Multipartite entanglement sudden death and birth in randomized hypergraph states

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We introduce and analyze the entanglement properties of randomized hypergraph (RH) states, as an extended notion of the randomization procedure in the quantum logic gates for the usual graph states, recently proposed in the literature. The probabilities of applying imperfect generalized controlled-Z gates simulate the noisy operations over the qubits. We obtain entanglement measures as negativity, concurrence, and genuine multiparticle negativity, and show that entanglement exhibits a nonmonotonic behavior in terms of the randomness parameters, which is a consequence of the nonuniformity of the associated hypergraphs, reinforcing the claim that the entanglement of randomized graph states is monotonic since they are related to 2-uniform hypergraphs. Moreover, we observed the phenomena of entanglement sudden death and entanglement sudden birth in RH states. This work reveals a connection between the nonuniformity of hypergraphs and loss of entanglement.

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I. INTRODUCTION

Multiparticle entanglement is a resource in quantum information processing (QIP), providing advantages over its classical counterparts, and is relevant in many physical applications [1,2]. Hypergraph states (HS) concern a family of multiparticle quantum states that are genuinely maximally entangled [3–8] and have been the subject of intense research [9–11]. This family generalizes the well-known family of graph states (GS), which are 2-uniform HS [12], that rely on the center of interest in quantum information protocols [13], show the potential for foundational studies [14], and extremely violate local realism [15,16].

Controlled systems that demand quantum measurements, such as quantum logic gates, still present some unavoidable degree of noise and the application of the quantum gates not always can be performed with success. While 1-qubit gates can be built with fidelities higher than 99%, 2-qubit entangling gates hardly reach 93%, and this becomes worse for 3-qubit gates, and so on [17–20]. So, an understanding of random quantum states and their generation in the presence of noise are of current interest [21]. Quantum decoherence resulting from low-fidelity gates, environmental factors, and interqubit interactions [22] represents a significant challenge in quantum computing. Indeed, one of the main goals of QIP is to assure a way to perform universal quantum computation in the most feasibly robust way [23]. A demonstration of the deleterious effects of decoherence on entangled states is the entanglement sudden death (ESD), which refers to the complete loss of entanglement in a finite time, regardless of the asymptotic nature of the loss of system coherence [24,25]. The phenomenon of ESD has been observed experimentally in different physical systems [26–30], and is something we want to avoid for any practical purpose of using entangled states.

Most of the studies investigating the ESD phenomenon concern bipartite systems. For multiparticle systems, we must take into account the concept of genuine multiparticle entanglement (GME) [31]. In this paper, we discuss noisy multipartite gates by introducing the concept of randomized hypergraph (RH) states as a generalization of the concept of randomized graph (RG) states [32]. We show that this randomization procedure of nonuniform hypergraphs leads to RH states that break the monotonic behavior of bipartite entanglement (BE) and GME as a function of the randomness parameters, reinforcing the claim raised in Ref. [32] that the entanglement of RG states is monotonic, since these latter states are 2-uniform HS. Moreover, we also show the manifestation of ESD, as well as entanglement sudden birth (ESB) [33], in RH states as a function of the randomness parameters. We note that these cases occur for nonuniform hypergraphs, the ones where the hyperedges of different cardinalities are present.

II. HYPERGRAPH STATES

A hypergraph $H = (V, E)$ is defined as a pair consisting of a finite set of vertices $V = \{v_1, v_2, \ldots, v_n\}$ and a set of hyperedges $E \subset 2^V$, with 2^V the power set of V [34]. A hyperedge that encloses k vertices has cardinality k and if a hypergraph has all hyperedges with cardinality k, the hypergraph is k-uniform. An ordinary graph is a 2-uniform hypergraph. If a hypergraph F is obtained from H by deleting hyperedges, in such a way that $V_F = V_H$, then F is called a subhypergraph of H [35]. In this latter case, we say that F spans H. Examples of hypergraphs are shown in Fig. 1. Given a hypergraph H on n vertices, the correspondent HS, denoted by $|H\rangle$, is obtained by assigning a qubit for every vertex of the hypergraph in the state $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$, in such a way that the initial state is $|+\rangle^{\otimes n}$, followed by the application of a nonlocal multiqubit

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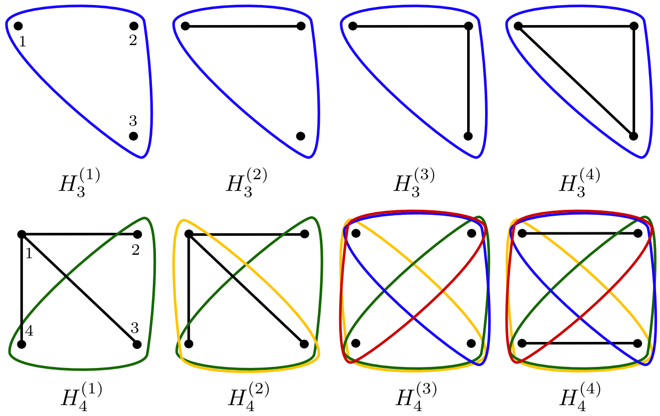

FIG. 1. Hypergraphs that are of interest in this paper. The hypergraphs $H_4^{(1)}$, $H_4^{(2)}$, and $H_4^{(3)}$ are cases of special interest because their reduced single-qubit matrices are maximally mixed [5]. $H_n^{(i)}$ represents the ith hypergraph on n vertices.

gate C_e acting on the Hilbert spaces associated with vertices $v_i \in e$, which is a $2^{|e|} \times 2^{|e|}$ diagonal matrix given by $C_e = \mathbb{1} - 2|1\dots1\rangle\langle1\dots1|$, where $e \in E$ represents a hyperedge, $|e|$ is the cardinality of the hyperedge e, and $\mathbb{1}$ is the identity matrix. Thus, a HS $|H\rangle$ is a pure quantum state defined as [7]

$$|H\rangle = \prod_{e\in E} C_e|+\rangle^{\otimes n}. \tag{1}$$

III. RANDOMIZED HYPERGRAPH STATES

Instead of applying the perfect usual C_e gate to obtain a pure HS, we define a randomization operator R_P which introduces the application of probabilistic gates $\Lambda^e_{p_{|e|}}$ to the state $|+\rangle^{\otimes n}$, where $P = \{p_k\}_{k=1}^n$ is a set of probabilities (or randomness parameters). Thus, we consider a noise implementation of the generalized C_e gate, in which with probability $p_{|e|}$ the gate succeeds (a hyperedge is created among the qubits) and with probability $(1 - p_{|e|})$ the gate fails, which has the same effect as an identity operator [32]. This allowed us to simulate the presence of noise in the construction of the states that can be used for quantum computational universality for hypergraphs [36]. All the results are obtained concerning the randomness parameter $p_{|e|}$, which can be implicitly linked with time and is deliberately left indeterminate to accommodate various possible noise behaviors. As an example, $p_{|e|} = e^{-\kappa_{|e|}t}$, where t might represent the time duration over which a quantum gate is actively applied or attempted and $\kappa_{|e|}$ is the decay constant. The decay could then signify a reduction in the probability of the success of gate as time progresses.

To exemplify the procedure, consider the hypergraph $H_3^{(2)}$ in Fig. 1. The randomization procedure produces the following state,

$$\begin{aligned} R_P\big(\big|H_3^{(2)}\big\rangle\big) &= \Lambda^{\{1,2\}}_{p_2} \circ \Lambda^{\{1,2,3\}}_{p_3}(|+++\rangle\langle+++|) \\ &= p_2p_3|F_1\rangle\langle F_1| + (1-p_2)p_3|F_2\rangle\langle F_2| \\ &\quad + p_2(1-p_3)|F_3\rangle\langle F_3| \\ &\quad + (1-p_2)(1-p_3)|F_4\rangle\langle F_4|, \end{aligned} \tag{2}$$

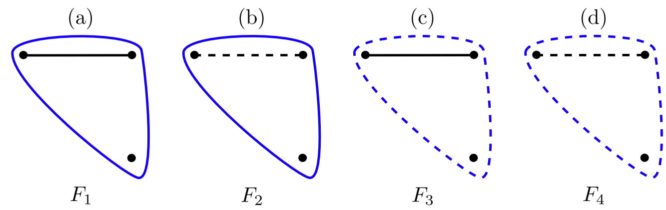

FIG. 2. Process of randomization of the $|H_3^{(2)}\rangle$. Dashed lines represent noisy C_e gates, in which with probability $p_{|e|}$ the gate succeeds and with probability $(1 - p_{|e|})$ the gate fails. When the gate fails, it has the same effect as an identity operator. Continuum lines are a successful creation of hyperedges. (a) F_1 subhypergraph with probability p_2p_3; (b) F_2 subhypergraph with probability $(1 - p_2)p_3$; (c) F_3 subhypergraph with probability $p_2(1 - p_3)$; and (d) F_4 subhypergraph with probability $(1 - p_2)(1 - p_3)$ [see Eq. (2)].

with p_2 and p_3 the randomness parameters for the application of the hyperedges of cardinalities 2 and 3, respectively. The spanning subhypergraphs F_i, $i = 1, \dots, 4$, are schematically shown in Fig. 2. Since all the gates $\Lambda^e_{p_{|e|}}$ commute, the order of application of these gates does not need to be specified. Due to the application of this randomization operator, we end up having randomized mixed states, contrary to the pure states that HS usually represent. These RH states will be denoted as ρ_H^P. Now we give a formal definition of RH states.

Definition III.1 (Randomized hypergraph state). Let $|H\rangle$ be a hypergraph state. Its randomization operator is defined as

$$R_P(|H\rangle) = \sum_{F \text{ spans } H} \prod_{p_k\in P} p_k^{|E_{k,F}|}(1-p_k)^{|E_{k,H}\setminus E_{k,F}|}|F\rangle\langle F|, \tag{3}$$

where F are the spanning subhypergraphs of H, $E_{k,H}$ and $E_{k,F}$ are the sets of hyperedges of H and F, respectively, and $P = \{p_k\}_{k=1}^n$ is the set of randomness parameters for hyperedges of cardinality k. The resulting hypergraph state $\rho_H^P := R_P(|H\rangle)$ is the randomized version of $|H\rangle$.

Thus applying the above definition, we can map an initially pure hypergraph state $|H\rangle$ onto a mixture of all its spanning subhypergraph states with probability p_k, $k = 1, \dots, n$, making the role of a parameter controlling the amount of entanglement of a RH state that connects the two extremes cases, for $p_k = 0$, an empty hypergraph, and $p_k = 1$ for a pure HS. It is important to emphasize that the randomization is related only to the hyperedges of the hypergraph initially given.

Physically, it is crucial to know the equivalent classes between states, since those states share the same entanglement properties. Two n-qubit HS $|H\rangle$ and $|H'\rangle$ are said to be local unitary (LU) equivalent, and so belong to the same entanglement class, if and only if they can be connected by local unitaries $U_1, \dots, U_n$, in such a way that $|H\rangle = U_1 \otimes \dots \otimes U_n|H'\rangle$. This property turns out to be quite important, since if we identify LU equivalent classes of HS, it is possible to know if they are also equivalent under stochastic local operations and classical communication [5]. A deeper analysis of the equivalent classes of HS up to six qubits can be found in Ref. [5]. As in the case of RG states generated from two LU equivalent GS, RH states generated from two LU equivalent HS are, in general, not LU equivalent. This stems from the fact that the randomization process produces mixed states

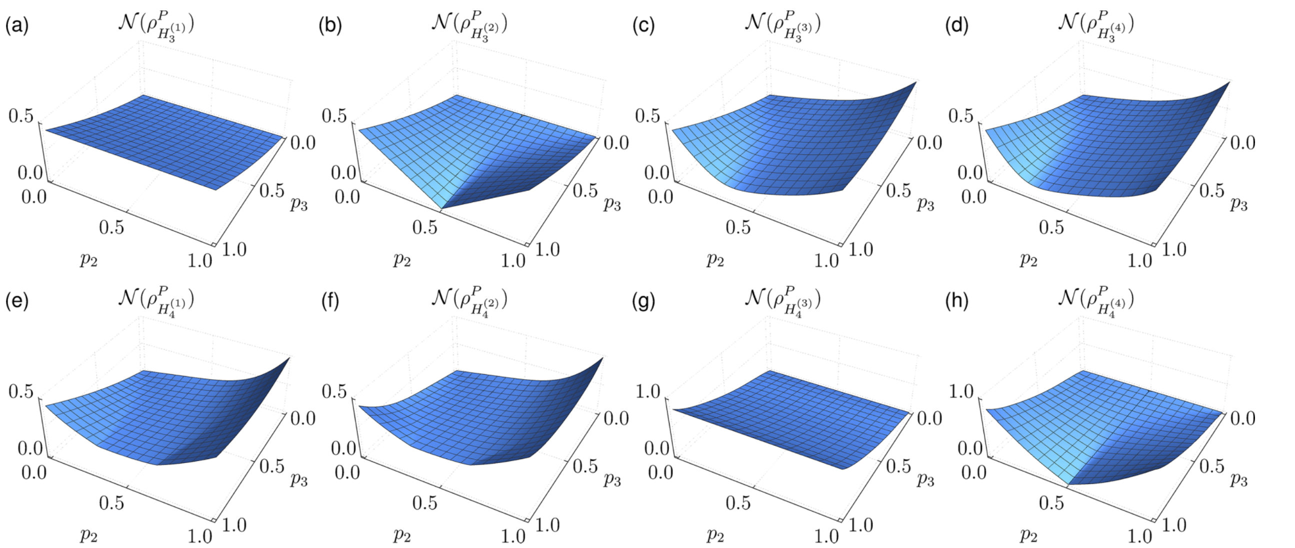

FIG. 3. Negativity for RH states listed in Fig. 1. The notation "$\{v_1, \ldots\}|\{u_1, \ldots\}$" represents the bipartition used to calculate the negativity. (a)–(d) Negativity of 3-qubit RH states for bipartition $\{3\}|\{1, 2\}$. (e), (f) Negativity of 4-qubit RH states for bipartition $\{2\}|\{1, 3, 4\}$. (g), (h) Negativity of 4-qubit RH states for bipartition $\{1, 2\}|\{3, 4\}$.

with different ranks and therefore they can no longer be transformed using LU operations. This fact is an attractive property of RH states, since the purpose of the randomization is to precisely simulate the noise degree during the experimental construction of these states in a laboratory. An interesting explanation concerning the experimental implementation using hypergraph states can be found in Ref. [37].

IV. BIPARTITE ENTANGLEMENT

To illustrate our findings concerning the entanglement properties of RH states, we have chosen hypergraphs shown in Fig. 1. Since RH states are mixed states, we use the negativity [38] and the concurrence [39] to quantify the amount of BE. The negativity of a bipartite state ρ is defined as $\mathcal{N}(\rho) = (\|\rho^{\Gamma_A}\| - 1)/2$, where ρ^{Γ_A} is the partial transpose of ρ with respect to subsystem A, and $\|X\| = \sqrt{X^\dagger X}$ is the trace norm. The concurrence for a mixed bipartite state ρ is defined as $C(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$, where λ_i are the eigenvalues, in decreasing order, of the non-Hermitian matrix $\rho\tilde{\rho}$, with $\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$.

In Fig. 3 we show the results for the negativity for the RH states in Fig. 1 as a function of the randomness parameters p_2 and p_3. We can observe that the negativity shows a nonmonotonic behavior for all cases but $\rho^P_{H_3^{(1)}}$ and $\rho^P_{H_4^{(3)}}$ [see Figs. 3(a) and 3(g), respectively], which show a monotonic behavior. This is an interesting result because in Ref. [32] an observation was made of a monotonic behavior of negativity in terms of the randomness parameter for all the RG states studied there. Moreover, the question was raised regarding whether the monotonic behavior of the entanglement is a common feature for all RG states. This monotonic behavior of negativity was also observed in Ref. [40] for multiqubit randomized entangled states obtained by using random Ising-type entangling operators. However, our results show that the answer to this question is in the negative for RH states. This behavior can be understood by noting that the monotonic behavior of the negativity stems from the fact that $\rho^P_{H_3^{(1)}}$ and $\rho^P_{H_4^{(3)}}$ are states associated with 3-uniform hypergraphs, in which there is no mixture of hyperedges of different cardinalities. We can say there is a sort of competition for entanglement when applying entangling gates of different cardinalities. Thus, the break of the monotonic behavior of the BE in RH states is a feature present in those states that are associated with nonuniform hypergraphs and such behavior could not be observed in RG states, since any graph is a 2-uniform hypergraph.

In Fig. 4, we present the results for the concurrence of the RH states in Fig. 1. As the negativity, the concurrence also manifests a nonmonotonic behavior of the BE for nonuniform HS. More interesting is the presence of ESD and ESB, as we can see large regions where the concurrence vanishes. In Table I, we show the values of p_2 for the manifestation of ESD and ESB for the RH states for the case $p_3 = 1$. For instance, consider Fig. 4(c) where we can observe the manifestation of ESD at $p_2 = 0.239$ and ESB at $p_2 = 0.761$. So, again we observe a sort of competition for the entanglement because of the application of entangling gates of different cardinalities, but now with impressive results. We emphasize that the concurrence is the entanglement measure that shows the most anomalous behavior for the entanglement results. It

TABLE I. Values of p_2 for which the ESD and ESB occur for 3- and 4-qubit RH states listed in Fig. 1 for $p_3 = 1$.

RH state	ESD	ESB	RH state	ESD	ESB
$\rho^P_{H_3^{(1)}}$			$\rho^P_{H_4^{(1)}}$	0.397	
$\rho^P_{H_3^{(2)}}$	0.500	0.500	$\rho^P_{H_4^{(2)}}$	0.397	
$\rho^P_{H_3^{(3)}}$	0.239	0.761	$\rho^P_{H_4^{(3)}}$		
$\rho^P_{H_3^{(4)}}$	0.315	0.707	$\rho^P_{H_4^{(4)}}$	0.288	

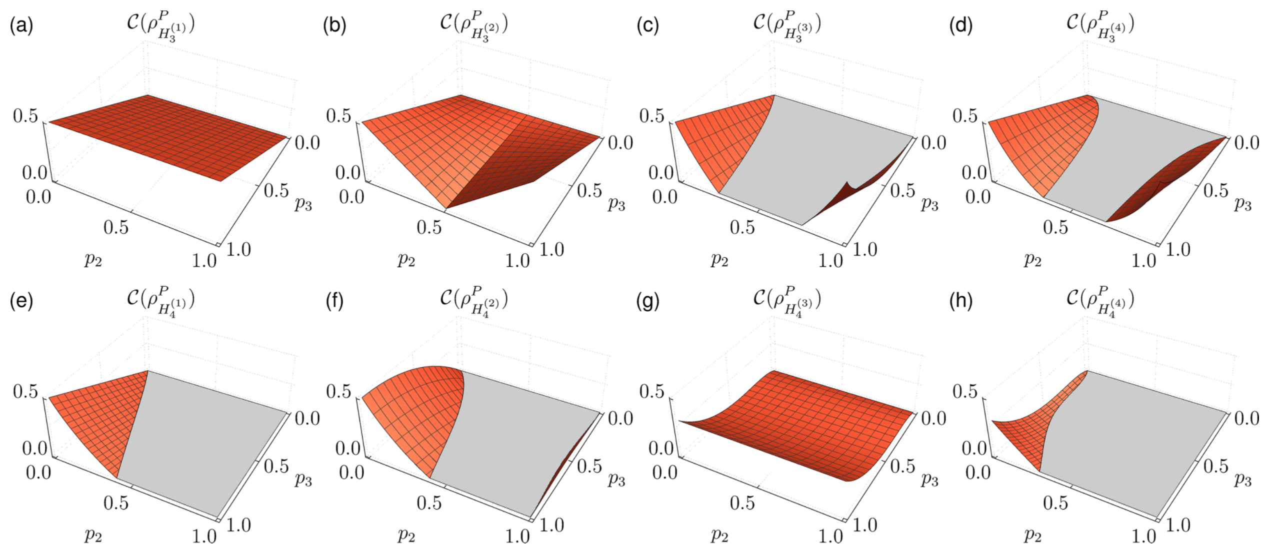

FIG. 4. Concurrence of RH states listed in Fig. 1. (a)–(d) Concurrence of 3-qubit RH states between qubits 1 and 3; (e), (f) concurrence of 4-qubit RH states between qubits 3 and 4; and (g), (h) concurrence of 4-qubit RH states between qubits 1 and 3. The light gray part indicates the zero value of concurrence.

is somehow expected since concurrence is calculated using two-qubit reduced density matrices.

V. GENUINE MULTIPARTICLE ENTANGLEMENT

Since concurrence and negativity are not able to quantify the amount of GME in RH states, we employ the concept of genuine multiparticle negativity (GMN) [41,42], to quantify the amount of GME. It is well known that HS are GME states [5], which are quite useful for quantum information protocols [43–45]. The GMN is a versatile tool to characterize multiparticle entanglement, where its implementation is based on fully decomposed witnesses $\mathcal{W}$, and it can be directly computed using semidefinite programming. One defines the GMN $\tilde{N}_g(\rho)$ by means of the optimization problem

$$\tilde{N}_g(\rho) = -\min \operatorname{tr}(\rho \mathcal{W}), \tag{4}$$

subject to $\mathcal{W} = P_m + Q_m^{T_m}$, with $0 \leqslant P_m \leqslant \mathbb{1}$ and $0 \leqslant Q_m \leqslant \mathbb{1}$, for all the partitions $\{m\}|\{\tilde{m}\}$, where $\{\tilde{m}\}$ is the complement of the part $\{m\}$, and T_m is the partial transpose with respect to m. The results for the GMN of RH states were obtained with the help of the online program PPTMIXER [46], and are shown in Fig. 5. As was the case for BE, the GMN for the RH states of Fig. 1 also shows a nonmonotonic behavior for all cases except for $\rho^P_{H_3^{(1)}}$ and $\rho^P_{H_4^{(3)}}$, which are related to uniform hypergraphs. It is interesting to comment that the RH state $\rho^P_{H_3^{(2)}}$ manifests bipartite and genuine multiparticle ESD for

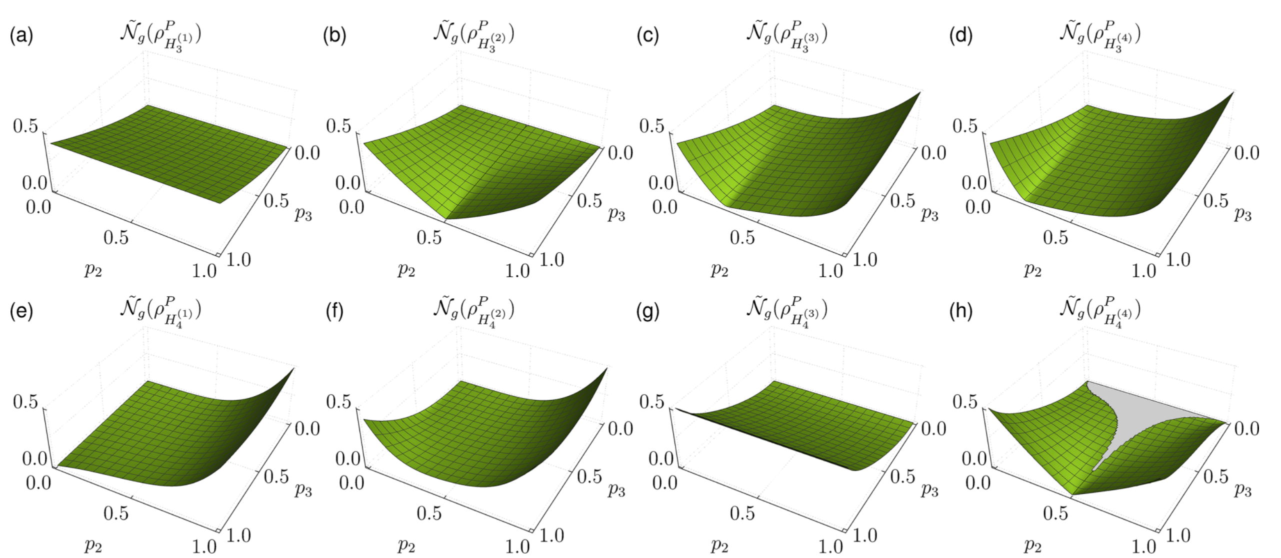

FIG. 5. GMN of RH states listed in Fig. 1. (a)–(d) GMN for 3-qubit RH states. (e)–(h) GMN for 4-qubit RH states. The light gray part indicates the zero value of GMN.

$p_2 = 0.5$ for all values of p_3. The RH state $\rho^P_{H_4^{(4)}}$ seems to be the most fragile one as the GMN (and concurrence) is the most affected one by the variation of the randomness parameters, as it shows a large region of null entanglement [see Figs. 4(h) and 5(h)].

VI. CONCLUSION

To summarize, we have introduced the concept of RH states. These randomized mixed states have a nonmonotonic behavior of BE and GME in terms of the randomness parameters, a feature not observed in RG states [32]. The nonmonotonic behavior of entanglement occurs in RH states associated with nonuniform hypergraphs, while the monotonic behavior is observed only in uniform RH states, reinforcing the claim that the entanglement in all RG states is generally monotonic in terms of the randomness parameter. Moreover, to support this conclusion we computed the entanglement properties for all the 27 equivalence classes under LU transformations for HS of four qubits [5] and the break of monotonicity of entanglement for nonuniform hypergraphs was observed. Moreover, the interesting phenomena of bipartite and multiparticle ESD and ESB for some RH states were observed. Such behavior is as strong as the difference between the cardinalities of the hyperedges. These results and related issues will be reported in a future work [47].

This work reveals a connection between the nonuniformity of hypergraphs and loss of entanglement, changing the way we can view ESD in multiparticle states. Observing the results for BE and GME, we can observe that the ESD for GME can be less frequent, a counterintuitive result that was recently observed in Ref. [48]. A precise description of ESD for multiparticle systems remains a challenging topic and there is still sparse literature. Such an intricate interplay between noise-induced perturbations and the capacity for entanglement regeneration emphasizes the resilience of the system under decoherence, potentially contributing to the development of more robust and fault-tolerant QIP techniques.

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