspects of quantum states. Being a positive distribution, the Q function enables us to define entropic-like quantities, such as the Wehrl and Rényi entropies. The latter are useful for quantifying the localization of the distribution within the corresponding classical phase space [4]. Consequently, they are instrumental in characterizing physical properties, including aspects like light polarization [5], nonclassical behavior of bosonic modes [6], entanglement detection [7], bipartite quantum correlations [8], quantum phase transitions [9], and more. Moreover, Q-function-based entropies are invariant under appropriate group transformations and have also been applied to describe the complexity of quantum states [10]. These qualities make the Q function a valuable tool for analyzing quantum systems, especially due to its feasibility for direct experimental assessment [11].

Discrete quasiprobability distributions, i.e., quasiprobability distributions in discrete phase spaces, retain most of the fundamental properties of continuous representations and provide nonredundant descriptions of finite-dimensional quantum systems [12,13]. Additionally, discrete phase-space methods are often more suitable for faithfully representing nonsymmetric multipartite states. Detecting phase-space localization and delocalization is particularly valuable in the case of multipartite systems, where discrete quasidistributions corresponding to pure states become sparser in the presence of quantum correlations [14,15]. Thus, the degree of delocalization also indicates the full amount of existing quantum correlations. Interestingly, there is a direct connection between certain features of the discrete quasiprobabilities [15] and algebraic properties of multipartite states, which are related to their entanglement characteristics [16].

Additional information about multipartite correlations [17] can be extracted by analyzing outcomes of local measurements [18–21] (see also Ref. [22] and references therein). Particularly, the idea of localizable entanglement [19,20] involves maximizing the average entanglement resulting from probabilistic measurements of one or several particles. Unfortunately, in all the discussed schemes, various types of optimization procedures are built-in, limiting their applications to a reduced number of particles or very specific states, such as some classes of states that are symmetric under permutations of particles [23].

In this paper, we propose a method to quantify multipartite correlations based on the results of deterministic measurements [24]. This protocol involves performing a single-qubit measurement on a given N-qubit state, probabilistically projecting it into two $(N-1)$-partite states. The choice of the qubit measurement basis ensures that the resulting pure $N-1$ qubit states have the same amount of correlations, which are further maximized according to the selected measure. As our correlation measure, we use the quadratic discrete Rényi entropy, which, as we demonstrate, is the only measure invariant under arbitrary (not only Clifford) local transformations. Additionally, the quadratic Rényi entropy is related to a previously introduced entanglement measure [16], providing a geometric interpretation of the latter. We show that there is a significant distinction between probabilistic and deterministic approaches. In particular, certain optimizations can be analytically carried out in the deterministic measurement scheme, allowing us to analyze the macroscopic limit involving a large number of qubits for a broad class of both symmetric and nonsymmetric states. We analytically demonstrate that

*Corresponding author: andrei.klimov@academicos.udg.mx

appropriately normalized discrete entropy decreases during the process of deterministic measurements for real (and locally equivalent) N-qubits states and introduce the robustness parameter in terms of the entropies of the original and output states. The robustness strongly depends on the type of correlations stored in the original state and characterizes their structural stability during the optimal transfer of correlations via local measurements.

In Sec. II, we extend the concept of Rényi entropy to the multipartite case by utilizing the discrete Q function and discuss its application for characterizing correlations within pure N-qubit states. Section III focuses on applying the introduced measure for detecting multipartite correlations during deterministic measurements. The results are discussed in Sec. IV.

II. DISCRETE RÉNYI ENTROPY

The N-qubit Hilbert space $\mathcal{H}_{2^N} = \mathcal{H}_2^{\otimes N}$ is spanned by the computational basis

$$|\lambda\rangle = |l_1, \dots, l_N\rangle, \quad l_i = 0, 1, \quad \sum_\lambda |\lambda\rangle\langle\lambda| = \hat{I}, \tag{1}$$

where $\hat{I}$ is the identity operator in $\mathcal{H}_{2^N}$. The corresponding discrete phase-space is a $2^N \times 2^N$ grid, labeled by a pair of N tuples: (α, β), $[\alpha = (a_1, \dots, a_N)$, $\beta = (b_1, \dots, b_N)$, $a_i, b_i \in \mathbb{Z}_2]$ [12–14]. The discrete Q function is defined as an average value of the density matrix over a set of discrete coherent states [14,25],

$$|\alpha,\beta\rangle = \otimes\Pi_{i=1}^N \sigma_z^{a_i}\sigma_x^{b_i}|\mathbf{n}_0\rangle_i = \otimes\Pi_{i=1}^N |a_i, b_i\rangle_i, \tag{2}$$

where $\sigma_{z,x}^{(i)}$ are the Pauli operators corresponding to the ith qubit and $|\mathbf{n}_0\rangle_i$ is the fiducial state fixed by the Bloch vector $\mathbf{n}_0 = (1, 1, 1)/\sqrt{3}$, so that is the usual spin coherent state with

$$\Pi_{i=1}^N |\mathbf{n}_0\rangle_i = |\xi\rangle, \quad \xi = 2^{-1/2}(\sqrt{3} - 1)e^{i\pi/4}, \tag{3}$$

$$|\mathbf{n}_0\rangle_i\langle\mathbf{n}_0| = \frac{1}{2}\big(\hat{I}_i + \mathbf{n}_0^{(i)} \cdot \hat{\sigma}^{(i)}\big), \tag{4}$$

so that $|\langle a', b'|a, b\rangle|^2 = 3^{-1}(2\delta_{aa'}\delta_{bb'} + 1)$. The set of projectors on (2) form an informationally complete set,

$$\sum_{\alpha,\beta} |\alpha, \beta\rangle\langle\alpha, \beta| = 2^N\,\hat{I},$$

which allows us to define the discrete Q function according to

$$Q_\rho(\alpha, \beta) = \langle\alpha, \beta|\hat{\rho}|\alpha, \beta\rangle = \mathrm{Tr}\big(\otimes\Pi_{i=1}^N \hat{q}_i(a_i, b_i)\hat{\rho}\big), \tag{5}$$

$$\hat{q}_i(a_i, b_i) = |a_i, b_i\rangle_i\langle a_i, b_i|, \quad \sum_{\alpha,\beta} Q_\rho(\alpha, \beta) = 2^N, \tag{6}$$

where $\hat{\rho}$ is the density matrix. Thus, the Q function of a pure factorized state $|\psi\rangle = \otimes\Pi_{i=1}^N|\psi_i\rangle_i$ is a product of the single-qubit Q functions,

$$Q_\psi(\alpha, \beta) = \otimes\Pi_{i=1}^N Q_i(a_i, b_i), \quad Q_i(a_i, b_i) = |\langle\psi_i|a_i, b_i\rangle|^2. \tag{7}$$

In particular, the Q function of the fiducial N-qubit state $|\xi\rangle$ is

$$Q_\xi(\alpha, \beta) = 3^{-[h(\alpha)+h(\beta)+h(\alpha+\beta)]/2}, \tag{8}$$

where $0 \leqslant h(\alpha) = \sum_{i=1}^N a_i \leqslant N$, is the weight of the N tuple $\alpha = (a_1, \dots, a_N)$ and the sum of N tuples in $h(\alpha + \beta)$ is taken by mod 2.

It is worth noting that the discrete P symbols can be defined as the dual representation of $\hat{\rho}$,

$$\hat{\rho} = \sum_{\alpha,\beta} |\alpha, \beta\rangle\langle\alpha, \beta|P_\rho(\alpha, \beta), \tag{9}$$

$$P_\rho(\alpha, \beta) = \mathrm{Tr}\big(\otimes\Pi_{i=1}^N \hat{p}_i(a_i, b_i)\hat{\rho}\big),$$
$$2\hat{p}_i(a_i, b_i) = 3\hat{q}_i(a_i, b_i) - \hat{I}_i, \tag{10}$$

so that the average value of an operator $\hat{f}$ is computed in the standard way as a convolution of Q and P symbols,

$$\langle\hat{f}\rangle = \sum_{\alpha,\beta} P_\rho(\alpha, \beta)Q_f(\alpha, \beta) = \sum_{\alpha,\beta} P_f(\alpha, \beta)Q_\rho(\alpha, \beta). \tag{11}$$

The discrete Q function provides a nonredundant representation of an N-qubit state. In what follows we focus on pure states $\hat{\rho} = |\psi\rangle\langle\psi|$ so that $Q_\rho(\alpha, \beta) \equiv Q_\psi(\alpha, \beta)$. $Q_\psi(\alpha, \beta)$ is always positive and application of local Clifford transformations leads to a rearranging of values $Q_\psi(\alpha, \beta)$ in the discrete grid (α, β). Much like the phase-space representation for continuous symmetries, the discrete Q function can be conveniently expressed by expanding the discrete coherent states (2) in the logical basis (1),

$$|\alpha, \beta\rangle = \mathcal{N}\sum_\mu (-1)^{\mu\alpha}\xi^{h(\mu+\beta)}|\mu\rangle, \tag{12}$$

where $\mu\alpha = \sum_i m_i a_i \pmod 2$, $\alpha = (a_1, \dots, a_N)$, $\mu = (m_1, \dots, m_N)$, $\mathcal{N} = (1 + |\xi|^2)^{-N/2}$, and ξ is the constant defined in (3), so that

$$Q_\psi(\alpha, \beta) = \mathcal{N}^2\sum_{\mu,\nu} (-1)^{(\mu+\nu)\alpha}\xi^{h(\mu+\beta)}\xi^{*h(\nu+\beta)}\rho_{\mu\nu}, \tag{13}$$

with $\rho_{\mu\nu} = \langle\nu|\psi\rangle\langle\psi|\mu\rangle$. For instance, the Q function of an arbitrary symmetric state $|\psi\rangle_S$, expanded in the Dicke basis $\{|D_k\rangle, k = 0, \dots, N\}$,

$$|D_k^{(N)}\rangle = \sqrt{\frac{k!(N-k)!}{N!}}\sum_{\text{perm}} P(|1\rangle_1\cdots|1\rangle_k|0\rangle_{k+1}\cdots|0\rangle_{\mathrm{N}}), \tag{14}$$

with

$$|\psi\rangle_{\mathrm{S}} = \sum_{k=0}^N a_{kN}\big|D_k^{(N)}\big\rangle, \tag{15}$$

depends of three discrete variables $h(\alpha)$, $h(\beta)$, $h(\alpha + \beta)$ and takes the form

$$Q_\psi(\alpha, \beta) = \mathcal{N}^2\left|\sum_{k=0}^N \frac{a_{kN}}{k!\sqrt{C_k^N}}\left.\frac{d^kF_{p_1,p_2,p_3}(z)}{dz^k}\right|_{z=0}\right|^2,$$

where

$$F_{p_1,p_2,p_3}(z) = \frac{(1 - \xi z)^{p_1}(\xi + z)^{p_2}(\xi - z)^{p_3}}{(1 + \xi z)^{p_1+p_2+p_3-N}},$$
$$h(\alpha) = p_1 + p_3, \quad h(\alpha + \beta) = p_1 + p_2,$$
$$h(\beta) = p_2 + p_3.$$

For instance, for the W state $|D_1\rangle$ we have,

$$Q_{D_1}(\alpha,\beta) = N^{-1}\mathcal{N}^2|\xi|^{2+2h(\beta)} \times |N + (\xi^{-2}-1)h(\alpha) - (\xi^{-2}+1)h(\alpha+\beta)|^2.$$

The quadratic Rényi entropy is defined as

$$S = -\ln\delta_\psi^{(N)}, \quad \delta_\psi^{(N)} = \sum_{\alpha,\beta} Q_\psi^2(\alpha,\beta) \tag{16}$$

and characterizes the degree of localization of the discrete Q function in the discrete phase-space. Higher values of S, and consequently, smaller values of $\delta_\psi^{(N)}$, correspond to sparser distributions, which one can relate with more correlated pure states. It is important to note that $\delta_\psi^{(N)}$ takes on a product form for factorized states, $\delta_{\psi_1\psi_2}^{(N)} = \delta_{\psi_1}^{(N)}\delta_{\psi_2}^{(N)}$. For an arbitrary pure single-qubit state $\delta_\psi^{(1)}$ assumes the same value, specifically $\delta_\psi^{(1)} = 4/3$. This property is a distinctive characteristic of the fiducial state (3).

It is noteworthy that $\delta_\rho^{(N)}$ (for an arbitrary state) can be expressed, as discussed in Ref. [15], in terms of purities associated with all possible bipartitions [16] (see also Appendix A), specifically,

$$\delta_\rho^{(N)} = \left(\frac{2}{3}\right)^N \left(1 + \sum_{i_1=1}^{N} \mathrm{Tr}_{i_1}(\mathrm{Tr}_{i_2\ldots i_N}\hat\rho)^2 + \sum_{i_1\neq i_2} \mathrm{Tr}_{i_1 i_2}(\mathrm{Tr}_{i_3\ldots i_N}\hat\rho)^2 + \cdots + \sum_{i_1\neq i_2\neq\cdots\neq i_{N-1}} \mathrm{Tr}_{i_1\ldots i_{N-1}}(\mathrm{Tr}_{i_N}\hat\rho)^2 + \mathrm{Tr}_{12\ldots N}\hat\rho^2\right). \tag{17}$$

The representation (17) shows that $\delta_\psi^{(N)}$ remains invariant under any local transformation, not limited to local Clifford transformations, as can be directly inferred from (16) (see also Appendix A where a direct proof of the invariance of $\delta_\psi^{(N)}$ is provided). In fact, this particular property sets apart the second moment of the discrete Q function (16) from all the other moments $\sum_{\alpha,\beta} Q_\psi^p(\alpha,\beta)$, $p>2$, as these moments maintain invariance solely under local Clifford transformations (see Appendix B).

As a consequence of (7) it becomes evident that the maximum value of $\delta_\psi^{(N)}$ for pure states is reached for factorized states with $\delta_{\max}^{(N)} = (4/3)^N$. To provide a meaningful measure that normalizes $\delta_\psi^{(N)}$, we introduce

$$\tilde\delta_\psi^{(N)} = (3/4)^N \delta_\psi^{(N)}. \tag{18}$$

This normalization ensures that for any factorized state, $\tilde\delta_\psi^{(N)} = 1$, representing its maximum value. Consequently, the deviation $\tilde\delta_\psi^{(N)}$ not only serves as a good witness for quantum correlations in pure N-qubit states but also characterizes the "amount" of such correlations contained in the state.

A relevant parameter that characterizes the extent of nonlocalization of the Q function for pure states is the asymptotic behavior of $\tilde\delta_\psi^{(N)}$ in the limit $N \gg 1$, $\tilde\delta_\psi^{(\infty)} = \lim_{N\to\infty}\tilde\delta_\psi^{(N)}$. Our numerical observations suggest that $\tilde\delta_\rho^{(\infty)}$ is related to the number of distinct multipartite correlations in a strongly entangled N-qubit state (i.e., when the size of nonfactorized clusters contained in the state is $\approx N$). Indeed, $\tilde\delta_\psi^{(\infty)}$ remains a constant nonzero value in cases where a state contains only a few k-partite correlations, as observed in examples such as $|\mathrm{GHZ}\rangle$ or low excited Dicke states (14), $|D_k^N\rangle$, $k \ll N$:

$$\tilde\delta_{D_1}^{(N)} = \frac{1}{2} + \frac{1}{2N}, \quad \tilde\delta_{\mathrm{GHZ}}^{(N)} = \frac{1}{2} + \frac{1}{2^N},$$
$$\tilde\delta_{D_{k\ll N/2}}^{(N)} = \frac{(2k-1)!!}{2^k k!} + \frac{k(2k-3)!!}{2^k(k-1)!N} + O(N^{-2}).$$

In the opposite case, when the number of types of correlations is $\approx N^l$, $0 < l \leqslant 1$, one has $\tilde\delta_\psi^{(\infty)} = 0$ [actually $\tilde\delta_\rho^{(N)} \sim O(N^{-l})$, $N \gg 1$].

Furthermore, for parameter-dependent states, $\tilde\delta_\rho^{(\infty)}$ appears to reach its minimum value for maximally correlated states within its own class. For example, consider the state $|GHZ\rangle_a \sim |0\ldots0\rangle + a|1\ldots1\rangle$, where a is a real parameter. In this case, one obtains

$$\tilde\delta_{\mathrm{GHZ}_a}^{(\infty)} = \frac{1+a^4}{(1+a^2)^2},$$

whose minimum is reached at $a=1$.

For a real superposition $\approx|D_1\rangle + a|\mathrm{GHZ}\rangle$ we can readily derive the expression for $\tilde\delta_a^{(N)}$ as follows:

$$\tilde\delta_a^{(N)} = \frac{1}{2} + \frac{1}{2N(1+a^2)^2} + \frac{a^2(4+a^2)}{2^N(1+a^2)^2}.$$

Interestingly, $\tilde\delta^{(\infty)}$ is independent of the parameter a of the state.

In a more sophisticated case of a real superposition of a Dicke state with two excitations and the GHZ state $\approx|k=2,N\rangle + a|\mathrm{GHZ}\rangle$, one obtains

$$\tilde\delta_a^{(\infty)} = \frac{(1+2a^2)^2+2}{8(1+a^2)^2},$$

with the minimum $\tilde\delta^{(\infty)} = 1/3$ at $a = 1/\sqrt{2}$.

For a general symmetric state (15) after some algebraic manipulation, we arrive at the following expression:

$$\tilde\delta_\psi^{(N)} = \sum \frac{a_{k_1N}a_{k_2N}^* a_{k_3N}a_{k_4N}^*}{\sqrt{C_{k_1}^N C_{k_2}^N C_{k_3}^N C_{k_4}^N}} \delta_{k_1+k_3,k_2+k_4} G_{k_1k_2k_3k_4}^{(N)}, \tag{19}$$

where $G_{k_1k_2k_3k_4} = G_{k_3k_2k_1k_4} = G_{k_1k_4k_3k_2}$ is a hypergeometric function, whose explicit form is given in Appendix C.

For any Dicke state $|k,N\rangle$, $\tilde\delta_{D_k}^{(N)}$ takes the form

$$\tilde\delta_{D_k}^{(N)} = \left(C_k^N\right)^{-1} {}_3F_2\left(\begin{matrix}-k, k-N, \frac{1}{2}\\ 1, 1\end{matrix}; 1\right).$$

In particular, for highly correlated state $|k = N/2, N\rangle$, $\tilde\delta_{N/2}^{(N)}$ acquires the asymptotic form

$$\tilde\delta_{N/2}^{(N)} \approx \frac{2}{\sqrt{N\pi}}[1 + O(N^{-1})],$$

with $\tilde\delta_{N/2}^{(\infty)} = 0$, as expected.

For several nonsymmetric states, it is reasonably straightforward to compute the parameter $\tilde\delta_\psi^{(N)}$ using Eq. (13). For instance, for the domain-wall state $|DW_N\rangle \sim |000\ldots\rangle +$

$|100\ldots\rangle + |110\ldots\rangle + \cdots$ we obtain

$$\tilde{\delta}_{DW}^{(N)} = \frac{5N + 2^{2-N} - 3}{(N+1)^2}, \quad \tilde{\delta}_{DW}^{(\infty)} = 0$$

for the singlet state with even N, $|S_N\rangle \sim \prod_{k=0}^{N/2-1}(|0\rangle_{2k+1}|1\rangle_{2k+2} - |1\rangle_{2k+1}|0\rangle_{2k+2})$,

$$\tilde{\delta}_{\mathrm{S}}^{(N)} = \left(\frac{3}{4}\right)^{N/2};$$

while for the closed cluster state,

$$|C_{\mathrm{N}}\rangle = \frac{1}{2^{N/2}} \sum_{\mu} (-1)^{\sum_{i=1}^{N} \mu_i \mu_{i+1}} |\mu\rangle, \quad \mu_{N+1} = \mu_1,$$

one gets,

$$\tilde{\delta}_{\mathrm{C}}^{(N)} = \frac{1}{2^N}(F_{N-1} + F_{N+1} + 1),$$

where F_N are Fibonacci numbers. The exponential decay of $\tilde{\delta}_{C}^{(N)}$ with respect to N,

$$\tilde{\delta}_{\mathrm{C}}^{(N)} \sim \left(\frac{1+\sqrt{5}}{4}\right)^N,$$

indicates a presence of a large number of different types of correlations in the cluster state.

Finally, we introduce a normalized Rényi entropy as

$$\tilde{S}^{(N)} = -\ln \tilde{\delta}_{\psi}^{(N)}, \tag{20}$$

which attains its minimum value $\tilde{S}^{(N)} = 0$ for factorized states and can be considered as a specific measure of quantum correlations.

III. DETERMINISTIC CORRELATION MEASURE

The core concept behind deterministic correlation extraction involves optimizing single-qubit projective measurements on a pure multipartite state in such a way that the total amount of correlations in both outcomes becomes equal. This enables us to deterministically convert higher-order correlations into their lower-order counterparts by implementing these optimal measurements.

In multipartite systems one can establish a hierarchy of quantum correlations [26], which can be converted into each other by employing adequate measurement schemes. In principle, correlations contained in an N-partite systems can be (partially) captured by an $(N-1)$-partite systems obtained as a result of a suitable local von Neumann measurement. A deterministic measurement protocol consists in choosing a convenient measurement basis for a specific qubit that allows us to (probabilistically) map an initial N-qubit state into outcome pure $(N-1)$-qubit states obeying the same global characteristics. In our case such a characteristic will be the total amount of correlations described by the parameter $\tilde{\delta}_{\psi}^{(N)}$, Eq. (18).

Let us consider a pure N-qubit state $|\psi\rangle$ and assume that the ith qubit is measured. These local measurements give rise to two possible outcomes:

(a) When projecting into the state

$$|\theta, \phi\rangle_i = \cos\frac{\theta}{2}|0\rangle_i + e^{i\phi}\sin\frac{\theta}{2}|1\rangle_i, \tag{21}$$

the resulting $(N-1)$-partite pure state

$$|\psi'_{\theta,\phi}\rangle = {}_i\langle\theta, \phi|\psi\rangle P_{\theta,\phi}^{-1/2} \tag{22}$$

is detected with the probability $P_{\theta,\phi} = \mathrm{Tr}_{N-1}({}_i\langle\theta, \phi|\psi\rangle\langle\psi|\theta, \phi\rangle_i)$.

(b) On the other hand, the projection into the orthogonal state $|\theta+\pi, \phi\rangle_i$ leads to the outcome $|\psi'_{\theta+\pi,\phi}\rangle$ with the probability $P_{\theta+\pi,\phi} = 1 - P_{\theta,\phi}$. In the deterministic scenario, both output states $|\psi'_{\theta,\phi}\rangle$ and $|\psi'_{\theta+\pi,\phi}\rangle$ share the same values of $\tilde{\delta}_{\psi'}^{(N-1)}(\theta,\phi) = \tilde{\delta}_{\psi'}^{(N-1)}(\theta+\pi,\phi) = \tilde{\delta}_{d}^{(N-1)}$, and consequently, they possess the same Rényi entropy; that is,

$$\tilde{S}_{\theta_d,\phi_d}^{(N-1)} = \tilde{S}_{\theta_d+\pi,\phi_d}^{(N-1)} = \tilde{S}_{d}^{(N-1)}, \tag{23}$$

where (θ_d, ϕ_d) is the measurement direction leading to (23). Our objective is to determine the maximum value of the deterministic Rényi entropy,

$$\tilde{S}_{\max}^{(N-1)} = \max_{\theta,\phi} \tilde{S}_{d}^{(N-1)},$$

which is equivalent to finding the minimum value of the parameter $\tilde{\delta}_{d}^{(N-1)}$. This approach describes the optimal projection of the initial state into two distinct sets of $(N-1)$-qubit states, where each set possesses both the equal and the maximum possible amount of correlations.

The difference between the original value $\tilde{\delta}_{d}^{(N)}$ and $\min \tilde{\delta}_{d}^{(N-1)} \equiv \tilde{\delta}_{d,\min}^{(N-1)}$, suitably normalized, characterizes the robustness of the original state under measurements that deterministically maximize the output correlations.

At first glance, the protocol described above appears to require a complex optimization procedure facing challenges similar to those encountered in well-established multipartite entanglement measures [18,26–28]. However, in Appendix D, we prove the following

Theorem 1. For an arbitrary pure state with real coefficients (and locally equivalent),

$$|\psi\rangle = \sum_{\mu} a_{\mu}|\mu\rangle, \quad a_{\mu} = a_{m_1 m_2 \ldots m_{\mathrm{N}}} \in \mathbb{R}, \quad \sum_{\mu} a_{\mu}^2 = 1, \tag{24}$$

it satisfies

$$\min_{\theta,\phi} \tilde{\delta}_{d}^{(N-1)} = \tilde{\delta}_{\psi'}^{(N-1)}\Big(\theta = \pm\frac{\pi}{4}, \phi = \frac{\pi}{2}\Big) = \tilde{\delta}_{d,\min}^{(N-1)}, \tag{25}$$

where $|\psi'_{\theta,\phi}\rangle$ is defined in (22).

This implies that an optimal deterministic projection can be performed on an arbitrary qubit. Consequently, for real symmetric states (15) the minimization (25) is global. However, for nonsymmetric states with real coefficients (and locally equivalent states), one should still carry out the minimization over the entire set of qubits.

The outcome of Theorem 1 enables us to establish the following central theorem (see Appendixes E and F).

Theorem 2. For any deterministically projected qubit of the state (24) the following inequality holds:

$$\tilde{\delta}_{d,\min}^{(N-1)} \geqslant \tilde{\delta}_{\psi}^{(N)}, \tag{26}$$

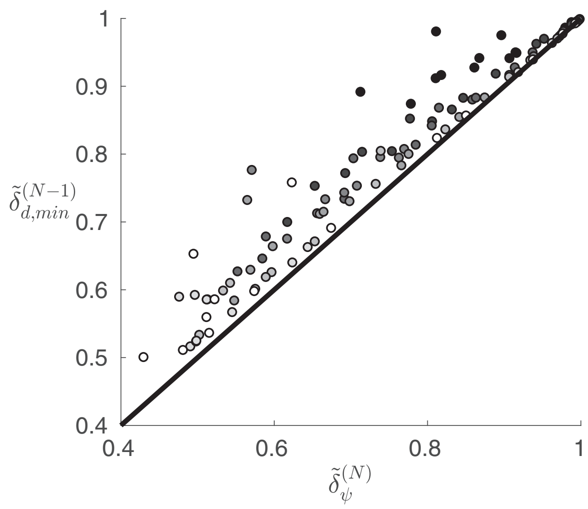

FIG. 1. The parameters $\tilde{\delta}^{(N-1)}_{d,\min}$ and $\tilde{\delta}^{(N)}_{\psi}$ for N-qubit states $\cos\epsilon|0\dots0\rangle+\sin\epsilon|RS\rangle$, where $|RS\rangle$ are complex Haar random states, ϵ is uniformly distributed in $[0,\pi]$; the darkest points correspond to the highest values of $2\leqslant N\leqslant 10$.

and

$$\lim_{N\to\infty}\tilde{\delta}^{(N-1)}_{d,\min}=\tilde{\delta}^{(\infty)}_{\psi}. \tag{27}$$

Therefore, the total amount of quantum correlations stored in a real state and quantified with the normalized Rényi entropy (20) cannot be increased in the deterministic projection scheme. The numeric simulations suggest that $S^{(N-1)}_{\max}\leqslant S^{(N)}$ holds true for N-qubit states (24) with $a_\mu\in\mathbb{C}$. In Fig. 1 we plot $\tilde{\delta}^{(N-1)}_{d,\min}$ and $\tilde{\delta}^{(N)}_{\psi}$ for N-qubit states generated as $\cos\epsilon|0\dots0\rangle+\sin\epsilon|RS\rangle$, where $|RS\rangle$ are Haar random states with complex coefficients, ϵ is uniformly distributed in the segment $[0,\pi]$, and the darkest points correspond to the highest values of N, $2\leqslant N\leqslant 10$. It can observe that the relation (26) is always fulfilled.

However, the relative variation of the entropy is sensitive to the types of correlations contained in the original N-qubit state. In particular, the global robustness r^N_ψ of quantum correlations can be characterized by the rate of change of the Rényi entropy with respect to its limit value at $N\to\infty$:

$$r^N_\psi=1-I^N_\psi, \tag{28}$$

$$I^N_\psi=\left(\tilde{\delta}^{(N)}_\psi-\tilde{\delta}^{(\infty)}_\psi\right)\left(\tilde{s}^{(N)}-\tilde{s}^{(N-1)}_{\max}\right), \tag{29}$$

where

$$\tilde{s}^{(N)}=-\ln\left(\tilde{\delta}^{(N)}_\psi-\tilde{\delta}^{(\infty)}_\psi\right),$$

and the relation (27) was taken into account. The instability parameter I^N_ψ is always positive and enables us to analytically analyze the asymptotic behavior of the robustness in the large-N limit.

The parameter I^N_ψ for some typical N-qubit states with real coefficients at $N\gg 1$ (up to the leading term) is presented in Table I.

The parameter r^N_ψ (28) provides a valuable insight into the structural stability of the state under measurements that deterministically transfer the maximum amount of quantum correlations. In particular, the GHZ state is very stable, since it is optimally projected into the mutually orthogonal $(N-1)$-qubit states $\approx|0\dots0\rangle\pm i|1\dots1\rangle$, which carry essentially the same amount and type of correlations as the original state for a large number of qubits. This is reflected in the exponentially fast decay of the instability parameter I^N_ψ with N. A similar behavior is observed for the singlet states, which is optimally projected into the $(N-2)$ singlet state with a factorized single qubit; and for the cluster states which are deterministically projectable into locally equivalent $(N-1)$-qubit cluster states,

$$|C_{\rm N}\rangle\to\hat{U}^{(\pm)}_{i-1}\hat{U}^{(\pm)}_{i+1}|C_{N-1}\rangle,\quad \hat{U}^{(\pm)}_j=\exp\left(\pm i\pi\sigma^{(j)}_z/4\right),$$

when the ith qubit is measured.

The symmetric Dicke states and the domain-wall state are considerably asymptotically less stable under the deterministic projections. In particular, the domain-wall state is optimally projected into unbalanced superposition of a factorized and correlated states:

$$|DW_{N-1}\rangle\to(|0_{N-1}\rangle\pm i\sqrt{N}|DW_{N-1}\rangle)(N+1)^{-1/2}.$$

In a similar way $|D^N_1\rangle$ is projectable into a superposition of $|D^{N-1}_1\rangle$ and a nonexcited state:

$$\left|D^{N-1}_1\right\rangle\to\sqrt{1-N^{-1}}\left|D^{N-1}_1\right\rangle\pm iN^{-1/2}|0_{N-1}\rangle,$$

and in general,

$$\left|D^{N-1}_k\right\rangle\to\sqrt{\frac{N-k}{N}}\left|D^{N-1}_k\right\rangle\pm i\sqrt{\frac{k}{N}}\left|D^{N-1}_{k-1}\right\rangle.$$

The most unstable among them is the state $|D^{2p}_p\rangle$, for which the projection is performed into a balanced superposition of

TABLE I. The parameters I^N_ψ, $\tilde{\delta}^{(N)}_\psi$, $\tilde{\delta}^{(N-1)}_{d,\min}$ for different N-qubit states in the $N\gg 1$ limit.

State	$\tilde{\delta}^{(N)}_\psi$	$\tilde{\delta}^{(N-1)}_{d,\min}$	I^N_ψ
$\lvert\mathrm{GHZ}\rangle$	$\frac{1}{2}+\frac{1}{2^N}$	$\frac{1}{2}+\frac{1}{2^{N-1}}$	$\frac{\ln 2}{2^N}$
$\lvert D^N_{k\ll N/2}\rangle$	$\frac{(2k-1)!!}{2^k k!}+\frac{k(2k-3)!!}{2^k(k-1)!N}$	$\frac{(2k-1)!!}{2^k k!}+\frac{(5k-2)(2k-3)!!}{2^k(k-1)!N}$	$\frac{k}{2k-1}\frac{(2k-1)!!}{2^{k+2}(k-1)!N}\ln\frac{5k-2}{k}$
$\lvert D^N_{N/2}\rangle$	$\frac{2}{\sqrt{N\pi}}$	$\frac{3}{\sqrt{N\pi}}$	$\frac{2\ln(3/2)}{\sqrt{\pi N}}$
$\lvert S\rangle$	$(\frac{3}{4})^{N/2}$	$(\frac{3}{4})^{N/2-1}$	$(\frac{\sqrt{3}}{2})^N\ln\frac{4}{3}$
$\lvert DW\rangle$	$\frac{5N-13}{N^2}$	$\frac{5N-11}{N^2}$	$\frac{2}{N^2}$
$\lvert C\rangle$	$(\frac{1+\sqrt{5}}{4})^N$	$(\frac{1+\sqrt{5}}{4})^{N-1}$	$(\frac{1+\sqrt{5}}{4})^N\ln(\sqrt{5}-1)$

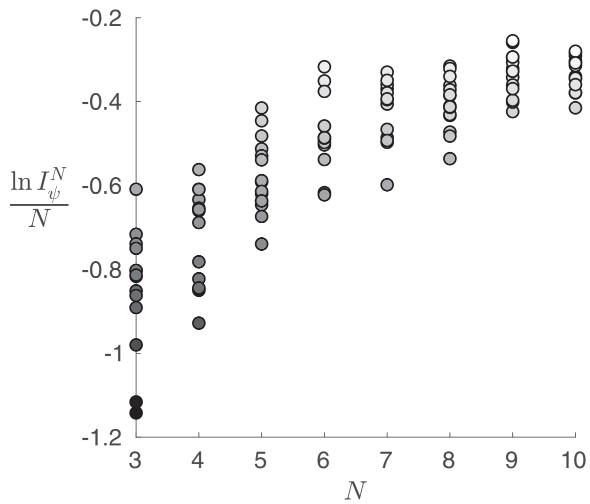

FIG. 2. $\ln I_\psi^N/N$, where the instability parameter (29) is computed for pure random Haar states as a function of the number N of qubits.

two orthogonal states with similar correlation characteristics $|D_p^{2p}\rangle \to |D_p^{2p-1}\rangle \pm i|D_{p-1}^{2p-1}\rangle$.

In Fig. 2 we plot the normalized instability parameter (29) for pure Haar random states and for different number of qubits $2 \leqslant N \leqslant 10$. Since the random states are highly correlated, it is expected that $\tilde{\delta}_\psi^{(\infty)} = 0$.

IV. SUMMARY

In this paper we employ a discrete analog of the Rényi entropy to assess the robustness of multipartite correlations in N-qubit systems when subjected to deterministic measurements in the asymptotic limit of a large number of particles, $N \gg 1$.

We have shown that the total amount of correlation stored in N-qubit system and characterized by the discrete Rényi entropy can be used for detection of N-partite correlations in the framework of deterministic measurements. Both the behavior of $\tilde{\delta}_\psi^{(N)}$ in the limit of a large number of particles and its change during projective deterministic measurements provides a valuable information about the structure of multipartite correlations.

The obtained results, particularly Theorem 2, indicate that the total amount of quantum correlations always decreases when they are transferred from N to $N-1$ qubit states with real coefficients (and locally equivalent states) through deterministic single-qubit measurements. This is in contrast with the behavior of the average (geometric) entanglement measure in the process of probabilistic measurements [20]. Therefore, as a monotone function of the number of projections, the robustness (28) assesses the stability of a multipartite state during the optimal measurement-based correlation transfer and distinguishes pure states with different types of quantum correlations.

The advantage of the present measure lies in its capability for analytical optimization of the measurement protocol for a broad class of N-qubit states. This analytical approach enabled us to derive a qualitative result Eq. (26) that can be extended to arbitrary nonsymmetric states.

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APPENDIX A

In this Appendix we prove that

$$\tilde{\delta}_\psi^{(N)} = \left(\frac{3}{4}\right)^N \sum_{\alpha,\beta} Q_\psi^2(\alpha,\beta), \tag{A1}$$

is invariant under local transformations.

First, we observe that, for an arbitrary pure N-qubit state

$$|\psi\rangle = \sum_\mu a_\mu |\mu\rangle, \quad \mu = (m_1, \dots, m_{\mathrm{N}}), \quad m_i = 0, 1, \tag{A2}$$

the sum (A1) is reduced to the following expression:

$$\tilde{\delta}_\psi^{(N)} = \sum_{\mu_1,\mu_2,\mu_3,\mu_4} a_{\mu_1} a_{\mu_2}^* a_{\mu_3} a_{\mu_4}^* \prod_{i=1}^N \delta_{m_{1,i}+m_{2,i}+m_{3,i}+m_{4,i},0} \times \delta_{(1-m_{1,i})m_{2,i}(1-m_{3,i}),0}\delta_{m_{1,i}(1-m_{2,i})m_{3,i},0} 2^{2m_{1,i}m_{3,i}-m_{1,i}-m_{3,i}}, \tag{A3}$$

where the summations and multiplications are taken by mod 2 and we employ the standard summation technic for functions defined on N-tuples,

$$\sum_\beta f(\beta) = \prod_{i=1}^N \sum_{\beta_i=0}^1 f(\beta_i), \tag{A4}$$

we have also used the explicit form of the fiducial state parameter (3) while summing over the coherent states (12). Observe that Eq. (A3) can be conveniently represented as

$$\tilde{\delta}_\psi^{(N)} = \sum_{\mu_1,\mu_2,\mu_3,\mu_4} a_{\mu_1} a_{\mu_2}^* a_{\mu_3} a_{\mu_4}^* g_{\mu_1\mu_2\mu_3\mu_4}^{(N)}, \tag{A5}$$

$$g_{\mu_1\mu_2\mu_3\mu_4}^{(N)} = 2^{-h(\mu_1+\mu_3)} \delta_{h(\mu_4)+h(\mu_2),h(\mu_1)+h(\mu_3)} \times \delta_{h(\mu_1+\mu_2)+h(\mu_2+\mu_3),h(\mu_1+\mu_3)} \delta_{\mu_1+\mu_2,\mu_3+\mu_4}, \tag{A6}$$

where $h(\mu) = \sum_{i=1}^N m_i$.

The state (A2) transformed by a local unitary,

$$U = \otimes \prod_i u^{(i)} |\psi\rangle,$$

$$u^{(i)} = \begin{bmatrix} \cos\frac{\theta_i}{2} e^{\frac{i}{2}(\phi_i+\eta_i)}, & i\sin\frac{\theta_i}{2} e^{\frac{i}{2}(\phi_i-\eta_i)} \\ i\sin\frac{\theta_i}{2} e^{-\frac{i}{2}(\phi_i-\eta_i)}, & \cos\frac{\theta_i}{2} e^{-\frac{i}{2}(\phi_i+\eta_i)} \end{bmatrix}, \tag{A7}$$

takes the form

$$|\psi_{\mathrm{U}}\rangle = \otimes \prod_i u^{(i)} |\psi\rangle = \sum_\mu a'_\mu |\mu\rangle, \tag{A8}$$

$$a'_\mu = \sum_\nu a_\nu \prod_i \langle m_i | u^{(i)} | n_i \rangle, \quad \nu = (n_1, \dots, n_{\mathrm{N}}), \tag{A9}$$

where

$$\langle m_i|u^{(i)}|n_i\rangle = \big((1-m_i)u^{(i)}_{1,1} + m_i u^{(i)}_{2,1}\big)(1-n_i) + \big((1-m_i)u^{(i)}_{1,2} + m_i u^{(i)}_{2,2}\big)n_i.$$

After performing extensive but straightforward calculations, we simplify the sum in Eq. (A5) with a_μ given in (A9) over $\mu_1, \mu_2, \mu_3, \mu_4$ to the following expression:

$$\sum_{m_{1,i},m_{2,i},m_{3,i},m_{4,i}=0}^{1} \langle m_{1,i}|u^{(i)}|n_{1,i}\rangle \langle m_{2,i}|u^{(i)}|n_{2,i}\rangle^* \times \langle m_{3,i}|u^{(i)}|n_{3,i}\rangle \langle m_{4,i}|u^{(i)}|n_{4,i}\rangle^* \times \prod_{i=1}^{N} \delta_{m_{1,i}+m_{2,i}+m_{3,i}+m_{4,i},0}\delta_{(1-m_{1,i})m_{2,i}(1-m_{3,i}),0} \times \delta_{m_{1,i}(1-m_{2,i})m_{3,i},0} 2^{2m_{1,i}m_{3,i}-m_{1,i}-m_{3,i}},$$

which, remarkably, does not depend on the angles θ_i, φ_i, η_i of local transformations, i.e., the above equation is reduced to

$$\prod_{i=1}^{N} \delta_{n_{1,i}+n_{2,i}+n_{3,i}+n_{4,i},0}\delta_{(1-n_{1,i})n_{2,i}(1-n_{3,i}),0} \times \delta_{n_{1,i}(1-n_{2,i})n_{3,i},0} 2^{2n_{1,i}n_{3,i}-n_{1,i}-n_{3,i}}.$$

This leads to the conclusion that the expression for $\tilde{\delta}^{(N)}_{\psi_U}$ remains the same, i.e.,

$$\tilde{\delta}^{(N)}_{\psi_{\mathrm{U}}} = \tilde{\delta}^{(N)}_{\psi}.$$

On the other hand, we represent the projector $|\alpha,\beta\rangle\langle\alpha,\beta|$ as follows using the factorization property (5) and the relation (10):

$$|\alpha,\beta\rangle\langle\alpha,\beta| = \left(\frac{1}{3}\right)^N \left[2^N \prod_{i=1}^{N} \otimes\hat{p}_i(a_i,b_i) + 2^{N-1}\sum_{j=1}^{N}\prod_{i\neq j}^{N} \otimes\hat{p}_i(a_i,b_i) + 2^{N-2}\sum_{j\neq k=1}^{N}\prod_{i\neq k\neq j}^{N} \otimes\hat{p}_i(a_i,b_i) + \cdots + \hat{I}\right].$$

Consequently, the $Q_\rho(\alpha,\beta)$ can be expressed in terms of the P functions (9) of the reduced density matrices according to

$$Q_\rho(\alpha,\beta) = \left(\frac{1}{3}\right)^N \left[2^N P_\rho(\alpha,\beta) + 2^{N-1}\sum_{j=1}^{N} P_{\mathrm{tr}_j\rho}(\alpha',\beta') + 2^{N-2}\sum_{j\neq k=1}^{N} P_{\mathrm{tr}_{j,k}\rho}(\alpha'',\beta'') + \cdots + 1\right],$$

where, for instance, $\mathrm{tr}_{j,k}\hat{\rho}$ is the reduced density matrix obtained by tracing over the jth and kth qubits, and (α'',β'') represents the $(N-2)$-tuples obtained by excluding the elements a_j, b_j, a_k, b_k from the N-tuple (α,β). As a result, we arrive at the following expression:

$$\sum_{\alpha,\beta} Q^2_\rho(\alpha,\beta) = \left(\frac{2}{3}\right)^N \sum_{\alpha,\beta} Q_\rho(\alpha,\beta)\Big[P_\rho(\alpha,\beta) + 2^{-1}\sum_{j=1}^{N} P_{tr_j\rho}(\alpha',\beta') + 2^{-2}\sum_{j\neq k=1}^{N} P_{tr_{j,k}\rho}(\alpha'',\beta'') + \cdots + 2^{-N}\Big],$$

This precisely corresponds to Eq. (17) in the main text, as established in Eq. (11).

APPENDIX B

In this Appendix we show that the higher moments of the discrete Q function, $\sum_{\alpha,\beta} Q^p_\rho(\alpha,\beta)$, $p > 2$, are not invariant under local transformations. This lack of invariance is evident even when considering the fiducial state, $\hat{\rho} = |\xi\rangle\langle\xi|$. To simplify our analysis, we focus on local transformations (A7) where $\phi_i, \eta_i = 0$. Consequently, the Q function of the transformed state becomes

$$Q_{U\rho U^\dagger}(\alpha,\beta) = \prod_{i=1}^{N} \left|\langle\xi|_i \hat{u}_i \sigma_z^{a_i}\sigma_x^{b_i}|\xi\rangle_i\right|^2 = \left(\frac{1}{3}\right)^{pN} \prod_{i=1}^{N}\big[(2+\cos\theta_i)^{(1-a_i)(1-b_i)} \times (1-\sin\theta_i)^{a_i(1-b_i)}\big] \times \big[(2-\cos\theta_i)^{(1-a_i)b_i}(1+\sin\theta_i)^{a_ib_i}\big].$$

This leads us to the following expression:

$$\sum_{\alpha,\beta} Q^p_\rho(\alpha,\beta) = \left(\frac{1}{3}\right)^{pN} \prod_{i=1}^{N} [(2+\cos\theta_i)^p + (1+\sin\theta_i)^p + (2-\cos\theta_i)^p + (1-\sin\theta_i)^p],$$

where the summation formula (A4) was employed. It is evident that the above expression is independent on the angle θ_i only for $p = 1, 2$.

APPENDIX C

An integral form of the function $G_{k_1k_2k_3k_4}$ appearing in Eq. (19) is obtained by using an integral representation of the δ functions in Eq. (A5),

$$G^{(N)}_{k_1k_2k_3k_4} = \sum_{\mu_1,\mu_2,\mu_3,\mu_4} g^{(N)}_{\mu_1\mu_2\mu_3\mu_4} \times \delta_{h(\mu_1),k_1}\delta_{h(\mu_2),k_2}\delta_{h(\mu_3),k_3}\delta_{h(\mu_1+\mu_2+\mu_3),k_4}, \quad \text{(C1)}$$

where $g^{(N)}_{\mu_1\mu_2\mu_3\mu_4}$ is defined in (A6) and which leads to

$$G^{(N)}_{k_1k_2k_3k_4} = \int_{|\omega_i|=1} \frac{d\omega}{(2\pi i)^4} \times \frac{\left[1 + \frac{1}{2}(\omega_1+\omega_3)(\omega_2+\omega_4) + \omega_1\omega_2\omega_3\omega_4\right]^N}{\omega_1^{k_1+1}\omega_2^{k_2+1}\omega_3^{k_3+1}\omega_4^{k_4+1}},$$

$$d\omega = d\omega_1 d\omega_2 d\omega_3 d\omega_4. \quad \text{(C2)}$$

Taking into account the restriction $\delta_{k_1+k_3,k_2+k_4}$,

$$G^{(N)}_{k_1k_2k_3k_4} = \delta_{k_1+k_3,k_2+k_4}\tilde{G}^{(N)}_{k_1k_2k_3},$$

we arrive at the following expression:

$$\tilde{G}^{(N)}_{k_1k_3k_2} = \int \frac{d\omega}{(2\pi i)^3}\frac{\left[1+\frac{1}{2}(\omega_1+\omega_3)(1+\omega_2)+\omega_1\omega_2\omega_3\right]^N}{\omega_1^{k_1}\omega_2^{k_2}\omega_3^{k_3}}, \quad d\omega = d\omega_1 d\omega_2 d\omega_3,$$

which is convenient for asymptotic estimations in the limit $N \gg 1$.

The explicit integration gives

$$\tilde{G}^{(N)}_{k_1k_2k_3} = \frac{N!2^{-|\beta-\alpha|}}{\alpha!(N-\beta)!|k_2-k_1|!|k_3-k_2|!} \times {}_4F_3\left(\begin{matrix}-\alpha, \beta-N, 1+|\beta-\alpha|/2, 1/2+|\beta-\alpha|/2\\ \beta-\gamma+1, \gamma-\alpha+1, |\beta-\alpha|+1\end{matrix};1\right),$$

where $\alpha = \min(k_1,k_2,k_3,k_4)$, $\beta = \max(k_1,k_2,k_3,k_4)$, γ is the middle value of the ordered set $\{k_i,k_j,k_k\}$, $i, j, k = 1, 2, 3$, and ${}_4F_3$ is a hypergeometric function.

APPENDIX D

In this Appendix we prove Theorem 1 of the main text.

Let us consider an arbitrary pure state with real coefficients,

$$|\psi\rangle = \sum_{\mu} a_\mu|\mu\rangle, \quad a_\mu = a_{m_1,m_2,\ldots,m_N} \in \mathbb{R}, \quad \sum_\mu a_\mu^2 = 1. \tag{D1}$$

Let us project the ith qubit on the state $|\varphi\rangle_i = \cos\theta|0\rangle_i + e^{i\phi}\sin\theta|1\rangle_i$ to obtain,

$$|\psi'\rangle = {}_i\langle\varphi|\psi\rangle = \cos\theta\sum_{\mu'} b_{\mu'}|\mu'\rangle + e^{-i\phi}\sin\theta\sum_{\mu'} c_{\mu'}|\mu'\rangle, \tag{D2}$$

where μ' are elements of an $(N-1)$-tuple, and

$$b_{\mu'} = a_{m_1,\ldots,0_i,\ldots,m_N}, \quad c_{\mu'} = a_{m_1,\ldots,1_i,\ldots,m_N}. \tag{D3}$$

The norm of the state (D2) is

$$\mathcal{M}^2(\theta,\phi) = \sum_{\mu'}|\cos\theta b_{\mu'} + e^{-i\phi}\sin\theta c_{\mu'}|^2.$$

Thus, $\delta^{(N-1)}_{\psi'}(\theta,\phi) = \sum_{\alpha',\beta'} Q^2_{\psi'}(\alpha',\beta')$ for the normalized projected state (D2) is

$$\delta^{(N-1)}_{\psi'}(\theta,\phi) = \frac{1}{\mathcal{M}^4(\theta,\phi)}\sum_{\alpha',\beta'}(\cos^2\theta Q_{\rho_{11}} + \frac{1}{2}\sin 2\theta\left(e^{i\phi}Q_{\rho_{12}} + e^{-i\phi}Q_{\rho_{21}}\right) + \sin^2\theta Q_{\rho_{22}})^2,$$

where $\rho_{jk} = |\varphi_j\rangle\langle\varphi_k|$, $j, k = 1, 2$ and

$$|\varphi_1\rangle = \sum_{\mu'} b_{\mu'}|\mu'\rangle, \quad |\varphi_2\rangle = \sum_{\mu'} c_{\mu'}|\mu'\rangle. \tag{D4}$$

The condition for deterministic measurements is that

$$\delta^{(N-1)}_{\psi'}(\theta,\phi) = \delta^{(N-1)}_{\psi'}(\theta+\pi/2,\phi) = \delta^{(N-1)}_d(\theta,\phi).$$

The minimum value of $\delta^{(N-1)}_d(\theta,\phi)$ can be determined by using the Lagrange multipliers method, i.e., looking for the extrema of the function

$$F(\theta,\phi) = \delta^{(N-1)}_{\psi'}(\theta,\phi) - \lambda\left[\delta^{(N-1)}_{\psi'}(\theta,\phi) - \delta^{(N-1)}_{\psi'}(\theta+\pi/2,\phi)\right].$$

It is easy to see that the extremum of $F(\theta,\phi)$ is reached at $\lambda = 1/2$, $\theta = \pi/4$, $\phi = \pi/2$ with,

$$\partial_\theta F|_{\theta=\pm\pi/4,\phi=\pi/2,\lambda=1/2} = \partial_\phi F|_{\theta=\pm\pi/4,\phi=\pi/2,\lambda=1/2} = \partial_\lambda F|_{\theta=\pm\pi/4,\phi=\pi/2,\lambda=1/2} = 0.$$

In other words, the minimum values of $\delta^{(N-1)}_{\psi'}$ for the projected state $|\psi'\rangle = {}_i\langle\varphi|\psi\rangle$ are achieved by projecting onto the states $(|0\rangle \pm i|1\rangle)_i/\sqrt{2}$.

APPENDIX E

In this Appendix we prove Theorem 2 of the main text.

Let us observe that $\tilde{\delta}^{(N)}_\psi$ for the state (D1) can be represented in the form (A5), but in terms of the coefficients $b_{\mu'}$ and $c_{\mu'}$ defined in (D3):

$$\tilde{\delta}^{(N)}_\psi = \sum_{\mu_1',\mu_2',\mu_3',\mu_4'}[b_{\mu_1'}b_{\mu_2'}b_{\mu_3'}b_{\mu_4'} + 2c_{\mu_1'}c_{\mu_2'}b_{\mu_3'}b_{\mu_4'} + c_{\mu_1'}c_{\mu_2'}c_{\mu_3'}c_{\mu_4'}]g^{(N-1)}_{\mu_1'\mu_2'\mu_3'\mu_4'}, \tag{E1}$$

where the summation is on the $N-1$ tuples, $\mu' = (m_1,\ldots,m_{i-1},m_{i+1},\ldots,m_N)$ and $g^{(N)}_{\mu_1\mu_2\mu_3\mu_4}$ is given in (A6).

On the other hand, the state (D1) optimally projected to the i-th qubit state $(|0\rangle + i|1\rangle)$,

$${}_i(\langle 0| - i\langle 1|)|\psi\rangle = \sum_{\mu'}\left(b_{\mu'} - ic_{\mu'}\right)|\mu'\rangle,$$

is automatically normalized,

$$\sum_{\mu'}|b_{\mu'} - ic_{\mu'}|^2 = \sum_{\mu'} b^2_{\mu'} + c^2_{\mu'} = 1.$$

Therefore, we obtain the normalized measure (18) before and after the optimal deterministic projection:

$$\tilde{\delta}^{(N-1)}_{\psi'} = \sum_{\mu_1',\mu_2',\mu_3',\mu_4'}[b_{\mu_1'}b_{\mu_2'}b_{\mu_3'}b_{\mu_4'} + 4c_{\mu_1'}c_{\mu_2'}b_{\mu_3'}b_{\mu_4'} + c_{\mu_1'}c_{\mu_2'}c_{\mu_3'}c_{\mu_4'} + -b_{\mu_1'}c_{\mu_2'}b_{\mu_3'}c_{\mu_4'} - c_{\mu_1'}b_{\mu_2'}c_{\mu_3'}b_{\mu_4'}] \times g^{(N-1)}_{\mu_1'\mu_2'\mu_3'\mu_4'}. \tag{E2}$$

The difference between the normalized measures (18) before, as given by Eq. (E1), and after, as given by Eq. (E2), the optimal deterministic projection can be expressed as an explicitly positive quantity,

$$\tilde{\delta}_{\psi'}^{(N-1)} - \tilde{\delta}_{\psi}^{(N)} = \sum_{\alpha',\beta'} 4[X(\alpha',\beta')W(\alpha',\beta') - Y(\alpha',\beta')Z(\alpha',\beta')]^2 \geqslant 0, \tag{E3}$$

where

$$\langle \alpha',\beta'|\varphi_1\rangle = Z(\alpha',\beta') + iW(\alpha',\beta'),$$
$$\langle \alpha',\beta'|\varphi_2\rangle = X(\alpha',\beta') + iY(\alpha',\beta'),$$

and $|\varphi_{1,2}\rangle$ are defined in (D4).

Interestingly, Eq. (E3) can be represented in a more compact form as follows:

$$\tilde{\delta}_{\psi'}^{(N-1)} - \tilde{\delta}_{\psi}^{(N)} = \sum_{\alpha',\beta'} Q^2_{\rho'}(\alpha',\beta'),$$

where

$$\hat{\rho}' = \mathrm{Tr}_i(|\psi\rangle\langle\psi|\sigma_{yi}).$$

APPENDIX F

In this Appendix we prove that $\lim_{N\to\infty}\tilde{\delta}_{d,\min}^{(N-1)} = \lim_{N\to\infty}\tilde{\delta}_{\psi}^{(N)}$.

In Eq. (E2), rewritten as

$$\tilde{\delta}_{d,min}^{(N-1)} - \tilde{\delta}_{\psi}^{(N)} = 2\sum_{\mu_1',\mu_2',\mu_3',\mu_4'} c_{\mu_1'}c_{\mu_2'}b_{\mu_3'}b_{\mu_4'} \times \left(g^{(N-1)}_{\mu_1'\mu_2'\mu_3'\mu_4'} - g^{(N-1)}_{\mu_1'\mu_3'\mu_2'\mu_4'}\right), \tag{F1}$$

the coefficients $c_{\mu'}$ and $b_{\mu'}$ satisfy the following invariance properties under permutation of elements of $N-1$ tuples,

$$c_{\Pi_{i,j}\mu_1'}c_{\mu_2'}b_{\mu_3'}b_{\mu_4'} = c_{\mu_1'}c_{\Pi_{i,j}\mu_2'}b_{\Pi_{i,j}\mu_3'}b_{\Pi_{i,j}\mu_4'},$$
$$c_{\mu_1'}c_{\Pi_{i,j}\mu_2'}b_{\mu_3'}b_{\mu_4'} = c_{\Pi_{i,j}\mu_1'}c_{\mu_2'}b_{\Pi_{i,j}\mu_3'}b_{\Pi_{i,j}\mu_4'}, \tag{F2}$$

$$c_{\mu_1'}c_{\mu_2'}b_{\Pi_{i,j}\mu_3'}b_{\mu_4'} = c_{\Pi_{i,j}\mu_1'}c_{\Pi_{i,j}\mu_2'}b_{\mu_3'}b_{\Pi_{i,j}\mu_4'},$$
$$c_{\mu_1'}c_{\mu_2'}b_{\mu_3'}b_{\Pi_{i,j}\mu_4'} = c_{\Pi_{i,j}\mu_1'}c_{\Pi_{i,j}\mu_2'}b_{\Pi_{i,j}\mu_3'}b_{\mu_4'}, \tag{F3}$$

where $\Pi_{i,j}$ denotes a permutation of the ith and jth components of each μ_k' $(k=1,2,3,4)$, i.e., $\Pi_{i,j}\mu_{k,i}' = \mu_{k,j}'$. These properties are guaranteed by the presence of the δ functions that depend only on the appropriate weights $h(\mu_k')$. As a consequence, the sum in (F1) is bounded by its symmetrization as follows:

$$\tilde{\delta}_{d,\min}^{(N-1)} - \tilde{\delta}_{\psi}^{(N)} \leqslant 2\sum_{k_1,k_2,k_3,k_4=0}^{N-1} \frac{C_{k_1}C_{k_2}B_{k_3}B_{k_4}}{\sqrt{C^{N-1}_{k_1}C^{N-1}_{k_2}C^{N-1}_{k_3}C^{N-1}_{k_4}}} \times \left(G^{(N-1)}_{k_1k_2k_3k_4} - G^{(N-1)}_{k_1k_3k_2k_4}\right), \tag{F4}$$

where $G_{k_1k_2k_3k_4}$ is defined in (C1) and the following definitions were used:

$$C_{h(\mu_j')} = \sqrt{C^{N-1}_{h(\mu_j')}}\sum_{\mu'}\delta_{h(\mu'),k_j}c_{\mu_j'},$$

$$B_{h(\mu_j')} = \sqrt{C^{N-1}_{h(\mu_j')}}\sum_{\mu'}\delta_{h(\mu'),k_j}b_{\mu_j'}, \quad j=1,2,3,4$$

such that

$$\sum_{\mu'}\left(b^2_{\mu'} + c^2_{\mu'}\right) = \sum_{\mu'}\left[\frac{C^2_{h(\mu')}}{C^{N-1}_{h(\mu')}} + \frac{B^2_{h(\mu')}}{C^{N-1}_{h(\mu')}}\right] = \sum_{k=0}^{N-1}\left(C_k^2 + B_k^2\right) = 1,$$

with the notation $h(\mu_{k_j}') = k_j$, $j=1,2,3,4$, $0\leqslant k_j\leqslant N-1$, and C_k^{N-1} are the binomial coefficients.

Using the representation (C) it can be shown that

$$G^{(N-1)}_{k_1k_2k_3k_4} - G^{(N-1)}_{k_1k_3k_2k_4} = r^{(N-1)}_{k_1k_2k_3k_4}O(N^{-1/2}),$$

where $|r^{(N-1)}_{k_1k_2k_3k_4}|\leqslant 1$. This implies that

$$\lim_{N\to\infty}\left(\tilde{\delta}_{d,\min}^{(N-1)} - \tilde{\delta}_{\psi}^{(N)}\right) = 0.$$

In other words, as N approaches infinity, the difference between $\tilde{\delta}_{d,\min}^{(N-1)}$ and $\tilde{\delta}_{\psi}^{(N)}$ converges to zero.

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