

Approximation scheme and non-Hermitian renormalization for the description of atom-field-system evolution

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Interactions between a source of light and atoms are ubiquitous in nature. The study of them is interesting on the fundamental level as well as for applications. They are at the core of quantum information processing tasks and quantum thermodynamics protocols. However, even for a two-level atom interacting with a field in rotating wave approximation there exists no exact solution. This touches upon a basic problem in quantum field theory, where we can only calculate the transitions in the time asymptotic limits (i.e., minus and plus infinity), while we are not able to trace the evolution. In this paper we want to get more insight into the time evolution of a total system of a two-level atom and a continuous-mode quantum field. We propose an approximation, which we are able to apply systematically to each order of Dyson expansion, resulting in a greatly simplified formula for the evolution of the combined system at any time. Our tools include a proposed *non-Hermitian* renormalization method. As a sanity check, by applying our framework, we derive the known optical Bloch equations.

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I. INTRODUCTION

In quantum information processing manipulating two-level systems (qubit), such as trapped ions [1–4], interacting with coherent control laser fields is vitally important for realization of qubit operations since such manipulations constitute the basic components of quantum gates. Within the framework of chemical applications the coherent control field is a fundamental tool in manipulation of ultracold molecules, used for instance in laser cooling, photodissociation, and photoassociation, which recently has gained a fast-growing attention [5–13]. Coherent control fields also play an important role in quantum thermodynamic processes such as shortcuts to adiabaticity technique in which the external field is designed for minimization or maximization of crucial quantities such as time or energy cost of the process [14–18].

However, the control fields are usually treated classically. Considering the field to be classical causes us to be oblivious to some fundamental quantum effects of the field on the quantum gate. These quantum effects may consist of entanglement of the field with the qubit or spontaneous emission and also the Lamb shift due to the vacuum effects [19,20]. The problem is that even the evolution of the two-level atom interacting with light is not exactly solvable, even in rotating wave approximation. In the case of a coherent state of the field, one often uses for description of the atom an approximation known as optical Bloch equations [21]. The description of dynamics of both atom and field is even more problematic. The exact solution exists just for the initial vacuum state of light [22,23].

This is related to the general problem in quantum field theory where the behavior of the system is examined using the S matrix (scattering matrix) which relates the initial state of

the system to its final state in asymptotic limits ($t_{\text{initial}} \rightarrow -\infty$ and $t_{\text{final}} \rightarrow \infty$) [24]. Thus, little is known about the system during the intermediate times.

The aim of this paper is to propose an approximation scheme, that allows for analytic examination of time evolution of a total system of atom and field for all times, in all orders of Dyson series. As a validation of our approach, we then show that the obtained formulas for the approximate evolution reproduce the well-known optical Bloch equations.

We consider the interaction of a two-level atom with a continuous-mode quantum field by taking all modes of the field directly into consideration. Using a renormalization method we will derive a greatly simplified formula for the evolution of the whole system at any time such that it depends only on normally ordered creation and annihilation operators of the field, and two parameters: the decay rate of the atom and the Lamb shift in atomic frequency. We will finally show that the optical Bloch equations [21] directly and rigorously emerge from our formalism by successively applying our approximation without any further assumption. Apart from the proposed approximation scheme, we heavily use the “dissipative renormalization” that we introduce in this paper, which can be of separate interest. Namely, we shift from interaction to self-Hamiltonian a non-Hermitian operator, the Hermitian part of which corresponds to standard renormalization (related to Lamb shift) while the anti-Hermitian part corresponds to decay rate.

Our paper is organized as follows. In Sec. II, we briefly explain the approximation used in our calculations as well as the renormalization scheme. In Sec. III, we renormalize the Hamiltonian of a continuous-mode electric field acting on a two-level atom. Later, in Sec. IV, we provide justifications for our approximation making use of simpler models such as Friedrichs-Lee [22,23,25]. We compute the explicit evolution of the renormalized S propagator (Theorem 1), get the

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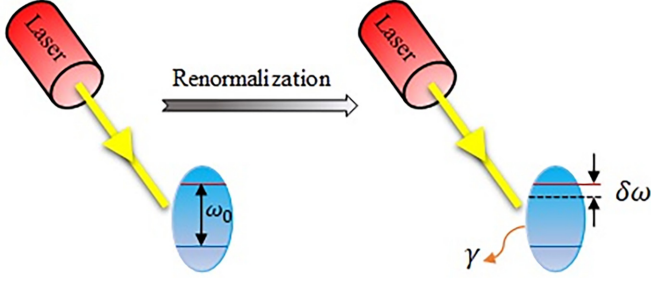


FIG. 1. Interaction of a two-level atom with a quantized continuous-mode laser field. After renormalization the renormalized free Hamiltonian of the atom reads $H_{0r} = (\omega_0 + \delta\omega - i\gamma)|1\rangle\langle 1|$ where ω_0 is the atomic frequency, γ the decay rate due to spontaneous emission, $\delta\omega$ the Lamb Shift, and $|1\rangle$ the excited state of the atom. As is seen the free Hamiltonian is non-Hermitian. The renormalized interaction Hamiltonian becomes $H_{I_r} = H_I - (\delta\omega - i\gamma)|1\rangle\langle 1|$ which is non-Hermitian.

differential equation for S , and check our results with particular examples in Sec. V. In Sec. VI, we illustrate our results from Theorem 1 by means of two opposite examples: the most classical case in which the laser starts in a coherent state, and when its initial state is a continuous superposition of a photon in different modes. For the former, we obtained the well-known optical Bloch equations which serve as a sanity check. Finally, in Sec. VII, we present the conclusions of our paper and pose possible future research directions.

II. OUTLINE OF THE RESULTS

As schematically depicted in Fig. 1 we will consider the interaction of a two-level atom with a quantized continuous-mode laser field. The Hamiltonian will be renormalized, which gives rise to a non-Hermitian free Hamiltonian for the atom and consequently a non-Hermitian interaction Hamiltonian:

$$H_{A_r} = \hbar(\omega_0 + \delta\omega - i\gamma)|1\rangle\langle 1|, \quad (1)$$

$$H_{I_r} = H_I - \hbar(\delta\omega - i\gamma)|1\rangle\langle 1|, \quad (2)$$

where ω_0 is the atomic frequency and γ the decay rate of the atom and $\delta\omega$ is the ‘‘Lamb shift’’ in atomic frequency. From now on, we use the notation in which $\hbar = 1$. The role of performing renormalization is the following. The Hermitian part is more or less standard. It is just to put the infinities (or cutoff dependent terms), emerging from commutation relations at each order of Dyson expansion, into a single parameter—the Lamb shift, whose value we assume can be taken from higher-order theory—i.e., quantum electrodynamics. The non-Hermitian part is a trick that in a sense separates spontaneous emission from the evolution, and allows us to arrive at a simplified form of the evolution. At the moment it is a technical tool, whose deeper interpretation is still awaited. Due to the emerged non-Hermitian interaction Hamiltonian the evolution of the atom-field in the interaction picture becomes *nonunitary*, i.e.,

$$|\psi_{I_r,AF}(t)\rangle = e^{-iH_{I_r}t}|\psi_{AF}(0)\rangle. \quad (3)$$

We then suppose that the probability amplitude of the atom in the excited state is of exponential decay form, namely, we shall assume that the long-time behavior of the survival amplitude is just an exponential decay. As will be seen in the following in order for this assumption to hold we must apply an approximation which, in turn, leads to a solvable model reproducing the standard second-order approximation to the decay rate γ and the Lamb shift $\delta\omega$. In fact, within the approximation regime, for long times, the probability amplitude of the atom in the excited state decays exponentially.

Applying the approximation the contribution of all terms, which appear after normal ordering of the creation and annihilation operators of the field in Dyson series, is pushed to the renormalization and the renormalized evolution deals only with normally ordered terms of creation and annihilation operators of the field, being therefore in principle exactly solvable by means of the coherent-state basis.

III. DISSIPATIVE RENORMALIZATION

In this section we first introduce the general form of the Hamiltonian of the interaction of a two-level atom with a continuous-mode field. Then by adding and subtracting two terms from the Hamiltonian we will renormalize the Hamiltonian and using this renormalized Hamiltonian we will formulate the *nonunitary* evolution of the atom-field in the interaction picture. For an atom interacting with a continuous-mode field the Hamiltonian, in the rotating wave approximation, reads ($\hbar = 1$)

$$H = H_0 + H_I, \quad (4)$$

where $H_0 = H_A + H_F$ with

$$H_A \equiv \omega_0|1\rangle\langle 1|, \quad H_F \equiv \sum_{\lambda} \int d^3\vec{k} \omega_{\vec{k}\lambda} a_{\vec{k}\lambda}^{\dagger} a_{\vec{k}\lambda}, \quad (5)$$

H_A and H_F being the Hamiltonians of the atom and the field respectively and

$$H_I = \sum_{\lambda} \int d^3\vec{k} [f(\omega_{\vec{k}\lambda})\sigma^+ a_{\vec{k}\lambda} + f^*(\omega_{\vec{k}\lambda})\sigma^- a_{\vec{k}\lambda}^{\dagger}], \quad (6)$$

where the modes are labeled by a continuous wave vector \vec{k} and a polarization label $\lambda = \pm 1$, $a_{\vec{k}\lambda}$ ($a_{\vec{k}\lambda}^{\dagger}$) are the field annihilation (creation) operators of mode (\vec{k}, λ) , σ^- (σ^+) is the atom lowering (raising) operator, and

$$f(\omega_{\vec{k}\lambda}) = i\sqrt{\frac{\omega_{\vec{k}\lambda}}{2(2\pi)^3\epsilon_0}} \vec{e}_{\vec{k}\lambda} \cdot \vec{D} \quad (7)$$

is the coupling constant in which $\vec{e}_{\vec{k}\lambda}$ is the electric-field unit vector and \vec{D} is the atomic dipole moment vector [26]. In the following, for ease of calculations, $\int d^3\vec{k}$ will be denoted by the symbol $\int dk$ and the polarization index λ is also dropped. The Hamiltonian of the atom-field given in Eq. (4) can be decomposed into two non-Hermitian renormalized parts:

$$H = H_{0r} + H_{I_r}, \quad (8)$$

where $H_{0r} = H_{A_r} + H_F$ with

$$H_{A_r} \equiv \Omega|1\rangle\langle 1|, \quad H_F \equiv \int dk \omega_k a_k^{\dagger} a_k, \quad (9)$$

and

$$H_I = \int dk [f(\omega_k)\sigma^+ a_k + f^*(\omega_k)\sigma^- a_k^\dagger] + ib|1\rangle\langle 1|, \quad (10)$$

with

$$b = i\delta\omega + \gamma, \quad \Omega = \omega_A - i\gamma, \quad \omega_A = \omega_0 + \delta\omega, \quad (11)$$

where γ is the decay rate and $\delta\omega$ is the Lamb shift (see Fig. 1 for more detail). The interaction Hamiltonian in the interaction picture reads¹

$$\begin{aligned} \tilde{H}_I(t) &= e^{iH_0 t} H_I e^{-iH_0 t} \\ &= \int dk [f(\omega_k, t)\sigma^+ a_k + f'(\omega_k, t)\sigma^- a_k^\dagger] + ib|1\rangle\langle 1|, \end{aligned} \quad (12)$$

where

$$f(\omega, t) = f(\omega)e^{-i(\omega - \omega_A + i\gamma)t}, \quad f'(\omega, t) = f^*(\omega)e^{i(\omega - \omega_A + i\gamma)t}. \quad (13)$$

It is also convenient to use the notation

$$\begin{aligned} A(t) &\equiv \int dk f(\omega_k, t)a_k, \\ A'(t) &\equiv \int dk f'(\omega_k, t)a_k^\dagger, \end{aligned} \quad (14)$$

so that

$$\tilde{H}_I(t) = \sigma^+ A(t) + \sigma^- A'(t) + ib|1\rangle\langle 1|. \quad (15)$$

The operators A and A' satisfy the commutation relations (inherited from canonical commutation relations)

$$[A(t), A'(s)] = F^r(t - s), \quad (16)$$

where

$$F^r(t) = e^{i(\omega_A - i\gamma)t} F(t), \quad F(t) \equiv \int dk e^{-i\omega_k t} |f(\omega_k)|^2. \quad (17)$$

For the (immediate) proof, see Lemma 4 in Appendix D.

Up to now, motivated by (i) the fact due to interaction with the field that the frequency of the atom is shifted and (ii) the fact that an atom in the excited state, at least in some time regimes, exhibits exponential decay (whose rate we denoted by γ), we defined non-Hermitian renormalized Hamiltonian H_0 , which accounts for the above two effects.

As will be seen in the following, dividing the Hamiltonian into non-Hermitian parts in this manner, and applying suitable approximation, we will arrive at a remarkably simplified evolution of the S propagator.

IV. APPROXIMATION SCHEME

Here we will propose an approximation scheme in which we shall assume that the long-time behavior of the survival amplitude is just exponential decay. As will be seen below,

¹One could note that the interaction picture here corresponds to a non-Hermitian operator (H_0). This is not a problem, since interaction pictures can also be defined for these operators, even though for these cases one no longer transforms as $\tilde{A}(t) = U^\dagger A U$ but $\tilde{A}(t) = U^{-1} A U$.

this approximation is equivalent to the following approximation:

$$F^r(t) \approx b\delta(t), \quad t > 0. \quad (18)$$

Applying this approximation allows us to remove the contribution of *all* non-normally ordered terms from the evolution of the S propagator of the atom-field. In fact, the contribution of non-normally ordered terms will be contained just in two terms—the Lamb shift, and the decay rate. We shall first motivate the approximation in a simpler model, that is exactly solvable—the Friedrichs-Lee model [22,23,25]. Then in the next section the full model will be presented.

A. Friedrichs-Lee model

We will consider the Friedrichs-Lee model to justify our approximation [22,23,25]. Consider the field to be initially in the vacuum state. Therefore the evolution of the atom-field leaves invariant the sector of Hilbert space spanned by the vectors: $|e\rangle \equiv |1\rangle \otimes |\{0\}\rangle$, $|f\rangle \equiv \int dk |0\rangle \otimes f(\omega_k) a_k^\dagger |\{0\}\rangle$ where $|1\rangle$ ($|0\rangle$) is the excited (ground) state of the atom and $|\{0\}\rangle$ ($|\{1_k\}\rangle$) is the vacuum state of the field [the state with one photon in the k mode with an arbitrary $f(\omega_k)$]. Thus the interaction Hamiltonian, given in Eq. (6), restricted to this sector reads

$$H_I = |e\rangle\langle f| + |f\rangle\langle e|, \quad |f\rangle = \int dk f(\omega_k) |0\rangle |\{1_k\}\rangle, \quad (19)$$

where $\{\omega_k\}$ denotes the complete set of frequencies that specify the states in each excited mode of the field. The evolution in the interaction picture $U_I(t) = e^{iH_0 t} e^{-iH t}$ satisfies the integral equation

$$U_I(t) = \mathbb{I} - i \int_0^t ds e^{iH_0 s} (|e\rangle\langle f| + |f\rangle\langle e|) e^{-iH_0 s} U_I(s) \quad (20)$$

where H and H_0 are defined in Eqs. (4) and (5), respectively (notice that the polarization index λ has been dropped for ease of calculations). Here we define the matrix elements

$$K(t) = \langle e|U_I(t)|e\rangle, \quad M(t) = \langle f_i|U_I(t)|e\rangle \quad (21)$$

and the ket

$$|f_i\rangle = \int dk e^{i\omega_k t} f(\omega_k) |0\rangle |\{1_k\}\rangle. \quad (22)$$

Now inserting Eq. (20) into Eq. (21) one obtains

$$\begin{aligned} K(t) &= 1 - i \int_0^t ds e^{i\omega_0 s} M(s), \\ M(t) &= -i \int_0^t ds e^{-i\omega_0 s} F(t - s) K(s), \end{aligned} \quad (23)$$

where $F(t)$ is defined in Eq. (17). Using the definition of the Laplace transform

$$\mathcal{F}(z) = \int_0^\infty dt e^{-zt} F(t) \quad (24)$$

and its properties one can readily transform Eq. (23) into

$$\mathcal{K}(z) = \frac{1}{z} - i \frac{1}{z} \mathcal{M}(z - i\omega_0), \quad \mathcal{M}(z) = -i \mathcal{F}(z) \mathcal{K}(z + i\omega_0), \quad (25)$$

which finally gives

$$\mathcal{K}(z) = \frac{1}{z + \mathcal{F}(z - i\omega_0)}. \quad (26)$$

Now, finding evolution of relevant matrix elements (21) amounts to finding suitable inverse Laplace transforms.

So far we have just recalled how one can (in principle) solve dynamics if we are in the single excitation sector. However, in this paper, we are not interested in exact solutions for such a restricted problem, and we want to go beyond the single excitation regime. To this end we are going to support the approximation of (18). The first step towards the approximation will be to assume that if the initial state of the atom is excited and that of the field is vacuum, then for times long enough, the population is exponentially decaying:

$$\langle e|e^{-iHt}|e\rangle \simeq e^{(-i\omega_A - \gamma)t} \quad (27)$$

where ω_A is the renormalized atomic frequency, and γ is the decay rate. Note that, by definition of the renormalized atom Hamiltonian H_{A_r} of (9), we have

$$e^{(-i\omega_A - \gamma)t} = \langle e|e^{-iH_{A_r}t}|e\rangle. \quad (28)$$

In the next section we will therefore solve the Friedrichs-Lee model in this interaction picture, which will lead us to our crucial approximation (18).

B. Dissipative renormalization scheme in the Friedrichs-Lee model

The (nonunitary) renormalized interaction picture evolution $U_{I_r}(t) \equiv e^{iH_0 t} e^{-iHt}$ reads

$$U_{I_r}(t) = 1 - i \int_0^t ds e^{iH_0 s} [|e\rangle\langle f| + |f\rangle\langle e| + ib|1\rangle\langle 1|] e^{-iH_0 s} U_{I_r}(s), \quad (29)$$

where H and H_0 are defined in Eqs. (8) and (9), respectively. The relevant matrix element can be expressed in terms of the renormalized interaction picture:

$$\langle e|e^{-iHt}|e\rangle = e^{(-i\omega_A - \gamma)t} \langle e|U_{I_r}(t)|e\rangle. \quad (30)$$

We introduce the notation

$$K^r(t) = \langle e|U_{I_r}(t)|e\rangle, \quad M^r(t) = \langle f_i|U_{I_r}(t)|e\rangle, \quad (31)$$

where $|f_i\rangle$ is defined as before. Inserting Eq. (29) into Eq. (31) one obtains

$$K^r(t) = 1 - i \int_0^t ds e^{i\Omega s} M^r(s) ds + i(\delta\omega - i\gamma) \int_0^t ds K^r(s), \quad (32)$$

$$M^r(t) = -i \int_0^t ds e^{-i\Omega s} F(t-s) K^r(s). \quad (33)$$

Applying the Laplace transform we get

$$\begin{aligned} \mathcal{K}^r(z) &= \frac{1}{z} - i \frac{1}{z} \mathcal{M}^r(z - i\Omega) + i(\delta\omega - i\gamma) \frac{1}{z} \mathcal{K}^r(z), \\ \mathcal{M}^r(z) &= -i \mathcal{F}(z) \mathcal{K}^r(z + i\Omega), \end{aligned} \quad (34)$$

which finally gives

$$\mathcal{K}^r(z) = \frac{1}{z + \mathcal{F}(z - i\omega_A - \gamma) - i\delta\omega - \gamma}. \quad (35)$$

Thus, for example, the evolution of the probability amplitude at which the atom stays excited is given by

$$\langle e|e^{-iHt}|e\rangle = e^{(-i\omega_A - \gamma)t} K^r(t), \quad (36)$$

where $K^r(t)$ is the inverse Laplace transform of $\mathcal{K}^r(z)$ from Eq. (35).

C. Justification of the approximation

Now we will propose the approximation, mentioned above in Eq. (18), that will later be carried out to all orders of Dyson series in the general case. According to Eqs. (27) and (36) this approximation means that for long times we have $K^r(t) \simeq 1$. In order to translate it into the picture of the Laplace transform, we can use the Tauberian theorem [27]

$$\mathcal{Y}(z) \simeq z^{-n} \text{ as } z \rightarrow 0 \iff Y(t) \simeq \frac{n}{\Gamma(n+1)} t^{(n-1)} \quad (37)$$

as $t \rightarrow \infty$, where $\Gamma(x)$ is the Gamma function. Putting $n = 1$ we conclude that $K^r(t) \simeq 1$ for long times if and only if $\mathcal{K}^r(z) \simeq 1/z$ for small z . In Eq. (35) this is equivalent to the condition

$$\mathcal{F}(z - i\omega_A - \gamma) \simeq \gamma + i\delta\omega \quad (38)$$

or equivalently

$$\mathcal{F}^r(z) \simeq b. \quad (39)$$

Taking the inverse Laplace transform of Eq. (38) our weak-coupling approximation (27), in the time domain, becomes

$$F^r(t) \approx b\delta(t), \quad t > 0. \quad (40)$$

And for negative times we have (see Appendix A)

$$F^r(t) \approx b^*\delta(t), \quad t < 0. \quad (41)$$

So far this approximation was considered for long times. Now, we propose to allow for *substitution*

$$F^r(t) \rightarrow b\delta(t), \quad \text{for } t > 0, \quad (42)$$

$$F^r(t) \rightarrow b^*\delta(t) \quad \text{for } t < 0, \quad (43)$$

under time integrals. We expect this approximation is valid for relatively small coupling, but stronger than typical weak-coupling scenarios as encountered in quantum optics, leaving room for non-Markovian effects. In Appendix B we also show that in typical time integrals, used in derivation of our main result, our approximation still holds with a good accuracy for coupling weak enough.

In the rest of the paper we shall apply the approximation to get simplified equations of motion for the spin boson model. In particular, we shall validate the resulting equations by showing that they reproduce the well-known optical Bloch equations.

We finish this section by showing that the obtained values of γ and $\delta\omega$ are consistent with the assumption that, for times long enough, we have Markovian evolution. In order

to compute the values of γ and $\delta\omega$ we substitute Eq. (17) into Eq. (39) and using dispersion relation $\omega = kc$ we get (with speed of light $c = 1$)

$$\mathcal{F}^r(z) = \int_0^\infty dt e^{-(z_r - \gamma)t} e^{i(\omega_A - \omega - z_i)t} \int d\omega 4\pi\omega^2 |f(\omega)|^2, \quad (44)$$

where $z_r \equiv \Re(z)$, $z_i \equiv \Im(z)$. As we can clearly see, the only way for this integral not to be divergent for $t \rightarrow \infty$ is that $z_r - \gamma > 0$. Considering that, we have

$$\mathcal{F}^r(z) = \int d\omega 4\pi\omega^2 \frac{|f(\omega)|^2}{z + i(\omega - \omega_A) - \gamma}. \quad (45)$$

For small z (and consequently γ) we have

$$\begin{aligned} \frac{1}{z - i\omega_A - \gamma + i\omega} &= \frac{z_r - \gamma + i(\omega_A - \omega - z_i)}{|z_r - \gamma|^2 + |\omega_A - \omega - z_i|^2} \\ &= \frac{z_r - \gamma}{|z_r - \gamma|^2 + |\omega_A - \omega - z_i|^2} \\ &\quad + i \frac{\omega_A - \omega - z_i}{|z_r - \gamma|^2 + |\omega_A - \omega - z_i|^2} \\ &\approx \pi\delta(\omega_A - \omega) + iP \left(\frac{1}{\omega_A - \omega} \right), \end{aligned} \quad (46)$$

where P is the Cauchy principal value. Now substituting Eqs. (39) and (46) into Eq. (45) one gets

$$\begin{aligned} \gamma + i\delta\omega + O(z) &\approx 4\pi^2\omega_A^2 |f(\omega_A)|^2 \\ &\quad + iP \int d\omega \frac{4\pi\omega^2}{\omega_A - \omega} |f(\omega)|^2, \end{aligned} \quad (47)$$

which gives

$$\gamma \approx 4\pi^2\omega_A^2 |f(\omega_A)|^2, \quad \delta\omega \approx P \int d\omega \frac{4\pi\omega^2}{\omega_A - \omega} |f(\omega)|^2. \quad (48)$$

We have just obtained the standard Markovian decay rate, as it should be in Eq. (27).

V. EVOLUTION COMING FROM DISSIPATIVE RENORMALIZATION AND APPROXIMATION

A. Renormalized S propagator

The renormalization method introduced in Sec. III enables us to greatly simplify the formula for the the S -propagator evolution of the atom-field. As will be seen below after the renormalization the evolution of the S matrix, in our approximation regime, can be surprisingly fully determined by the normal ordered terms, the Lamb shift and the decay rate. In fact, the Lamb shift and the decay rate account for the contribution of all non-normal ordered terms in Dyson series. Using Eqs. (4)–(17) the time-evolution operator, in the interaction picture with respect to $H_{0,}$, takes the form

$$\begin{aligned} U_I(t, 0) &= \mathbb{1} - i \int_0^t dt_1 \tilde{H}_I(t_1) \\ &\quad + (-i)^2 \int_0^t dt_1 \int_0^{t_1} dt_2 \tilde{H}_I(t_1) \tilde{H}_I(t_2) + \dots \end{aligned} \quad (49)$$

Applying now the approximation given in Eqs. (40) and (41) the evolution of S -propagator elements, i.e., $S_{I_r, ij}(t, 0) \equiv \langle i|U_I(t, 0)|j\rangle$, will take the simple form given by our main theorem.

Theorem 1. The evolution of S -propagator elements $S_{I_r, ij}(t, 0) \equiv \langle i|U_I(t, 0)|j\rangle$ of the whole system is given by

$$S_{I_r, 11}(t, 0) \simeq \mathbb{I}_F + \sum_{n=1} (-i)^{2n} \int_0^t dt_1 \dots \int_0^{t_{2n-1}} dt_{2n} A'(t_2) \dots A'(t_{2n}) A(t_1) \dots A(t_{2n-1}), \quad (50)$$

$$S_{I_r, 00}(t, 0) \simeq \mathbb{I}_F + \sum_{n=1} (-i)^{2n} \int_0^t dt_1 \dots \int_0^{t_{2n-1}} dt_{2n} A'(t_1) \dots A'(t_{2n-1}) A(t_2) \dots A(t_{2n}), \quad (51)$$

$$S_{I_r, 01}(t, 0) \simeq \sum_{n=1} (-i)^{2n-1} \int_0^t dt_1 \dots \int_0^{t_{2n-2}} dt_{2n-1} A'(t_1) \dots A'(t_{2n-1}) A(t_2) \dots A(t_{2n-2}), \quad (52)$$

$$S_{I_r, 10}(t, 0) \simeq \sum_{n=1} (-i)^{2n-1} \int_0^t dt_1 \dots \int_0^{t_{2n-2}} dt_{2n-1} A'(t_2) \dots A'(t_{2n-2}) A(t_1) \dots A(t_{2n-1}), \quad (53)$$

where \mathbb{I}_F is the identity matrix in the field space.

Here we present a sketch for the proof (see Appendix D for the whole proof).

Sketch of the proof

Here we illustrate how the contribution from non-normally ordered terms, from different orders in Dyson series, cancel each other out. For the first order, in Dyson series with the

initial excited state of the atom $|1\rangle$, the surviving term is

$$\langle 1|\tilde{H}_I^1|1\rangle \longrightarrow ib, \quad (54)$$

where \tilde{H}_I was defined in Eq. (15) and the superscript 1 indicates the order in Dyson series. Now integrating over time we get

$$(-i)\langle 1|\int_0^t dt_1 \tilde{H}_I^1|1\rangle = \underbrace{bt}_{(I)}. \quad (55)$$

For the second order, the surviving terms are

$$\langle 1 | \tilde{H}_r^1 \tilde{H}_r^2 | 1 \rangle \longrightarrow a_1 a_2^\dagger + (ib)^2 = a_2^\dagger a_1 + \delta_{12} + (ib)^2. \quad (56)$$

Now integrating over time we get

$$\langle 1 | \int_0^t dt_1 \int_0^{t_1} dt_2 \tilde{H}_r^1 \tilde{H}_r^2 | 1 \rangle = \int_0^t dt_1 \int_0^{t_1} dt_2 \left\{ A'(t_2) A(t_1) + F^r(t_1 - t_2) + (ib)^2 \right\}, \quad (57)$$

where $F(t_1 - t_2)$ comes from the Dirac delta $\delta_{12} := \delta(\omega_1 - \omega_2)$ on the right-hand side of Eq. (56). Now exploiting Eq. (40) and the discussion in Sec. IV C we have the following approximation under the time integral for all times:

$$F^r(t_i - t_j) \approx b\delta(t_i - t_j). \quad (58)$$

Therefore, we have the following substitution:

$$\delta(\omega_i - \omega_j) \Rightarrow F^r(t_i - t_j) \Rightarrow b\delta(t_i - t_j). \quad (59)$$

It should be noted that Eq. (59) means that integrating $F^r(t_i - t_j)$ over time should give the following equality:

$$\int_0^t dt_i \int_0^{t_i} dt_j F^r(t_i - t_j) \approx \int_0^t dt_i \int_0^{t_i} dt_j b\delta(t_i - t_j) = bt. \quad (60)$$

In Appendix B we show that for an Ohmic distribution of $f(\omega)$ with cutoff on the frequency one can safely approximate the integral by bt . Then Eq. (57) reads

$$\begin{aligned} (-i)^2 \langle 1 | \int_0^t dt_1 \int_0^{t_1} dt_2 \tilde{H}_r^1 \tilde{H}_r^2 | 1 \rangle &= (-i)^2 \int_0^t dt_1 \int_0^{t_1} dt_2 \{ A'(t_2) A(t_1) + b\delta(t_1 - t_2) + (ib)^2 \} \\ &= (-i)^2 \int_0^t dt_1 \int_0^{t_1} dt_2 A'(t_2) A(t_1) - \underbrace{bt}_{(I)} + \underbrace{b^2 \int_0^t dt_1 \int_0^{t_1} dt_2}_{(II)}. \end{aligned} \quad (61)$$

For the third order, the surviving terms are

$$\langle 1 | \tilde{H}_r^1 \tilde{H}_r^2 \tilde{H}_r^3 | 1 \rangle \longrightarrow ib(a_1 a_2^\dagger + a_2 a_3^\dagger) + (ib)^3 = ib(a_2^\dagger a_1 + a_3^\dagger a_2 + \delta_{12} + \delta_{23}) + (ib)^3. \quad (62)$$

The contribution of $a_1 a_3^\dagger$ is zero. Generally the terms $(i\delta\omega + \gamma)a_i a_j^\dagger$ with $j > i + 1$ (like $a_1 a_3^\dagger$) vanish. For example,

$$\sigma^+ a_1 (i\delta\omega + \gamma) | 1 \rangle \langle 1 | \sigma^- a_3^\dagger = (i\delta\omega + \gamma) a_1 \underbrace{| 1 \rangle \langle 0 |}_{\sigma^+} \underbrace{| 1 \rangle | v \rangle \langle v |}_{| 1 \rangle \langle 1 | \otimes | v \rangle \langle v |} \underbrace{| 0 \rangle \langle 1 |}_{\sigma^-} a_3^\dagger = 0, \quad (63)$$

where $|v\rangle$ is the vacuum state of the field. Now integrating over time we get

$$\begin{aligned} (-i)^3 \langle 1 | \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \tilde{H}_r^1 \tilde{H}_r^2 \tilde{H}_r^3 | 1 \rangle &= (-i)^3 \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \{ ib[A'(t_2) A(t_1) + A'(t_3) A(t_2)] \\ &\quad + F^r(t_1 - t_2) + F^r(t_2 - t_3) \} + (ib)^3. \end{aligned} \quad (64)$$

Therefore, using Eq. (58) we get

$$\begin{aligned} (-i)^3 \langle 1 | \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \tilde{H}_r^1 \tilde{H}_r^2 \tilde{H}_r^3 | 1 \rangle &= (-i)^3 \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \{ ib[A'(t_2) A(t_1) + A'(t_3) A(t_2)] \\ &\quad + b\delta(t_1 - t_2) + b\delta(t_2 - t_3) \} + (ib)^3 \\ &= \left\{ \underbrace{-b \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 [A'(t_2) A(t_1) + A'(t_3) A(t_2)]}_{(III)} \right. \\ &\quad \left. - \underbrace{2b^2 \int_0^t dt_1 \int_0^{t_1} dt_2}_{(II)} + \underbrace{b^3 \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3}_{(IV)} \right\}. \end{aligned} \quad (65)$$

As we can see, the terms with the same number (or color) already cancel out or they will with terms from higher orders. ■

In the Schrödinger picture, the evolution of S -propagator elements $S_{lm}(t, 0)$ reads

$$S_{lm}(t, 0) = e^{-l(i\omega_A + \gamma)t} e^{-iH_F t} S_{l,lm}(t, 0), \quad l, m = 0, 1 \quad (66)$$

where H_F is the free Hamiltonian of the field given in Eq. (9). Using coherent states of the form $|\{\alpha\}\rangle \equiv |\alpha_1, \alpha_2, \dots, \alpha_n\rangle$ and Theorem 1 the matrix elements $S_{l,r,ij}^{\beta\alpha}(t, 0) \equiv \langle \{\beta\}, i | S_{l,r}(t, 0) | j, \{\alpha\} \rangle$ can also be readily obtained as

$$S_{l,r,11}^{\beta\alpha}(t, 0) \simeq \langle \{\beta\} | \{ \alpha \} \rangle \left(1 + \sum_{n=1}^{l-1} (-i)^{2n} \int_0^t dt_1 \dots \int_0^{t_{2n-1}} dt_{2n} A'_\beta(t_2) A_\alpha(t_1) \dots A'_\beta(t_{2n}) A_\alpha(t_{2n-1}) \right), \quad (67)$$

$$S_{I_r,00}^{\beta\alpha}(t, 0) \simeq \langle \{\beta\} | \{\alpha\} \rangle \left(1 + \sum_{n=1} (-i)^{2n} \int_0^t dt_1 \dots \int_0^{t_{2n-1}} dt_{2n} A'_\beta(t_1) A_\alpha(t_2) \dots A'_\beta(t_{2n-1}) A_\alpha(t_{2n}) \right), \quad (68)$$

$$S_{I_r,01}^{\beta\alpha}(t, 0) \simeq \langle \{\beta\} | \{\alpha\} \rangle \left(\sum_{n=1} (-i)^{2n-1} \int_0^t dt_1 \dots \int_0^{t_{2n-2}} dt_{2n-1} A'_\beta(t_1) A_\alpha(t_2) \dots A'_\beta(t_{2n-1}) A_\alpha(t_{2n-2}) \right), \quad (69)$$

$$S_{I_r,10}^{\beta\alpha}(t, 0) \simeq \langle \{\beta\} | \{\alpha\} \rangle \left(\sum_{n=1} (-i)^{2n-1} \int_0^t dt_1 \dots \int_0^{t_{2n-2}} dt_{2n-1} A'_\beta(t_2) A_\alpha(t_1) \dots A'_\beta(t_{2n-2}) A_\alpha(t_{2n-1}) \right), \quad (70)$$

where

$$A_\alpha(t) \equiv e^{i\Omega t} f_\alpha(t), \quad A'_\beta(t) \equiv e^{-i\Omega t} f_\beta^*(t), \quad (71)$$

with

$$f_\alpha(t) = \int dk f(\omega_k) \alpha(\omega_k) e^{-i\omega_k t}. \quad (72)$$

In the Schrödinger picture, the matrix elements $S_{lm}^{\beta\alpha}(t, 0) \equiv \langle \{\beta\}, l | S(t, 0) | m, \{\alpha\} \rangle$, $l, m = 0, 1$ are obtained as

$$S_{lm}^{\beta\alpha}(t, 0) = e^{-l\gamma t} e^{-i(l\omega_A + \Theta)t} \langle \{\beta\}, l | S_{I_r}(t, 0) | m, \{\alpha\} \rangle, \quad (73)$$

where $\Theta = \int dk \omega_k \beta^*(\omega_k) \alpha(\omega_k)$.

Now that using our approximation we computed the S propagator of the dynamics of the total system we are in a position to derive a differential equation for the evolution of the S propagator, which is the content of the following section.

B. S-propagator differential equation

The (nonunitary) renormalized interaction picture evolution operator $U_I^r(t)$ satisfies the following differential equation:

$$\frac{d}{dt} U_I^r(t) = -i\tilde{H}_{I_r}(t) U_I^r(t). \quad (74)$$

The result obtained above is the derivation of the useful approximation for the map $U_I^r(t)$ given by a much simpler map $S_I(t)$ which can be treated as a certain weak-coupling approximation leading to normal order expression in terms of field operators. This approximate dynamics satisfies the following differential equation (notice the unusual ordering of the operators):

$$\frac{d}{dt} S_I(t) = -i[\sigma^+ S_I(t) A(t) + \sigma^- A'(t) S_I(t)], \quad S_I(0) = 1. \quad (75)$$

The explicit form of $S_I(t)$ can be obtained using the Dyson series expansion for Eq. (75) and the properties of σ^\pm . It corresponds to Eqs. (50)–(53). In the Schrödinger picture we have

$$\begin{aligned} \frac{d}{dt} S(t) &= \frac{d}{dt} (e^{-iH_{0r}t} S_{I_r}(t)) \\ &= -iH_{0r} e^{-iH_{0r}t} S_{I_r}(t) + e^{-iH_{0r}t} \frac{d}{dt} S_{I_r}(t) \\ &= -iH_{0r} S(t) - ie^{-iH_{0r}t} [\sigma^+ e^{iH_{0r}t} S(t) A(t) \end{aligned}$$

$$\begin{aligned} &+ \sigma^- A'(t) e^{iH_{0r}t} S(t)] \\ &= -iH_{0r} S(t) - i[\sigma^+ S(t) \int dk f(\omega_k) e^{-i\omega_k t} a_{\omega_k} \\ &+ \sigma^- \int dk f^*(\omega_k) a_{\omega_k}^\dagger S(t)]. \quad (76) \end{aligned}$$

A useful representation of $S_{I_r}(t)$ can be written in terms of the partial matrix element

$$S_{I_r}^{\beta\alpha}(t) \equiv \langle \{\beta\} | S_{I_r}(t) | \{\alpha\} \rangle \quad (77)$$

with respect to the coherent states $|\{\alpha\}\rangle, |\{\beta\}\rangle$ defined as before. Each of them satisfies the following evolution equation for the 2×2 matrix $S_I^{\beta\alpha}(t) = [S_{I_r,ij}^{\beta\alpha}(t)]$, $i, j = 0, 1$:

$$\frac{d}{dt} S_I^{\beta\alpha}(t) = -i[A_\alpha(t) \sigma^+ + A'_\beta(t) \sigma^-] S_I^{\beta\alpha}(t), \quad (78)$$

and the initial value $S_{I_r}^{\beta\alpha}(0) = \langle \{\beta\} | \{\alpha\} \rangle$.

C. Particular examples

(1) The simplest object is the survival amplitude of the atomic excited state in the vacuum field defined as $\langle \{0\}, 1 | e^{-iHt} | 1, \{0\} \rangle$. Then

$$\begin{aligned} \langle \{0\}, 1 | e^{-iHt} | 1, \{0\} \rangle &= e^{(-i\omega_A - \gamma)t} \langle \{0\}, 1 | U_r(t) | 1, \{0\} \rangle \\ &= e^{(-i\omega_A - \gamma)t} S_{11}^{00} \\ &\simeq e^{(-i\omega_A - \gamma)t}, \quad (79) \end{aligned}$$

where the last line comes from our Theorem 1 in which we assumed that our approximation [Eq. (18)] holds for long time. Equation (79) reproduces the Wigner-Weisskopf result.

(2) Under the evolution for initial state $|1, \{0\}\rangle$ the norm of the state at time t is approximately preserved (see Appendix E):

$$\langle \psi(t) | \psi(t) \rangle = \langle \{0\}, 1 | S^\dagger(t, 0) S(t, 0) | 1, \{0\} \rangle \approx 1. \quad (80)$$

(3) Taking $\beta_k = \alpha_k$ where α represents the initial state of the field we can compute the final state of the atom $|\psi(t)\rangle$ in the improved semiclassical approximation which assumes that the total state remains a product state of the atom and the freely evolving field $|\alpha_r\rangle$:

$$|\psi(t)\rangle = e^{t(-i\omega_A - \gamma)|1\rangle\langle 1|} S_{I_r}^{\alpha\alpha}(t) |\psi(0)\rangle = U_\alpha(t) |\psi(0)\rangle, \quad (81)$$

where

$$U_\alpha(t) = e^{t(-i\omega_A - \gamma)|1\rangle\langle 1|} S_{I_r}^{\alpha\alpha}(t). \quad (82)$$

The differential equations corresponding to Eq. (81) take the form

$$\frac{d}{dt}|\psi(t)\rangle = -i[(\omega_A - i\gamma)|1\rangle\langle 1| + f_\alpha(t)\sigma^+ + f_\alpha^*(t)\sigma^-]|\psi(t)\rangle \quad (83)$$

and

$$\frac{d}{dt}U_\alpha(t) = -i[(\omega_A - i\gamma)|1\rangle\langle 1| + f_\alpha(t)\sigma^+ + f_\alpha^*(t)\sigma^-]U_\alpha(t). \quad (84)$$

(4) The most general situation is described by the initial field state written in the Glauber P representation:

$$\rho^{\text{field}}(0) = \int \mathcal{D}\alpha P_0(\alpha)|\alpha\rangle\langle\alpha| \quad (85)$$

where we use symbolic notation for the functional integral properly defined by a limit procedure. Here, the functional $P(\alpha)$ takes values in 2×2 matrices. Then we can compute the (2×2 matrix valued) Husimi Q function for the final state, $Q_t(\beta) \equiv \langle\{\beta\}|\rho(t)|\{\beta\}\rangle$, as

$$Q_t(\beta) = e^{-iH_0^t} \left\{ \int \mathcal{D}\alpha P_0(\alpha) S_I^{\beta\alpha}(t) |\psi(0)\rangle \langle\psi(0)| [S_I^{\alpha\beta}(t)]^\dagger \right\} \times (e^{-iH_0^t})^\dagger. \quad (86)$$

The above expression is, in principle, computable for numerous examples of initial states. Finally, equipped with the S -propagator evolution, below we apply our formalism to

$$\begin{aligned} |\psi(t)\rangle &= S_{I_r}(t)|0, h\rangle = S_{I_r,00}(t)|0, h\rangle + S_{I_r,10}(t)|1, h\rangle \\ &= |0, h\rangle - \int_0^t dt_1 \int_0^{t_1} dt_2 A'(t_1)A(t_2)|0, h\rangle - i \int_0^t dt_1 A(t_1)|1, h\rangle \\ &= |0, h\rangle - \int_0^t dt_1 \int_0^{t_1} dt_2 \int dk' \int dk f'(\omega_{k'}, t_1) f(\omega_k, t_2) h(\omega_k) |0, \{1_{k'}\}\rangle - i \int_0^t dt_1 \int dk f(\omega_k, t_1) h(\omega_k) |1, \{0\}\rangle. \end{aligned} \quad (88)$$

The state of the atom at time t is given by $\tilde{\rho}_A(t) = \text{Tr}_F(|\psi(t)\rangle\langle\psi(t)|)$. As our dynamics is trace preserving (in the Schrödinger picture), we are only interested in the population of the excited level of the atom. Then, in the Schrödinger picture we have

$$\begin{aligned} p_e(t) &= e^{-2\gamma t} \left| \int_0^t dt_1 \int dk f(\omega_k, t_1) h(\omega_k) \right|^2 \\ &= e^{-2\gamma t} \frac{1}{\sqrt{\pi}\sigma} \left| \int dk f(\omega_k) \frac{e^{\gamma t} e^{-i(\omega_k - \omega_A)t} - 1}{\gamma - i(\omega_k - \omega_A)} \right. \\ &\quad \left. \times e^{-\frac{(\omega_k - \omega_A)^2}{(2\sigma^2)}} \right|^2, \end{aligned} \quad (89)$$

where $f(\omega_k) = \kappa\sqrt{\omega_k}$. In Fig. 2 we can see the numerical solution for $p_e(t)$. It is clear that the population decays exponentially as $t \rightarrow \infty$ as ensured by our approximation. However, by decreasing σ , one can see that more time is needed for the atom to decay. The reason is that the state of the field approaches a sharp state of one photon in one specific

different examples and then derive a master equation for the evolution of the reduced state of the atom which as a confirmation reproduces the already well-known Gorini-Kossakowski-Lindblad-Sudarshan (GKLS) master equation [28,29] in the case of a field initially in a vacuum state, and Bloch equations in the case of a coherent state of the field.

VI. ILLUSTRATION

In this section we would like to apply our approximation to different settings and obtain the dynamics for the atom. First, we apply it to the case in which the initial state of the field consists of a continuous superposition of one photon in different frequencies. Later, we treat the evolution for the coherent state of the field and get the optical Bloch equations, which shows that our approach reproduces known results.

A. One photon

In this subsection, we consider that the initial state of the atom and field is given by

$$\rho(0) = |0\rangle\langle 0| \otimes |h\rangle\langle h|, \quad (87)$$

where $|0\rangle$ corresponds to the ground state of the atom and $|h\rangle = \int dk h(\omega_k) |\{1_k\}\rangle$ and $h(\omega_k) = \frac{1}{\pi^{1/4}\sqrt{\sigma}} e^{-(\omega_k - \omega_A)^2/(2\sigma^2)}$, which is a Gaussian function. One could extend our calculations for an arbitrary state of the atom but because it is not very illuminating, we restrict ourselves to this simpler case. As our state is pure, we calculate the following (in the interaction picture with respect to H_0):

mode, which will be delocalized in time and the atom will be interacting with the photon at all times. Because of this interaction, the atom will absorb the photon with almost constant probability for all values of time, making it impossible for the atom to decay.

B. Coherent state

In order to perform the calculations of this subsection, we first need to write the evolution for the propagator. Making use of Eq. (78) in the Schrödinger picture we have

$$U^{\beta\alpha}(t) \equiv \frac{1}{\mathcal{N}_{\beta\alpha}} e^{t(-i\omega_A - \gamma)|1\rangle\langle 1|} S_{I_r}^{\beta\alpha}(t), \quad (90)$$

where $\mathcal{N}_{\beta\alpha} \equiv \langle\{\beta\}|\{\alpha\}\rangle$. The now normalized Schrödinger picture propagator [$U^{\beta\alpha}(0) = \mathbb{I}$] satisfies the following evolution equation:

$$\begin{aligned} \frac{d}{dt}U^{\beta\alpha}(t) &= -i[(\omega_A - i\gamma)|1\rangle\langle 1| + f_\alpha(t)\sigma^+ + f_\beta^*(t)\sigma^-] \\ &\quad \times U^{\beta\alpha}(t), \end{aligned} \quad (91)$$

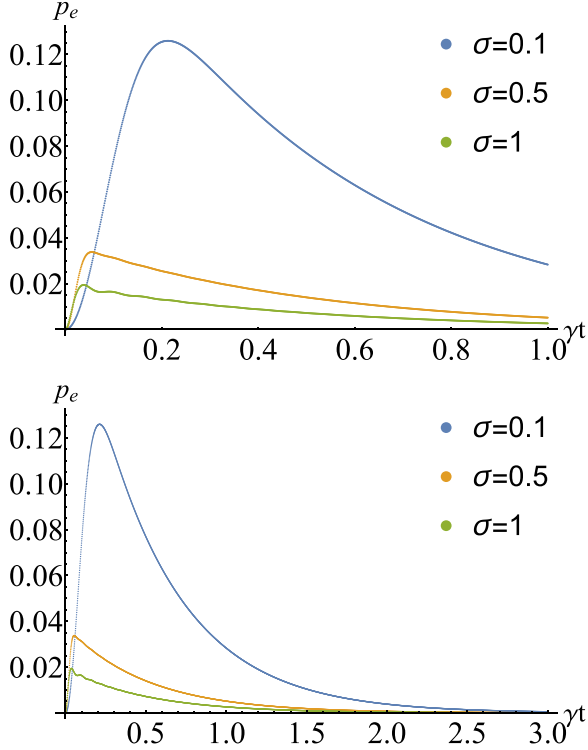


FIG. 2. Numerical plots of the population of the excited state of the atom as a function of γt for $\omega_A = 1$, $\gamma = 0.01$, and $\kappa = \sqrt{\frac{\gamma}{\pi\omega_A}}$ for different σ (the smaller σ the bigger the population and the upper curve in the plot) and for short (top) and long (bottom) time scales.

where $f_\alpha(t)$ is defined in Eq. (72). It is more convenient to define $\xi = \beta - \alpha$ and change the propagator parametrization to $U^{\alpha\beta}(t) \equiv U_\alpha^\xi(t)$. In this way, the evolution is given by

$$\begin{aligned} \frac{d}{dt}U_\alpha^\xi(t) &= -i[(\omega_A - i\gamma)|1\rangle\langle 1| + f_\alpha(t)\sigma^+ + f_\alpha^*(t)\sigma^- \\ &\quad + f_\xi^*(t)\sigma^-]U_\alpha^\xi(t) \\ &= -i[H_{A_r} + H_\alpha(t) + f_\xi^*(t)\sigma^-]U_\alpha^\xi(t) \end{aligned} \quad (92)$$

where

$$H_{A_r} \equiv (\omega_A - i\gamma)|1\rangle\langle 1|, \quad (93)$$

and

$$H_\alpha(t) \equiv f_\alpha(t)\sigma^+ + f_\alpha^*(t)\sigma^-. \quad (94)$$

We now move to the (nonunitary) interaction picture with respect to the Hamiltonian $H_{A_r} + H_\alpha(t)$. Therefore, the propagator on this picture will be

$$\tilde{U}_\alpha^\xi = \mathcal{T}e^{i\int_0^t dt'(H_{A_r} + H_\alpha(t'))} \mathcal{T}e^{-i\int_0^t dt'(H_{A_r} + H_\alpha(t') + f_\xi^*(t')\sigma^-)}. \quad (95)$$

Hence we can write (see Appendix C for the complete derivation)

$$\frac{d}{dt}\tilde{U}_\alpha^\xi(t) = -if_\xi^*(t)\tilde{\sigma}^-(t)\tilde{U}_\alpha^\xi(t), \quad (96)$$

in which

$$\tilde{\sigma}^-(t) = \mathcal{T}e^{i\int_0^t dt'(H_{A_r} + H_\alpha(t'))} \sigma^- \mathcal{T}e^{-i\int_0^t dt'(H_{A_r} + H_\alpha(t'))}. \quad (97)$$

Before further calculations, we make the following change $\gamma \rightarrow \frac{\gamma}{2}$ (so $b + b^* = \gamma$ now) for ease of notation. We rewrite Eq. (92) as

$$\begin{aligned} \frac{d}{dt}U_\alpha^\xi(t) &= -i[\omega_A P_1 + f_\alpha(t)\sigma^+ + f_\alpha^*(t)\sigma^-]U_\alpha^\xi(t) \\ &\quad - \left[\frac{\gamma}{2} P_1 + if_\xi^*(t)\sigma^- \right]U_\alpha^\xi(t), \end{aligned} \quad (98)$$

where $P_1 \equiv |1\rangle\langle 1|$. Defining $U_\alpha(t, 0)$ as

$$U_\alpha(t, 0) = \mathcal{T}e^{-i\int_0^t dt' H_\alpha(t')}, \quad (99)$$

where $H_\alpha(t) \equiv \omega_A P_1 + f_\alpha(t)\sigma^+ + f_\alpha^*(t)\sigma^-$, we can expand $U_\alpha^\xi(t)$ in Dyson series as

$$\begin{aligned} U_\alpha^\xi(t) &= U_\alpha(t, 0) \left[\mathbb{I} - \int_0^t ds_1 \left(\frac{\gamma}{2} \tilde{P}_1(s_1) + if_\xi^*(s_1)\tilde{\sigma}^-(s_1) \right) \right. \\ &\quad + \int_0^t ds_1 \int_0^{s_1} ds_2 \left(\frac{\gamma}{2} \tilde{P}_1(s_1) + if_\xi^*(s_1)\tilde{\sigma}^-(s_1) \right) \\ &\quad \times \left(\frac{\gamma}{2} \tilde{P}_1(s_2) + if_\xi^*(s_2)\tilde{\sigma}^-(s_2) \right) \left. \right] \\ &\quad + O(\gamma^3), \end{aligned} \quad (100)$$

where $\tilde{X} \equiv U_\alpha^\dagger(s_1, 0)XU_\alpha(s_1, 0)$. Then the reduced density matrix of the atom at time t , up to the first order in γ , reads

$$\begin{aligned} \rho_A(t) &= \Lambda(t)\rho_A(0) \\ &\approx U_\alpha(t, 0) \left\{ \rho_A(0) - \frac{\gamma}{2} \int_0^t ds_1 \left[\tilde{P}_1(s_1)\rho_A(0) \right. \right. \\ &\quad \left. \left. + \rho_A(0)\tilde{P}_1(s_1) \right] + \gamma \int_0^t ds_1 \tilde{\sigma}^-(s_1)\rho_A(0)\tilde{\sigma}^+(s_1) \right\} \\ &\quad \times U_\alpha^\dagger(t, 0), \end{aligned} \quad (101)$$

where we have applied the usual approximation for F^r (see Appendix F for all the calculations). Note that for the approximation, used above, to be valid it was assumed that we have a slow driving laser field with $f_\alpha(t)$ small enough such that $\tilde{\sigma}^-(t)$ is a sufficiently slowly varying function in the time scale of $1/\gamma$. Therefore the map $\Lambda(t)$, up to the first order in γ , may be written in the form

$$\begin{aligned} \Lambda(t)(\cdot) &= \mathcal{U}_\alpha(t, 0) \left[\mathbb{I} - \frac{\gamma}{2} \int_0^t ds_1 \{ \tilde{P}_1(s_1), (\cdot) \} \right. \\ &\quad \left. + \gamma \int_0^t ds_1 \tilde{\sigma}^-(s_1)(\cdot)\tilde{\sigma}^+(s_1) \right] + O(\gamma^2), \end{aligned} \quad (102)$$

where the superoperator $\mathcal{U}_\alpha(t, 0)[\cdot] = U_\alpha(t, 0)[\cdot]U_\alpha^\dagger(t, 0)$. Hence, inspired by Eq. (102) we conjecture that the following differential equation for the evolution of $\Lambda(t)$, in the interaction picture with respect to $H_\alpha(t)$ (defined at the beginning of the subsection), holds:

$$\frac{d\tilde{\Lambda}(t)(\cdot)}{dt} \approx -\frac{\gamma}{2} \{ \tilde{P}_1(t), (\cdot) \} + \gamma \tilde{\sigma}^-(t)(\cdot)\tilde{\sigma}^+(t). \quad (103)$$

Our conjecture is indeed true—at least, up to the second order (see the proof in Appendix F)—and we strongly think it will be preserved for all the orders. Finally, we would like to mention that in the simpler case in which the state starts in the vacuum state ($\alpha = 0$) one obtains that the evolution of the reduced state of the atom is given by the GKLS master equation [28,29].

VII. CONCLUSION

In this paper, we investigated the interaction of a two-level atom with a quantized continuous-mode laser field. Using a renormalized method we managed to write the evolution of the atom-field system in a form that depends only on normally ordered creation and annihilation operators of the field, the decay rate of the atom, and its natural frequency ω_A . In fact, the decay rate and the Lamb shift account for the contribution of all remaining terms, which appear after the normal ordering of the creation and annihilation operators of the field in the Dyson series. Furthermore, we showed that our approach reproduces the previously known optical Bloch equations. We have studied just a two-level atom, but the need for a generalization to a d -level system may arise naturally. This would be useful for studying typical setups of V systems or Λ systems [30,31].

Aside from the quantum optics realm, we reckon it may be beneficial to translate the results presented here to the framework of Feynman diagrams to understand better the evolution of the system at all times. In addition, the renormalization method introduced here may have some potential in dealing with cutoff terms that arise in the evolution as well as giving some flavor in dealing with divergences that appear in open quantum systems in general.

Last but not least will be the investigation of quantumness traces of gravitational field when interacting with gravitational wave detectors [32–34]. It is of great interest to know what the quantumness effects of the gravitational wave are when interacting with the lengths of the arms of gravitational wave detectors. We hope that our proposed study can prove useful also in the above context.

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APPENDIX A: PROOF OF EQ. (41)

Employing Eqs. (17) and (40) we have

$$F^r(t) \approx e^{\gamma t} h(t) \approx b \delta(t),$$

in which

$$h(t) \equiv \int dk e^{-i(\omega_k - \omega_A)t} |f(\omega_k)|^2.$$

Then for $t < 0$, denoting $u = -t$ we get

$$\begin{aligned} F^r(t) &= F^r(-u) \\ &= e^{-\gamma u} h^*(u) \\ &= e^{-2\gamma u} [e^{\gamma u} h(u)]^* \\ &\approx e^{-2\gamma u} b^* \delta(u) \\ &= e^{2\gamma t} b^* \delta(-t) \end{aligned}$$

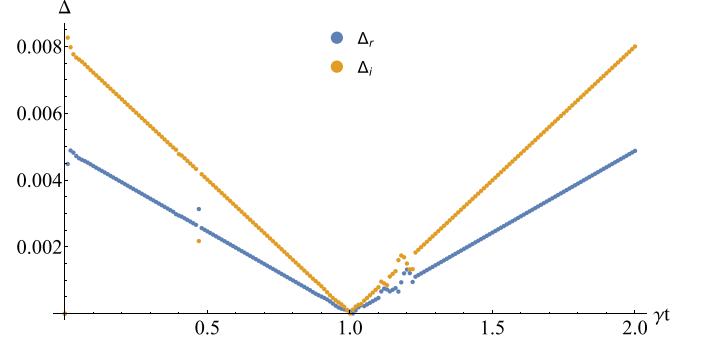


FIG. 3. $\Delta(t)$ vs γt for $\gamma = 0.01$, $\omega_A = 1$, and $\omega_c = 5$. Here $\Delta_r(t) \equiv \text{Re}\Delta(t) - \gamma t$ and $\Delta_i(t) \equiv \text{Im}\Delta(t) - \delta\omega t$. The upper line corresponds to $\Delta_i(t)$ and the lower one to $\Delta_r(t)$.

$$\begin{aligned} &= e^{2\gamma t} b^* \delta(t) \\ &= b^* \delta(t), \end{aligned} \quad (\text{A1})$$

where in the last equality we dropped the term $e^{2\gamma t}$ just because inside the time integrals this term is equal to 1 due to the presence of the Dirac delta.

APPENDIX B: JUSTIFICATION OF THE SUBSTITUTION IN EQS. (42) AND (43)

In Fig. 3 we illustrate that for times up to $\gamma t = 2$, with the decay rate $\gamma = 0.01$, our substitution introduced in Eqs. (42) and (43) is valid with an error of the order of 10^{-3} magnitude. Here $\Delta(t)$ is defined as

$$\Delta(t) \equiv \int_0^t dt_i \int_0^{t_i} dt_j F^r(t_i - t_j) - bt, \quad (\text{B1})$$

with $b = \gamma + i\delta\omega$ and the one-dimensional spectral density

$$f(\omega) = \alpha \sqrt{\omega} e^{\frac{-|\omega|}{2\omega_c}}, \quad (\text{B2})$$

in which

$$\alpha^2 \equiv \frac{\gamma}{\pi \omega_A} e^{\frac{\omega_A}{\omega_c}} \quad (\text{B3})$$

where ω_c is the cutoff frequency, ω_A is the atomic frequency, and we have used the formula obtained for γ given in Eq. (48).

APPENDIX C: PROOF OF EQ. (96)

Assume that

$$\frac{d}{dt} U(t) = [Z(t) + A(t)]U(t) \quad (\text{C1})$$

and

$$\frac{d}{dt} U_0(t) = Z(t)U_0(t). \quad (\text{C2})$$

Using the fact that $U_0^{-1}(t)U_0(t) = \mathbb{I}$ we have

$$\begin{aligned} \frac{d}{dt} [U_0^{-1}(t)U_0(t)] &= \frac{d}{dt} [U_0^{-1}(t)]U_0(t) + U_0^{-1}(t) \frac{d}{dt} U_0(t) \\ &= \left(\frac{d}{dt} U_0^{-1}(t) + U_0^{-1}(t)Z(t) \right) U_0(t) = 0, \end{aligned} \quad (\text{C3})$$

which gives

$$\frac{d}{dt} U_0^{-1}(t) = -U_0^{-1}(t)Z(t). \quad (\text{C4})$$

Using the above equation we can write

$$\begin{aligned}
\frac{d}{dt}\tilde{U}(t) &= \frac{d}{dt}[U_0^{-1}(t)U(t)] \\
&= \frac{d}{dt}U_0^{-1}(t)U(t) + U_0^{-1}(t)\frac{d}{dt}U(t) \\
&= -U_0^{-1}(t)Z(t)U_0(t)U_0^{-1}(t)U(t) + U_0^{-1}(t)[Z(t) + A(t)]U_0(t)U_0^{-1}(t)U(t) \\
&= U_0^{-1}(t)A(t)U_0(t)\tilde{U}(t) \\
&= A'(t)\tilde{U}(t),
\end{aligned} \tag{C5}$$

which completes the proof.

APPENDIX D: PROOF OF THEOREM 1

Here the main result of the paper, i.e., Theorem 1, will be proved. As a matter of fact, we shall only prove Eq. (50), and the three others are proved analogously. The proof of Eq. (50) is given below in Proposition 2 of this Appendix. We shall need the following notation. Let $I = \{(i_1, i_1 + 1), \dots, (i_l, i_l + 1)\}$, be a set of pairs of indices where $i_1 < i_2 - 1, i_2 < i_3 - 1, \dots, i_{l-1} \leq i_l - 1$. Then we will denote

$$\begin{aligned}
X_I(AA') &= A(t_{i_1})A'(t_{i_1+1}) \dots A(t_{i_l})A'(t_{i_l+1}), \quad X_I(aa^\dagger) = a_{i_1}a_{i_1+1}^\dagger \dots a_{i_l}a_{i_l+1}^\dagger, \\
X_I(F) &= F(t_{i_1} - t_{i_1+1}) \dots F(t_{i_l} - t_{i_l+1}), \quad X_I(\delta) = \delta(t_{i_1} - t_{i_1+1}) \dots \delta(t_{i_l} - t_{i_l+1}).
\end{aligned} \tag{D1}$$

We note that normal ordering of creation annihilation operators translates into normal ordering of A and \tilde{A} operators, so that, in particular, we have

$$:X_I: = A'(t_{i_1+1})A'(t_{i_1+2}) \dots A'(t_{i_l+1})A(t_{i_1})A(t_{i_2}) \dots A(t_{i_l}), \tag{D2}$$

where “: x :” denotes normal ordering. For a set of pairs of neighboring natural numbers,

$$I \sim [n] \tag{D3}$$

means that I is a set of pairs chosen from the set $[n] \equiv \{1, \dots, n\}$. (The set I may not be the set of all pairs.) For example, $\{(1, 2), (5, 6)\} \approx [7]$.

Finally we shall denote

$$\int dt^n \equiv \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \dots \int_0^{t_{n-2}} dt_{n-1} \int_0^{t_{n-1}} dt_n. \tag{D4}$$

Having set the needed notation, we present the following lemma.

Lemma 1.

$$\langle 1|\tilde{H}^1\tilde{H}^2 \dots \tilde{H}^{n-1}\tilde{H}^n|1\rangle = \begin{cases} A(t_1)A'(t_2) \dots A(t_{n-1})A'(t_n) & \text{for } n \text{ even,} \\ 0 & \text{for } n \text{ odd,} \end{cases} \tag{D5}$$

$$\langle 0|\tilde{H}^1\tilde{H}^2 \dots \tilde{H}^{n-1}\tilde{H}^n|0\rangle = \begin{cases} A'(t_1)A(t_2) \dots A'(t_{n-1})A(t_n) & \text{for } n \text{ even,} \\ 0 & \text{for } n \text{ odd,} \end{cases} \tag{D6}$$

$$\langle 0|\tilde{H}^1\tilde{H}^2 \dots \tilde{H}^{n-1}\tilde{H}^n|1\rangle = \begin{cases} A'(t_1)A(t_2) \dots A(t_{n-1})A'(t_n) & \text{for } n \text{ odd,} \\ 0 & \text{for } n \text{ even,} \end{cases} \tag{D7}$$

$$\langle 1|\tilde{H}^1\tilde{H}^2 \dots \tilde{H}^{n-1}\tilde{H}^n|0\rangle = \begin{cases} A(t_1)A'(t_2) \dots A'(t_{n-1})A(t_n) & \text{for } n \text{ odd,} \\ 0 & \text{for } n \text{ even.} \end{cases} \tag{D8}$$

Proof. Using Eq. (15) where $\tilde{H}_{I_r} \equiv \tilde{H}$ we have

$$\tilde{H}^1\tilde{H}^2 \dots \tilde{H}^{n-1}\tilde{H}^n = \sum_{s_1, s_2, \dots, s_n = \pm} A^{(s_1)}(t_1) \dots A^{(s_n)}(t_n) \sigma^{s_1} \dots \sigma^{s_n}, \tag{D9}$$

where

$$A^{(s_j)}(t_j) = \begin{cases} A(t_j) & \text{for } s_j = +, \\ A'(t_j) & \text{for } s_j = -. \end{cases} \tag{D10}$$

Since $\sigma^+\sigma^+|1\rangle = 0$ and $\sigma^-\sigma^-|1\rangle = 0$, hence

$$\langle 1|\sigma^{s_1} \dots \sigma^{s_n}|1\rangle = \begin{cases} 1 & \text{if } s_1 \dots s_n = + - \dots + -, \\ 0 & \text{otherwise.} \end{cases} \tag{D11}$$

This means that first n must be even in order for $\langle 1|\sigma^{s_1} \dots \sigma^{s_n}|1\rangle$ to be nonzero. Secondly, Eq. (D11) tells us that for even n there is only one nonzero term in the sum (D9), namely, the term

$$A(t_1)A'(t_2) \dots A(t_{n-1})A'(t_n). \tag{D12}$$

This proves Eq. (D5). Equations (D6)–(D8) can also be proved in the same way. ■

Proposition 1.

$$\langle 1 | \tilde{H}_l^1 \dots \tilde{H}_l^n | 1 \rangle = \sum_{k=0}^n (ib)^k \overbrace{\sum_{p_1, \dots, p_{(n-k)/2}} A(t_{p_1}) A'(t_{p_1+1}) \dots A(t_{p_{(n-k)/2}}) A'(t_{p_{(n-k)/2}+1})}^{\mathcal{A}_k} \equiv \sum_{k=0}^n (ib)^k \mathcal{A}_k, \quad (\text{D13})$$

where the sum runs over $1 \leq p_1 < p_2 - 1, p_2 < p_3 - 1, \dots, p_{(n-k)/2-1} < p_{(n-k)/2} \leq n - 1$. See Fig. 4 for an illustration.

Proof. Let us first rewrite \tilde{H}_l , defined in Eq. (15), as

$$\tilde{H}_l = H_l + ib|1\rangle\langle 1|. \quad (\text{D14})$$

Then using Eq. (D14) we have

$$\begin{aligned} \tilde{H}^1 \tilde{H}^2 \dots \tilde{H}^{n-1} \tilde{H}^n &= \sum_{j_1, j_2, j_3, \dots, j_m}^n (ib)^{(j_1+j_2+\dots+j_m)} \mathcal{H}_l(i_1) |1\rangle\langle 1|^{j_1} \mathcal{H}_l(i_2) |1\rangle\langle 1|^{j_2} \mathcal{H}_l(i_3) |1\rangle\langle 1|^{j_3} \dots |1\rangle\langle 1|^{j_m-1} \mathcal{H}_l(i_m) |1\rangle\langle 1|^{j_m} \\ &= \sum_{k=0}^n (ib)^k \mathcal{H}_l(i_1) |1\rangle\langle 1| \mathcal{H}_l(i_2) |1\rangle\langle 1| \mathcal{H}_l(i_3) |1\rangle\langle 1| \dots |1\rangle\langle 1| \mathcal{H}_l(i_m) |1\rangle\langle 1|, \end{aligned} \quad (\text{D15})$$

where

$$\mathcal{H}_l(i_s) = \overbrace{H_l^{i_1+j_1+\dots+i_{s-1}+j_{s-1}+s-1} \dots H_l^{i_1+j_1+\dots+i_{s-1}+j_{s-1}+i_s+s-1}}^{i_s \text{ times}}, \quad s = 1, 2, 3, \dots, n \quad \text{and} \quad i_s = 0, 1, 2, 3, \dots (i_0 = 0), \quad (\text{D16})$$

$$|1\rangle\langle 1|^{j_s} = \overbrace{|1\rangle\langle 1| \dots |1\rangle\langle 1|}^{j_s \text{ times}}, \quad j_s = \{0, 1, 2, 3, \dots, n\}, (j_0 = 0), \quad (\text{D17})$$

and in the first equality $n = \sum_{s=1}^m (i_s + j_s)$ and m is the number of times the pattern $\mathcal{H}_l(i_s)(ib|1\rangle\langle 1|)^{j_s}$ is repeated and in the second equality $k = \sum_{s=1}^m j_s$. Now using Lemma 1 we have

$$\begin{aligned} \langle 1 | \tilde{H}^1 \dots \tilde{H}^n | 1 \rangle &= \sum_{k=0}^n (ib)^k \overbrace{A(t_1) A'(t_2) \dots A(t_{i-1}) A'(t_i)}^{\text{even}} \overbrace{A(t_{i+j+1}) A'(t_{i+j+2}) \dots A(t_{l-1}) A'(t_l)}^{\text{even}} \\ &\quad \times \overbrace{A(t_{l+m+1}) A'(t_{l+m+2}) \dots A(t_{p-1}) A'(t_p) \dots}^{\text{even}} \end{aligned} \quad (\text{D18})$$

And since t_i and t_{i+1} are subsequent times in Eq. (D18), we can write

$$\langle 1 | \tilde{H}^1 \dots \tilde{H}^n | 1 \rangle = \sum_{k=0}^n (ib)^k \sum_{i_1, \dots, i_{(n-k)/2}} A(t_{i_1}) A'(t_{i_1+1}) \dots A(t_{i_{(n-k)/2}}) A'(t_{i_{(n-k)/2}+1}), \quad (\text{D19})$$

where the sum runs over $1 \leq i_1 < i_2 - 1, i_2 < i_3 - 1, \dots, i_{(n-k)/2-1} < i_{(n-k)/2} \leq n - 1$. ■

Example 1. As an example consider the fourth order, $n = 4$, in Dyson series (see Fig. 5). We have

$$\langle 1 | \tilde{H}^1 \tilde{H}^2 \tilde{H}^3 \tilde{H}^4 | 1 \rangle = \underbrace{A(t_1) A'(t_2) A(t_3) A'(t_4)}_{\mathcal{A}_0} + (ib)^2 \underbrace{[A(t_1) A'(t_2) + A(t_2) A'(t_3) + A(t_3) A'(t_4)]}_{\mathcal{A}_2} + (ib)^4, \quad (\text{D20})$$

where $\mathcal{A}_1 = \mathcal{A}_3 = 0$ and $\mathcal{A}_4 = 1$.

Lemma 2. For even number p we have

$$\begin{aligned} a_1 a_2^\dagger \dots a_{p-1} a_p^\dagger &= \sum_{\substack{I, J \sim [p] \\ |I|+|J|=p/2 \\ I \cap J = \emptyset}} a_{i_1+1}^\dagger a_{i_2+1}^\dagger \dots a_{i_l+1}^\dagger a_{i_1} a_{i_2} \dots a_{i_l} \delta(\omega_{j_1} - \omega_{j_1+1}) \dots \delta(\omega_{j_m} - \omega_{j_m+1}) + \mathcal{R}(a, a^\dagger), \\ &\equiv \sum_{\substack{I, J \sim [p] \\ |I|+|J|=p/2 \\ I \cap J = \emptyset}} :X_I(a, a^\dagger): X_J(\delta) + \mathcal{R}(a, a^\dagger) \end{aligned} \quad (\text{D21})$$

where $a_i \equiv a(\omega_i)$, $I = \{(i_1, i_1 + 1), \dots, (i_l, i_l + 1)\}$, $J = \{(j_1, j_1 + 1), \dots, (j_m, j_m + 1)\}$ and $l = |I|$, $m = |J|$ are the number of pairs in I and J , respectively. In other words we divide the set of pairs $(1, 2), (3, 4), \dots, (p-1, p)$ into two disjoint subsets I and J , and sum up over all such possible divisions. Moreover, $\mathcal{R}(a, a^\dagger)$ is the sum of terms of the form

$$a_{i_1}^\dagger a_{i_2}^\dagger \dots a_{i_l}^\dagger a_{i_1} a_{i_2} \dots a_{i_l} \delta(\omega_{j_1} - \omega_{j_1'}) \dots \delta(\omega_{j_m} - \omega_{j_m'}), \quad (\text{D22})$$

where all indices i', i'', j' , and j'' are distinct from each other, and at least one of δ 's exhibits jump, i.e., $|j'_s - j''_s| > 1$ for some s .

Proof. This can be proven making use of Wick’s theorem. ■

Example 2. As an example consider the fourth and six orders, $n = 4$ and 6, in Dyson series. We have

$$a_1 a_2^\dagger a_3 a_4^\dagger = a_2^\dagger a_4^\dagger a_1 a_3 + a_4^\dagger a_3 \delta(\omega_1 - \omega_2) + a_2^\dagger a_1 \delta(\omega_3 - \omega_4) + \delta(\omega_1 - \omega_2) \delta(\omega_3 - \omega_4) + \underbrace{a_2^\dagger a_3 \delta(\omega_1 - \omega_4)}_{\mathcal{R}(a, a^\dagger)}, \tag{D23}$$

$$\begin{aligned} a_1 a_2^\dagger a_3 a_4^\dagger a_5 a_6^\dagger &= a_2^\dagger a_4^\dagger a_6^\dagger a_1 a_3 a_5 + \delta(\omega_1 - \omega_2) \delta(\omega_3 - \omega_4) \delta(\omega_5 - \omega_6) + a_2^\dagger a_4^\dagger a_1 a_3 \delta(\omega_5 - \omega_6) + a_2^\dagger a_6^\dagger a_1 a_5 \delta(\omega_3 - \omega_4) \\ &+ a_4^\dagger a_6^\dagger a_3 a_5 \delta(\omega_1 - \omega_2) + (a_6^\dagger a_5 \delta(\omega_3 - \omega_4) + a_4^\dagger a_3 \delta(\omega_5 - \omega_6)) \delta(\omega_1 - \omega_2) + a_2^\dagger a_1 \delta(\omega_5 - \omega_6) \delta(\omega_3 - \omega_4) \\ &+ \underbrace{a_2^\dagger a_4^\dagger a_3 a_5 \delta(\omega_1 - \omega_6) + a_2^\dagger a_4^\dagger a_1 a_5 \delta(\omega_3 - \omega_6) + a_2^\dagger a_6^\dagger a_3 a_5 \delta(\omega_1 - \omega_4) + a_4^\dagger a_5 \delta(\omega_1 - \omega_2) \delta(\omega_3 - \omega_6)}_{\in \mathcal{R}(a, a^\dagger)} \\ &+ \underbrace{[a_2^\dagger a_3 \delta(\omega_5 - \omega_6) + a_2^\dagger a_5 \delta(\omega_3 - \omega_6)] \delta(\omega_1 - \omega_4) + a_2^\dagger a_5 \delta(\omega_1 - \omega_6) \delta(\omega_3 - \omega_4)}_{\in \mathcal{R}(a, a^\dagger)}. \end{aligned} \tag{D24}$$

For \mathcal{A}_k defined in Proposition 1 the following lemma can also be proved.

Lemma 2.1.

$$\mathcal{A}_k = \sum_{\substack{I, J \sim [n] \\ |I|+|J|=(n-k)/2 \\ I \cap J = \emptyset}} :X_I(A', A): X_J(F^r) + \mathcal{R}_k(A', A, F^r), \tag{D25}$$

where

$$X_I(A', A) = A(t_{i_1}) A'(t_{i_1+1}) \dots A(t_{i_l}) A'(t_{i_l+1}), \quad X_J(F) = F^r(t_{j_1} - t_{j_1+1}) \dots F^r(t_{j_m} - t_{j_m+1}). \tag{D26}$$

The object $\mathcal{R}_k(A, A', F)$ consists of the terms of the form

$$A(t_{i'_1}) A'(t_{i'_2}) \dots A'(t_{i'_l}) A(t_{i''_1}) A(t_{i''_2}) \dots A(t_{i''_{l'}}) F^r(t_{j'_1} - t_{j'_1}) \dots F^r(t_{j'_m} - t_{j'_m}), \tag{D27}$$

where all indices $i', i'', j',$ and j'' are distinct from each other, and at least one of F^r ’s exhibits jump, i.e., $|j'_s - j''_s| > 1$ for some s .

Proof. In Lemma 2 we have proved the same relation for operators a and a^\dagger , where we have only exploited the canonical commutation relation, and the fact that $\delta(\omega_i - \omega_j)$ is a scalar. Now, operators A and \tilde{A} satisfy the same commutation relation, with $F^r(t_i - t_j)$ in place of the Dirac delta. Since F is also scalar, we obtain

$$A(t_{p_1}) A'(t_{p_1+1}) \dots A(t_{p_{(n-k)/2}}) A'(t_{p_{(n-k)/2}+1}) \equiv X_S(A, \tilde{A}) \equiv \sum_{\substack{I \cup J = S \\ I \cap J = \emptyset}} :X_I(A, \tilde{A}): X_J(F^r) + \mathcal{R}_S(A, \tilde{A}). \tag{D28}$$

Here $\mathcal{R}_S(A, A', F^r)$ has the same feature as $\mathcal{R}(a, a^\dagger, \delta)$ (i.e., it is the sum of the terms with “jump”). Therefore we have

$$\begin{aligned} \mathcal{A}_k &= \sum_{p_1, \dots, p_{(n-k)/2}} A(t_{p_1}) A'(t_{p_1+1}) \dots A(t_{p_{(n-k)/2}}) A'(t_{p_{(n-k)/2}+1}) \equiv \sum_{\substack{S \sim [n] \\ |S|=(n-k)/2}} X_S \\ &= \sum_{\substack{S \sim [n] \\ |S|=(n-k)/2}} \sum_{\substack{I \cup J = S \\ I \cap J = \emptyset}} :X_I(A, \tilde{A}): X_J(F^r) + \mathcal{R}_S(A, \tilde{A}) \\ &= \sum_{\substack{I, J \sim [n] \\ |I|+|J|=(n-k)/2}} :X_I(A', A): X_J(F^r) + \mathcal{R}_k(A', A, F^r) \end{aligned} \tag{D29}$$

where $\mathcal{R}_k(A, \tilde{A}, F^r) = \sum_S \mathcal{R}_S$, and by definition it has the same feature as \mathcal{R}_S , i.e., it is the sum of terms with jumps. ■

Example 3. As an example, consider the fourth order, $n = 4$, in Dyson series. We have

$$\begin{aligned} \mathcal{A}_0 &= A(t_1) A'(t_2) A(t_3) A'(t_4) = \\ &= A'(t_2) A'(t_4) A(t_1) A(t_3) + A'(t_2) A(t_1) F^r(t_3 - t_4) + A'(t_4) A(t_3) F^r(t_1 - t_2) + \underbrace{A'(t_2) A(t_3) F^r(t_1 - t_4)}_{\mathcal{R}(A, A', F^r)}. \end{aligned} \tag{D30}$$

We shall now apply the approximation introduced in Eq. (18), $D(t_i - t_j) = b\delta(t_i - t_j) - F^r(t_i - t_j) \approx 0$, then

$$\int dt_i \dots \int dt_j F^r(t_i - t_j) \approx \int dt_i \dots \int dt_j b\delta(t_i - t_j). \tag{D31}$$



FIG. 4. Schematic illustration of distribution of $(ib)^k$ and the pairs $A(t_i)A'(t_{i+1})$ for a general n . Each empty circle denotes one ib and every two full circles denote one pair $A(t_i)A'(t_{i+1})$. Thus the total number of the circles is n and number of the empty circles is k and the number of pairs is $(n - k)/2$.

Therefore we have the following rule:

$$F^r(t_i - t_j) \Rightarrow b\delta(t_i - t_j). \quad (\text{D32})$$

In the following the rule (D32) will be applied.

Accordingly, here we define $\mathcal{A}_k^{\text{appr}}$ and $\mathcal{R}_k^{\text{appr}}(A, A', F^r)$ as follows:

$$\mathcal{A}_k^{\text{appr}} \equiv \sum_{I, J} : X_I(A', A) : X_J(\delta), \quad (\text{D33})$$

in which the sum runs over sets I of J of indices, satisfying $I = \{i_1, i_1 + 1, \dots, i_l, i_l + 1\}$, $J = \{j_1, j_1 + 1, \dots, j_m, j_m + 1\}$, $I \cup J = \{1, 3, \dots, (n - k)/2\}$, $I \cap J = \{\emptyset\}$, $l + m = (n - k)/2$. $\mathcal{R}_k^{\text{appr}}(A, A', F^r)$ is like $\mathcal{R}_k(A, A', F^r)$ in Eq. (D25) where $F^r(t_{j_m} - t_{j_m+1})$ is replaced by $\delta(t_{j_m} - t_{j_m+1})$. Therefore using Lemma 5 we have

$$\int dt^{n-k} \mathcal{R}_k^{\text{appr}}(A, A', F^r) = 0. \quad (\text{D34})$$

Now defining

$$\langle 1 | \tilde{H}^1 \dots \tilde{H}^n | 1 \rangle^{\text{appr}} \quad (\text{D35})$$

we get

$$\begin{aligned} \langle 1 | S(0, t) | 1 \rangle^{\text{appr}} &= \sum_{n=0}^{\infty} (-i)^n \int dt^n \langle 1 | \tilde{H}^1 \dots \tilde{H}^n | 1 \rangle^{\text{appr}} \\ &= \sum_{n=0}^{\infty} (-i)^n \int dt^n \sum_{k=0}^n (ib)^k \mathcal{A}_k^{\text{appr}}. \end{aligned} \quad (\text{D36})$$

Before proceeding with the above equation, here we introduce a useful diagrammatic notation for time integrals.

1. Diagrams and tableaux

The integrand of each integral in $\langle 1 | S(0, T) | 1 \rangle^{\text{appr}}$ in Eq. (D36) which are $(ib)^k \mathcal{A}_k^{\text{appr}}$ is a sequence of ibs , the pairs of $A'(t_{i+1})A(t_i)$ and $b\delta(t_{j_m} - t_{j_m+1})$ s. We will schematically illustrate each ib by an empty circle and every pair of $A'(t_{i+1})A(t_i)$ by two green circles and each $b\delta(t_{j_m} - t_{j_m+1})$ by two brown circles. Therefore each integrand can be represented by a sequence of three such types of objects. We will accomplish this in three steps. As an example consider

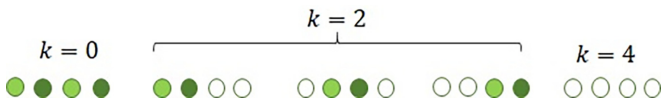


FIG. 5. Schematic illustration of distribution of $(ib)^k$ and the pairs $A(t_i)A'(t_{i+1})$ for $n = 4$.

the following integral whose integrand has been schematically illustrated in Fig. 6 (step I):

$$\begin{aligned} \mathcal{B} &= \int dt^{34} b^{14} A(t_2)A'(t_3)A(t_7)A'(t_8)A(t_9)A'(t_{10})A(t_{13})A'(t_{14}) \\ &\quad \times A(t_{19})A'(t_{20})A(t_{21})A'(t_{22})A(t_{23})A'(t_{24})A(t_{28})A'(t_{29}) \\ &\quad \times A(t_{30})A'(t_{31})A(t_{33})A'(t_{34}). \end{aligned} \quad (\text{D37})$$

After commuting, the operators through many normally ordered terms are produced. For instance, one of them is the following (step II):

$$\begin{aligned} \mathcal{B}_i &= \int dt^{34} b^{19} A'(t_3)A'(t_8)A'(t_{22})A'(t_{24})A'(t_{34})A(t_2)A(t_7) \\ &\quad \times A(t_{21})A(t_{23})A(t_{33})\delta(t_9 - t_{10})\delta(t_{13} - t_{14})\delta(t_{19} - t_{20}) \\ &\quad \times \delta(t_{28} - t_{29})\delta(t_{30} - t_{31}). \end{aligned} \quad (\text{D38})$$

And then from Lemma 5.1 we know that any $\delta(t_j - t_{j+1})$ reduces the time integral by 1. Hence (step III),

$$\begin{aligned} \mathcal{B}_i &= \int dt^{29} b^{19} A'(t_3)A'(t_8)A'(t_{19})A'(t_{21})A'(t_{29})A(t_2)A(t_7) \\ &\quad \times A(t_{18})A(t_{20})A(t_{28}). \end{aligned} \quad (\text{D39})$$

Then we encode the positions of these three types of circles in the final distribution, i.e., in step III by a diagram μ and a tableau y (see Fig. 7). The diagram μ is used to show the general form of the integrand, i.e., the distribution of bs and the pairs $A'A$ and the tableau y , which is the diagram μ filled with the sequences of zeros (for ibs) and ones [for $b\delta(t_j - t_{j+1})$ s], demonstrates the specific form of the integrand which means that it precisely encodes the positions of ibs and $b\delta(t_j - t_{j+1})$ s. The number of blocks in each row $|\mu_j|$ shows the final number of time integrals in step III. After each row there exists a pair of $A'(t_i)A(t_{i+1})$ and $P(\mu) = (|\mu_1| + 1, |\mu_1| + |\mu_2| + 3, |\mu_1| + |\mu_2| + |\mu_3| + 5, \dots, \sum_{j=1}^l |\mu_j| + 2l - 1)$ completely determines the position of $A(t_i)$ (hence the position of the pair). Note that if a row is empty, this means that the pairs $A'(t_{i+1})A(t_i)$ come one after another. It is seen that for $l = 5$ pairs of $A'A$ there will be $l + 1 = 6$ rows in diagram $\mu = (\mu_1, \mu_2, \dots, \mu_{l+1})$ (see Fig. 7). In this example we have $P(\mu) = (1 + 1, 1 + 3 + 3, 1 + 3 + 9 + 5, 1 + 3 + 9 + 0 + 7, 1 + 3 + 9 + 0 + 6 + 9, 1 + 3 + 9 + 0 + 6 + 0 + 11) = (2, 7, 18, 20, 28)$.

2. Proof of the main result

Proposition 2.

$$\begin{aligned} \langle 1 | S_I(t, 0) | 1 \rangle^{\text{appr}} &= \sum_{l=0}^{\infty} (-1)^l \sum_{\mu \vdash l+1} b^{|\mu|} \\ &\quad \times \int dt^{|\mu|+2l} : X_{P(\mu)}(A, A') : \sum_{y \in \mu} (-1)^{w(y)}, \end{aligned} \quad (\text{D40})$$

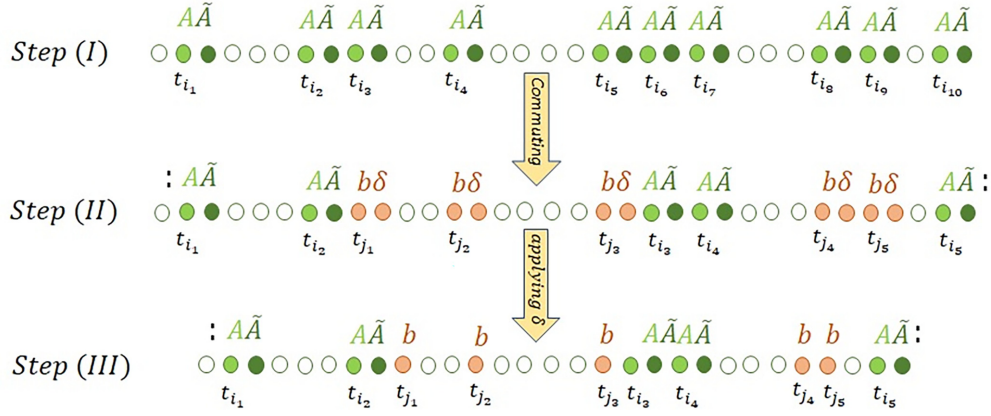


FIG. 6. Schematic illustration of distribution of the integrand containing $(ib)^k$ (empty circles), $b\delta(t_{j_m} - t_{j_{m+1}})$ (every two brown circles), and the pairs $A(t_{i_i})A'(t_{i_{i+1}})$ (every two green circles) for $n = 34$ and $k = 14$. Step I: The original integrand. Step II: Commuting the operators through. Step III: Applying $\delta(t_{j_m} - t_{j_{m+1}})$. “: x :” denotes normal ordering.

where

$$P(\mu) = \{(P_1, P_1 + 1), \dots, (P_l, P_l + 1)\},$$

$$\text{with } P_r = \sum_{j=1}^r |\mu_j| + 2r - 1, \quad r = 1, \dots, l. \quad (\text{D41})$$

The second sum runs over all μ with $l + 1$ rows (denoted here by $\mu \vdash l + 1$) and $y \in \mu$ means that y is μ filled with sequences of zeros and ones showing the positions of ibs and F 's after commuting the pairs of $A(t_{i_i})A'(t_{i_{i+1}})$. The index $P(\mu)$ shows the position of the pairs (see Figs. 6 and 7 for more detail). $|\mu|$ is the number of blocks in μ and $w(y)$ is the number of ones in y .

Proof. Substituting Eq. (D33) into Eq. (D36) we get

$$\begin{aligned} \langle 1|S_I(t, 0)|1 \rangle^{\text{appr}} &= \sum_{n=0}^{\infty} (-i)^n \int dt^n \sum_{k=0}^n (ib)^k A_k^{\text{appr}} \\ &= \sum_{n=0}^{\infty} \int dt^n \sum_{k=0}^n (-i)^{n-k} b^k \\ &\quad \times \sum_{\substack{I, J \sim [n], \\ I \cap J = \emptyset, \\ |I| + |J| = (n-k)/2}} :X_I(A, A') : X_J(\delta) \end{aligned}$$

$$\begin{aligned} &= \sum_{n=0}^{\infty} \sum_{k, l, m} (-1)^{l+m} b^{k+m} \\ &\quad \times \sum_{\substack{I, J \sim [n], \\ I \cap J = \emptyset, \\ |I| = l, |J| = m}} \int dt^n :X_I(A, A') : X_J(\delta), \end{aligned} \quad (\text{D42})$$

where the second sum in the last equality runs over all k such that $n - k$ is even and $l, m \geq 0$ such that $m + l = (n - k)/2$ where $X_I(A', A)$ and $X_J(\delta)$ were introduced in Eqs. (D26) and (D33), respectively, and “: x :” denotes normal ordering. Now Eq. (D42) may be rewritten as

$$\begin{aligned} \langle 1|S_I(t, 0)|1 \rangle^{\text{appr}} &= \sum_{k, l, m=0}^{\infty} (-1)^{l+m} b^{k+m} \sum_{\substack{I, J \sim \mathbb{N}, \\ I \cap J = \emptyset, \\ |I| = l, |J| = m}} \\ &\quad \times \int dt^{m+l+2k} :X_I(A, A') : X_J(\delta). \end{aligned} \quad (\text{D43})$$

We can now further rewrite it using diagrams:

$$\begin{aligned} \langle 1|S_I(t, 0)|1 \rangle^{\text{appr}} &= \sum_{l=0}^{\infty} (-1)^l \sum_{\mu \vdash l+1} b^{|\mu|} \sum_{y \in \mu} (-1)^{w(y)} \\ &\quad \times \int dt^{|\mu|+2l} :X_{P(\mu)}(A, A') :, \end{aligned} \quad (\text{D44})$$

where the second sum runs over all μ with $l + 1$ rows (denoted here by $\mu \vdash l + 1$) and $y \in \mu$ means that y is μ already filled with sequences of zeros and ones showing the positions of ibs and F 's commuting the pairs of $A(t_{i_i})A'(t_{i_{i+1}})$ and $P(\mu)$ for μ with $l + 1$ rows is given by

$$P(\mu) = \{(P_1, P_1 + 1), \dots, (P_l, P_l + 1)\},$$

$$\text{with } P_r = \sum_{j=1}^r |\mu_j| + 2r - 1. \quad (\text{D45})$$

Here $|\mu|$ is the number of blocks in μ and $w(y)$ is the number of ones in y . Since $X_{P(\mu)}$ depends only on μ and not on y ,

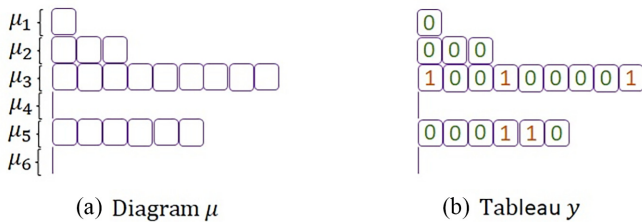


FIG. 7. For l pairs of $A'A$ there will be $l + 1$ rows in diagram $\mu = (\mu_1, \mu_2, \dots, \mu_{l+1})$. (a) The diagram μ shows how bs and the pairs $A'A$ are distributed in Fig. 6. As can be seen $P(\mu) = (2, 7, 18, 20, 28)$ completely determines the position of $A(t_{i_i})$ in step III of Fig. 6. (b) The tableau y , which is the diagram μ filled with the sequences of zeros and ones, specifies the positions of ibs and $b\delta$'s in step III of Fig. 6.

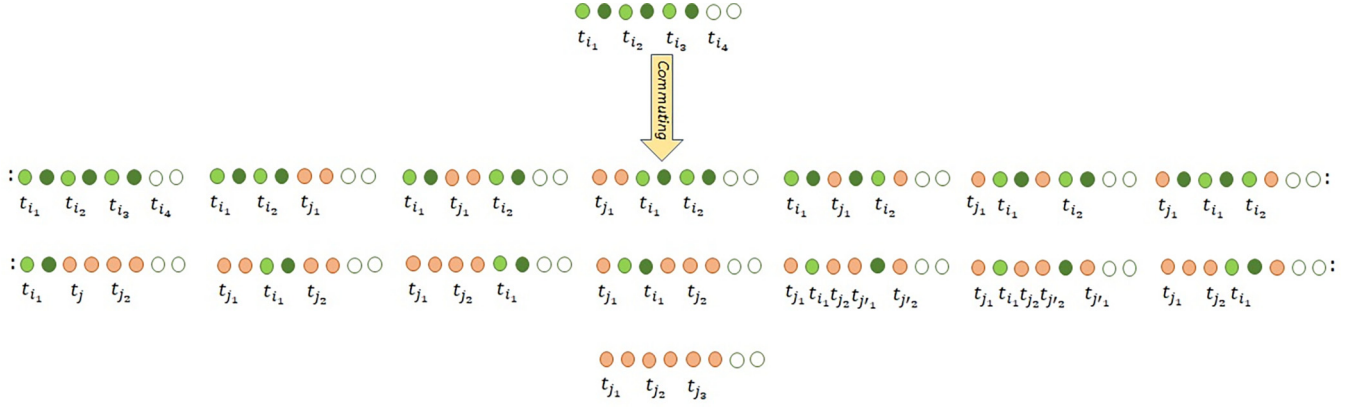


FIG. 8. Schematic illustration of distribution of $(ib)^2$ (empty circles), $b\delta(t_{j_m} - t_{j_{m+1}})$ (every two brown circles), and the pairs $A(t_i)A'(t_{i+1})$ (every two green circles) for the term $b^2 A(t_1)A'(t_2)A(t_3)A'(t_4)A(t_5)A'(t_6)$ (steps I and II). For step III from every two brown circles one of them is removed. “: x :” denotes normal ordering.

Eq. (D42) may be rewritten as

$$\langle 1|S_I(t, 0)|1\rangle^{\text{appr}} = \sum_{l=0}^{\infty} (-1)^l \sum_{\mu^{l+1}} b^{|\mu|} \times \int dt^{|\mu|+2l} :X_{P(\mu)}(A, A') : \sum_{y \in \mu} (-1)^{w(y)}. \quad (\text{D46})$$

Lemma 3. For all μ apart from empty one $|\mu| = 0$ we have

$$\sum_{y \in \mu} (-1)^{w(y)} = 0. \quad (\text{D47})$$

Proof. ■

$$\begin{aligned} \sum_{y \in \mu} (-1)^{w(y)} &= \sum_{m=0}^{|\mu|} (-1)^m \frac{(|\mu| + l)!}{l! m! (|\mu| - m)!} \\ &= \frac{(|\mu| + l)(|\mu| + l - 1) \dots (|\mu| + 1)}{l!} \\ &\times \sum_{m=0}^{|\mu|} (-1)^m \frac{|\mu|!}{m! (|\mu| - m)!} \end{aligned}$$

$$\begin{aligned} &= \frac{(|\mu| + l)(|\mu| + l - 1) \dots (|\mu| + 1)}{l!} \\ &\times \sum_{m=0}^{|\mu|} (-1)^m \binom{|\mu|}{m}. \end{aligned} \quad (\text{D48})$$

From the binomial theorem [35] we know that if x and y are variables and $n \in \mathbb{N}$, then

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k. \quad (\text{D49})$$

Now choosing $x = 1$ and $y = -1$ we get

$$\begin{aligned} 0 &= [1 + (-1)]^n = \sum_{k=0}^n \binom{n}{k} 1^{n-k} (-1)^k = \sum_{k=0}^n \binom{n}{k} (-1)^k \\ &= \sum_{m=0}^{|\mu|} (-1)^m \binom{|\mu|}{m}. \end{aligned} \quad (\text{D50})$$

We are now in a position to prove a proposition, that gives us Eq. (50) from Theorem 1. The other equations from this theorem are obtained analogously.

Proposition 3.

$$\begin{aligned} \langle 1|S_I(t, 0)|1\rangle^{\text{appr}} &= \sum_{l=0}^{\infty} (-i)^{2l} \int dt^{2l} A'(t_2)A'(t_4) \dots A'(t_{2l}) \\ &\times A(t_1)A(t_3) \dots A(t_{2l-1}). \end{aligned} \quad (\text{D51})$$

Proof. Inserting Eq. (D47) from Lemma 3 into Eq. (D40) we obtain the required result. We see that only one diagram is left in the sum over μ —the trivial one—with no blocks (let us call it μ_{null}). This diagram has very simple $P(\mu_{\text{null}}) = (1, 2), (3, 4), \dots, (2l - 1, 2l)$. This gives

$$\langle 1|S_I(t, 0)|1\rangle^{\text{appr}} = \sum_{l=0}^{\infty} (-i)^{2l} :X_{P(\mu_{\text{null}})}(A, A') :, \quad (\text{D52})$$

which by definition of X gives the required result. ■

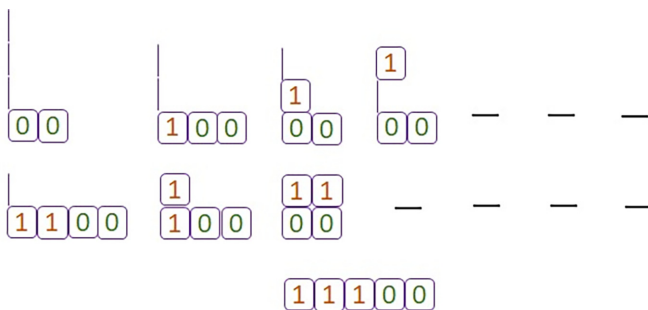


FIG. 9. The tableau y for the term $b^2 A(t_1)A'(t_2)A(t_3)A'(t_4)A(t_5)A'(t_6)$. The dash indicates that this term lies in $\mathcal{R}(A, A')$, hence its integral over time vanishes.

3. Example

Example 4. As an example consider the term $b^2 a_1 a_2^\dagger a_3 a_4^\dagger a_5 a_6^\dagger$ in the eighth order of Dyson series. In Fig. 8 it is illustrated how $(ib)^2$, the pairs $A'A$ and $b\delta$ s, are distributed after commuting the operators through. Then taking the integral over time we get the following terms:

$$\int dt^8 b^2 A'(t_2)A'(t_4)A'(t_6)A(t_1)A(t_3)A(t_5) = \int dt^8 b^2 : X_{P_1, P_2, P_3} :, \quad (D53)$$

where $X_{P_1, P_2, P_3} = A'(t_2)A'(t_4)A'(t_6)A(t_1)A(t_3)A(t_5)$, $P_1 = \mu_1 + 2 \times 1 - 1 = 0 + 2 - 1 = 1$, $P_2 = \mu_1 + \mu_2 + 2 \times 2 - 1 = 0 + 0 + 4 - 1 = 3$ and $P_3 = \mu_1 + \mu_2 + \mu_3 + 2 \times 3 - 1 = 0 + 0 + 0 + 6 - 1 = 5$ (see Fig. 9 for more detail).

$$\int dt^8 b^2 A'(t_2)A'(t_4)A(t_1)A(t_3)F(t_5 - t_6) \simeq \int dt^8 b^2 A'(t_2)A'(t_4)A(t_1)A(t_3)\delta(t_5 - t_6) = \int dt^7 b^2 : X_{P_1, P_2} :, \quad (D54)$$

where $X_{P_1, P_2} = A'(t_2)A'(t_4)A(t_1)A(t_3)$, $P_1 = \mu_1 + 2 \times 1 - 1 = 0 + 2 - 1 = 1$ and $P_2 = \mu_1 + \mu_2 + 2 \times 2 - 1 = 0 + 0 + 4 - 1 = 3$.

$$\int dt^8 b^2 A'(t_2)A'(t_6)A(t_1)A(t_5)F(t_3 - t_4) \simeq \int dt^8 b^2 A'(t_2)A'(t_6)A(t_1)A(t_5)\delta(t_3 - t_4) = \int dt^7 b^2 : X_{P_1, P_2} :, \quad (D55)$$

where $X_{P_1, P_2} = A'(t_2)A'(t_6)A(t_1)A(t_5)$, $P_1 = \mu_1 + 2 \times 1 - 1 = 0 + 2 - 1 = 1$ and $P_2 = \mu_1 + \mu_2 + 2 \times 2 - 1 = 0 + 1 + 4 - 1 = 4$.

$$\int dt^8 b^2 A'(t_4)A'(t_6)A(t_3)A(t_5)F(t_1 - t_2) \simeq \int dt^8 b^2 A'(t_4)A'(t_6)A(t_3)A(t_5)\delta(t_1 - t_2) = \int dt^7 b^2 : X_{P_1, P_2} :, \quad (D56)$$

where $X_{P_1, P_2} = A'(t_4)A'(t_6)A(t_3)A(t_5)$, $P_1 = \mu_1 + 2 \times 1 - 1 = 1 + 2 - 1 = 2$ and $P_2 = \mu_1 + \mu_2 + 2 \times 2 - 1 = 1 + 0 + 4 - 1 = 4$.

$$\int dt^8 b^2 A'(t_2)A(t_1)F(t_3 - t_4)F(t_5 - t_6) \simeq \int dt^8 b^2 A'(t_2)A(t_1)\delta(t_3 - t_4)\delta(t_5 - t_6) = \int dt^6 b^2 : X_{P_1} :, \quad (D57)$$

where $X_{P_1} = A'(t_2)A(t_1)$ and $P_1 = \mu_1 + 2 \times 1 - 1 = 0 + 2 - 1 = 1$.

$$\int dt^8 b^2 A'(t_4)A(t_3)F(t_1 - t_2)F(t_5 - t_6) \simeq \int dt^8 b^2 A'(t_4)A(t_3)\delta(t_1 - t_2)\delta(t_5 - t_6) = \int dt^6 b^2 : X_{P_1} :, \quad (D58)$$

where $X_{P_1} = A'(t_4)A(t_3)$ and $P_1 = \mu_1 + 2 \times 1 - 1 = 1 + 2 - 1 = 2$.

$$\int dt^8 b^2 A'(t_6)A(t_5)F(t_1 - t_2)F(t_3 - t_4) \simeq \int dt^8 b^2 A'(t_6)A(t_5)\delta(t_1 - t_2)\delta(t_3 - t_4) = \int dt^6 b^2 : X_{P_1} :, \quad (D59)$$

where $X_{P_1} = A'(t_6)A(t_5)$ and $P_1 = \mu_1 + 2 \times 1 - 1 = 2 + 2 - 1 = 3$.

$$\int dt^8 b^2 F(t_1 - t_2)F(t_3 - t_4)F(t_5 - t_6) \simeq \int dt^8 b^2 \delta(t_1 - t_2)\delta(t_3 - t_4)\delta(t_5 - t_6) = \int dt^5 b^2. \quad (D60)$$

4. Auxiliary lemmas

Lemma 4.

$$[A(t_i), A'(t_j)] = F^r(t_i - t_j), \quad (D61)$$

where

$$F^r(t) = e^{i(\omega_A - i\gamma)t} F(t), \quad F(t) \equiv \int dk e^{-i\omega_k t} |f(\omega_k)|^2. \quad (D62)$$

Proof. Using Eq. (14) we have

$$\begin{aligned} [A(t_i), A'(t_j)] &= \left[\int dk f(\omega_k, t_i) a_k, \int dk' f'(\omega_{k'}, t_j) a_{k'}^\dagger \right] = \int dk \int dk' f(\omega_k, t_i) f'(\omega_{k'}, t_j) [a_k, a_{k'}^\dagger] \\ &= \int dk \int dk' f(\omega_k, t_i) f'(\omega_{k'}, t_j) \delta(\omega_k - \omega_{k'}) = \int dk e^{i(\omega_A - i\gamma)(t_i - t_j)} e^{-i\omega_k(t_i - t_j)} |f(\omega_k)|^2 \\ &= e^{i(\omega_A - i\gamma)(t_i - t_j)} \int dk e^{-i\omega_k(t_i - t_j)} |f(\omega_k)|^2 = F^r(t_i - t_j), \end{aligned} \quad (D63)$$

where $F^r(t)$ is given by Eq. (17) as

$$F^r(t) \equiv e^{i(\omega_A - i\gamma)t} \int dk e^{-i\omega_k t} |f(\omega_k)|^2. \quad (\text{D64})$$

■

Lemma 5.

$$\int_0^t dt_1 \dots \int_0^{t_{i-1}} dt_i \dots \int_0^{t_{j-1}} dt_j \dots \int_0^{t_{k-1}} dt_k Z(t_1, \dots, t_k) \delta(t_i - t_j) = 0, \quad \text{for } j > i + 1, \quad (\text{D65})$$

where $Z(t_1, \dots, t_k)$ is an arbitrary function over t_1, \dots, t_k .

Proof. We use the change of variables as $u = t_i - t_j$ and then

$$du = -dt_j, \quad t_j = 0 : u = t_i, \quad \text{and} \quad t_j = t_{j-1} : u = t_i - t_{j-1}, \quad (\text{D66})$$

so we keep t_i and u . Now we define

$$\begin{aligned} K(t_1, \dots, t_{j-1}, t_{j+1}, \dots, t_k) &=: \int_0^{t_{j-1}} dt_j Z(t_1, \dots, t_k) \delta(t_i - t_j) \\ &= \int_{t_i - t_{j-1}}^{t_i} du \delta(u) Z(t_1, \dots, t_i, \dots, t_{j-1}, t_i - u, t_{j+1}, \dots, t_k) \\ &= \begin{cases} 1 & \text{for } t_i = t_{j-1}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \quad (\text{D67})$$

Since the function $K(t_1, \dots, t_{j-1}, t_{j+1}, \dots, t_k)$ is only nonzero at one point then its integral over t_i and t_{j-1} is zero, thus

$$\int_0^t dt_1 \dots \int_0^{t_{i-1}} dt_i \dots \int_0^{t_{j-2}} dt_{j-1} K(t_1, \dots, t_{j-1}, t_{j+1}, \dots, t_k) = 0, \quad (\text{D68})$$

which completes the proof. ■

Lemma 5.1.

$$\int_0^t dt_1 \int_0^{t_1} dt_2 \delta(t_1 - t_2) = \int_0^t dt_1, \quad (\text{D69})$$

where t_1 and t_2 are subsequent times in Dyson series.

Proof. We use the following change of variables: $u = t_1 - t_2$ and then

$$du = -dt_2, \quad t_2 = 0 : u = t_1, \quad \text{and} \quad t_2 = t_1 : u = 0. \quad (\text{D70})$$

So keeping u and t_1 we have

$$\int_0^t dt_1 \int_0^{t_1} du \delta(u) = \int_0^t dt_1, \quad (\text{D71})$$

which completes the proof. ■

Remark 1. Lemma 5 states that the Dirac delta $\delta(t_i - t_j)$ with times that are not neighboring causes the expression to vanish and Lemma 5.1 states that, for neighboring times, the Dirac delta $\delta(t_i - t_j)$ removes one integral.

APPENDIX E: PROOF OF EQ. (80), I.E., CHECKING NORMALIZATION OF THE TOTAL STATE FOR INITIAL VACUUM

First let us see if the renormalized time-evolution operator given in Eq. (49) preserves the norm of the state vector. Using Eqs. (14) and (50)–(53) for the initial states $|\{0\}\rangle$ and $|\{1_k\}\rangle$ (one photon in each mode) of the field, respectively, we get

$$\begin{aligned} S_{I_r}(t, 0)|1, \{0\}\rangle &= S_{I_r,11}(t, 0)|1, \{0\}\rangle + S_{I_r,01}(t, 0)|0, \{0\}\rangle \\ &= |1, \{0\}\rangle - i \int_0^t dt_1 A'(t_1)|0, \{0\}\rangle \\ &= |1, \{0\}\rangle - i \int_0^t dt_1 \int dk f'(\omega_k, t_1)|0, \{1_k\}\rangle. \end{aligned} \quad (\text{E1})$$

For the Schrödinger picture, since $U(t) = e^{-iH_0^r t} U_{I_r}(t)$ using Eq. (66) we have

$$S(t, 0)|1, \{0\}\rangle = e^{-iH_0^r t} S_{I_r}(t, 0)|1, \{0\}\rangle = e^{-\gamma t |\{1\}\rangle \langle \{1\}|} V \left[e^{-\gamma t} |1, \{0\}\rangle - i \int_0^t dt_1 \int dk f'(\omega_k, t_1) |0, \{1_k\}\rangle \right], \quad (\text{E2})$$

where

$$V = e^{-i(\omega_A|1\rangle\langle 1| + H_F)t} \quad (\text{E3})$$

is a unitary operator and H_F is defined in Eq. (9). Therefore using Eq. (E2), we get

$$\langle \{0\}, 1 | S^\dagger(t, 0) S(t, 0) | 1, \{0\} \rangle = e^{-2\gamma t} + \mathcal{D}(t, 0) \approx 1, \quad (\text{E4})$$

where $\mathcal{D}(t, 0)$ is obtained as

$$\begin{aligned} \mathcal{D}(t, 0) &= \int dk |f(\omega_k)|^2 \int_0^t dt_1 e^{-\gamma t_1} e^{-i(\omega_k - \omega_A)t_1} \int_0^t dt'_1 e^{-\gamma t'_1} e^{i(\omega_k - \omega_A)t'_1} \\ &= \int dk |f(\omega_k)|^2 \left(\frac{e^{-\gamma t} e^{-i(\omega_k - \omega_A)t} - 1}{-\gamma - i(\omega_k - \omega_A)} \right) \left(\frac{e^{-\gamma t} e^{i(\omega_k - \omega_A)t} - 1}{-\gamma + i(\omega_k - \omega_A)} \right) \\ &= \int dk |f(\omega_k)|^2 \frac{e^{-2\gamma t} + 1 - e^{-\gamma t} (e^{i(\omega_k - \omega_A)t} + e^{-i(\omega_k - \omega_A)t})}{\gamma^2 + (\omega_k - \omega_A)^2} \\ &= (e^{-2\gamma t} + 1) \int dk \frac{|f(\omega_k)|^2}{\gamma^2 + (\omega_k - \omega_A)^2} - e^{-\gamma t} \int dk |f(\omega_k)|^2 \frac{e^{i(\omega_k - \omega_A)t} + e^{-i(\omega_k - \omega_A)t}}{\gamma^2 + (\omega_k - \omega_A)^2} \\ &= (e^{-2\gamma t} + 1) \int d\omega \frac{4\pi\omega^2 |f(\omega)|^2}{\gamma^2 + (\omega - \omega_A)^2} - e^{-\gamma t} \int d\omega 4\pi\omega^2 |f(\omega)|^2 \frac{e^{i(\omega - \omega_A)t} + e^{-i(\omega - \omega_A)t}}{\gamma^2 + (\omega - \omega_A)^2}. \end{aligned} \quad (\text{E5})$$

Now we assume that $4\pi\omega^2 |f(\omega)|^2$ is a slow varying function such that under the Lorentzian distribution $\frac{\gamma}{\gamma^2 + (\omega - \omega_A)^2}$ we have $4\pi\omega^2 |f(\omega)|^2 \approx 4\pi\omega_A^2 |f(\omega_A)|^2$. Therefore we get

$$\begin{aligned} \mathcal{D}(t, 0) &\approx \frac{4\pi\omega_A^2 |f(\omega_A)|^2}{\gamma} \left(\pi(e^{-2\gamma t} + 1) - e^{-\gamma t} \int d\omega \frac{\gamma}{\gamma^2 + (\omega - \omega_A)^2} (e^{i(\omega - \omega_A)t} + e^{-i(\omega - \omega_A)t}) \right) \\ &= \frac{4\pi\omega_A^2 |f(\omega_A)|^2}{\gamma} \left[\pi(e^{-2\gamma t} + 1) - e^{-\gamma t} \left(e^{-i\omega_A t} \int d\omega \frac{\gamma e^{i\omega t}}{\gamma^2 + (\omega - \omega_A)^2} + e^{i\omega_A t} \int d\omega \frac{\gamma e^{-i\omega t}}{\gamma^2 + (\omega - \omega_A)^2} \right) \right] \\ &= \frac{4\pi\omega_A^2 |f(\omega_A)|^2}{\gamma} [\pi(e^{-2\gamma t} + 1) - e^{-\gamma t} (e^{-i\omega_A t} \pi e^{-\gamma t} e^{i\omega_A t} + e^{i\omega_A t} \pi e^{-\gamma t} e^{-i\omega_A t})] \\ &= \frac{4\pi\omega_A^2 |f(\omega_A)|^2}{\gamma} (e^{-2\gamma t} + 1 - 2e^{-2\gamma t}) \\ &= 1 - e^{-2\gamma t}, \end{aligned} \quad (\text{E6})$$

in which we used the relation for Fourier transform of Lorentzian distribution

$$\int_{-\infty}^{\infty} dx \frac{c}{c^2 + (x - a)^2} e^{\pm ixt} = \pi e^{-c|t|} e^{\pm iat}, \quad (\text{E7})$$

and in the last equality of Eq. (E6) we used Eq. (48).

APPENDIX F: EXPANDING $\Lambda(t)$ UP TO THE SECOND ORDER

In order to confirm our conjecture let us now expand $\tilde{\rho}_A(t)$ up to the second order. First, let us calculate (101) with all the details as follows:

$$\begin{aligned} \rho_A(t) &= \Lambda(t)\rho_A(0) = \int \mathcal{D}\xi e^{-\|\xi\|^2} U_\alpha^\xi(t) \rho_A(0) (U_\alpha^\xi(t))^\dagger \\ &= \int \mathcal{D}\xi e^{-\|\xi\|^2} U_\alpha(t, 0) \left[\mathbb{I} - \int_0^t ds_1 \left(\frac{\gamma}{2} \tilde{\mathbb{P}}_1(s_1) + i f_\xi^*(s_1) \tilde{\sigma}^-(s_1) \right) \right] \rho_A(0) \\ &\quad \times \left[\mathbb{I} - \int_0^t ds_2 \left(\frac{\gamma}{2} \tilde{\mathbb{P}}_1(s_2) - i f_\xi(s_2) \tilde{\sigma}^+(s_2) \right) \right] U_\alpha^\dagger(t, 0) \\ &= U_\alpha(t, 0) \left[\rho_A(0) - \int_0^t ds_1 \frac{\gamma}{2} (\tilde{\mathbb{P}}_1(s_1) \rho_A(0) + \rho_A(0) \tilde{\mathbb{P}}_1(s_1)) \right. \\ &\quad \left. + \int \mathcal{D}\xi e^{-\|\xi\|^2} \int_0^t ds_1 \int_0^t ds_2 f_\xi^*(s_1) f_\xi(s_2) \tilde{\sigma}^-(s_1) \rho_A(0) \tilde{\sigma}^+(s_2) \right] U_\alpha^\dagger(t, 0) \end{aligned}$$

$$\begin{aligned}
&= U_\alpha(t, 0) \left[\rho_A(0) - \frac{\gamma}{2} \int_0^t ds_1 (\tilde{\mathcal{P}}_1(s_1) \rho_A(0) + \rho_A(0) \tilde{\mathcal{P}}_1(s_1)) \right. \\
&\quad \left. + \int_0^t ds_1 \int_0^{s_1} ds_2 e^{\gamma(s_1-s_2)} e^{-i\omega_A(s_1-s_2)} F^r(s_1-s_2) \tilde{\sigma}^-(s_1) \rho_A(0) \tilde{\sigma}^+(s_2) \right] U_\alpha^\dagger(t, 0) \\
&\approx U_\alpha(t, 0) \left[\rho_A(0) - \frac{\gamma}{2} \int_0^t ds_1 (\tilde{\mathcal{P}}_1(s_1) \rho_A(0) + \rho_A(0) \tilde{\mathcal{P}}_1(s_1)) \right. \\
&\quad \left. + \int_0^t ds_1 \int_0^{s_1} ds_2 \gamma e^{\gamma(s_1-s_2)} e^{-i\omega_A(s_1-s_2)} \delta(s_1-s_2) \tilde{\sigma}^-(s_1) \rho_A(0) \tilde{\sigma}^+(s_2) \right] U_\alpha^\dagger(t, 0) \\
&= U_\alpha(t, 0) \left[\rho_A(0) - \frac{\gamma}{2} \int_0^t ds_1 (\tilde{\mathcal{P}}_1(s_1) \rho_A(0) + \rho_A(0) \tilde{\mathcal{P}}_1(s_1)) + \gamma \int_0^t ds_1 \tilde{\sigma}^-(s_1) \rho_A(0) \tilde{\sigma}^+(s_1) \right] U_\alpha^\dagger(t, 0). \tag{F1}
\end{aligned}$$

For the second-order expansion we use Eq. (100) and we get

$$\begin{aligned}
\rho_A(t) &= \Lambda(t) \rho_A^\alpha(0) = \int \mathcal{D}\xi e^{-\|\xi\|^2} U_\alpha^\xi(t) \rho_A(0) (U_\alpha^\xi(t))^\dagger \\
&= U_\alpha(t, 0) \left[\rho_A^\alpha(0) - \int_0^t ds_1 \frac{\gamma}{2} \left(\tilde{\mathcal{P}}_1(s_1) \rho_A^\alpha(0) + \rho_A^\alpha(0) \tilde{\mathcal{P}}_1(s_1) \right) \right. \\
&\quad - \int \mathcal{D}\xi e^{-\|\xi\|^2} \int_0^t ds_1 \int_0^{s_1} ds_2 f_\xi^*(s_1) f_\xi(s_2) \tilde{\sigma}^-(s_1) \rho_A^\alpha(0) \tilde{\sigma}^+(s_2) + \frac{\gamma^2}{4} \int_0^t ds_1 \int_0^{s_1} ds_1' \tilde{\mathcal{P}}_1(s_1) \rho_A^\alpha(0) \tilde{\mathcal{P}}_1(s_1') \\
&\quad - \frac{\gamma}{2} \int \mathcal{D}\xi e^{-\|\xi\|^2} \int_0^t ds_1 \int_0^{s_1} ds_1' \int_0^{s_1'} ds_2' (f_\xi^*(s_1) f_\xi(s_2') \tilde{\sigma}^-(s_1) \rho_A^\alpha(0) \tilde{\sigma}^+(s_2') \tilde{\mathcal{P}}_1(s_1') \\
&\quad + f_\xi^*(s_1) f_\xi(s_2') \tilde{\mathcal{P}}_1(s_1') \tilde{\sigma}^-(s_2') \rho_A^\alpha(0) \tilde{\sigma}^+(s_1) + f_\xi^*(s_1) f_\xi(s_1') \tilde{\sigma}^-(s_1) \rho_A^\alpha(0) \tilde{\mathcal{P}}_1(s_2') \tilde{\sigma}^+(s_1') \\
&\quad + f_\xi^*(s_1) f_\xi(s_1') \tilde{\sigma}^-(s_1') \tilde{\mathcal{P}}_1(s_2') \rho_A^\alpha(0) \tilde{\sigma}^+(s_1)) + \frac{\gamma^2}{4} \int_0^t ds_1 \int_0^{s_1} ds_1' (\rho_A^\alpha(0) \tilde{\mathcal{P}}_1(s_1') \tilde{\mathcal{P}}_1(s_1) + \tilde{\mathcal{P}}_1(s_1) \tilde{\mathcal{P}}_1(s_1') \rho_A^\alpha(0)) \\
&\quad + \int \mathcal{D}\xi e^{-\|\xi\|^2} \int_0^t ds_1 \int_0^{s_1} ds_1' \int_0^{s_1'} ds_2 \\
&\quad \left. \times \int_0^{s_2} ds_2' f_\xi^*(s_1) f_\xi(s_2) f_\xi^*(s_1') f_\xi(s_2') \tilde{\sigma}^-(s_1) \tilde{\sigma}^-(s_1') \rho_A^\alpha(0) \tilde{\sigma}^+(s_2') \tilde{\sigma}^+(s_2) \right] U_\alpha^\dagger(t, 0). \tag{F2}
\end{aligned}$$

Now applying the approximation for $F^r(t-s)$ carefully we get

$$\begin{aligned}
\rho_A(t) &= \Lambda(t) \rho_A(0) \approx U_\alpha(t, 0) \left[\rho_A(0) - \int_0^t ds_1 \frac{\gamma}{2} (\tilde{\mathcal{P}}_1(s_1) \rho_A(0) + \rho_A(0) \tilde{\mathcal{P}}_1(s_1)) + \gamma \int_0^t ds_1 \tilde{\sigma}^-(s_1) \rho_A(0) \tilde{\sigma}^+(s_1) \right. \\
&\quad + \frac{\gamma^2}{4} \int_0^t ds_1 \int_0^{s_1} ds_1' \tilde{\mathcal{P}}_1(s_1) \rho_A(0) \tilde{\mathcal{P}}_1(s_1') \\
&\quad - \frac{\gamma^2}{2} \int_0^t ds_1' \int_0^{s_1'} ds_2' (\tilde{\sigma}^-(s_2') \rho_A(0) \tilde{\sigma}^+(s_2') \tilde{\mathcal{P}}_1(s_1') + \tilde{\mathcal{P}}_1(s_1') \tilde{\sigma}^-(s_2') \rho_A(0) \tilde{\sigma}^+(s_2')) \\
&\quad - \frac{\gamma^2}{2} \int_0^t ds_1 \int_0^{s_1} ds_2' (\tilde{\sigma}^-(s_1) \rho_A(0) \tilde{\mathcal{P}}_1(s_2') \tilde{\sigma}^+(s_1) + \tilde{\sigma}^-(s_1) \tilde{\mathcal{P}}_1(s_2') \rho_A(0) \tilde{\sigma}^+(s_1)) \\
&\quad + \frac{\gamma^2}{4} \int_0^t ds_1 \int_0^{s_1} ds_1' (\rho_A(0) \tilde{\mathcal{P}}_1(s_1') \tilde{\mathcal{P}}_1(s_1) \\
&\quad + \tilde{\mathcal{P}}_1(s_1) \tilde{\mathcal{P}}_1(s_1') \rho_A(0)) + \gamma^2 \int_0^t ds_1 \int_0^{s_1} ds_1' \tilde{\sigma}^-(s_1) \tilde{\sigma}^-(s_1') \rho_A(0) \tilde{\sigma}^+(s_1) \tilde{\sigma}^+(s_1') \left. \right] U_\alpha^\dagger(t, 0). \tag{F3}
\end{aligned}$$

Now writing the solution of Eq. (103) to the second order in Dyson series, and going back to the Schrödinger picture, we obtain

$$\begin{aligned} \Lambda(t)(\cdot) = \mathcal{U}_\alpha(t, 0) & \left[\mathbb{I} - \frac{\gamma}{2} \int_0^t ds_1 \{ \tilde{\mathbb{P}}_1(s_1), (\cdot) \} + \gamma \int_0^t ds_1 \tilde{\sigma}^-(s_1)(\cdot) \tilde{\sigma}^+(s_1) + \frac{\gamma^2}{4} \int_0^t ds_2 \left\{ \tilde{\mathbb{P}}_1(s_2), \int_0^{s_2} ds_1 \{ \tilde{\mathbb{P}}_1(s_1), (\cdot) \} \right\} \right. \\ & - \frac{\gamma^2}{2} \int_0^t ds_2 \int_0^{s_2} ds_1 \left\{ \tilde{\mathbb{P}}_1(s_2), \tilde{\sigma}^-(s_1)(\cdot) \tilde{\sigma}^+(s_1) \right\} - \frac{\gamma^2}{2} \int_0^t ds_2 \int_0^{s_2} ds_1 \tilde{\sigma}^-(s_2) \{ \tilde{\mathbb{P}}_1(s_1), (\cdot) \} \tilde{\sigma}^+(s_2) \\ & \left. + \gamma^2 \int_0^t ds_1 \int_0^{s_1} ds_2 \tilde{\sigma}^-(s_1) \tilde{\sigma}^-(s_2)(\cdot) \tilde{\sigma}^+(s_2) \tilde{\sigma}^+(s_1) \right] + O(\gamma^3). \end{aligned} \quad (\text{F4})$$

Hence for the evolution of the state of the atom, in the Schrödinger picture, we have

$$\begin{aligned} \rho_A(t) = \Lambda(t) \rho_A^\alpha(0) = \mathcal{U}_\alpha(t, 0) & \left[\mathbb{I} - \frac{\gamma}{2} \int_0^t ds_1 \{ \tilde{\mathbb{P}}_1(s_1), \rho_A^\alpha(0) \} + \gamma \int_0^t ds_1 \tilde{\sigma}^-(s_1) \rho_A^\alpha(0) \tilde{\sigma}^+(s_1) \right. \\ & + \frac{\gamma^2}{4} \int_0^t ds_2 \int_0^{s_2} ds_1 (\tilde{\mathbb{P}}_1(s_2) \tilde{\mathbb{P}}_1(s_1) \rho_A^\alpha(0) + \rho_A^\alpha(0) \tilde{\mathbb{P}}_1(s_1) \tilde{\mathbb{P}}_1(s_2)) + \frac{\gamma^2}{4} 2 \int_0^t ds_2 \int_0^{s_2} ds_1 \tilde{\mathbb{P}}_1(s_1) \rho_A^\alpha(0) \tilde{\mathbb{P}}_1(s_2) \\ & - \frac{\gamma^2}{2} \int_0^t ds_2 \int_0^{s_2} ds_1 (\tilde{\mathbb{P}}_1(s_2) \tilde{\sigma}^-(s_1) \rho_A^\alpha(0) \tilde{\sigma}^+(s_1) + \tilde{\sigma}^-(s_1) \rho_A^\alpha(0) \tilde{\sigma}^+(s_1) \tilde{\mathbb{P}}_1(s_2)) \\ & - \frac{\gamma^2}{2} \int_0^t ds_2 \int_0^{s_2} ds_1 (\tilde{\sigma}^-(s_2) \tilde{\mathbb{P}}_1(s_1) \rho_A^\alpha(0) \tilde{\sigma}^+(s_2) + \tilde{\sigma}^-(s_2) \rho_A^\alpha(0) \tilde{\mathbb{P}}_1(s_1) \tilde{\sigma}^+(s_2)) \\ & \left. + \gamma^2 \int_0^t ds_2 \int_0^{s_2} ds_1 \tilde{\sigma}^-(s_2) \tilde{\sigma}^-(s_1) \rho_A^\alpha(0) \tilde{\sigma}^+(s_1) \tilde{\sigma}^+(s_2) \right] + O(\gamma^3). \end{aligned} \quad (\text{F5})$$

Then comparing Eqs. (F3) and (F5) we see that they are the same. The only term which is nontrivial is

$$\frac{\gamma^2}{4} 2 \int_0^t ds_2 \int_0^{s_2} ds_1 \tilde{\mathbb{P}}_1(s_1) \rho_A^\alpha(0) \tilde{\mathbb{P}}_1(s_2) \quad (\text{F6})$$

but checking the integration limits we can see that $2 \int_0^t ds_2 \int_0^{s_2} ds_1 \tilde{\mathbb{P}}_1(s_1) \rho_A^\alpha(0) \tilde{\mathbb{P}}_1(s_2)$ is equal to $\int_0^t ds_2 \int_0^t ds_1 \tilde{\mathbb{P}}_1(s_1) \rho_A^\alpha(0) \tilde{\mathbb{P}}_1(s_2)$ as we want. Therefore, up to second order in γ , we see that the evolution is given by the map $\mathcal{T}e^{\tilde{\Lambda}(t)}$ where

$$\tilde{\Lambda}(t) = -\frac{\gamma}{2} \int_0^t ds_1 \{ \tilde{\mathbb{P}}_1(s_1), (\cdot) \} + \gamma \int_0^t ds_1 \tilde{\sigma}^-(s_1)(\cdot) \tilde{\sigma}^+(s_1). \quad (\text{F7})$$

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