# Equioverlapping measurements as extensions of symmetric informationally complete positive operator valued measures

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(Received 24 September 2023; accepted 2 January 2024; published 24 January 2024)

Equioverlapping measurements, as a natural class of quantum measurements with the appealing property of equal overlap between any pair of measurement operators, generalize both the notions of equiangular tight frames (including von Neumann measurements) and symmetric informationally complete positive operator valued measures (SIC-POVMs). The structures of equioverlapping measurements in general dimensions are rather subtle and complicated. In this work, we reveal some structural properties of equioverlapping measurements which may be useful for constructing and classifying equioverlapping measurements. In particular, we obtain two bounds for the number of measurement operators in an equioverlapping measurement. We further illustrate how equioverlapping measurements go beyond SIC-POVMs in a nontrivial fashion with some illuminating examples in lower dimensions (two, three, and four). Finally, we present the challenging problem of fully classifying equioverlapping measurements and discuss some related perspectives.

DOI: 10.1103/PhysRevA.109.012218

## I. INTRODUCTION

In quantum mechanics, traditionally, a quantum measurement in a system Hilbert space often refers to a von Neumann measurement, which is determined by an orthonormal basis (or the corresponding set of projection operators) of the system Hilbert space [1–3]. With modern development, now a general measurement is often described by a positive operator valued measure (POVM) $E = \{E_\alpha : \alpha = 1, 2, \dots\}$, which consists of non-negative operators (called measurement operators) $E_\alpha$ in a system Hilbert space summing to the identity operator $\mathbf{1}$, i.e., $E_\alpha \geqslant 0$ and $\sum_\alpha E_\alpha = \mathbf{1}$ (resolution of identity) [4–10]. If the POVM $E$ is performed in a system state $\rho$, then the probability of obtaining the outcome labeled by $\alpha$ is $p_\alpha = \mathrm{tr}(E_\alpha \rho)$, as postulated by Born's probability rule. In this work, we will study a special class of quantum measurements, so-called equioverlapping measurements (see Definition 4 in Sec. II), investigate the structures of such measurements, and illuminate how these measurements relate to other measurements, particularly the celebrated symmetric informationally complete positive operator valued measures (SIC-POVMs; see Definition 1 in Sec. II).

Despite the simplicity in its formal mathematical definition, the notion of POVMs is rather versatile and has played a crucial role in studies of quantum foundations and applications, and many features of POVMs with certain special structures remain unexplored or unknown [11,12]. For instance, a highly symmetric class of POVMs, so-called SIC-POVMs, has attracted considerable interest in quantum information theory [11–23]. However, its existence in every dimension, although widely believed and rather convincingly supported by much analytical and numerical evidence, remains an elusive and outstanding conjecture (Zauner's conjecture) [11,12,22,23]. Quite recently, semi-SIC-POVMs (see Definition 1 in Sec. II with item 3 dropped) were introduced to relax the equal-trace condition in SIC-POVMs, and some remarkable new phenomena appeared [24].

Ever since the 1970s, equiangular lines and equiangular tight frames (see Definition 2 in Sec. II) have been extensively and intensively studied in the fields of signal processing, frame theory, and combinatorial geometry [25–34]. Equiangular tight frames can be equivalently formulated in the language of POVMs and are essentially equivalent to equiangular measurements (see Definition 3 in Sec. II). We emphasize that the notions of equioverlapping and equiangularity, although closely related, are quite different. This will be elaborated in Sec. II.

For SIC-POVMs and equiangular lines (as well as equiangular tight frames), a key feature is the requirement of equal angles: the angles between all pairs of constituent elements are equal. However, in quantum mechanics, the *overlaps* between operators, as evidenced in Born's probability rule $p_\alpha = \mathrm{tr}(E_\alpha \rho)$ and in the correlation $c_{\alpha\beta}(\rho) = \mathrm{tr}(E_\alpha \rho E_\beta) = \mathrm{tr}(E_\alpha \sqrt{\rho})(E_\beta \sqrt{\rho})^\dagger$, appear frequently and play an important role in quantum measurements. Indeed, the most basic structure of a Hilbert space lies in the inner product (scalar product), which is just the overlap (rather than angle) between vectors. Consequently, from both theoretical and practical perspectives, it is desirable to study overlaps between

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measurement operators, which encode important information about quantum measurements. This leads to the notion of equioverlapping measurements [35], which is closely related to equiangular tight frames and SIC-POVMs, but with fundamental and subtle distinctions. In fact, the former simultaneously generalizes the latter two notions. Some basic properties and classification of equioverlapping measurements in a qubit system were addressed in Ref. [35]. However, general properties and the classification of equioverlapping measurements remain largely unexplored. Our purpose here is to reveal some structural features of equioverlapping measurements, illuminate how they go beyond SIC-POVMs in a nontrivial fashion, and pose the very challenging problem of fully classifying equioverlapping measurements.

The remainder of this work is arranged as follows. In Sec. II, we recall basic aspects of SIC-POVMs, equiangular tight frames, equiangular measurements, and equioverlapping measurements, which are closely related yet subtly different. In Sec. III, we prove several structural properties of equioverlapping measurements, which may be useful in classifying and constructing equioverlapping measurements. We establish two upper bounds for the number of measurement operators in an equioverlapping measurement. In Sec. IV, we present some nontrivial examples of equioverlapping measurements in lower dimensions, which are neither von Neumann measurements nor SIC-POVMs. Finally, we summarize the results and discuss some perspectives in Sec. V. In the Appendixes, we present detailed proofs of the main results, as summarized in Propositions 1–6.

## II. FROM SIC-POVMS TO EQUIOVERLAPPING MEASUREMENTS

In this section, we review, in a rigorous way, the detailed definitions of several important classes of measurements and make a careful comparison between them: SIC-POVMs, equiangular measurements, and equioverlapping measurements. We also discuss the equivalence between equiangular measurements and equiangular tight frames, which include von Neumann measurements as special instances.

Recall that in the modern formalism, a quantum measurement in a quantum system described by a $d$-dimensional Hilbert space $\mathbb{C}^d$ is mathematically represented by a POVM

$$E = \{E_\alpha : \alpha = 1, 2, \ldots, m\}$$

consisting of a family of distinct non-negative operators $E_\alpha$ summing to the identity, i.e., $E_\alpha \geqslant 0$ and

$$\sum_{\alpha=1}^{m} E_\alpha = \mathbf{1}_d \text{ (identity operator on } \mathbb{C}^d).$$

A key point for POVMs is that the number $m$ of measurement operators $E_\alpha$ may be any natural number.

POVMs generalize the traditional quantum measurements (von Neumann measurements, Lüders measurements) which are mathematically described by spectral decompositions of observables [1–3,36]. Notice that compared with a von Neumann measurement $\Pi = \{\Pi_\alpha = |\psi_\alpha\rangle\langle\psi_\alpha| : \alpha = 1, 2, \ldots, d\}$, the number $m$ of measurement operators in a POVM is not necessarily $d$ (it may even be 1 or infinity), and furthermore, the measurement operators may not be projective or rank one and are not orthogonal in general [4–9]. In this work, we will assume that both the dimension $d$ and the number $m$ of measurement operators are finite. To exclude the trivial case, we also assume that $d \geqslant 2$.

A symmetric informationally complete POVM is a special type of POVM notable for its extraordinary properties. In this context, "symmetric" means that the measurement operators are both equiangular and equidistant in the Hilbert space, and "informationally complete" means that any state can be expanded along the measurement operators. The precise meaning is as follows [12].

*Definition 1. SIC-POVM.* A symmetric informationally complete POVM in $\mathbb{C}^d$ is a POVM $E = \{E_\alpha : \alpha = 1, 2, \ldots, d^2\}$ such that the following are true:

(1) *Informational completeness.* The measurement operators $E_\alpha$ span the whole operator (matrix) space acting on $\mathbb{C}^d$ and thus in particular also span the state (pure or mixed) space in $\mathbb{C}^d$.

(2) *Equal overlap.* $\mathrm{tr}(E_\alpha E_\beta) = b$ is a constant independent of $\alpha \neq \beta$.

(3) *Equal trace.* $\mathrm{tr}E_\alpha = t$ is a constant independent of $\alpha$.

(4) *Rank one.* All measurement operators $E_\alpha$ are rank one in the sense that $E_\alpha = t|\psi_\alpha\rangle\langle\psi_\alpha|$ for some pure states (unit norm vectors) $|\psi_\alpha\rangle$ in $\mathbb{C}^d$ and a common constant $t$.

It turns out that the parameters $b$ and $t$ in a SIC-POVM are uniquely determined by the system dimension $d$ as [12]

$$b = \frac{1}{d^2(d+1)}, \quad t = \frac{1}{d}.$$

Moreover, the angles between any two different measurement operators in a SIC-POVM are equal (i.e., equiangularity), that is,

$$|\langle\psi_\alpha|\psi_\beta\rangle|^2 = \frac{\mathrm{tr}(E_\alpha E_\beta)}{\mathrm{tr}E_\alpha \mathrm{tr}E_\beta} = \frac{1}{d+1}, \quad \alpha \neq \beta.$$

For a concrete example, it can be readily checked that $E = \{E_\alpha = \frac{1}{3}|\psi_\alpha\rangle\langle\psi_\alpha| : \alpha = 1, 2, \ldots, 9\}$ is a SIC-POVM in $\mathbb{C}^3$, where

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle),$$
$$|\psi_2\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle),$$
$$|\psi_3\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |2\rangle),$$
$$|\psi_4\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{2\pi i/3}|1\rangle),$$
$$|\psi_5\rangle = \frac{1}{\sqrt{2}}(|1\rangle + e^{2\pi i/3}|2\rangle),$$
$$|\psi_6\rangle = \frac{1}{\sqrt{2}}(|2\rangle + e^{2\pi i/3}|0\rangle),$$
$$|\psi_7\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{-2\pi i/3}|1\rangle),$$
$$|\psi_8\rangle = \frac{1}{\sqrt{2}}(|1\rangle + e^{-2\pi i/3}|2\rangle),$$
$$|\psi_9\rangle = \frac{1}{\sqrt{2}}(|2\rangle + e^{-2\pi i/3}|0\rangle).$$

TABLE I. Comparison between the concepts of equioverlapping measurement, equiangular measurement, and SIC-POVM in a $d$-dimensional system.

| | POVM $E = \{E_\alpha : \alpha = 1, 2, \dots, m\}$ | | |
|---|---|---|---|
| Requirements | Equioverlapping measurement | Equiangular measurement | SIC-POVM |
| Number of measurement operators $m$ | $d \leqslant m \leqslant d^2$ | $d \leqslant m \leqslant d^2$ | $m = d^2$ |
| Equal trace: $\mathrm{tr}E_\alpha = \mathrm{tr}E_\beta, \ \ \forall\, \alpha, \beta$ | No | Yes | Yes |
| Equal overlap: $\mathrm{tr}E_\alpha E_\beta = b, \ \ \forall\, \alpha \neq \beta$ | Yes | Yes | Yes |
| Equal angle: $\frac{\mathrm{tr}E_\alpha E_\beta}{\mathrm{tr}E_\alpha \mathrm{tr}E_\beta} = c, \ \ \forall\, \alpha \neq \beta$ | No | Yes | Yes |

Due to their structural symmetry and informational completeness, SIC-POVMs have many significant applications in quantum information theory, although the construction of a SIC-POVM in a general dimension remains an outstanding open problem [22,23].

If the condition of equal trace (i.e., item 3 in Definition 1) is dropped, then we come to the notion of semi-SIC-POVMs [24].

Equiangularity in a SIC-POVM is closely linked to the concept of equiangular tight frames.

*Definition 2. Equiangular tight frame.* A family of vectors $\{c_\alpha|\psi_\alpha\rangle : \alpha = 1, 2, \dots, m\}$ in $\mathbb{C}^d$ (with $|\psi_\alpha\rangle$ being unit norm vectors and $c_\alpha \neq 0$ being constants) is called an equiangular tight frame if the following are true:

(1) *Equiangularity.* $|\langle\psi_\alpha|\psi_\beta\rangle|^2 = c$ is a constant independent of $\alpha \neq \beta$.

(2) *Tightness.* $\sum_{\alpha=1}^m |c_\alpha|^2 |\psi_\alpha\rangle\langle\psi_\alpha| = \gamma \mathbf{1}_d$ for some positive constant $\gamma$.

Since condition (2) above can be rewritten as

$$\sum_{\alpha=1}^{m} \frac{|c_\alpha|^2}{\gamma} |\psi_\alpha\rangle\langle\psi_\alpha| = \mathbf{1}_d,$$

up to a multiplicative constant, equiangular tight frames are equivalent to equiangular measurements defined as follows.

*Definition 3. Equiangular measurement.* A POVM $E = \{E_\alpha : \alpha = 1, 2, \dots, m\}$ in $\mathbb{C}^d$ is called an equiangular measurement if the following are true:

(1) *Equiangularity.* $\mathrm{tr}(E_\alpha E_\beta) = c\,\mathrm{tr}(E_\alpha)\mathrm{tr}(E_\beta)$, with $c$ being a constant independent of $\alpha \neq \beta$.

(2) *Rank one.* All measurement operators $E_\alpha$ are rank one in the sense that $E_\alpha = t_\alpha|\psi_\alpha\rangle\langle\psi_\alpha|$ for some pure states $|\psi_\alpha\rangle$ and positive constants $t_\alpha$.

As a simple example, $E = \{E_\alpha = \frac{2}{3}|\psi_\alpha\rangle\langle\psi_\alpha| : \alpha = 1, 2, 3\}$ is an equiangular measurement (but not a SIC-POVM) in $\mathbb{C}^2$ for any $\theta \in [0, 2\pi)$, where

$$|\psi_1\rangle = |0\rangle,$$
$$|\psi_2\rangle = \tfrac{1}{2}(|0\rangle - e^{i\theta}\sqrt{3}|1\rangle),$$
$$|\psi_3\rangle = \tfrac{1}{2}(|0\rangle + e^{i\theta}\sqrt{3}|1\rangle).$$

In an equiangular measurement, the trace of any measurement operator $\mathrm{tr}E_\alpha = (1 - cd)/(1 - c)$ is a constant independent of $\alpha$. Here $c = \mathrm{tr}(E_\alpha E_\beta)/(\mathrm{tr}E_\alpha \mathrm{tr}E_\beta) = (m - d)/d(m - 1)$ is a constant independent of $\alpha \neq \beta$. This value reaches the Welch bound [37], which is a lower bound on the maximal overlap (cross correlation) between vectors in a set first given by Welch and subsequently studied by many authors [38–40]. Furthermore, due to the equiangular property, $\mathrm{tr}(E_\alpha E_\beta)$ is also a constant independent of $\alpha \neq \beta$.

It is known that the number of elements, denoted as $m$, in any equiangular tight frame or equiangular measurement in $\mathbb{C}^d$ is constrained within the range $d \leqslant m \leqslant d^2$ [25]. However, it should be noted that an equiangular tight frame (equiangular measurement) may not exist for certain values of $m$ between $d$ and $d^2$. For instance, there is no equiangular tight frame consisting of five vectors in $\mathbb{C}^3$ [29]. When $m = d^2$, an equiangular measurement reduces to a SIC-POVM, whose existence in a general dimension still stands as an elusive and outstanding conjecture known as Zauner's conjecture [11,12,22,23], despite the widespread belief in and substantial supporting evidence for the truth of this conjecture.

Equiangularity of an equiangular measurement (including SIC-POVM) implies that the measurement has equal overlap and equal trace [35]. If we drop the equal-trace condition and retain only the equal-overlap condition, we arrive at the concept of "equioverlapping measurements", which opens the door to a broad class of measurements with intricate structures [35].

*Definition 4. Equioverlapping measurement.* A POVM $E = \{E_\alpha : \alpha = 1, 2, \dots, m\}$ in $\mathbb{C}^d$ is called an equioverlapping measurement if the following are true:

(1) *Equal overlap.* $\mathrm{tr}(E_\alpha E_\beta) = b$ is a constant independent of $\alpha \neq \beta$.

(2) *Rank one.* All measurement operators $E_\alpha$ are rank one in the sense that $E_\alpha = t_\alpha|\psi_\alpha\rangle\langle\psi_\alpha|$ for some pure states $|\psi_\alpha\rangle$ and positive constants $t_\alpha$.

Some basic features and classification of equioverlapping measurements in dimension 2 are discussed in Ref. [35]. More examples of equioverlapping measurements will be given in Sec. IV.

For the convenience of comparison, we list the basic requirements of various measurements in Table I. These measurements constitute a hierarchical structure with the following strict inclusion relations: {SIC-POVM} $\subset$ {equiangular measurement} $\subset$ {equioverlapping measurement} $\subset$ {POVM}. This is further illustrated in Fig. 1.

## III. GENERAL PROPERTIES OF EQUIOVERLAPPING MEASUREMENTS

In this section, we study the general properties of equioverlapping measurements. For latter convenience, we first recall some preliminary results from Ref. [35], which are summarized as the following two lemmas.

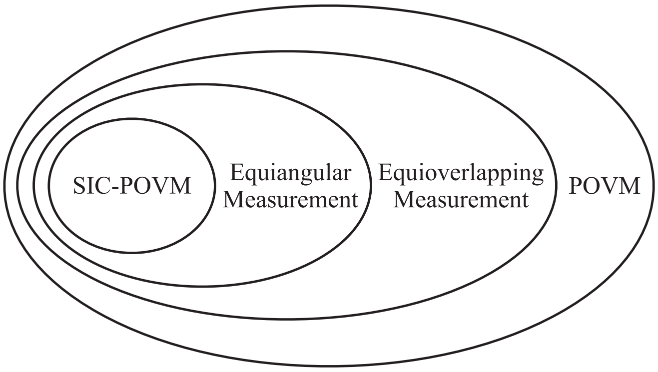


FIG. 1. Hierarchy of measurements: any SIC-POVM is an equiangular measurement, which in turn is an equioverlapping measurement, which in turn is a POVM. The converse is not true in general.

*Lemma 1.* Let $\{E_\alpha : \alpha = 1, 2, \dots, m\}$ be an equioverlapping measurement in $\mathbb{C}^d$ such that

$$\mathrm{tr}E_\alpha = t_\alpha, \quad \alpha = 1, 2, \dots, m,$$
$$\mathrm{tr}(E_\alpha E_\beta) = b, \quad \alpha \neq \beta.$$

Then we have the following statements.

(1) For any $\alpha$, $\mathrm{tr}E_\alpha$ can take at most two possible values. More precisely, $t_\alpha \in \{t_-, t_+\}$ with

$$t_- = \tfrac{1}{2}[1 - \sqrt{1 - 4(m-1)b}], \tag{1}$$

$$t_+ = \tfrac{1}{2}[1 + \sqrt{1 - 4(m-1)b}]. \tag{2}$$

(2) Let $k = \#\{\alpha : \mathrm{tr}E_\alpha = t_-\}$ be the number of measurement operators $E_\alpha$ such that $\mathrm{tr}E_\alpha = t_-$. Then $0 \leqslant k \leqslant m$, and

$$m - 2d = (2k - m)\sqrt{1 - 4(m-1)b}. \tag{3}$$

In the following, we identify any rank one operator with a complex line determined by the operator in the system Hilbert space; for example, the measurement operator $E_\alpha = t_\alpha|\psi_\alpha\rangle\langle\psi_\alpha|$ (with $t_\alpha \neq 0$ fixed) corresponds to the line $l = \{c_\alpha|\psi_\alpha\rangle : c_\alpha \in \mathbb{C}\}$ in $\mathbb{C}^d$.

*Lemma 2.* Let $E = \{E_\alpha : \alpha = 1, 2, \dots, m\}$ be an equioverlapping measurement in $\mathbb{C}^d$. Then $E$ is either a set of equiangular lines or a union of two disjoint sets of equiangular lines. In the latter case, we can separate $E$ into two disjoint sets as

$$E = L_- \cup L_+, \quad L_- \cap L_+ = \emptyset,$$

with $L_- = \{E_\alpha : t_\alpha = t_-\}$ and $L_+ = \{E_\alpha : t_\alpha = t_+\}$, both of which are sets of equiangular lines. Moreover, we have

$$\cos^2(\theta_-) = \frac{b}{t_-^2} = \frac{2b}{1 - 2(m-1)b - \sqrt{1 - 4(m-1)b}},$$

$$\cos^2(\theta_+) = \frac{b}{t_+^2} = \frac{2b}{1 - 2(m-1)b + \sqrt{1 - 4(m-1)b}},$$

$$\cos^2(\theta_0) = \frac{b}{t_- t_+} = \frac{1}{m-1},$$

where $\theta_-$ ($\theta_+$) denotes the angle between any two lines both in $L_-$ ($L_+$), while $\theta_0$ denotes the angle between any line in $L_-$ and any line in $L_+$. In particular,

$$\theta_- \leqslant \theta_0 \leqslant \theta_+.$$

From Lemma 2, we know that the angles between any two measurement operators in an equioverlapping measurement can take at most three distinct values. More precisely, we have the following:

(1) If $k = 0$ or $m$, then the equioverlapping measurement is a set of equiangular lines, and the angle can only take one value.

(2) If $k = 1$ or $m - 1$, then the equioverlapping measurement is a union of a set of equiangular lines and a set consisting of a single element. The angle then can take two values: either $\theta_0$ and $\theta_+$ (corresponding to the case with $k = 1$) or $\theta_0$ and $\theta_-$ (corresponding to the case with $k = m - 1$).

(3) If $2 \leqslant k \leqslant m - 2$, then the angles take on three different values: $\theta_-$, $\theta_0$, and $\theta_+$.

In case 2, the equioverlapping measurement is a biangular measurement, which is defined as follows.

*Definition 5. Biangular measurement.* A POVM $E = \{E_\alpha : \alpha = 1, 2, \dots, m\}$ in $\mathbb{C}^d$ is called a biangular measurement if the following are true:

(1) *Biangularity*. The angles between the measurement operators $E_\alpha$ only take two values, i.e., $\#\{\mathrm{tr}(E_\alpha E_\beta)/\mathrm{tr}(E_\alpha)\mathrm{tr}(E_\beta) : \alpha \neq \beta\} = 2$.

(2) *Rank one*. All measurement operators $E_\alpha$ are rank one in the sense that $E_\alpha = t_\alpha|\psi_\alpha\rangle\langle\psi_\alpha|$ for some pure states $|\psi_\alpha\rangle$ and positive constants $t_\alpha$.

Biangular measurements have a close connection to the concept of biangular lines, as discussed in Refs. [41–47], where the angles between the lines are constrained to exactly two values. When a collection of biangular lines forms a tight frame, it naturally induces a corresponding biangular measurement.

In Ref. [35], it was shown that the number $m$ of measurement operators in any equioverlapping measurement in $\mathbb{C}^d$ is bounded as $d \leqslant m < d^2 + d$. Here we improve the upper bound to $m \leqslant d^2$. For this purpose, we first establish the following result.

*Proposition 1.* Let $E = \{E_\alpha : \alpha = 1, 2, \dots, m\}$ be an equioverlapping measurement in $\mathbb{C}^d$, where $\mathrm{tr}(E_\alpha E_\beta) = b$ for $\alpha \neq \beta$, then we have the following statements.

(1) $b = 0$ if and only if $m = d$. This corresponds to a von Neumann measurement.

(2) $b \leqslant t_-^2 \leqslant t_+^2$, with $t_-$ and $t_+$ defined by Eqs. (1) and (2). Furthermore, $b = t_-^2$ if and only if $m = d$ or $d + 1$, while $t_- = t_+$ if and only if $m = 2d$. If $0 < k < m$, we have $b < t_-^2 < t_+^2$.

(3) If $m > d$, then

$$\frac{1}{m^2} \leqslant b \leqslant \frac{1}{4(m-1)}.$$

Moreover, $b = 1/m^2$ if and only if $m = d + 1$, and $b = 1/4(m-1)$ if and only if $m = 2d$.

The proof is given in Appendix A. The above proposition reveals some intrinsic relations between various parameters $m$, $d$, and $b$ in an equioverlapping measurement.

*Proposition 2.* Let $\{E_\alpha : \alpha = 1, 2, \dots, m\}$ be an equioverlapping measurement in $\mathbb{C}^d$; then

$$d \leqslant m \leqslant d^2. \tag{4}$$

For the proof, see Appendix B.

According to Eq. (3), equioverlapping measurements can generally be categorized into two classes based on the parameter $m$ (the number of measurement operators). Specifically, when $m \neq 2d$, the overlap parameter $b$ between any two distinct measurement operators can take only some discrete values. In contrast, when $m = 2d$, the overlap $b$ can take continuous values within the interval $(1/4d^2, 1/4(2d-1)]$. The results are summarized as follows.

*Proposition 3.* Let $E = \{E_\alpha : \alpha = 1, 2, \ldots, m\}$ be an equioverlapping measurement with $\mathrm{tr}(E_\alpha E_\beta) = b$ for $\alpha \neq \beta$. Let $k = \#\{\alpha : \mathrm{tr}E_\alpha = t_-\}$ be the number of measurement operators $E_\alpha$ such that $\mathrm{tr}E_\alpha = t_-$.

(1) If $m \neq 2d$, then $b$ can take only some discrete values in the form of

$$b = \frac{(k-d)(k+d-m)}{(m-1)(m-2k)^2}.$$

Moreover, if $m = d$, then $k = 0$; if $d < m < 2d$, then $k < m - d$, and if $2d < m \leqslant d^2$, then $m - d < k \leqslant m$.

(2) If $m = 2d$, then $b$ can take continuous values in $(1/4d^2, 1/4(2d-1)]$. Moreover, if $b = 1/4(2d-1)$, then $k = 2d$, and if $b < 1/4(2d-1)$, then $k = d$.

For the proof, see Appendix C. The above proposition displays some surprising features of an equioverlapping measurement in the sense that the overlap parameter $b$ takes only discrete values for the general case $m \neq 2d$ and takes continuous values in the critical case $m = 2d$.

Next, we discuss some methods for constructing new equioverlapping measurements from existing ones: the complementary trick and tensor product.

If we represent any state $|\psi\rangle \in \mathbb{C}^d$ as a $d$-dimensional column vector, then the equioverlapping measurement $E = \{E_\alpha = t_\alpha|\psi_\alpha\rangle\langle\psi_\alpha| : \alpha = 1, 2, \ldots, m\}$ in $\mathbb{C}^d$ induces a $d \times m$ matrix

$$X = (\sqrt{t_1}|\psi_1\rangle, \ldots, \sqrt{t_m}|\psi_m\rangle) \in M_{d\times m}(\mathbb{C}),$$

where rows of $X$ collectively form an orthonormal system in $\mathbb{C}^m$ (considered to be a complex vector space of row vectors) since $XX^\dagger = \sum_{\alpha=1}^{m} t_\alpha|\psi_\alpha\rangle\langle\psi_\alpha| = \mathbf{1}_d$. When $m > d$, it is always possible to extend the matrix $X$ to a complete orthonormal basis by adding $m - d$ additional row vectors, typically through methods such as the Gram-Schmidt orthogonalization procedure. This extended basis enables us to construct a complementary measurement in $\mathbb{C}^{m-d}$ denoted as $E'$ when $m > d + 1$.

*Proposition 4.* Let $E = \{E_\alpha = t_\alpha|\psi_\alpha\rangle\langle\psi_\alpha| : \alpha = 1, 2, \ldots, m\}$ be an equioverlapping measurement in $\mathbb{C}^d$, with $\mathrm{tr}(E_\alpha E_\beta) = b$ for $\alpha \neq \beta$. Let $k = \#\{\alpha : \mathrm{tr}E_\alpha = t_-\}$ be the number of measurement operators $E_\alpha$ such that $\mathrm{tr}E_\alpha = t_-$. When $m > d + 1$, we can always construct another equioverlapping measurement $E' = \{E'_\alpha : \alpha = 1, 2, \ldots, m\}$ in $\mathbb{C}^{d'}$, with $d' = m - d$ and the same overlap value $b$. Moreover, $k + k' = m$, where $k' = \#\{\alpha : \mathrm{tr}E'_\alpha = t_-\}$.

For the proof, see Appendix D. Proposition 4 not only provides a way of constructing new equioverlapping measurements from existing ones but also implies a strong constraint on the number of measurement operators in an equioverlapping measurement. To highlight this point, we summarize the result as follows.

*Proposition 5.* Let $E = \{E_\alpha = t_\alpha|\psi_\alpha\rangle\langle\psi_\alpha| : \alpha = 1, 2, \ldots, m\}$ be an equioverlapping measurement in $\mathbb{C}^d$ with $m > d + 1$; then

$$m \leqslant (m-d)^2. \tag{5}$$

Putting it alternatively, for any natural numbers $r > 1$ and $d > r(r-1)$, an equioverlapping measurement in $\mathbb{C}^d$ with $d + r$ measurement operators does not exist.

For the proof, see Appendix E. It is interesting to compare the upper bounds in inequalities (4) and (5). Although the proof of Proposition 5 relies on Proposition 4 (see Appendix E), if we assume Proposition 5, then the upper bound in inequality (4) follows readily. Here is a simple argument. Suppose that $E = \{E_\alpha : \alpha = 1, 2, \ldots, m\}$ is an equioverlapping measurement in $\mathbb{C}^d$. If $m = d$ or $d + 1$, then it is trivially true that $m \leqslant d^2$ since $d \geqslant 2$. If $m > d + 1$, then by Proposition 4, an equioverlapping measurement $E' = \{E'_\alpha : \alpha = 1, 2, \ldots, m\}$ in $\mathbb{C}^{d'}$ with $d' = m - d$ exists. Since, in this dimension $d'$, it is always true that $m > d' + 1$, it then follows from Proposition 5 that $m \leqslant (m - d')^2 = [m - (m-d)]^2 = d^2$, which is consistent with Proposition 2.

To illustrate the power of Proposition 5, let us consider some special instances.

(1) If $m = d + 2$, then from inequality (5), we have $m \leqslant 4$; thus, $d + 2 \leqslant 4$, i.e., $d \leqslant 2$. Consequently, for $d \geqslant 3$, an equioverlapping measurement in $\mathbb{C}^d$ with $m = d + 2$ measurement operators does not exist.

(2) If $m = d + 3$, then from inequality (5), we have $m \leqslant 9$; thus, $d + 3 \leqslant 9$, i.e., $d \leqslant 6$. Consequently, for $d \geqslant 7$, an equioverlapping measurement in $\mathbb{C}^d$ with $m = d + 3$ measurement operators does not exist.

For construction of new equioverlapping measurements from existing ones, a natural question arises about the construction of the direct sum or tensor product. In general, the direct sum of equioverlapping measurements is not an equioverlapping measurement, as discussed in Ref. [35]. For the tensor product, the answer is as follows.

*Proposition 6.* Let $E = \{E_\alpha : \alpha = 1, 2, \ldots, m\}$ be an equioverlapping measurement in $\mathbb{C}^d$, with $\mathrm{tr}(E_\alpha E_\beta) = b$ for $\alpha \neq \beta$, and let $F = \{F_\mu : \mu = 1, 2, \ldots, n\}$ be an equioverlapping measurement in $\mathbb{C}^{d'}$, with $\mathrm{tr}(F_\mu F_\nu) = b'$ for $\mu \neq \nu$; then

$$E \otimes F = \{E_\alpha \otimes F_\mu : \alpha = 1, 2, \ldots, m;\ \mu = 1, 2, \ldots, n\}$$

is an equioverlapping measurement in $\mathbb{C}^d \otimes \mathbb{C}^{d'} = \mathbb{C}^{dd'}$ if and only if both $E$ and $F$ are von Neumann measurements.

The proof is given in Appendix F. The above proposition shows that the tensor product of equioverlapping measurements cannot be an equioverlapping measurement except in the rather trivial case of von Neumann measurements and amends an incorrect observation in Ref. [35]. This excludes an easy way of constructing equioverlapping measurements and indicates certain difficulty and complexity in constructing equioverlapping measurements.

## IV. ILLUSTRATIVE EXAMPLES

Clearly, von Neumann measurements are trivial examples of equioverlapping measurements, and SIC-POVMs are prominent examples of equioverlapping measurements. In this section, we present some nontrivial examples of equioverlapping measurements which, in general, are not SIC-POVMs. This shows that the class of equioverlapping measurements is significantly broader than the class of SIC-POVMs in a concrete way.

*Example 1*. Consider a qubit system $\mathbb{C}^2$ (thus, $d = 2$) with the measurement $E = \{E_\alpha = t_\alpha|\psi_\alpha\rangle\langle\psi_\alpha| : \alpha = 1, 2, 3, 4\}$. Here

$$|\psi_1\rangle = |0\rangle,$$
$$|\psi_2\rangle = r|0\rangle + \sqrt{1-r^2}|1\rangle,$$
$$|\psi_3\rangle = \frac{1}{\sqrt{3}}\big(|0\rangle - e^{i\theta}\sqrt{2}|1\rangle\big),$$
$$|\psi_4\rangle = \frac{1}{\sqrt{3}}\big(|0\rangle - e^{-i\theta}\sqrt{2}|1\rangle\big),$$

with $b \in (1/16, 1/12]$,

$$r = \frac{2\sqrt{b}}{1-\sqrt{1-12b}}, \quad \cos\theta = \frac{\sqrt{1-8b-\sqrt{1-12b}}}{4\sqrt{b}},$$

and

$$t_\alpha = \mathrm{tr}E_\alpha = \begin{cases} \frac{1}{2}(1-\sqrt{1-12b}), & \alpha = 1, 2, \\ \frac{1}{2}(1+\sqrt{1-12b}), & \alpha = 3, 4. \end{cases}$$

It can be straightforwardly checked that

$$\sum_{\alpha=1}^{4} E_\alpha = \mathbf{1}_2, \quad \mathrm{tr}(E_\alpha E_\beta) = b, \quad \alpha \neq \beta,$$

which shows that $E$ is, indeed, an equioverlapping measurement in $\mathbb{C}^2$. In this case, the number $m$ of measurement operators equals $2d = 4$ with $d = 2$. Since the equal-overlap parameter $b$ assumes continuous values in the interval $(1/16, 1/12]$, we have constructed a continuous family of equioverlapping measurements. In particular, when $b = 1/12$, we come to a SIC-POVM. For any other $b$, the equioverlapping measurement $E$ is not a SIC-POVM since $\mathrm{tr}E_1 = \mathrm{tr}E_2 \neq \mathrm{tr}E_3 = \mathrm{tr}E_4$.

*Example 2*. Consider a qutrit system $\mathbb{C}^3$ (thus, $d = 3$) with the measurement $E = \{E_\alpha = t_\alpha|\psi_\alpha\rangle\langle\psi_\alpha| : \alpha = 1, 2, \dots, 7\}$. Here

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{2\pi i/3}|1\rangle),$$
$$|\psi_2\rangle = \frac{1}{\sqrt{2}}(|1\rangle + e^{2\pi i/3}|2\rangle),$$
$$|\psi_3\rangle = \frac{1}{\sqrt{2}}(|2\rangle + e^{2\pi i/3}|0\rangle),$$
$$|\psi_4\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{-2\pi i/3}|1\rangle),$$
$$|\psi_5\rangle = \frac{1}{\sqrt{2}}(|1\rangle + e^{-2\pi i/3}|2\rangle),$$
$$|\psi_6\rangle = \frac{1}{\sqrt{2}}(|2\rangle + e^{-2\pi i/3}|0\rangle),$$
$$|\psi_7\rangle = \frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + |2\rangle),$$

and

$$t_\alpha = \mathrm{tr}E_\alpha = \begin{cases} \frac{2}{5}, & \alpha = 1, 2, 3, 4, 5, 6, \\ \frac{3}{5}, & \alpha = 7. \end{cases}$$

It can be straightforwardly checked that

$$\sum_{\alpha=1}^{7} E_\alpha = \mathbf{1}_3, \quad \mathrm{tr}(E_\alpha E_\beta) = \frac{1}{25}, \quad \alpha \neq \beta,$$

which shows that $E$ is an equioverlapping measurement in $\mathbb{C}^3$. It is worth noting that $|\psi_1\rangle, \dots, |\psi_6\rangle$ are derived from a set of vectors associated with a SIC-POVM in $\mathbb{C}^3$, while $|\psi_7\rangle$ represents the average of the three remaining SIC-POVM vectors. Angles between any two vectors from $\{|\psi_\alpha\rangle : \alpha = 1, 2, \dots, 7\}$ take only two distinct values since

$$|\langle\psi_\alpha|\psi_\beta\rangle|^2 = \tfrac{1}{4}, \quad \alpha \neq \beta \in \{1, 2, \dots, 6\},$$
$$|\langle\psi_\alpha|\psi_7\rangle|^2 = \tfrac{1}{6}, \quad \alpha \in \{1, 2, \dots, 6\}.$$

Thus, this equioverlapping measurement $E$ is not a SIC-POVM but a biangular measurement.

*Example 3*. Considering the equioverlapping measurement $E$ in Example 2 and following Proposition 4, we now construct a complementary measurement $E'$ in $\mathbb{C}^4$ (noting that $7 - 3 = 4$) as follows. By the proof of Proposition 4 in Appendix D, we first construct the $3 \times 7$ matrix

$$X = (\sqrt{t_1}|\psi_1\rangle, \dots, \sqrt{t_7}|\psi_7\rangle) = \begin{pmatrix} \frac{1}{\sqrt{5}} & 0 & \frac{e^{i\theta}}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 & \frac{e^{2i\theta}}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{e^{i\theta}}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 & \frac{e^{2i\theta}}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ 0 & \frac{e^{i\theta}}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 & \frac{e^{2i\theta}}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix},$$

which can be complemented to the $7 \times 7$ unitary matrix

$$A = \begin{pmatrix} X \\ X' \end{pmatrix},$$

with

$$X' = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{e^{i\theta}}{\sqrt{6}} & 0 & \frac{e^{2i\theta}}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-e^{i\theta}}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{10}} & \frac{e^{i\theta}}{\sqrt{10}} & 0 & \frac{-\sqrt{3}ie^{2i\theta}}{\sqrt{10}} & \frac{\sqrt{3}i}{\sqrt{10}} & \frac{e^{i\theta}}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{-\sqrt{3}i}{\sqrt{10}} & \frac{e^{2i\theta}}{\sqrt{10}} & \frac{\sqrt{3}ie^{i\theta}}{\sqrt{10}} & \frac{e^{2i\theta}}{\sqrt{10}} & 0 & \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{30}} & \frac{\sqrt{7}e^{i(\theta+\varphi)}}{\sqrt{30}} & \frac{\sqrt{3}e^{i\theta}}{\sqrt{10}} & \frac{e^{2i\theta}}{\sqrt{30}} & \frac{-2}{\sqrt{30}} & \frac{-\sqrt{7}e^{i(\theta-\varphi)}}{\sqrt{30}} & \frac{-1}{\sqrt{30}} \end{pmatrix},$$

where $\theta = 2\pi i/3$ and $e^{i\varphi} = -(1 + 3\sqrt{3}i)/2\sqrt{7}$. Now by the proof of Proposition 4, a complementary equioverlapping measurement in $\mathbb{C}^4$ can be constructed as

$$E' = \{E'_\alpha = t'_\alpha|\psi'_\alpha\rangle\langle\psi'_\alpha| : \alpha = 1, 2, \dots, 7\},$$

where

$$|\psi'_1\rangle = \frac{\sqrt{10}}{6}|0\rangle + \frac{1}{\sqrt{6}}|1\rangle - \frac{i}{\sqrt{2}}|2\rangle + \frac{\sqrt{2}}{6}|3\rangle,$$
$$|\psi'_2\rangle = \frac{\sqrt{10}}{6}|0\rangle + \frac{1}{\sqrt{6}}|1\rangle + \frac{1}{\sqrt{6}}e^{i\theta}|2\rangle + \frac{\sqrt{14}}{6}e^{i\varphi}|3\rangle,$$
$$|\psi'_3\rangle = \frac{1}{\sqrt{2}}|2\rangle - \frac{i}{\sqrt{2}}|3\rangle,$$

$$|\psi_4'\rangle = \frac{\sqrt{10}}{6}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{6}}|2\rangle + \frac{\sqrt{2}}{6}|3\rangle,$$
$$|\psi_5'\rangle = \frac{\sqrt{10}}{6}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle - \frac{\sqrt{2}}{3}|3\rangle,$$
$$|\psi_6'\rangle = \frac{\sqrt{10}}{6}|0\rangle - \frac{1}{\sqrt{6}}|1\rangle - \frac{1}{\sqrt{6}}e^{-i\theta}|2\rangle + \frac{\sqrt{14}}{6}e^{-i\varphi}|3\rangle,$$
$$|\psi_7'\rangle = \frac{\sqrt{15}}{6}|0\rangle - \frac{1}{2}|1\rangle - \frac{1}{2}|2\rangle + \frac{\sqrt{3}}{6}|3\rangle,$$

and

$$t_\alpha' = \mathrm{tr}E_\alpha' = \begin{cases} \frac{3}{5}, & \alpha = 1,2,3,4,5,6, \\ \frac{2}{5}, & \alpha = 7. \end{cases}$$

It can be straightforwardly checked that

$$\sum_{\alpha=1}^{7} E_\alpha' = \mathbf{1}_4,$$
$$\mathrm{tr}(E_\alpha' E_\beta') = \frac{1}{25}, \quad \alpha \neq \beta,$$

which show that the complementary measurement $E'$ is, indeed, an equioverlapping measurement in $\mathbb{C}^4$. The angles between any two measurement operators take two distinct values,

$$|\langle\psi_\alpha'|\psi_\beta'\rangle|^2 = \tfrac{1}{9}, \quad \alpha \neq \beta \in \{1,2,\dots,6\},$$
$$|\langle\psi_\alpha'|\psi_7'\rangle|^2 = \tfrac{1}{6}, \quad \alpha \in \{1,2,\dots,6\}.$$

Thus, this equioverlapping measurement $E'$ is a biangular measurement.

## V. SUMMARY

In this work, we have made a careful comparison between various measurements with some special structures: SIC-POVMs, equiangular measurements, and equioverlapping measurements. We have proved several basic properties of equioverlapping measurements, which set a variety of constraints for the parameters in the measurements. We have shown how equioverlapping measurements generalize both equiangular tight frames and SIC-POVMs and have illustrated these notions through some explicit examples in lower dimensions. We have discussed methods for deriving new equioverlapping measurements from established ones and have presented examples of equioverlapping measurements going beyond SIC-POVMs.

These results may be useful in further investigations of equioverlapping measurements.

The following questions arise naturally in this context:

(1) For which size $m$ (number of measurement operators) does a nontrivial equioverlapping measurement exist?

(2) How to classify and construct equioverlapping measurements in a general dimension?

(3) What is the use of equioverlapping measurements?

Question 1 is closely related to Zauner's conjecture on the existence of SIC-POVMs. Since this conjecture is widely believed to be true and SIC-POVMs are special cases of equioverlapping measurements, and moreover since equiangular measurements with $m = d + 1$ always exist, we tend to believe that some other non-trivial equioverlapping measurements exist in every dimension. Even if Zauner's conjecture is not true, it is still possible that nontrivial equioverlapping measurements exist in the absence of a SIC-POVM, and work on equioverlapping measurements may help clarify certain aspects of Zauner's conjecture.

Question 2 is of basic importance and considerable significance since equioverlapping measurements are natural extensions of SIC-POVMs, and SIC-POVMs, apart from their own intrinsic interest, have found many applications [11–23]. However, even for qutrit systems (i.e., dimension $d = 3$), the issues of classifying and constructing equioverlapping measurements are rather complicated, and we still do not have a complete classification of all equioverlapping measurements in $\mathbb{C}^3$, although we have constructed some families, which will be fully treated in a separate work.

Question 3 is worthy of further investigation because equioverlapping measurements capture overlaps (cross correlations) between measurement operators and generalize the notion of SIC-POVMs, which have been extensively and intensively investigated. Due to their intrinsic nature related to Born's rule and close connections to SIC-POVMs, we expect equioverlapping measurements may shed new light on SIC-POVMs and may play an interesting and useful role in the theory of quantum measurements.

## ACKNOWLEDGMENTS

This work was supported by the National Key R&D Program of China, Grant No. 2020YFA0712700, and the National Natural Science Foundation of China, Grant No. 12271325.

## APPENDIX A: PROOF OF PROPOSITION 1

Here we present a detailed proof of Propositions 1. For item 1, noting that there are at most $d$ mutually orthogonal vectors in $\mathbb{C}^d$, if $b = 0$ (orthogonality), then we conclude that $m = d$. Conversely, if $m = d$, then from Eq. (3) we have $-d = (2k - d)\sqrt{1 - 4(d-1)b}$, which is equivalent to

$$2k\sqrt{1-4(d-1)b} = d[\sqrt{1-4(d-1)b} - 1].$$

Since, obviously, the above left-hand side $\geqslant 0$ while the above right-hand side $\leqslant 0$,

$$0 = 2k\sqrt{1-4(d-1)b} = d[\sqrt{1-4(d-1)b} - 1] = 0,$$

which implies that $b = 0$ and $k = 0$. In this case, the measurement $E$ reduces to a von Neumann measurement.

For item 2, it follows from Eqs. (1) and (2) that $t_- \leqslant t_+$. Furthermore, $t_- = t_+$ if and only if $\sqrt{1-4(m-1)b} = 0$. This, combined with Eq. (3), means that $k = m = 2d$. We now proceed to establish $b \leqslant t_-^2$. For this purpose, we consider the following two cases.

(1) $m = d$. In this case, from item 1, we have $b = t_- = 0$.

(2) $m > d$. We consider the subcases $k \leqslant 1$ and $k > 1$ separately.

When $k \leqslant 1$, we have $2k - m \neq 0$ since $d \geqslant 2$. By Eqs. (1) and (3), we have

$$b = \frac{(d-k)(m-k-d)}{(m-1)(m-2k)^2}, \quad t_- = \frac{m-k-d}{m-2k}.$$

Since $b > 0$ and $k \leqslant 1$, we obtain $m - k - d \geqslant 1$. It follows that

$$\begin{aligned} t_-^2 - b &= \left(\frac{m-k-d}{m-2k}\right)^2 \left(1 - \frac{d-k}{(m-1)(m-k-d)}\right) \\ &= \left(\frac{m-k-d}{m-2k}\right)^2 \frac{m^2 - mk - md - m + 2k}{(m-1)(m-k-d)} \\ &= \left(\frac{m-k-d}{m-2k}\right)^2 \frac{m(m-k-d-1) + 2k}{(m-1)(m-k-d)} \\ &\geqslant 0. \end{aligned}$$

The last inequality becomes an equality if and only if $m - k - d - 1 = 0$ and $k = 0$; thus, $m = d + 1$.

When $k > 1$, there are at least two different measurement operators, say, $E_1$ and $E_2$, such that $\mathrm{tr}E_1 = \mathrm{tr}E_2 = t_-$ and $\mathrm{tr}(E_1E_2) = b$. Noting that $\mathrm{tr}(E_1^2) = \mathrm{tr}(E_2^2) = t_-^2$ and by the Cauchy-Schwarz inequality, we have

$$b = \mathrm{tr}(E_1E_2) \leqslant \sqrt{\mathrm{tr}\big(E_1^2\big)}\sqrt{\mathrm{tr}\big(E_2^2\big)} = t_-t_- = t_-^2.$$

Here the equality holds if and only if a number $r$ exists such that $E_1 = rE_2$. But from $\mathrm{tr}E_1 = \mathrm{tr}E_2$ we conclude that $r = 1$ and thus $E_1 = E_2$. However $E_1 \neq E_2$, and consequently, the above inequality is strict, i.e., $b < t_-^2$.

For item 3, since $0 < b \leqslant t_-^2$, substituting Eq. (1) into the above inequality and after straightforward manipulation, we come to $1/m^2 \leqslant b$. The lower bound $1/m^2$ is reached if and only if $m = d + 1$.

The inequality $b \leqslant 1/4(m-1)$ comes from

$$t_\alpha = \mathrm{tr}E_\alpha = \sum_{\beta=1}^{m} \mathrm{tr}(E_\alpha E_\beta) = t_\alpha^2 + (m-1)b$$

and the fact that $t_\alpha = \mathrm{tr}E_\alpha \geqslant 0$ are real numbers. Furthermore, $b = 1/4(m-1)$ if and only if $t_- = t_+$, that is, $m = 2d$ in view of Eq. (3).

## APPENDIX B: PROOF OF PROPOSITION 2

We proceed to prove that $E_1, E_2, \ldots, E_m$ are linearly independent in the real $d^2$-dimensional space of operators (matrices) acting on $\mathbb{C}^d$, and thus, $m \leqslant d^2$.

First, if $E$ is an equiangular measurement, i.e., $k = 0$ or $m$, then from the result concerning equiangular lines we have $m \leqslant d^2$ [25].

If $0 < k < m$, then from Proposition 1 we have

$$t_+^2 > t_-^2 > b.$$

Assume that there exist $m$ real numbers $c_\alpha$ such that

$$\sum_{\alpha=1}^{m} c_\alpha E_\alpha = 0; \tag{B1}$$

we need to prove that all $c_\alpha$ are zero, and thus, $E_1, E_2, \ldots, E_m$ are linearly independent. By Eq. (B1),

$$0 = \mathrm{tr}\left(\sum_{\alpha=1}^{m} c_\alpha E_\alpha E_\beta\right) = \left(\sum_{\alpha=1}^{m} c_\alpha\right) b + c_\beta(t_-^2 - b)$$

for any $\beta = 1, 2, \ldots, k$. This implies that

$$c_\beta = \frac{\left(\sum_{\alpha=1}^{m} c_\alpha\right) b}{b - t_-^2} \tag{B2}$$

is a constant independent of $\beta$, which will be denoted by $c_-$. Similarly,

$$c_\gamma = \frac{\left(\sum_{\alpha=1}^{m} c_\alpha\right) b}{b - t_+^2}, \quad \gamma = k+1, k+2, \ldots, m$$

is also a constant independent of $\gamma$, which will be denoted by $c_+$. From

$$0 = \mathrm{tr}\left(\sum_{\alpha=1}^{m} c_\alpha E_\alpha\right) = \left(\sum_{\alpha=1}^{k} c_\alpha\right)(t_- - t_+) + \left(\sum_{\alpha=1}^{m} c_\alpha\right) t_+,$$

we have

$$c_- = \frac{\left(\sum_{\alpha=1}^{m} c_\alpha\right) t_+}{k(t_+ - t_-)}. \tag{B3}$$

Comparing Eqs. (B2) and (B3) (noting that $c_\beta = c_-$), we have

$$\frac{\left(\sum_{\alpha=1}^{m} c_\alpha\right) b}{b - t_-^2} = \frac{\left(\sum_{\alpha=1}^{m} c_\alpha\right) t_+}{k(t_+ - t_-)},$$

which implies that either $\sum_{\alpha=1}^{m} c_\alpha = 0$ or

$$\frac{b}{b - t_-^2} = \frac{t_+}{k(t_+ - t_-)}.$$

In the latter case, the left-hand side of the equation is less than zero while the right-hand side is greater than zero, which is a contradiction. Therefore, $\sum_{\alpha=1}^{m} c_\alpha$ must be equal to zero, which means

$$c_- = \frac{\left(\sum_{\alpha=1}^{m} c_\alpha\right) b}{b - t_-^2} = 0, \quad c_+ = \frac{\left(\sum_{\alpha=1}^{m} c_\alpha\right) b}{b - t_+^2} = 0.$$

It follows that all $c_\alpha$ are zero. Consequently, $E_1, E_2, \ldots, E_m$ are linearly independent, which implies that $m \leqslant d^2$.

## APPENDIX C: PROOF OF PROPOSITION 3

For item 1, if $m \neq 2d$, then from Eq. (3), we have

$$b = \frac{(k-d)(k+d-m)}{(m-1)(m-2k)^2},$$

which shows that $b$ can take only discrete values.

According to Proposition 1, $m = d$ if and only if $b = 0$. Since $b = 0$, by Eq. (3), we obtain $k = 0$.

When $m > d$, based on the above conclusion, the value of $b$ is necessarily greater than zero, which implies that

$$(k-d)(k+d-m) > 0. \tag{C1}$$

In addition, Eq. (3) implies that

$$(m-2d)(2k-m) > 0 \tag{C2}$$

unless $m = 2d$. When $d < m < 2d$, $k$ must be less than $m/2$, i.e., $k < m/2 < d$. Then inequality (C1) implies that $k < m - d$ since $k < d$. Similarly, when $m > 2d$, from inequalities (C1) and (C2) we have $k > m/2 > d$ and $k > m - d$. The upper bound $k \leqslant m$ is evident.

For item 2, the situation is quite different when $m = 2d$, which implies that the left-hand side of Eq. (3) is zero. In this case, either $1 - 4(m-1)b = 0$, or $2k - m = 0$. When $b = 1/4(m-1) = 1/4(2d-1)$, $t_-$ and $t_+$ degenerate to the same value, 1/2, which means all $\mathrm{tr}E_\alpha$ are the same. Meanwhile, if $1 - 4(m-1)b > 0$, then $m - 2k = 0$, i.e., $m = 2k$. In this case, $b$ can take continuous values in the interval $(1/4d^2, 1/4(2d-1)]$. Notice that the case with $b = 1/4d^2$ is excluded since $E_\alpha$ are distinct for different $\alpha$.

## APPENDIX D: PROOF OF PROPOSITION 4

Let

$$X = (\sqrt{t_1}|\psi_1\rangle, \dots, \sqrt{t_m}|\psi_m\rangle) \in M_{d\times m}(\mathbb{C}).$$

From $\sum_{\alpha=1}^m E_\alpha = \mathbf{1}_d$, we have

$$XX^\dagger = \sum_{\alpha=1}^m t_\alpha|\psi_\alpha\rangle\langle\psi_\alpha| = \mathbf{1}_d \in M_{d\times d}(\mathbb{C}),$$

which means that the rows of $X$ can be viewed as $d$ mutually orthogonal unit vectors in $\mathbb{C}^m$ (considered as a complex vector space of row vectors). When $m > d+1$, it is always possible to add $m - d$ ($\geqslant 2$) unit vectors, typically through methods such as the Gram-Schmidt orthogonalization process, such that they form an orthonormal basis of $\mathbb{C}^m$. That is, $X$ can be dilated to an $m \times m$ unitary matrix

$$A = \begin{pmatrix} \sqrt{t_1}|\psi_1\rangle & \cdots & \sqrt{t_m}|\psi_m\rangle \\ \sqrt{t_1'}|\psi_1'\rangle & \cdots & \sqrt{t_m'}|\psi_m'\rangle \end{pmatrix} \in M_{m\times m}(\mathbb{C}),$$

with $|\psi_1'\rangle, \dots, |\psi_m'\rangle$ being unit vectors in $\in \mathbb{C}^{m-d}$. Since $A$ is unitary, we find that

$$AA^\dagger = \begin{pmatrix} \sum_{\alpha=1}^m t_\alpha|\psi_\alpha\rangle\langle\psi_\alpha| & \sum_{\alpha=1}^m \sqrt{t_\alpha t_\alpha'}|\psi_\alpha\rangle\langle\psi_\alpha'| \\ \sum_{\alpha=1}^m \sqrt{t_\alpha t_\alpha'}|\psi_\alpha'\rangle\langle\psi_\alpha| & \sum_{\alpha=1}^m t_\alpha'|\psi_\alpha'\rangle\langle\psi_\alpha'| \end{pmatrix} = \mathbf{1}_m \in M_{m\times m}(\mathbb{C}).$$

Because

$$\sum_{\alpha=1}^m t_\alpha|\psi_\alpha\rangle\langle\psi_\alpha| = \sum_{\alpha=1}^m E_\alpha = \mathbf{1}_d,$$

we have

$$\sum_{\alpha=1}^m t_\alpha'|\psi_\alpha'\rangle\langle\psi_\alpha'| = \mathbf{1}_{m-d}.$$

Therefore, we can construct a POVM $E' = \{E_\alpha' = t_\alpha'|\psi_\alpha'\rangle\langle\psi_\alpha'| : \alpha = 1, 2, \dots, m\}$ in $\mathbb{C}^{m-d}$, with $m - d \geqslant 2$. Furthermore, from $A^\dagger A = \mathbf{1}_m \in M_{m\times m}(\mathbb{C})$, we have

$$t_\alpha\langle\psi_\alpha|\psi_\alpha\rangle + t_\alpha'\langle\psi_\alpha'|\psi_\alpha'\rangle = 1, \quad \alpha = 1, 2, \dots, m, \quad \text{(D1)}$$

$$\sqrt{t_\alpha t_\beta}\langle\psi_\alpha|\psi_\beta\rangle + \sqrt{t_\alpha' t_\beta'}\langle\psi_\alpha'|\psi_\beta'\rangle = 0, \quad \alpha \neq \beta. \quad \text{(D2)}$$

From Eq. (D2), we know that

$$\begin{aligned} \mathrm{tr}(E_\alpha' E_\beta') &= t_\alpha' t_\beta' |\langle\psi_\alpha'|\psi_\beta'\rangle|^2 \\ &= t_\alpha t_\beta |\langle\psi_\alpha|\psi_\beta\rangle|^2 \\ &= \mathrm{tr}(E_\alpha E_\beta) = b \end{aligned}$$

is a constant independent of $\alpha \neq \beta$. Consequently, $E'$ is an equioverlapping measurement in $\mathbb{C}^{m-d}$ with $m$ elements. Moreover, from Eq. (D1), we find that

$$\mathrm{tr}E_\alpha' = t_\alpha' = 1 - t_\alpha = 1 - \mathrm{tr}E_\alpha, \quad \alpha = 1, 2, \dots, m,$$

which implies that if $\mathrm{tr}E_\alpha = t_-$ ($t_+$), then $\mathrm{tr}E_\alpha' = 1 - t_-(t_+) = t_+$ ($t_-$). Because there are $m - k$ measurement operators in $E$ with trace $t_+$, we have $\#\{\alpha : E_\alpha' = t_-\} = m - k$.

## APPENDIX E: PROOF OF PROPOSITION 5

By Proposition 4, we can construct an equioverlapping measurement $E'$ in $\mathbb{C}^{m-d}$ with $m$ elements. Now from Proposition 2, it is necessary that $m \leqslant (m-d)^2$.

If $m = d + r$, then from inequality (5), we have

$$m \leqslant (m-d)^2 = r^2;$$

thus, $d + r \leqslant r^2$, i.e., $d \leqslant r(r-1)$. Consequently, for $d > r(r-1)$, an equioverlapping measurement in $\mathbb{C}^d$ with $m = d + r$ measurement operators does not exist.

## APPENDIX F: PROOF OF PROPOSITION 6

First, it is clear that if $E$ and $F$ are von Neumann measurements in $\mathbb{C}^d$ and $\mathbb{C}^{d'}$, respectively, then $E \otimes F$ is also a von Neumann measurement in $\mathbb{C}^{dd'}$ and thus is an equioverlapping measurement.

Conversely, if $E \otimes F$ is an equioverlapping measurement, we need to show that $b = b' = 0$, which implies that both $E$ and $F$ are von Neumann measurements. To prove this, we compute

$$\mathrm{tr}(E_\alpha \otimes F_\mu)(E_\beta \otimes F_\nu) = \begin{cases} bb', & \alpha \neq \beta, \mu \neq \nu, \\ b(t_\mu')^2, & \alpha \neq \beta, \mu = \nu, \\ t_\alpha^2 b', & \alpha = \beta, \mu \neq \nu, \end{cases} \quad \text{(F1)}$$

where $t_\alpha = \mathrm{tr}E_\alpha$ and $t_\mu' = \mathrm{tr}F_\mu$. If $E \otimes F$ is an equioverlapping measurement, then from Eq. (F1) we have

$$bb' = b(t_\mu')^2 = t_\alpha^2 b', \quad \forall\, \alpha, \mu.$$

If $b' \neq 0$, then $t_\alpha^2 = b$ is a constant independent of $\alpha$, which implies that $E$ is an equiangular measurement. But such an equiangular measurement does not exist in view of Lemma 1 (noting that any equiangular measurement is an equioverlapping measurement). Consequently, $b' = 0$, which implies that $F$ is a von Neumann measurement. Similarly, $b = 0$, and $E$ is a von Neumann measurement.

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