Reversibility-unitality-disturbance tradeoff in quantum channels

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In this work, we employ the squared Hilbert-Schmidt norm of the Gram matrix for a quantum channel as a reversibility quantifier of this channel, which is shown to be complementary to the entropy of this channel, and derive a complementary relation between the reversibility of a quantum channel and its complementary channel. For a natural unitality measure of a quantum channel, we show that it is equivalent to the entropy of the corresponding complementary channel. By quantifying the disturbance of a quantum channel as the decrease of correlations in a maximally entangled state locally passing through this channel, we eventually establish a reversibility-unitality-disturbance triality relation. To illustrate and compare these quantities, we further evaluate them for some prototypical channels associated with some special quantum information processing tasks and computations, such as the quantum teleportation channel, DQC1 channel, Mach-Zehnder interferometry channel, dephrasure channel and so on, including both unital and nonunital cases.

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I. INTRODUCTION

Quantum channels are fundamental ingredients and important instruments for quantum information processing tasks. To gain information about a physical system, one has to make a measurement on it, which inevitably causes disturbance to the measured system. The characterization and quantification of information gain, state disturbance, and their tradeoff relations induced by a quantum measurement are important subjects of quantum information theory and have practical significance in quantum information processing tasks, such as quantum cryptography [1–3], where the security of information transmission relies on the delicate balance between information gain and disturbance. Untill now, numerous efforts have been made to quantify the information transfer between system and its surroundings. By exploiting state-channel duality and state-channel interactions, various features of quantum channels have been studied in the literature [4–16]. In terms of these information quantities, a variety of complementary relations have been presented from pairwise tradeoffs [17–34] to triplewise tradeoffs [35–39]. Every well-justified tradeoff relation can provide valuable insights into the fundamental limits and possibilities of manipulating quantum information while considering the associated disturbances. In this work, we establish an information conservation relation by dividing the total information between system and environment into reversible information (quantified by reversibility), recoverable classical information (quantified by unitality), and disturbed information (quantified by reduced correlations).

A quantum process described by a quantum channel $\mathcal{E}$ is reversible if there exists a channel $\mathcal{R}$ such that $\mathcal{R} \circ \mathcal{E}$ is the identity channel $\mathcal{I}$. Otherwise, it is irreversible. Irreversibility arises from the interaction between a system and its environment, leading to dissipation and decoherence. Both dissipation and decoherence limit perfect recovery of the original quantum state, introducing errors and noise in quantum information processing. From the perspective of information theory, reversibility can be understood as the ability of a quantum channel in preserving information. Following this line, some features of reversibility have been studied and applied to investigate information flow in a quantum channel [26–28].

A useful tool for studying quantum channels is the Gram matrix, which is defined by Gram via a pairwise inner product for a set of vectors [40] and extended to a set of operators via various overlaps [41–44]. In view of its simple yet useful structure, the Gram matrix has found significant applications in quantum information theory [45–59]. In this work, we introduce the Gram matrix of quantum channels from the perspective of Jamiołkowski-Choi isomorphism and employ the Hilbert-Schmidt norm of the Gram matrix to quantify the reversibility of quantum channels.

Unitality (nonunitality) quantify the ability of a quantum channel keeping (disturbing) the identity operator. The characterizations and applications of unital or nonunital channels have received lots of attention [60–67]. For example, the nonunital channels have been used to create quantum correlations [62] and nonunitality has been proved to provide a lower bound for the heat exchange in a Landauer erasure process [66], etc. It is well known that the entropy of all quantum states passing through a quantum channel is nondecreasing if and only if it is a unital channel [61,63,67]. Thus

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the nonunitality is closely related to information exchange between system and its surroundings and can be exploited to quantify information flow in a quantum channel. Another important quantity in our triality relation is the disturbance. Disturbance is a broader concept, which has been defined from many angles for different tasks [17–34]. In this work, we characterize the disturbance of a quantum channel as the decrease of total correlations encoded in the maximally entangled state caused by this channel.

The rest of the work is arranged as follows: In Sec. II, we quantify reversibility of quantum channels via the Hilbert-Schmidt norm of the corresponding Gram matrix and explore its connections with entropy of quantum channels. In Sec. III, we investigate the unitality and disturbance of quantum channels, study their fundamental properties, and establish a reversibility-unitality-disturbance triality relation. We evaluated these information quantities for several prototypical channels in Sec. IV and make a comparison with an existing triality relation in Sec. V. Finally, we conclude with a summary in Sec. VI.

II. REVERSIBILITY OF QUANTUM CHANNELS VIA GRAM MATRIX

In this section, we first introduce the Gram matrix of quantum channels and study its basic properties. Then we quantify the reversibility of quantum channels via the Hilbert-Schmidt norm of the corresponding Gram matrix. Finally, we discuss the connections between reversibility and various entropies of quantum channels. For convenience, we consider quantum channels with the same input and output system in this work.

A. Gram matrix of quantum channels

Let $\mathcal{H}$ be a d-dimensional Hilbert space, $L(\mathcal{H})$ be the real Hilbert space of all linear operators over $\mathcal{H}$, and $\mathcal{E}: L(\mathcal{H}) \to L(\mathcal{H})$ be a quantum channel satisfying

$$\mathcal{E}(\rho) = \sum_{j=1}^{n} E_j \rho E_j^\dagger, \tag{1}$$

with $\sum_{j=1}^{n} E_j^\dagger E_j = \mathbf{1}$ and $\mathbf{1}$ being the identity operator on $\mathcal{H}$. The Jamiołkowski-Choi state associated with channel $\mathcal{E}$ is [68,69]

$$J_{\mathcal{E}} = \mathcal{I} \otimes \mathcal{E}(|\Phi^+\rangle\langle\Phi^+|), \tag{2}$$

with $\mathcal{I}$ the identity channel, $|\Phi^+\rangle = 1/\sqrt{d}\sum_i |i\rangle \otimes |i\rangle$ and $\{|i\rangle : i = 1, 2, \ldots, d\}$ an orthonormal basis of $\mathcal{H}$. The set of postmeasurement vectors induced by local channel $\mathcal{E}$ on the state $|\Phi^+\rangle$ is

$$\{|\eta_j\rangle = \mathbf{1} \otimes E_j|\Phi^+\rangle : j = 1, 2, \ldots, n\}. \tag{3}$$

In this case, $J_{\mathcal{E}}$ can be rewritten as

$$J_{\mathcal{E}} = \sum_{j=1}^{n} |\eta_j\rangle\langle\eta_j| = (|\eta_1\rangle, |\eta_2\rangle, \ldots, |\eta_n\rangle)\begin{pmatrix}\langle\eta_1| \\ \langle\eta_2| \\ \vdots \\ \langle\eta_n|\end{pmatrix}. \tag{4}$$

It is obvious that all information of channel $\mathcal{E}$ is encoded in the set of vectors $\{|\eta_j\rangle : j = 1, 2, \ldots, n\}$. In the following, we focus on the postmeasurement vectors and define the Gram matrix of channel $\mathcal{E}$ as the Gram matrix of corresponding set of vectors $\{|\eta_j\rangle : j = 1, 2, \ldots, n\}$, i.e.,

$$G(\mathcal{E}, \{E_j\}) = (g_{jk}), \tag{5}$$

with matrix elements $g_{jk} = \langle\eta_j|\eta_k\rangle = \mathrm{tr}E_j^\dagger E_k/d$. By rewriting $G(\mathcal{E}, \{E_j\})$ as

$$G(\mathcal{E}, \{E_j\}) = \begin{pmatrix}\langle\eta_1| \\ \langle\eta_2| \\ \vdots \\ \langle\eta_n|\end{pmatrix}(|\eta_1\rangle, |\eta_2\rangle, \ldots, |\eta_n\rangle), \tag{6}$$

we obtain that $G(\mathcal{E}, \{E_j\})$ and $J_{\mathcal{E}}$ have the same nonzero eigenvalues.

We remark that the Gram matrix $G(\mathcal{E}, \{E_j\})$ is the same as the Gram matrix of $\mathcal{E}$ defined via quantum Fisher information by taking the quantum state as the maximally mixed state $\mathbf{1}/d$ and $G(\mathcal{E}, \{E_j\})$ satisfies the following nice properties [44]:

(i) $G(\mathcal{E}, \{E_j\})$ is an $n \times n$ positive-semidefinite matrix and $\mathrm{tr}G(\mathcal{E}, \{E_j\}) = 1$, i.e., $G(\mathcal{E}, \{E_j\})$ can be regarded as a quantum state in a fictitious system of dimension n. Moreover, $G(\mathcal{E}, \{E_j\})$ is a diagonal matrix if and only if $\mathrm{tr}E_j^\dagger E_k = 0$, $j \neq k$.

(ii) Let $\{E_j : j = 1, 2, \ldots, n\}$ and $\{E_k' : k = 1, 2, \ldots, m\}$ be two sets of Kraus operators of $\mathcal{E}$. Without loss of generality, suppose $n = m$, then they are connected by a unitary matrix $U = (u_{ij})$ satisfying $E_i = \sum_j u_{ij}^* E_j'$ [70]. In this case,

$$G(\mathcal{E}, \{E_j\}) = UG(\mathcal{E}, \{E_k'\})U^\dagger. \tag{7}$$

(iii) Let $\mathcal{E}_U$ and $\mathcal{E}_V$ be any two unitary channels with U, V unitary operators, $\mathcal{E}$ be a quantum channel with Kraus operators $\{E_j\}$, then

$$G(\mathcal{E}_U \circ \mathcal{E} \circ \mathcal{E}_V, \{UE_jV\}) = G(\mathcal{E}, \{E_j\}). \tag{8}$$

Here $\circ$ denotes the compound operation between maps.

(iv) Let $\mathcal{E}, \mathcal{F} : L(\mathcal{H}) \to L(\mathcal{H})$ be two channels with Kraus operators $\{E_j\}$ and $\{F_k\}$, respectively, $p_j \geqslant 0$, $p_1 + p_2 = 1$, then

$$\begin{aligned} &G(p_1\mathcal{E} + p_2\mathcal{F}, \{\sqrt{p_1}E_j\} \cup \{\sqrt{p_2}F_k\}) \\ &\quad \leqslant (p_1 + \sqrt{p_1p_2})G(\mathcal{E}, \{E_j\}) \oplus (p_2 + \sqrt{p_1p_2})G(\mathcal{F}, \{F_k\}). \end{aligned} \tag{9}$$

(v) Let $\mathcal{E} : L(\mathcal{H}_1) \to L(\mathcal{H}_1)$ be a quantum channel with Kraus operators $\{E_j\}$ and $\mathcal{F} : L(\mathcal{H}_2) \to L(\mathcal{H}_2)$ be a quantum channel with Kraus operators $\{F_k\}$, then

$$G(\mathcal{E} \otimes \mathcal{F}, \{E_j \otimes F_k\}) = G(\mathcal{E}, \{E_j\}) \otimes G(\mathcal{F}, \{F_k\}), \tag{10}$$

with $\{E_j \otimes F_k\}$ the Kraus operators of $\mathcal{E} \otimes \mathcal{F}$. In particular, when $\mathcal{F} = \mathcal{I}$ is the identity channel, we further obtain

$$G(\mathcal{E} \otimes \mathcal{I}, \{E_j \otimes \mathbf{1}\}) = G(\mathcal{E}, \{E_j\}). \tag{11}$$

(vi) Let $\mathcal{E}$ be a quantum channel with Kraus operators $\{E_j\}$ and $\mathcal{E}^\dagger$ be its dual channel satisfying $\mathcal{E}^\dagger(\sigma) = \sum_j E_j^\dagger \sigma E_j$ for any quantum state σ, then

$$G(\mathcal{E}^\dagger, \{E_j^\dagger\}) = G(\mathcal{E}, \{E_j\})^T, \tag{12}$$

with T the transpose operation relative to the representation basis of the Gram matrix.

Since the Gram matrix $G(\mathcal{E}, \{E_j\})$ encodes the structure information of $\mathcal{E}$, the characteristics of this Gram matrix naturally reflect the essential feature of channel $\mathcal{E}$. In the next section, we employ the Hilbert-Schmidt norm of $G(\mathcal{E}, \{E_j\})$ to quantify the reversibility of channel $\mathcal{E}$.

B. Reversibility of quantum channels

Let $\mathcal{E}(\rho) = \sum_{j=1}^{n} E_j \rho E_j^\dagger$ be a quantum channel on Hilbert space $L(\mathcal{H})$. Without loss of generality, we can suppose $n \leqslant d^2$ [70]. By appending zero operators to the list of Kraus operators $\{E_j : j = 1, 2, \ldots, n\}$, we can further suppose $n = d^2$. Let $V = (v_{jk})$ be a $d^2 \times d^2$ unitary matrix and $F_k = \sum_j v_{kj} E_j$, $k = 1, 2, \ldots, d^2$. Then $\sum_k F_k \rho F_k^\dagger = \sum_{jj'} (\sum_k v_{kj} v_{kj'}^*) E_j \rho E_{j'}^\dagger = \sum_j E_j \rho E_j^\dagger = \mathcal{E}(\rho)$ for any quantum state ρ. Thus $\{F_k : k = 1, 2, \ldots, d^2\}$ also constitutes a set of Kraus operators of $\mathcal{E}$. Based on this fact and property (ii) of the Gram matrix, we know that $G(\mathcal{E}, \{E_j\})$ can be written as a diagonal matrix by selecting appropriate Kraus operators $\{E_j\}$ for the given channel $\mathcal{E}$, i.e.,

$$G(\mathcal{E}, \{E_j\}) = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{d^2}) \tag{13}$$

for some probability distribution $\{\lambda_j : j = 1, 2, \ldots, d^2\}$. Thus the essential difference of different quantum channels in the sense of unitary equivalence is the spectral difference of the corresponding Gram matrices.

A channel $\mathcal{E}$ is called reversible if there is a quantum channel $\mathcal{R}$ such that $\mathcal{R} \circ \mathcal{E} = \mathcal{I}$. Among all quantum channels, the unitary channels $\mathcal{E}_U(\rho) = U \rho U^\dagger$ with U a unitary operator are the channels of the maximal reversibility, while the completely depolarizing channel $\mathcal{E}_{\mathrm{CD}}(\rho) = \mathbf{1}/d$ is the channel of the minimal reversibility. In this case, the Gram matrix of $\mathcal{E}_U$ is

$$G(\mathcal{E}_U, \{E_j\}) = \mathrm{diag}(1, 0, \ldots, 0), \tag{14}$$

with $E_1 = U, E_2 = \cdots = E_{d^2} = \mathbf{0}$ (the zero operator) and the Gram matrix of $\mathcal{E}_{\mathrm{CD}}$ is

$$G(\mathcal{E}_{\mathrm{CD}}, \{X_j\}) = \mathrm{diag}\left(\frac{1}{d^2}, \frac{1}{d^2}, \ldots, \frac{1}{d^2}\right), \tag{15}$$

with $\{X_j : j = 1, 2, \ldots, d^2\}$ being an orthonormal basis of $L(\mathcal{H})$. Thus, any function $f(\vec{\lambda})$ of probability vector $\vec{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_{d^2})$ satisfying

$$\begin{aligned} \arg\max_{\vec{\lambda}} f(\vec{\lambda}) &= \vec{e}_j, \quad j = 1, 2, \ldots, d^2, \\ \arg\min_{\vec{\lambda}} f(\vec{\lambda}) &= \left(\frac{1}{d^2}, \frac{1}{d^2}, \ldots, \frac{1}{d^2}\right) \end{aligned} \tag{16}$$

might be used to quantify the reversibility of quantum channels. Here $\vec{e}_j = (0, \ldots, 0, 1, 0, \ldots, 0)$ with the jth element being 1 and other elements being 0, $j = 1, 2, \ldots, d^2$. A kind of natural candidates are the unitarily invariant norms. For simplicity, we employ the Hilbert-Schmidt norm to quantify the reversibility of quantum channels.

For a quantum channel $\mathcal{E}$ with Kraus operators $\{E_j : j = 1, 2, \ldots, n\}$, the reversibility of $\mathcal{E}$ can be defined as

$$R(\mathcal{E}) = \|G(\mathcal{E}, \{E_j\})\|^2 = \frac{1}{d^2} \sum_{jk} |\mathrm{tr}(E_j^\dagger E_k)|^2, \tag{17}$$

where $\|A\|^2 = \mathrm{tr}(A^\dagger A)$ is the squared Hilbert-Schmidt norm of operator A. By the unitary invariance of Hilbert-Schmidt norm and the property (ii) of $G(\mathcal{E}, \{E_j\})$, we know that $R(\mathcal{E})$ is independent of the choice of Kraus operators. It turns out that

$$\frac{1}{d^2} \leqslant R(\mathcal{E}) \leqslant 1, \tag{18}$$

with $R(\mathcal{E}) = 1/d^2$ if and only if $\mathcal{E} = \mathcal{E}_{\mathrm{CD}}$ is the completely depolarizing channel and $R(\mathcal{E}) = 1$ if and only if $\mathcal{E}$ is a unitary channel.

To establish it, let $\{E_j : j = 1, 2, \ldots, d^2\}$ be the Kraus operators of $\mathcal{E}$ such that $G(\mathcal{E}, \{E_j\}) = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{d^2})$ is a diagonal matrix with $\{\lambda_j : j = 1, 2, \ldots, d^2\}$ a probability distribution. Since $R(\mathcal{E})$ is independent of the choice of Kraus operators, we have

$$R(\mathcal{E}) = \|G(\mathcal{E}, \{E_j\})\|^2 = \sum_j \lambda_j^2. \tag{19}$$

By the fact that $\lambda_j \geqslant 0$ and $\sum_j \lambda_j = 1$, we further obtain $1/d^2 \leqslant R(\mathcal{E}) \leqslant 1$ with the maximum achieved by the one-point distributions $\vec{e}_j$, $j = 1, 2, \ldots, d^2$, and the minimum achieved by the uniform distribution $(1/d^2, 1/d^2, \ldots, 1/d^2)$. The desired result follows from the fact that the quantum channel associated with the one-point distribution is a unitary channel and that associated with the uniform distribution is the completely depolarizing channel.

The quantity $R(\mathcal{E})$ distinguishes quantum channels from the unitary channels (the most reversible channels) to the completely depolarizing channel (the least reversible channel). Thus $R(\mathcal{E})$ characterizes the reversibility of quantum channel $\mathcal{E}$. Besides, $R(\mathcal{E})$ has the following desirable properties:

(i) For any quantum channels $\mathcal{E}_1, \mathcal{E}_2 : L(\mathcal{H}) \to L(\mathcal{H})$ and $p_j \geqslant 0, \sum_{j=1}^{2} p_j = 1$,

$$R(p_1\mathcal{E}_1 + p_2\mathcal{E}_2) \leqslant p_1 R(\mathcal{E}_1) + p_2 R(\mathcal{E}_2). \tag{20}$$

(ii) For any unital channel Φ (i.e., $\Phi(\mathbf{1}) = \mathbf{1}$),

$$R(\Phi \circ \mathcal{E}) \leqslant R(\mathcal{E}) \tag{21}$$

with $\circ$ being the compound operation between maps.

(iii) For any unitary channel $\mathcal{E}_V$ on $L(\mathcal{H})$ with V a unitary operator, we have

$$R(\mathcal{E}_V \circ \mathcal{E}) = R(\mathcal{E} \circ \mathcal{E}_V) = R(\mathcal{E}). \tag{22}$$

(iv) Let $\mathcal{E} : L(\mathcal{H}_1) \to L(\mathcal{H}_1)$ be a quantum channel with Kraus operators $\{E_j\}$ and $\mathcal{F} : L(\mathcal{H}_2) \to L(\mathcal{H}_2)$ be a quantum channel with Kraus operators $\{F_k\}$, then

$$R(\mathcal{E} \otimes \mathcal{F}) = R(\mathcal{E}) R(\mathcal{F}). \tag{23}$$

In particular, when $\mathcal{F} = \mathcal{I}$ is the identity channel on $L(\mathcal{H}_2)$, we further obtain

$$R(\mathcal{E} \otimes \mathcal{I}) = R(\mathcal{E}). \tag{24}$$

These properties can be easily verified by relation (32) in the following section and the properties of entropy of quantum channels [38].

Let $\mathcal{E} : L(\mathcal{H}) \to L(\mathcal{H}_b)$ be a quantum channel, $\mathcal{H}_b$ and $\mathcal{H}_c$ be Hilbert spaces of systems b and c with dimensions d_b and

d_c, respectively. In terms of Stinespring dilation, $\mathcal{E}$ can be represented as

$$\mathcal{E}(\rho) = \mathrm{tr}_c(V\rho V^\dagger), \tag{25}$$

with $V : \mathcal{H} \to \mathcal{H}_b \otimes \mathcal{H}_c$ being an isometry operator. Then

$$\tilde{\mathcal{E}}(\rho) = \mathrm{tr}_b(V\rho V^\dagger) \tag{26}$$

is the complementary channel of $\mathcal{E}$. Let $\{E_j\}$ be a set of Kraus operators of $\mathcal{E}$, i.e.,

$$\mathcal{E}(\rho) = \sum_j E_j\rho E_j^\dagger, \tag{27}$$

then $\tilde{\mathcal{E}}$ can be further represented as [71]

$$\tilde{\mathcal{E}}(\rho) = \sum_{jk} \mathrm{tr}\big(E_j\rho E_k^\dagger\big)|e_j\rangle\langle e_k| = \sum_\alpha \tilde{E}_\alpha\rho\tilde{E}_\alpha^\dagger, \tag{28}$$

with $\tilde{E}_\alpha = \sum_j |e_j\rangle\langle h_\alpha|E_j$, $\{|e_j\rangle : j = 1, 2, \dots, d_c\}$ being an orthonormal basis for $\mathcal{H}_c$ and $\{|h_\alpha\rangle : \alpha = 1, 2, \dots, d_b\}$ being an orthonormal basis for $\mathcal{H}_b$. It can be verified directly that the reversibility of $\tilde{\mathcal{E}}$ is

$$R(\tilde{\mathcal{E}}) = \mathrm{tr}\left[\mathcal{E}\left(\frac{\mathbf{1}}{d}\right)\right]^2. \tag{29}$$

In particular, when $\mathcal{E}$ is a unital channel, we have

$$R(\tilde{\mathcal{E}}) = \frac{1}{d}, \tag{30}$$

i.e., the reversibility of the complementary channels associated with all unital channels are the same.

We remark that the quantity $R(\mathcal{E})$ has been introduced in Ref. [72] as a unitarity measure of $\mathcal{E}$ from the angle of the purity of the corresponding Jamiołkowski-Choi state. Different from their work, we derive this quantity from the perspective of the Gram matrix.

In the following, we discuss the relations between reversibility and entropy of quantum channels and further clarify the essential differences between two common definitions of entropy for quantum channels [38,39].

C. Reversibility versus entropy of quantum channels

For a quantum channel $\mathcal{E}$ with Kraus operators $\{E_j\}$, the entropy of $\mathcal{E}$ can be defined as the entropy of the corresponding Jamiołkowski-Choi state $J_\mathcal{E}$ [38], i.e.,

$$S_1(\mathcal{E}) \triangleq S_L(J_\mathcal{E}) = 1 - \frac{1}{d^2}\sum_{jk}|\mathrm{tr}E_j^\dagger E_k|^2, \tag{31}$$

where $S_L(\rho) = 1 - \mathrm{tr}\rho^2$ is the linear entropy of quantum state ρ. It is obvious that

$$S_1(\mathcal{E}) = 1 - R(\mathcal{E}), \tag{32}$$

which implies a complementary relation

$$S_1(\mathcal{E}) + R(\mathcal{E}) = 1. \tag{33}$$

This further shows that $R(\mathcal{E})$ is a reasonable reversibility measure of quantum channel $\mathcal{E}$.

Another commonly used entropy measure for quantum channel $\mathcal{E}$ is defined as the average entropy of all output states [39], i.e.,

$$\begin{aligned} S_2(\mathcal{E}) &\triangleq \int S_L(\mathcal{E}(|\phi\rangle\langle\phi|))d\phi \\ &= 1 - \frac{1}{d(d+1)}\left(\sum_{jk}\left|\mathrm{tr}(E_j^\dagger E_k)\right|^2 + \mathrm{tr}[\mathcal{E}(\mathbf{1})]^2\right). \end{aligned} \tag{34}$$

By Eqs. (17) and (29), we get

$$S_2(\mathcal{E}) = 1 - \frac{d}{d+1}[R(\mathcal{E}) + R(\tilde{\mathcal{E}})], \tag{35}$$

which implies a complementary relation between the entropy and reversibility of a quantum channel $\mathcal{E}$ and its complementary channel $\tilde{\mathcal{E}}$, i.e.,

$$S_2(\mathcal{E}) + \frac{d}{d+1}[R(\mathcal{E}) + R(\tilde{\mathcal{E}})] = 1. \tag{36}$$

Comparing Eqs. (32) and (35), we see that $S_1(\mathcal{E})$ can distinguish $\mathcal{E}$ from its complementary channel $\tilde{\mathcal{E}}$, while $S_2(\mathcal{E})$ is the same for both $\mathcal{E}$ and $\tilde{\mathcal{E}}$, i.e., $S_2(\mathcal{E}) = S_2(\tilde{\mathcal{E}})$.

By the fact $0 \leqslant S_2(\mathcal{E}) \leqslant 1 - 1/d$, we further obtain an interesting complementary relation between the reversibility of quantum channel $\mathcal{E}$ and its complementary channel $\tilde{\mathcal{E}}$

$$\frac{d+1}{d^2} \leqslant R(\mathcal{E}) + R(\tilde{\mathcal{E}}) \leqslant \frac{d+1}{d}. \tag{37}$$

Moreover, the upper bound is achieved by any unitary channel and the lower bound is achieved by the completely depolarizing channel.

III. REVERSIBILITY-UNITALITY-DISTURBANCE TRADEOFF

In this section, we introduce a unitality measure and a disturbance measure for quantum channels, investigate their basic properties and establish a reversibility-unitality-disturbance tradeoff in a quantum channel.

A. Unitality of quantum channels

Let $\mathcal{E} : L(\mathcal{H}) \to L(\mathcal{H})$ be a quantum channel with Kraus operators $\{E_j : j = 1, 2, \dots, n\}$. $\mathcal{E}$ is called unital if it satisfies $\mathcal{E}(\mathbf{1}/d) = \mathbf{1}/d$. Otherwise, it is nonunital. A natural candidate of nonunitality is

$$N(\mathcal{E}) = \left\|\mathcal{E}\left(\frac{\mathbf{1}}{d}\right) - \frac{\mathbf{1}}{d}\right\|^2, \tag{38}$$

which can be expressed as

$$N(\mathcal{E}) = \mathrm{tr}\left[\mathcal{E}\left(\frac{\mathbf{1}}{d}\right)\right]^2 - \frac{1}{d}. \tag{39}$$

It is obvious that $0 \leqslant N(\mathcal{E}) \leqslant 1 - 1/d$ where $N(\mathcal{E}) = 0$ if and only if $\mathcal{E}$ is a unital channel and $N(\mathcal{E}) = 1 - 1/d$ if and only if $\mathcal{E}(\mathbf{1}/d)$ is a pure state. Thus the quantity

$$U(\mathcal{E}) = 1 - \frac{1}{d} - N(\mathcal{E}) = 1 - \mathrm{tr}\left[\mathcal{E}\left(\frac{\mathbf{1}}{d}\right)\right]^2 \tag{40}$$

can be seen as a unitality measure of $\mathcal{E}$.

Combining Eqs. (29), (32), and (40), we further obtain

$$U(\mathcal{E}) = S_1(\tilde{\mathcal{E}}), \tag{41}$$

which shows that the unitality of $\mathcal{E}$ describes the entropy of the corresponding complementary channel.

$U(\mathcal{E})$ has some desirable properties:

(i) $0 \leqslant U(\mathcal{E}) \leqslant 1 - 1/d$. Moreover, $U(\mathcal{E}) = 0$ if and only if $\mathcal{E}$ is a special entanglement-breaking channel satisfying $\mathcal{E}(\rho) = (\mathrm{tr}\rho)|\phi\rangle\langle\phi|$ for some pure state $|\phi\rangle\langle\phi|$, and $U(\mathcal{E})$ achieves the maximal value $1 - 1/d$ if and only if $\mathcal{E}$ is a unital channel.

(ii) $U(\cdot)$ is concave in the sense that

$$U(p_1\mathcal{E}_1 + p_2\mathcal{E}_2) \geqslant p_1 U(\mathcal{E}_1) + p_2 U(\mathcal{E}_2), \tag{42}$$

for $p_j \geqslant 0$, $p_1 + p_2 = 1$, and any quantum channels $\mathcal{E}_j$.

(iii) $U(\cdot)$ is unitarily invariant in the sense that

$$U(\mathcal{E}_V \circ \mathcal{E}) = U(\mathcal{E} \circ \mathcal{E}_V) = U(\mathcal{E}), \tag{43}$$

where $\mathcal{E}_V(\rho) = V\rho V^\dagger$ with V any unitary operator and $\circ$ is the compound operation between maps.

(iv) $U(\cdot)$ is nondecreasing in the sense that

$$U(\mathcal{F} \circ \mathcal{E}) \geqslant U(\mathcal{E}) \tag{44}$$

for any unital quantum channel $\mathcal{F}$.

(v) For any quantum channels $\mathcal{E}^a$, $\mathcal{E}^b$ on systems a and b, respectively, we have

$$1 - U(\mathcal{E}^a \otimes \mathcal{E}^b) = [1 - U(\mathcal{E}^a)][1 - U(\mathcal{E}^b)]. \tag{45}$$

In particular, when $\mathcal{E}^b = \mathcal{I}^b$ is an identity channel, we further obtain

$$1 - U(\mathcal{E}^a \otimes \mathcal{I}^b) = \frac{1 - U(\mathcal{E}^a)}{d_b} \tag{46}$$

with d_b the dimension of system b.

Now we sketch the proof of the above properties.

For item (i), by Eq. (40), we have $0 \leqslant U(\mathcal{E}) \leqslant 1 - 1/d$ and $U(\mathcal{E}) = 0$ if and only if $\mathcal{E}(\mathbf{1}/d)$ is a pure state and $U(\mathcal{E}) = 1 - 1/d$ if and only if $\mathcal{E}(\mathbf{1}/d) = \mathbf{1}/d$, i.e., $\mathcal{E}$ is a unital channel. Thus we only need to show that $\mathcal{E}(\mathbf{1}/d)$ is a pure state if and only if $\mathcal{E}$ is a special entanglement-breaking channel satisfying $\mathcal{E}(\rho) = (\mathrm{tr}\rho)|\phi\rangle\langle\phi|$ for some pure state $|\phi\rangle\langle\phi|$. The necessity is obvious and we remain to prove the sufficiency. Suppose $\mathcal{E}(\mathbf{1}/d) = \sum_j E_j E_j^\dagger/d$ is a pure state. Then $\sum_j E_j E_j^\dagger$ is a rank-one positive operator, which further implies that all $E_j E_j^\dagger$ are proportional to each other and thus are rank one. Therefore, E_j can be written as $|\phi\rangle\langle r_j|$ with $|\phi\rangle$ being a normalized vector and vectors $|r_j\rangle$ satisfying $\sum_j E_j^\dagger E_j = \sum_j |r_j\rangle\langle r_j| = \mathbf{1}$. At this time, for any quantum state ρ,

$$\mathcal{E}(\rho) = \sum_j E_j \rho E_j^\dagger = \sum_j |\phi\rangle\langle r_j|\rho|r_j\rangle\langle\phi| = (\mathrm{tr}\rho)|\phi\rangle\langle\phi|. \tag{47}$$

Items (ii)–(v) can be easily obtained from the basic properties of linear entropy.

B. Disturbance of quantum channels

Let ρ^{ab} be a quantum state on composite system ab, the total correlations of ρ^{ab} is usually quantified by the von Neumann mutual information. In the context of this work, we employ the quantum linear mutual information [73]

$$I(\rho^{ab}) = S_L(\rho^a) + S_L(\rho^b) - S_L(\rho^{ab}), \tag{48}$$

where ρ^a and ρ^b are the reduced states of ρ^{ab} on subsystems a and b, respectively, as a measure of total correlations of ρ^{ab}. By the subadditivity of linear entropy [74], $I(\rho^{ab})$ is always non-negative. Moreover, $I(\rho^{ab})$ achieves its maximum if and only if ρ^{ab} is a maximally entangled state.

When a local channel acts on a maximally entangled state, the correlations may be reduced. Thus the amount of the reduced correlations can be used to quantify the disturbance of information encoded in the correlations caused by this channel. Following the method in Ref. [75], we define the decorrelating capability of channel $\mathcal{E}$ as

$$D(\mathcal{E}) = I(J_{\mathcal{I}}) - I(J_{\mathcal{E}}) \tag{49}$$

by replacing the von Neumann mutual information with linear mutual information. Direct derivation shows that

$$D(\mathcal{E}) = S_L(J_{\mathcal{E}}) + N(\mathcal{E}), \tag{50}$$

which implies that the decorrelating power of $\mathcal{E}$ can be divided into two parts: irreversibility of $\mathcal{E}$ and nonunitality of $\mathcal{E}$.

$D(\mathcal{E})$ has the following properties.

(i) It holds that

$$0 \leqslant D(\mathcal{E}) \leqslant 2\left(1 - \frac{1}{d}\right). \tag{51}$$

Moreover, $D(\mathcal{E})$ achieves the minimum 0 if and only if $\mathcal{E}$ is a unitary channel, and $D(\mathcal{E})$ achieves the maximum $2(1 - 1/d)$ if and only if $\mathcal{E}$ is a special entanglement-breaking channel satisfying $\mathcal{E}(\rho) = (\mathrm{tr}\rho)|\phi\rangle\langle\phi|$ for some pure state $|\phi\rangle\langle\phi|$.

(ii) $D(\cdot)$ is unitarily invariant in the sense that

$$D(\mathcal{E}_V \circ \mathcal{E}) = D(\mathcal{E} \circ \mathcal{E}_V) = D(\mathcal{E}),$$

for any unitary channel $\mathcal{E}_V(\rho) = V\rho V^\dagger$ with V a unitary operator.

We now sketch the proof of the above properties.

For item (i), by Eq. (50), we have $D(\mathcal{E}) \geqslant 0$ and the equality holds if and only if $S_L(J_{\mathcal{E}}) = 0$ and $N(\mathcal{E}) = 0$, which further implies that $\mathcal{E}$ is a unitary channel. For the upper bound, by Eq. (49),

$$D(\mathcal{E}) \leqslant I(J_{\mathcal{I}}) = 2\left(1 - \frac{1}{d}\right) \tag{52}$$

and $D(\mathcal{E}) = 2(1 - 1/d)$ if and only if $I(J_{\mathcal{E}}) = 0$. If $\mathcal{E}$ is a quantum channel satisfying $\mathcal{E}(\rho) = (\mathrm{tr}\rho)|\phi\rangle\langle\phi|$ for some pure state $|\phi\rangle$, then

$$\begin{aligned} J_{\mathcal{E}} &= \mathcal{I} \otimes \mathcal{E}(|\Phi^+\rangle\langle\Phi^+|) = \frac{1}{d}\sum_{ij} |i\rangle\langle j| \otimes \mathcal{E}(|i\rangle\langle j|) \\ &= \frac{1}{d}\sum_{ij} \langle j|i\rangle |i\rangle\langle j| \otimes |\phi\rangle\langle\phi| = \frac{\mathbf{1}}{d} \otimes |\phi\rangle\langle\phi|, \end{aligned} \tag{53}$$

and

$$I(J_{\mathcal{E}}) = I\left(\frac{\mathbf{1}}{d} \otimes |\phi\rangle\langle\phi|\right) = 0. \tag{54}$$

Conversely, suppose $I(J_{\mathcal{E}})=0$, by the following expression of $I(J_{\mathcal{E}})$, i.e.,

$$\begin{aligned}I(J_{\mathcal{E}}) &= S_L\left(\frac{\mathbf{1}}{d}\right)+S_L\left[\mathcal{E}\left(\frac{\mathbf{1}}{d}\right)\right]-S_L(J_{\mathcal{E}})\\ &= \left(1-\frac{1}{d}\right)S_L\left[\mathcal{E}\left(\frac{\mathbf{1}}{d}\right)\right]+\left\|J_{\mathcal{E}}-\frac{\mathbf{1}}{d}\otimes\mathcal{E}\left(\frac{\mathbf{1}}{d}\right)\right\|^2\\ &= \left(1-\frac{1}{d}\right)U(\mathcal{E})+\left\|J_{\mathcal{E}}-\frac{\mathbf{1}}{d}\otimes\mathcal{E}\left(\frac{\mathbf{1}}{d}\right)\right\|^2,\end{aligned} \quad (55)$$

we have $U(\mathcal{E})=0$ and $J_{\mathcal{E}}=\mathbf{1}/d\otimes\mathcal{E}(\mathbf{1}/d)$, which further implies that $\mathcal{E}$ is a quantum channel satisfying $\mathcal{E}(\rho)=(\mathrm{tr}\rho)|\phi\rangle\langle\phi|$ for some pure state $|\phi\rangle\langle\phi|$.

For item (ii), let $\mathcal{E}_U$ and $\mathcal{E}_V$ be any two unitary channels, then $J_{\mathcal{E}_U\circ\mathcal{E}\circ\mathcal{E}_V}$ has the same nonzero eigenvalues with $G(\mathcal{E}_U\circ\mathcal{E}\circ\mathcal{E}_V,\{UE_jV\})$. By property (iii) of the Gram matrix, we further obtain

$$\begin{aligned}&I(J_{\mathcal{E}_U\circ\mathcal{E}\circ\mathcal{E}_V})\\ &= S_L\left(\frac{\mathbf{1}}{d}\right)+S_L\left[\mathcal{E}\left(\frac{\mathbf{1}}{d}\right)\right]-S_L(J_{\mathcal{E}_U\circ\mathcal{E}\circ\mathcal{E}_V})\\ &= S_L\left(\frac{\mathbf{1}}{d}\right)+S_L\left[\mathcal{E}\left(\frac{\mathbf{1}}{d}\right)\right]-S_L(G(\mathcal{E}_U\circ\mathcal{E}\circ\mathcal{E}_V,\{UE_jV\}))\\ &= S_L\left(\frac{\mathbf{1}}{d}\right)+S_L\left[\mathcal{E}\left(\frac{\mathbf{1}}{d}\right)\right]-S_L(G(\mathcal{E},\{E_j\}))\\ &= S_L\left(\frac{\mathbf{1}}{d}\right)+S_L\left[\mathcal{E}\left(\frac{\mathbf{1}}{d}\right)\right]-S_L(J_{\mathcal{E}})\\ &= I(J_{\mathcal{E}}),\end{aligned} \quad (56)$$

from which the desired result follows.

Furthermore, we illustrate the implication of $D(\mathcal{E})$ from the perspective of coherent information.

Let ρ be a quantum state on Hilbert space $\mathcal{H}$, $|\Psi\rangle$ be a purified state of ρ on composite Hilbert space $\mathcal{H}^a\otimes\mathcal{H}$ satisfying $\mathrm{tr}_a|\Psi\rangle\langle\Psi|=\rho$, $\mathcal{E}$ be a quantum channel on $L(\mathcal{H})$, then the coherent information of $\mathcal{E}$ in state ρ via linear entropy can be defined as [76]

$$I_C(\rho,\mathcal{E})=S_L(\mathcal{E}(\rho))-S_L(\mathcal{I}\otimes\mathcal{E}(|\Psi\rangle\langle\Psi|)), \quad (57)$$

which measures the amount of quantum information conveyed in quantum channel $\mathcal{E}$. The coherent information of $\mathcal{E}$ in maximally mixed state $\mathbf{1}/d$ is

$$I_C\left(\frac{\mathbf{1}}{d},\mathcal{E}\right)=S_L\left[\mathcal{E}\left(\frac{\mathbf{1}}{d}\right)\right]-S_L(J_{\mathcal{E}})=R(\mathcal{E})+U(\mathcal{E})-1, \quad (58)$$

which shows that the information conveyed by channel $\mathcal{E}$ can be divided into reversible information (quantified by $R(\mathcal{E})$) and recoverable classical information (quantified by $U(\mathcal{E})$). By Eqs. (50) and (58), we further obtain

$$D(\mathcal{E})=1-\frac{1}{d}-I_C\left(\frac{\mathbf{1}}{d},\mathcal{E}\right), \quad (59)$$

which implies that $D(\mathcal{E})$ essentially characterizes the disturbance of information caused by channel $\mathcal{E}$.

C. Reversibility-unitality-disturbance tradeoff

Until now, we have discussed three information quantities describing different characteristics of quantum channels: reversibility defined by Eq. (17), unitality defined by Eq. (40), and disturbance defined by Eq. (49).

Combining Eqs. (32), (40), and (50), we further obtain the following information conservation relation

$$R(\mathcal{E})+D(\mathcal{E})+U(\mathcal{E})=2-\frac{1}{d}. \quad (60)$$

In particular, when $\mathcal{E}$ is a unital channel, the above tradeoff relation reduces to

$$R(\mathcal{E})+D(\mathcal{E})=1. \quad (61)$$

IV. EXAMPLES

In this section, we evaluate the reversibility, unitality, and disturbance for some prototypical quantum channels. For simplicity, we denote the Gram matrix of a quantum channel $\mathcal{E}$ for the given Kraus operators $\{E_j\}$ as $G(\mathcal{E})$ rather than $G(\mathcal{E},\{E_j\})$.

First, we illustrate the reversibility, unitality, and disturbance of quantum channels for two important nonunital quantum channels and investigate their competitive behaviors.

Example 1. For the amplitude damping channel $\mathcal{E}_{\mathrm{AD}}(\rho)=\sum_{j=1}^2 E_j\rho E_j^\dagger$ with

$$E_1=\begin{pmatrix}1 & 0\\ 0 & \sqrt{1-p}\end{pmatrix},\quad E_2=\begin{pmatrix}0 & \sqrt{p}\\ 0 & 0\end{pmatrix},\quad 0\leqslant p\leqslant 1, \quad (62)$$

the Gram matrix of $\mathcal{E}_{\mathrm{AD}}$ is

$$G(\mathcal{E}_{\mathrm{AD}})=\frac{1}{2}\begin{pmatrix}2-p & 0\\ 0 & p\end{pmatrix}. \quad (63)$$

The reversibility, unitality, and disturbance of $\mathcal{E}_{\mathrm{AD}}$ are

$$\begin{aligned}R(\mathcal{E}_{\mathrm{AD}}) &= 1-p(2-p)/2,\\ U(\mathcal{E}_{\mathrm{AD}}) &= (1-p^2)/2,\\ D(\mathcal{E}_{\mathrm{AD}}) &= p,\end{aligned} \quad (64)$$

respectively. It is obvious that the noise parameter p of amplitude damping channel essentially reflects the disturbance degree of this channel and the reversibility and unitality of $\mathcal{E}_{\mathrm{AD}}$ are both decreasing with the noise parameter $p\in[0,1]$.

Example 2. Consider the measurement-preparation channel in a d-dimensional quantum system:

$$\mathcal{E}_{\mathrm{MP}}(\rho)=\sum_j \mathrm{tr}(\rho M_j)\sigma_j,$$

with σ_j quantum states and $M=\{M_j\}$ a positive operator-valued measure (POVM) satisfying $\sum_j M_j=\mathbf{1}$. It is obvious that

$$\{E_{jkl}=\sqrt{\lambda_{jl}}|\phi_{jl}\rangle\langle k|\sqrt{M_j}:\forall\ j,k,l\} \quad (65)$$

is a set of Kraus operators of $\mathcal{E}_{\mathrm{MP}}$ with $\sigma_j=\sum_l\lambda_{jl}|\phi_{jl}\rangle\langle\phi_{jl}|$ for any j. For these Kraus operators, the Gram matrix of $\mathcal{E}_{\mathrm{MP}}$ is

$$G(\mathcal{E}_{\mathrm{MP}})=\frac{1}{d}(\sqrt{\lambda_{jl}\lambda_{j'l'}}\langle k|\sqrt{M_j}\sqrt{M_{j'}}|k'\rangle\langle\phi_{j'l'}|\phi_{jl}\rangle). \quad (66)$$

The reversibility, unitality, and disturbance of $\mathcal{E}_{\rm MP}$ are

$$R(\mathcal{E}_{\rm MP}) = \frac{1}{d^2}\sum_{jj'}\mathrm{tr}(M_jM_{j'})\mathrm{tr}(\sigma_j\sigma_{j'}),$$

$$U(\mathcal{E}_{\rm MP}) = 1-\frac{1}{d^2}\sum_{jj'}\mathrm{tr}(M_j)\mathrm{tr}(M_{j'})\mathrm{tr}(\sigma_j\sigma_{j'}),$$

$$D(\mathcal{E}_{\rm MP}) = \frac{1}{d^2}\sum_{jj'}[\mathrm{tr}(M_j)\mathrm{tr}(M_{j'})-\mathrm{tr}(M_jM_{j'})]\mathrm{tr}(\sigma_j\sigma_{j'}) + 1-\frac{1}{d}, \tag{67}$$

respectively.

In the following, we consider two special cases. When $M_j = |\phi_j\rangle\langle\phi_j|$ and $\sigma_j = |\psi_j\rangle\langle\psi_j|$, $\mathcal{E}_{\rm MP}$ reduces to the entanglement-breaking channel $\mathcal{E}_{\rm EB}$ with

$$\mathcal{E}_{\rm EB}(\rho) = \sum_j |\psi_j\rangle\langle\phi_j|\rho|\phi_j\rangle\langle\psi_j| \tag{68}$$

for any quantum state ρ. In this case, the reversibility, unitality, and disturbance of $\mathcal{E}_{\rm EB}$ are

$$R(\mathcal{E}_{\rm EB}) = \frac{1}{d^2}\sum_{jj'}|\langle\phi_j|\phi_{j'}\rangle|^2|\langle\psi_j|\psi_{j'}\rangle|^2,$$

$$U(\mathcal{E}_{\rm EB}) = 1-\frac{1}{d^2}\sum_{jj'}|\langle\psi_j|\psi_{j'}\rangle|^2,$$

$$D(\mathcal{E}_{\rm EB}) = \frac{1}{d^2}\sum_{jj'}(1-|\langle\phi_j|\phi_{j'}\rangle|^2)|\langle\psi_j|\psi_{j'}\rangle|^2 + 1 - \frac{1}{d}, \tag{69}$$

respectively.

When $M = \mathcal{I}$ is the identity measurement, then the measurement-preparation channel is reduced to the channel

$$\mathcal{E}_{\rm MP}(\rho) = (\mathrm{tr}\rho)\sigma \tag{70}$$

for a fixed quantum state σ. In this case,

$$R(\mathcal{E}_{\rm MP}) = \frac{1}{d}\mathrm{tr}\sigma^2,\quad U(\mathcal{E}_{\rm MP}) = 1-\mathrm{tr}\sigma^2,$$

$$D(\mathcal{E}_{\rm MP}) = (1-1/d)(1+\mathrm{tr}\sigma^2), \tag{71}$$

from which we get that the reversibility and disturbance of $\mathcal{E}_{\rm MP}$ are both increasing with $\mathrm{tr}\sigma^2$, while the unitality of $\mathcal{E}_{\rm MP}$ is decreasing with $\mathrm{tr}\sigma^2$.

Next, we further consider several unital channels which play a crucial role in quantum information processing and computations. In this case, the reversibility-unitality-disturbance tradeoff is reduced to the reversibility-disturbance tradeoff

$$R(\mathcal{E}) + D(\mathcal{E}) = 1. \tag{72}$$

Thus we only need to evaluate and explain the reversibility for the following unital channels.

Example 3. For a Lüders measurement $\Pi = \{\Pi_j : j = 1, 2, \ldots, n\}$ in a d-dimensional quantum system, the Gram matrix of Π is

$$G(\Pi) = \frac{1}{d}\mathrm{diag}(\mathrm{tr}\Pi_1, \mathrm{tr}\Pi_2, \ldots, \mathrm{tr}\Pi_n), \tag{73}$$

and the reversibility of Π is

$$R(\Pi) = \frac{1}{d^2}\sum_{j=1}^{n}|\mathrm{tr}\Pi_j|^2. \tag{74}$$

It is easy to show that $1/d \leqslant R(\Pi) \leqslant 1$. Moreover $R(\Pi) = 1/d$ if and only if $\Pi = \Pi_{\rm vN}$ is a von Neumann measurement and $R(\Pi) = 1$ if and only if $\Pi = \mathcal{I}$ is the identity channel, which implies that von Neumann measurements $\Pi_{\rm vN}$ are the Lüders measurements with the minimal reversibility and the identity channel $\mathcal{I}$ is the Lüders measurement with the maximal reversibility. Thus $R(\Pi)$ provides a method for characterizing reversibility of all Lüders measurements which fits our intuition.

Example 4. For the random unitary channel $\mathcal{E}_{\rm RU}(\rho) = \sum_j p_jU_j\rho U_j^\dagger$ for any unitary operators U_j in a d-dimensional quantum system and any probability distribution $\{p_j\}$, the Gram matrix of $\mathcal{E}_{\rm RU}$ is

$$G(\mathcal{E}_{\rm RU}) = \frac{1}{d}\big(\sqrt{p_jp_k}\mathrm{tr}U_j^\dagger U_k\big), \tag{75}$$

and the reversibility of $\mathcal{E}_{\rm RU}$ is

$$R(\mathcal{E}_{\rm RU}) = \frac{1}{d^2}\sum_{jk}p_jp_k|\mathrm{tr}U_j^\dagger U_k|^2. \tag{76}$$

It is obvious that $1/d^2 \leqslant R(\mathcal{E}_{\rm RU}) \leqslant 1$ and $R(\mathcal{E}_{\rm RU}) = 1/d^2$ if and only if $\mathcal{E}_{\rm RU}$ is the completely depolarizing channel and $R(\mathcal{E}_{\rm RU}) = 1$ if and only if $\mathcal{E}_{\rm RU}$ is a unitary channel.

As an application, we further consider the quantum teleportation channel which is a special random unitary channel [77]. Suppose Alice and Bob share an entangled state ω^{ab} and Alice aims to transmit a quantum state ρ to Bob in terms of this entanglement resource. Then the teleportation protocol can be described as

$$\mathcal{E}_{\rm TEL}(\rho) = \sum_{j=0}^{3}p_j\sigma_j\rho\sigma_j, \tag{77}$$

where $p_j = \mathrm{tr}\omega^{ab}M_j$ with

$$M_j = (\sigma_j\otimes\mathbf{1})|\Phi^+\rangle\langle\Phi^+|(\sigma_j\otimes\mathbf{1}),$$

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle+|11\rangle). \tag{78}$$

For the Werner state

$$\omega^{ab} = \frac{1-\mu}{4}\mathbf{1}\otimes\mathbf{1} + \mu|\Psi^-\rangle\langle\Psi^-|,\quad -1/3 \leqslant \mu \leqslant 1, \tag{79}$$

with $|\Psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$ the singlet state, we have

$$R(\mathcal{E}_{\rm TEL}) = \frac{1+3\mu^2}{4}. \tag{80}$$

It has been shown that the fidelity of this protocol is larger than 2/3 (the best possible fidelity when Alice and Bob communicate only through a classical channel) if and only if $\mu > 1/3$ [78–80]. In this case, $R(\mathcal{E}_{\rm TEL}) > 1/3$, which might shed some light onto the teleportation protocol from the perspective of reversibility.

Example 5. For the phase-damping channel $\mathcal{E}_{\rm PD}=\sum_{j=1}^{2}E_j\rho E_j^\dagger$ with

$$E_1=\begin{pmatrix}1&0\\0&\sqrt{1-p}\end{pmatrix},\quad E_2=\begin{pmatrix}0&0\\0&\sqrt{p}\end{pmatrix},\quad 0\leqslant p\leqslant 1, \tag{81}$$

the Gram matrix of $\mathcal{E}_{\rm PD}$ is

$$G(\mathcal{E}_{\rm PD})=\frac{1}{2}\begin{pmatrix}2-p&\sqrt{p(1-p)}\\\sqrt{p(1-p)}&p\end{pmatrix}, \tag{82}$$

and the reversibility of $\mathcal{E}_{\rm PD}$ is

$$R(\mathcal{E}_{\rm PD})=1-p/2. \tag{83}$$

Example 6. For any $x\in[0,1/2)$, consider the channel

$$\mathcal{E}_x(\rho)=E_x\rho E_x+E_{1-x}\rho E_{1-x} \tag{84}$$

associated with the weak measurement $\{E_x,E_{1-x}\}$ with $E_x=\sqrt{1-x}\Pi_0+\sqrt{x}\Pi_1$. Here $\{\Pi_0,\Pi_1\}$ is a Lüders measurement in a d-dimensional system. In particular, when $x\to 0$, the weak measurement tends to the Lüders measurement $\{\Pi_0,\Pi_1\}$. The Gram matrix of $\mathcal{E}_x$ is

$$G(\mathcal{E}_x)=\frac{1}{d}\begin{pmatrix}dx+(1-2x){\rm tr}\Pi_0&d\sqrt{x(1-x)}\\d\sqrt{x(1-x)}&d(1-x)+(2x-1){\rm tr}\Pi_0\end{pmatrix} \tag{85}$$

and the reversibility of $\mathcal{E}_x$ is

$$R(\mathcal{E}_x)=1-\frac{2}{d^2}(1-2x)^2{\rm tr}\Pi_0{\rm tr}\Pi_1, \tag{86}$$

which shows that the reversibility of $\mathcal{E}_x$ is increasing with the measurement strength x.

Example 7. Recall that a SIC-POVM (symmetric informationally complete, positive operator-valued measure) is a set of d^2 rank-one operators $E_j=\frac{1}{d}|\phi_j\rangle\langle\phi_j|$, $j=1,2,\ldots,d^2$, satisfying [81]

$$|\langle\phi_j|\phi_k\rangle|^2=\begin{cases}1, & j=k\\ \frac{1}{d+1}, & j\neq k.\end{cases} \tag{87}$$

Any SIC-POVM naturally induces a channel

$$\mathcal{E}_{\rm SIC}(\rho)=\sum_j\sqrt{E_j}\rho\sqrt{E_j}. \tag{88}$$

By straightforward calculations, the Gram matrix of $\mathcal{E}_{\rm SIC}$ is

$$G(\mathcal{E}_{\rm SIC})=(g_{jk}), \tag{89}$$

with

$$g_{jk}=\begin{cases}\frac{1}{d^2}, & j=k\\ \frac{1}{d^2(d+1)}, & j\neq k.\end{cases} \tag{90}$$

As a special entanglement-breaking channel, the reversibility of $\mathcal{E}_{\rm SIC}$ is

$$R(\mathcal{E}_{\rm SIC})=\frac{2}{d(d+1)}. \tag{91}$$

Example 8. The Werner-Holevo channel

$$\mathcal{E}_{\rm WH}(\rho)=\frac{1}{d-1}(\mathbf{1}-\rho^T) \tag{92}$$

provides a counterexample to an additivity conjecture for output purity of channels [82]. Here ρ^T is the transpose of ρ in an orthonormal basis $\{|i\rangle : i=1,2,\ldots,d\}$ of $\mathcal{H}$. It is known that a Kraus representation of $\mathcal{E}_{\rm WH}$ is

$$\mathcal{E}_{\rm WH}(\rho)=\frac{1}{2(d-1)}\sum_{i,j}(|i\rangle\langle j|-|j\rangle\langle i|)\rho(|i\rangle\langle j|-|j\rangle\langle i|)^\dagger. \tag{93}$$

Direct calculations show that the Gram matrix of $\mathcal{E}_{\rm WH}$ is

$$G(\mathcal{E}_{\rm WH})=(g_{ijlm}), \tag{94}$$

with

$$g_{ijlm}=\frac{1}{d(d-1)}(\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl}) \tag{95}$$

and the reversibility of $\mathcal{E}_{\rm WH}$ is

$$R(\mathcal{E}_{\rm WH})=\frac{2}{d(d-1)}. \tag{96}$$

In particular, when $d=2$, the Werner-Holevo channel reduces to the unitary channel

$$\mathcal{E}_{\rm WH}(\rho)=({\rm tr}\rho)\mathbf{1}-\rho^T=\sigma_y\rho\sigma_y, \tag{97}$$

with σ_y the second Pauli matrix. In this case, the reversibility is

$$R(\mathcal{E}_{\rm WH})=1. \tag{98}$$

Example 9. Recall the channel induced by the model of the deterministic quantum computation with one bit (DQC1) [83] $\mathcal{E}_{\rm DQC1}(\rho)=E_1\rho E_1^\dagger+E_2\rho E_2^\dagger$ with Kraus operators

$$E_1=\begin{pmatrix}1&0\\0&u\end{pmatrix},\quad E_2=\begin{pmatrix}0&0\\0&v\end{pmatrix}. \tag{99}$$

Here $u={\rm tr}U/2^n$ and $|u|^2+|v|^2=1$ with U being the unitary operator on the n-qubit ancillary system. The purpose of DQC1 is to estimate the normalized trace of the unitary operator U, i.e., u. At this time, the Gram matrix of $\mathcal{E}_{\rm DQC1}$ is

$$G(\mathcal{E}_{\rm DQC1})=\frac{1}{2}\begin{pmatrix}1+|u|^2&u^*v\\uv^*&|v|^2\end{pmatrix}, \tag{100}$$

and the reversibility of $\mathcal{E}_{\rm DQC1}$ is

$$R(\mathcal{E}_{\rm DQC1})=\frac{1+|u|^2}{2}. \tag{101}$$

Example 10. Consider quantum channel induced by Mach-Zehnder interferometry [84]

$$\mathcal{E}_{\rm MZ}(\rho)={\rm tr}_b(U\rho\otimes\tau U^\dagger)=\sum_{jk}E_{jk}\rho E_{jk}^\dagger, \tag{102}$$

with Kraus operators

$$E_{jk}=\frac{e^{i\alpha}\delta_{jk}\sqrt{\lambda_j}}{2}(\sigma_3+i\sigma_2)+\frac{\sqrt{\lambda_j}\langle\phi_k|V|\phi_j\rangle}{2}(\sigma_3-i\sigma_2), \tag{103}$$

TABLE I. Comparison between reversibility $R(\mathcal{E})$, unitality $U(\mathcal{E})$, disturbance $D(\mathcal{E})$, fidelity $F(\mathcal{E})$, entropy $S_2(\mathcal{E})$, and disturbance $D'(\mathcal{E})$.

Information quantifiers	Mixing of channels	Minimum	Argmin	Maximum	Argmax
$R(\mathcal{E})$	Convex	$1/d^2$	$\mathcal{E}(\rho)=\mathbf{1}/d$	1	$\mathcal{E}(\rho)=U\rho U^\dagger, \forall\, U$
$U(\mathcal{E})$	Concave	0	$\mathcal{E}(\rho)=\|\phi\rangle\langle\phi\|, \forall\, \|\phi\rangle$	$1-1/d$	$\mathcal{E}(\mathbf{1})=\mathbf{1}$
$D(\mathcal{E})$	?	0	$\mathcal{E}(\rho)=U\rho U^\dagger, \forall\, U$	$2(1-1/d)$	$\mathcal{E}(\rho)=\|\phi\rangle\langle\phi\|, \forall\, \|\phi\rangle$
$F(\mathcal{E})$	Affine	$1/(d+1)$	$\mathcal{E}(\rho)=\sum_j E_j\rho E_j^\dagger, \mathrm{tr}E_j=0$	1	$\mathcal{E}(\rho)=\rho$
$D'(\mathcal{E})$	Convex	0	$\mathcal{E}(\rho)=\rho$	$2d/(d+1)$	$\mathcal{E}(\rho)=U\rho U^\dagger, \mathrm{tr}U=0$
$S_2(\mathcal{E})$	Concave	0	$\mathcal{E}(\rho)=U\rho U^\dagger, \forall\, U$	$1-1/d$	$\mathcal{E}(\rho)=\mathbf{1}/d$

$\tau=\sum_j \lambda_j|\phi_j\rangle\langle\phi_j|$ being the spectral decomposition of τ and δ_{jk} being the Kronecker delta function. Here,

$$U=U_{\mathrm{B}}^{ab}U_{\mathrm{M}}^{ab}V^{ab}U_{\mathrm{B}}^{ab},\quad U_{\mathrm{B}}^{ab}=U_B\otimes\mathbf{1}^b,$$
$$U_{\mathrm{M}}^{ab}=U_M\otimes\mathbf{1}^b,\quad V^{ab}=e^{i\alpha}|0\rangle\langle0|\otimes\mathbf{1}^b+|1\rangle\langle1|\otimes V, \tag{104}$$

with beam-splitter, mirror, and phase-shift unitary matrices being

$$U_{\mathrm{B}}=\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix},\quad U_{\mathrm{M}}=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad U_\alpha=\begin{pmatrix}e^{i\alpha}&0\\0&1\end{pmatrix}, \tag{105}$$

respectively. The Gram matrix of $\mathcal{E}_{\mathrm{MZ}}$ is

$$G(\mathcal{E}_{\mathrm{MZ}})=(g_{jklm}), \tag{106}$$

with

$$g_{jklm}=\frac{\sqrt{\lambda_j\lambda_l}}{2}(\delta_{jk}\delta_{lm}+\langle\phi_j|V^\dagger|\phi_k\rangle\langle\phi_m|V|\phi_l\rangle). \tag{107}$$

The reversibility of $\mathcal{E}_{\mathrm{MZ}}$ is

$$R(\mathcal{E}_{\mathrm{MZ}})=\frac{1+\mathcal{V}^2}{2}, \tag{108}$$

where $\mathcal{V}=|\mathrm{tr}\tau V|$ is a fringe visibility quantifier introduced by Englert [85]. It shows that the reversibility of the quantum channel induced by Mach-Zehnder interferometry reflects the wave feature of quantum system.

V. COMPARISONS

In this section, we compare the reversibility-unitality-disturbance triality relation established in this work with the fidelity-disturbance-entropy triality relation established in Ref. [39] qualitatively and quantitatively.

In Ref. [39], a fidelity-disturbance-entropy tradeoff relation is established as

$$2F(\mathcal{E})+D'(\mathcal{E})+S_2(\mathcal{E})=2, \tag{109}$$

with the fidelity being quantified as

$$F(\mathcal{E})=\int F(|\phi\rangle\langle\phi|,\mathcal{E}(|\phi\rangle\langle\phi|)d\phi=\frac{d+\sum_j|\mathrm{tr}E_j|^2}{d(d+1)}, \tag{110}$$

where $F(\rho,\sigma)=(\mathrm{tr}\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}})^2$ is the fidelity between quantum states ρ and σ, and $d\phi$ is the normalized Haar measure over all pure states, the disturbance being quantified as

$$D'(\mathcal{E})=\int\|\mathcal{E}(|\phi\rangle\langle\phi|)-|\phi\rangle\langle\phi|\|^2d\phi$$
$$=\frac{d^2+\|\mathcal{E}(\mathbf{1})-\mathbf{1}\|^2+\sum_{jk}\left|\mathrm{tr}\left(E_j^\dagger E_k\right)\right|^2-2\sum_j|\mathrm{tr}E_j|^2}{d(d+1)}, \tag{111}$$

and the entropy being quantified as $S_2(\mathcal{E})$.

To compare the triality relations (60) and (109) qualitatively, we summarize the basic features of related information quantities in Table I. From Table I, we can find the following facts:

(a) $R(\mathcal{E})$ and $F(\mathcal{E})$ characterize different aspects of $\mathcal{E}$ in conveying information. $R(\mathcal{E})$ measures the capability of $\mathcal{E}$ in preserving information, while $F(\mathcal{E})$ measures the capability of $\mathcal{E}$ in preserving quantum states.

(b) $D(\mathcal{E})$ and $D'(\mathcal{E})$ characterize different aspects of $\mathcal{E}$ in disturbing quantum system. $D(\mathcal{E})$ describes the deviation of $\mathcal{E}$ from all unitary channels, i.e., measures the disturbance of information, while $D'(\mathcal{E})$ describes the deviation of $\mathcal{E}$ from the identity channel, i.e., measures the disturbance of quantum states.

(c) $U(\mathcal{E})$ and $S_2(\mathcal{E})$ are complementary information quantities in the sense that $U(\mathcal{E})$ measures the recoverable classical information caused by $\mathcal{E}$, while $S_2(\mathcal{E})$ measures the information leakage to the environment caused by $\mathcal{E}$.

To illustrate the difference between the reversibility-unitality-disturbance triality and fidelity-disturbance-entropy triality quantitatively, we evaluate related information quantities for the dephrasure channel [86]

$$\mathcal{E}_{\mathrm{DEP}}(\rho)=(1-q)[(1-p)\rho+p\sigma_3\rho\sigma_3]+q(\mathrm{tr}\rho)|\phi\rangle\langle\phi|, \tag{112}$$

with $0\leqslant p,q\leqslant1$, $|\phi\rangle$ some pure state, and σ_3 the third Pauli operator. It is obvious that dephrasure channel is the mixture of the dephasing channel and the erasure channel. By direct calculations, we have

$$R(\mathcal{E}_{\mathrm{DEP}})=(1-q)^2[p^2+(1-p)^2]+\tfrac{1}{2}q,$$
$$U(\mathcal{E}_{\mathrm{DEP}})=\tfrac{1}{2}(1-q^2),$$
$$D(\mathcal{E}_{\mathrm{DEP}})=\tfrac{3}{2}q-\tfrac{1}{2}q^2+2p(1-p)(1-q)^2, \tag{113}$$

and

$$F(\mathcal{E}_{\mathrm{DEP}})=1-\tfrac{1}{2}q-\tfrac{2}{3}p(1-q),$$
$$D'(\mathcal{E}_{\mathrm{DEP}})=q^2+\tfrac{4}{3}p(1-q)(p+q-pq),$$
$$S_2(\mathcal{E}_{\mathrm{DEP}})=q-q^2+\tfrac{4}{3}p(1-p)(1-q)^2. \tag{114}$$

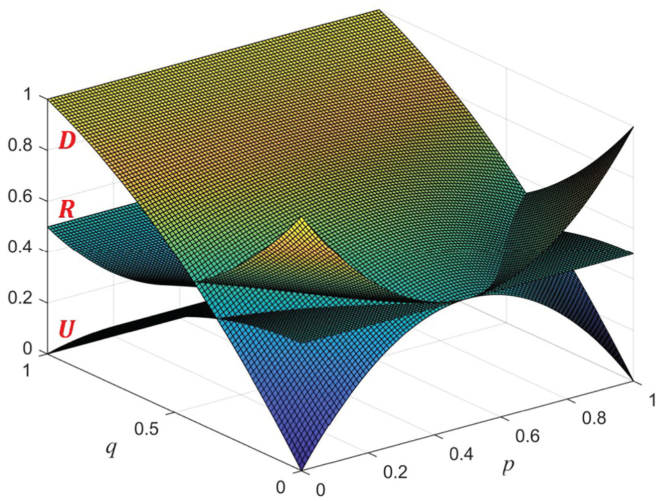

FIG. 1. The behaviors of the reversibility R, unitality U, and disturbance D for the dephrasure channel with parameters p, q.

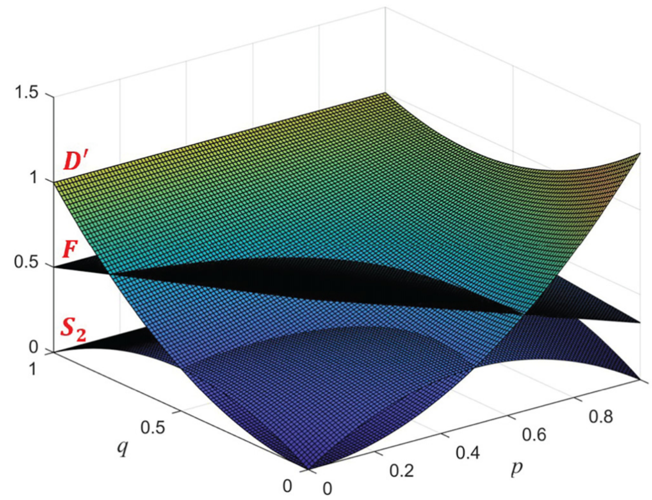

FIG. 2. The behaviors of the fidelity F, disturbance D', and entropy S_2 for the dephrasure channel with parameters p, q.

The behaviors of reversibility $R(\mathcal{E}_{\rm DEP})$, unitality $U(\mathcal{E}_{\rm DEP})$, disturbance $D(\mathcal{E}_{\rm DEP})$, and fidelity $F(\mathcal{E}_{\rm DEP})$, disturbance $D'(\mathcal{E}_{\rm DEP})$, entropy $S_2(\mathcal{E}_{\rm DEP})$ with parameters p, q are depicted in Figs. 1 and 2, respectively.

By simple analysis, we can obtain the following results:

(a) For the parameter p, $R(\mathcal{E}_{\rm DEP})$ is decreasing with $p \in [0, 1/2]$ and increasing with $p \in [1/2, 1]$ for any $q \in [0, 1]$, while $F(\mathcal{E}_{\rm DEP})$ is decreasing with $p \in [0, 1]$ for any $q \in [0, 1]$. For the parameter q, $R(\mathcal{E}_{\rm DEP})$ is decreasing with $q \in [0, 1 - 1/\{4[1 - 2p(1-p)]\}]$ and increasing with $p \in [1 - 1/\{4[1 - 2p(1-p)]\}, 1]$ for any $p \in [0, 1]$, while when $p \in [0, 3/4]$, $F(\mathcal{E}_{\rm DEP})$ is decreasing with $q \in [0, 1]$, otherwise $F(\mathcal{E}_{\rm DEP})$ is increasing with $q \in [0, 1]$.

(b) For the parameter p, $D(\mathcal{E}_{\rm DEP})$ is increasing with $p \in [0, 1/2]$ and decreasing with $p \in [1/2, 1]$ for any $q \in [0, 1]$, while $D'(\mathcal{E}_{\rm DEP})$ is increasing with $p \in [0, 1]$ for any $q \in [0, 1]$. For the parameter q, $D(\mathcal{E}_{\rm DEP})$ is increasing with $q \in [0, 1]$ for any $p \in [0, 1]$, while when $p \in [0, 1/2]$, $D'(\mathcal{E}_{\rm DEP})$ is increasing with $q \in [0, 1]$, otherwise $D'(\mathcal{E}_{\rm DEP})$ is decreasing with $q \in [0, (4p^2 - 2p)/(4p^2 - 4p + 3)]$ and increasing with $q \in [(4p^2 - 2p)/(4p^2 - 4p + 3), 1]$.

VI. SUMMARY

In this work, we have been devoted to studying and establishing an information conservation relation in quantum channels involving the reversibility, unitality, and disturbance of a quantum channel. The reversibility of a quantum channel is defined as the squared Hilbert-Schmidt norm of corresponding Gram matrix, which is shown to be complementary to the entropy of this channel via Jamiołkowski-Choi isomorphism. Based on this quantity, we have further derived a complementary relation between a quantum channel and its complementary channel. The unitality of a quantum channel is quantified as the entropy of its complementary channel with the unital channels being the channels of maximal unitality and a special kind of entanglement-breaking channels being the channels of minimal unitality. Moreover, the disturbance of a quantum channel is described as the reduce of correlations in maximally entangled state and is complementary to the coherent information of this channel in maximally mixed state. By detailed observations on these three information quantities, we eventually establish a reversibility-unitality-disturbance tradeoff, which provides a decomposition of information: reversible information, recoverable classical information, and disturbed information. Furthermore, we also evaluated these quantities for some prototypical channels and compared the reversibility-unitality-disturbance triality relation established in this work with the fidelity-disturbance-entropy triality relation established in Ref. [39]. We hope these results may shed light on complementary relations and have further applications in quantum information processing.

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