Parametrized multipartite entanglement measures

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We investigate parametrized multipartite entanglement measures from the perspective of k nonseparability in this paper. We present two types of entanglement measures in n-partite systems, q-k-ME concurrence ($q > 1,\ 2 \leqslant k \leqslant n$) and α-k-ME concurrence ($0 \leqslant \alpha < 1,\ 2 \leqslant k \leqslant n$), which unambiguously detect all k-nonseparable states in arbitrary n-partite systems. Rigorous proofs show that the proposed k-nonseparable measures satisfy all the requirements for being an entanglement measure including the entanglement monotone, strong monotone, convexity, vanishing on all k-separable states, and being strictly greater than zero for all k-nonseparable states. In particular, the q-2-ME concurrence and α-2-ME concurrence, renamed as q-GME concurrence and α-GME concurrence, respectively, are two kinds of genuine entanglement measures corresponding to the case where the systems are divided into bipartition ($k = 2$). The lower bounds of the two classes k-nonseparable measures are obtained by employing the approach that takes into account the permutationally invariant part of a quantum state. Furthermore, the relations between q-n-ME concurrence (α-n-ME concurrence) and global negativity are established. In addition, we discuss the degree of separability and elaborate on an effective detection method with concrete examples. Moreover, we compare the q-GME concurrence defined by us to other genuine entanglement measures.

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I. INTRODUCTION

Quantum entanglement as a physical resource is indispensable in such tasks as quantum cryptography [1–4], quantum teleportation [5–7], and quantum communication [8–10]. Moreover, it is recognized that entangled states are at the core of quantum information processing [11–13]. Therefore, the qualitative and quantitative study of multipartite quantum states is a matter of great importance, and in this paper we mainly focus on the quantitative description of entanglement of states.

Initially, bipartite systems were studied extensively, and a wide range of measures were found to quantify the entanglement of states. Concurrence is one of the well-known measures for bipartite quantum systems [14–17], and Wootters gave an analytical expression for arbitrary two-qubit quantum states in Ref. [15]. Furthermore, there are other methods, such as negativity [18,19], entanglement of formation [20,21], and Tsallis entropy of entanglement [22], that can also characterize the entanglement of quantum states commendably.

Many efforts have been made to detect multipartite entanglement [23–33], but no measure can be employed to calculate the entanglement of multipartite mixed states. In Ref. [34], Ma *et al.* put forth a measure of genuine multipartite entanglement (GME), termed GME concurrence, which can distinguish the genuinely entangled states from the others, in addition, they rendered a computable lower bound. Subsequently, Chen *et al.* [35] optimized the lower bound of Ref. [34]. To quantitatively characterize the entire hierarchy of k separability of states more precisely in n-partite systems, Hong *et al.* [36] advanced generalized measures called k-ME concurrence, where k runs from n to 2, and provided their two strong lower bounds. The GME concurrence [34] is a special case of the k-ME concurrence [36] when $k = 2$. It is acknowledged that multipartite entanglement (ME) is extremely complicated. Gao *et al.* [37] proposed that whether a state is k nonseparable can be determined by its permutationally invariant (PI) part, which dramatically reduces the dimension of the space to be considered.

In addition, some researchers devoted themselves to the study of parametrized measures. Yang *et al.* [38] introduced a parametrized entanglement monotone [39] called q concurrence ($q \geqslant 2$) for arbitrary bipartite systems, and presented the lower bound of q concurrence meanwhile. Later, Wei and Fei [40] came up with a generalized concurrence in terms of different ranges of the parameter named α concurrence ($0 \leqslant \alpha \leqslant \frac{1}{2}$). Shi [41] generalized the geometric mean of bipartite concurrences (GBC) defined by Li and Shang [42] to parametrized form, which is known as geometric mean of q concurrence (GqC).

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Motivated by the thoughts in Refs. [36–38,40], our aim in this paper is to define new ME measures from the point of k nonseparability utilizing the parametrized concurrence [38,40]. The calculation of entanglement for an n-partite mixed state is considerably difficult as it generally involves optimization procedures. We will try to give lower bounds on the k-nonseparable parametrized entanglement measures and provide an effective degree of separability for a convenient way to understand the structure of multipartite quantum states. Genuine multipartite entanglement is vital in quantum spin chains [43] and measurement-based quantum computing [44], hence we specifically mention the special case of $k=2$ (i.e., the genuine entanglement measure), and compare it with other available GME measures.

The content of this paper is arranged as follows. In Sec. II, we briefly introduce a few notions. In Sec. III, we put forward two classes of parametrized measures called q-k-ME concurrence C_{q-k} $(q>1)$ and α-k-ME concurrence $C_{\alpha-k}$ $(0\leqslant\alpha<1)$, respectively. When $k=2$, the corresponding two categories of genuine entanglement measures, respectively, are termed q-GME concurrence ($C_{q-\mathrm{GME}}$) and α-GME concurrence ($C_{\alpha-\mathrm{GME}}$). We verify that these measures defined by us conform with, simultaneously, the properties including nonnegativity, being strictly greater than zero for all k-nonseparable states, invariance under local unitary transformations, (strong) monotonicity, and convexity. In addition, C_{q-k} and $C_{q-\mathrm{GME}}$ satisfy the subadditivity as well, but $C_{\alpha-k}$ and $C_{\alpha-\mathrm{GME}}$ fail. The lower bound of C_{q-k} is obtained in Sec. IV by taking the maximum of C_{q-k} of the PI part of a quantum state and so is $C_{\alpha-k}$. Furthermore, we establish the relations between C_{q-n} ($C_{\alpha-n}$) and global negativity, which can be used to detect whether a quantum state is entangled or fully separable. Meanwhile, we discuss a specific example of mixing the W state with white noise and observe that the range detected by our results is larger than that of Ref. [29]. These two approaches, reflecting the degree of entanglement in Sec. V, exhibit different aspects of dominance in the detection of entanglement. The combination of these two methods can be used to detect k-nonseparable states more effectively. In Sec. VI, we compare $C_{q-\mathrm{GME}}$ with concurrence fill [61] and GqC [41] by a concrete example, which shows that they generate different entanglement orders as well as that $C_{q-\mathrm{GME}}$ is smoother at times. We conclude in Sec. VII.

II. PRELIMINARIES

Let ρ be an n-partite quantum state on Hilbert space $\mathcal{H}=\mathcal{H}_1\otimes\mathcal{H}_2\otimes\cdots\otimes\mathcal{H}_n$ with $\dim\mathcal{H}_i=d_i$, and $A_1|A_2|\cdots|A_k$ be a k partition $(2\leqslant k\leqslant n)$ of set $A=\{1,2,\ldots,n\}$ such that

$$\bigcup_{t=1}^{k}A_t=\{1,2,\ldots,n\},\quad A_t\bigcap A_{t'}=\emptyset \text{ when } t\neq t'. \tag{1}$$

An n-partite pure state $|\varphi\rangle$ on Hilbert space $\mathcal{H}$ is referred to as k separable $(2\leqslant k\leqslant n)$ [16] if there exists a splitting of the n parties into k parts $A_1,A_2,\ldots,A_k$ such that $|\varphi\rangle=\otimes_{t=1}^{k}|\varphi_t\rangle_{A_t}$ holds. An n-partite mixed state ρ is called k separable if it can be represented as a convex combination of k-separable pure states, that is, $\rho=\sum_i p_i|\varphi_i\rangle\langle\varphi_i|$, where $|\varphi_i\rangle$ could be k separable regarding different partitions fulfilling the above condition (1). Otherwise, the quantum state is called k nonseparable. If ρ is n separable, then it is called fully separable. If not, it is said to be entangled.

For any n-partite quantum state ρ, its PI part can be denoted as [37]

$$\rho^{\mathrm{PI}}=\frac{1}{n!}\sum_{j=1}^{n!}\Pi_j\rho\Pi_j^{\dagger}, \tag{2}$$

where the set $\{\Pi_j\}$ contains all of permutations of n particles.

A well-defined k-nonseparability measure $E(\rho)$ ought to meet the conditions as follows.

(M1) $E(\rho)=0$ for arbitrary k-separable quantum states.

(M2) $E(\rho)>0$ for arbitrary k-nonseparable quantum states.

(M3) (Invariance under local unitary transformations) $E(\rho)=E(U_{\mathrm{Local}}\rho U_{\mathrm{Local}}^{\dagger})$.

(M4) (Monotonicity) E is nonincreasing under local operations and classical communication (LOCC), i.e., $E[\Lambda_{\mathrm{LOCC}}(\rho)]\leqslant E(\rho)$. Moreover, several entanglement measures can obey a stronger condition called strong monotonicity, that E is average nonincreasing under LOCC, i.e., $E(\rho)\geqslant\sum_j p_jE(\sigma_j)$ with $\{p_j,\sigma_j\}$ being yielded after Λ_{LOCC} acts on state ρ.

(M5) Most entanglement measures also conform to convexity, $E(\sum_i p_i\rho_i)\leqslant\sum_i p_iE(\rho_i)$.

(M6) Further, there may be several entanglement measures that satisfy subadditivity, $E(\rho\otimes\sigma)\leqslant E(\rho)+E(\sigma)$.

For any bipartite pure state $|\varphi\rangle_{AB}$, Yang *et al.* [38] defined a parametrized bipartite entanglement measure, q concurrence $(q\geqslant 2)$, which is

$$C_q(|\varphi\rangle_{AB})=1-\mathrm{Tr}\big(\rho_A^q\big). \tag{3}$$

Then, Wei and Fei [40] also came up with a new bipartite entanglement measure in terms of different parameter ranges, α concurrence $(0\leqslant\alpha\leqslant\frac{1}{2})$, the form is

$$C_\alpha(|\varphi\rangle_{AB})=\mathrm{Tr}\big(\rho_A^\alpha\big)-1. \tag{4}$$

Here ρ_A is the reduced density operator of $|\varphi\rangle_{AB}$. In fact, from the perspective of being an entanglement measure, the above parameters' range can be extended to $q\in(1,+\infty)$ and $\alpha\in[0,1)$.

Bipartite systems are the simplest ones containing entanglement. However, multipartite entangled systems are extremely complicated owing to the multiple distinct ways in which a multipartite state can be entangled; to be more blunt, a multipartite state can be partially entangled rather than just genuinely multipartite entangled. Moreover, multipartite entangled states play an essential role in the applications of quantum information theory, such as quantum computation [45] and quantum secret sharing [46]. Therefore, it is necessary to gain a more refined structure of multipartite quantum states. Here we will generalize bipartite parametrized concurrence to multipartite quantum systems from the point of k nonseparability in the following sections.

III. NEW PARAMETRIZED ENTANGLEMENT MEASURES

We start by introducing new measures, q-k-ME concurrence ($q > 1$) and α-k-ME concurrence ($0 \leqslant \alpha < 1$), which are inspired by those proposed in Refs. [36–38,40].

Definition 1. For any n-partite pure state $|\varphi\rangle \in \mathcal{H}$, we define the q-k-ME concurrence as

$$C_{q-k}(|\varphi\rangle) = \min_{A} \frac{\sum_{t=1}^{k}\left[1 - \mathrm{Tr}\left(\rho_{A_t}^{q}\right)\right]}{k} \tag{5}$$

for any $q > 1$, and the α-k-ME concurrence as

$$C_{\alpha-k}(|\varphi\rangle) = \min_{A} \frac{\sum_{t=1}^{k}\left[\mathrm{Tr}\left(\rho_{A_t}^{\alpha}\right) - 1\right]}{k} \tag{6}$$

for any $0 \leqslant \alpha < 1$, respectively. Here ρ_{A_t} is the reduced density operator of subsystem A_t, and the minimum is done over all feasible k partitions $A = \{A_1|A_2|\cdots|A_k\}$ obeying condition (1).

The quantization of Definition 1 can be generalized to any n-partite mixed state ρ via convex-roof extension, the q-k-ME concurrence is defined as

$$C_{q-k}(\rho) = \inf_{\{p_i,|\varphi_i\rangle\}} \sum_i p_i C_{q-k}(|\varphi_i\rangle) \tag{7}$$

for any $q > 1$. Analogously, the α-k-ME concurrence is defined as

$$C_{\alpha-k}(\rho) = \inf_{\{p_i,|\varphi_i\rangle\}} \sum_i p_i C_{\alpha-k}(|\varphi_i\rangle) \tag{8}$$

for any $0 \leqslant \alpha < 1$. Here the infimum is taken over all viable pure decompositions $\rho = \sum_i p_i|\varphi_i\rangle\langle\varphi_i|$.

The following we will verify q-k-ME concurrence ($q > 1$) and α-k-ME concurrence ($0 \leqslant \alpha < 1$) satisfy the requirements of an entanglement measure.

Proposition 1. Both q-k-ME concurrence ($q > 1$) and α-k-ME concurrence ($0 \leqslant \alpha < 1$) fulfill the necessary conditions to be a reasonable entanglement measure.

The detailed proof is presented in Appendix A. We have that both kinds of parametrized multipartite entanglement measures possess the properties (M1) to (M5), in addition, q-k-ME concurrence also satisfy the property (M6).

A quantum state is genuinely multipartite entangled iff it is 2 nonseparable. In particular, for the special case $k = 2$, the formula (5) can be written as

$$C_{q-\mathrm{GME}}(|\varphi\rangle) = \min_{\gamma_i\in\gamma}\left[1 - \mathrm{Tr}\left(\rho_{A_{\gamma_i}}^{q}\right)\right], \tag{9}$$

which is called q-GME concurrence for any $q > 1$. Similarly, the formula (6) can be reduced to

$$C_{\alpha-\mathrm{GME}}(|\varphi\rangle) = \min_{\gamma_i\in\gamma}\left[\mathrm{Tr}\left(\rho_{\gamma_i}^{\alpha}\right) - 1\right], \tag{10}$$

which is termed α-GME concurrence for any $0 \leqslant \alpha < 1$. Here $\gamma = \{\gamma_i\}$ expresses the set of all feasible bipartitions.

For an arbitrary n-partite mixed state ρ, the q-GME concurrence is expressed as

$$C_{q-\mathrm{GME}}(\rho) = \inf_{\{p_i,|\varphi_i\rangle\}} \sum_i p_i C_{q-\mathrm{GME}}(|\varphi_i\rangle) \tag{11}$$

for any $q > 1$. Analogously, the α-GME concurrence is given by

$$C_{\alpha-\mathrm{GME}}(\rho) = \inf_{\{p_i,|\varphi_i\rangle\}} \sum_i p_i C_{\alpha-\mathrm{GME}}(|\varphi_i\rangle) \tag{12}$$

for any $0 \leqslant \alpha < 1$, where the infimum runs over all feasible pure decompositions of ρ.

The expressions (11) and (12) are particular cases of the formulas (7) and (8), respectively. It is natural that q-GME concurrence and α-GME concurrence satisfy the necessary requirements (M1) to (M5). Moreover, q-GME concurrence fulfills additivity as well. So they can be used to detect whether a quantum state is genuinely multipartite entangled.

Note that q-k-ME concurrence $C_{q-k}(|\varphi\rangle)$ tends to 1 as $q \to +\infty$ when the quantum state $|\varphi\rangle$ is k nonseparable. However, when $\alpha = 0$ and the state $|\varphi\rangle$ is k nonseparable, 0-k-ME concurrence C_{0-k} depends on the rank of the reduced density operator ρ_{A_t}, $C_{0-k}(|\varphi\rangle) = \frac{\sum_{t=1}^{k} r_{A_t}}{k} - 1$, where r_{A_t} denotes the rank of ρ_{A_t}, $t = 1, 2, \ldots, k$. Thus, the two entanglement measures C_{q-k} and $C_{\alpha-k}$ describe different aspects. When $q \to +\infty$, C_{q-k} will take two extremes, $C_{q-k} \to 1$ for any k-nonseparable pure state, whereas $C_{q-k} = 0$ for any k-separable pure state, which mean that the pure states are classified in terms of whether or not they are k separable, if the state is k nonseparable, the value of entanglement is unified to 1, otherwise, the value of entanglement is 0.

Since bipartite parametrized concurrence and Tsallis-$\hat{q}$ entanglement are equivalent for some particular $\hat{q}$, the $\hat{q}$-k-ME concurrence (or $\hat{q}$-GME concurrence) can be viewed as a sort of generalization of parametrized concurrence [38,40] and Tsallis entanglement [22] in some sense. It may provide a method to estimate Tsallis-$\hat{q}$ entanglement with a particular parameter $\hat{q}$ for any multipartite quantum state.

Concurrence is known to be one of the most widely used bipartite entanglement measures since it has analytic expressions for any two-qubit quantum states [15]. Moreover, Gour and Sanders [47] indicated that concurrence has important implications in remote entanglement distribution protocols, including entanglement swapping and remote preparation of bipartite entangled states. As far as the quantum networks are concerned, the distribution of long-range entanglement is crucial [48–50]. A statistical theory, concurrence percolation theory, was advanced, which implies that entanglement transmission can also be established when two infinitely distant nodes are connected via paths with only imperfectly entangled states, provided there are enough paths [48,50]. In addition, entanglement swapping, as a fundamental protocol in quantum information has many applications, such as the creation of multiparticle entangled states from singlets [51], entanglement purification [52], and so on. Thus, we can see that concurrence is not only a powerful mathematical tool, but also has physical operational interpretations.

The concurrence for any bipartite pure state $|\varphi\rangle_{AB}$ is associated with Tsallis-$\hat{q}$ entropy corresponding to $\hat{q} = 2$, i.e.,

$C(|\varphi\rangle_{AB})=\sqrt{2T_2(\rho_A)}$. Also the parametrized bipartite entanglement measures, q concurrence and α concurrence, are obtained from Tsallis-$\hat{q}$ entropy $\mathcal{T}_{\hat{q}}$, thus these formulas presented in Eqs. (5) and (6) can be uniformly expressed as

$$C_{\hat{q}-k}(|\varphi\rangle)=\min_{A}\frac{|\hat{q}-1|\sum_{t=1}^{k}\mathcal{T}_{\hat{q}}(\rho_{A_t})}{k} \tag{13}$$

for specific $\hat{q}$, where $\mathcal{T}_{\hat{q}}(\rho)=\frac{1}{\hat{q}-1}(1-\mathrm{Tr}\rho^{\hat{q}})$, $\hat{q}>0$, and $\hat{q}\neq 1$. Remarkably, the Tsallis entropy is a parametrized generalized form of the Boltzmann-Gibbs entropy [53], and the corresponding Tsallis statistical mechanics can be widely applied to long-range systems interaction, such as the optical lattice [54,55], trapped ion [56], spin-glass relaxation [57], and so on. However, Boltzmann-Gibbs statistical physics is based on a series of idealized assumptions about the motion of a large number of microscopic particles, and deals, in general, with systems in which there are no or negligible interactions between microscopic particles; of course, there are also cases where the interactions between particles in the system are taken into account, but this is limited to the treatment of weak short-range interactions between nearby particles. Therefore, the parametrized generalized form we define may be more flexible to realize long-range interactions between multipartite systems than the fashion given based on standard concurrence.

IV. LOWER BOUNDS OF q-k-ME CONCURRENCE AND α-k-ME CONCURRENCE

Compared with the bipartite systems, the structure of the multipartite systems is rather complicated. As a result, it is extremely difficult to give an analytical lower bound since the optimization procedure is involved in computing entanglement of multipartite quantum states. Therefore, we first employ the approach proposed by Gao *et al*. in Ref. [37], considering the PI part of quantum state ρ, to give the lower bounds of q-k-ME concurrence ($q>1$) and α-k-ME concurrence ($0\leqslant\alpha<1$).

Theorem 1. For any n-partite quantum state ρ, the q-k-ME concurrence $C_{q-k}(\rho)$ ($q>1$) is lower bounded by the maximum of q-k-ME concurrence of $\rho_U^{\mathrm{PI}}=(U\rho U^\dagger)^{\mathrm{PI}}$,

$$C_{q-k}(\rho)\geqslant\max_{U}C_{q-k}\big(\rho_U^{\mathrm{PI}}\big). \tag{14}$$

Analogically, the α-k-ME concurrence ($0\leqslant\alpha<1$) satisfies the relation as follows:

$$C_{\alpha-k}(\rho)\geqslant\max_{U}C_{\alpha-k}\big(\rho_U^{\mathrm{PI}}\big). \tag{15}$$

Here the maximum is taken all locally unitary transformations U. Please see Appendix B for the detailed proof.

From here on, we only need to take into account the space of the permutationally invariant quantum states, instead of the entire space, which broadly reduces the dimension of the space to be considered. The structure of the multipartite quantum state is extremely complicated, so the approach introduced in Ref. [37] provides great convenience for characterizing and detecting k separability of general quantum states.

In the following we will establish the relation between q-n-ME concurrence and global negativity [58]. Global negativity, a measure between subsystem p and the remaining subsystem, is given by

$$N^p=\frac{1}{d_p-1}(\|\rho^{T_p}\|_1-1)=-\frac{2}{d_p-1}\sum_i\lambda_i^{p-}, \tag{16}$$

where ρ^{T_p} is the partial transpose with respect to the subsystem p, $\|\cdot\|_1$ is trace norm, λ_i^{p-} is the negative eigenvalue of ρ^{T_p}, and d_p denotes the dimension of subsystem p. When $d_p=2$ ($p=1,2,\dots,n$), Eq. (16) can be reduced to $N^p=\|\rho^{T_p}\|_1-1=-2\sum_i\lambda_i^{p-}$.

It is especially emphasized here that we need to further impose stronger restrictions on the range of parameters q and α in the following content of this section.

The lower bound of q concurrence ($q\geqslant 2$) was derived by Yang *et al*. using the positive partial transpose (PPT) criterion and realignment criterion in Ref. [38]. The relation is

$$C_q(\rho_{AB})\geqslant\frac{\big[\max\big\{\|\rho^{T_A}\|_1^{q-1},\|\mathcal{R}(\rho)\|_1^{q-1}\big\}-1\big]^2}{m^{2q-2}-m^{q-1}}, \tag{17}$$

where ρ^{T_A} is partial transpose with regard to the subsystem A and $\mathcal{R}$ is a realignment operation [38], and m is obtained by taking the minimum of the dimensions of the two subsystems. On the basis of inequality (17), we show the connection between q-n-ME concurrence ($q\geqslant 2$) and global negativity in the following theorem.

Theorem 2. For any n-qubit quantum state ρ, the relation between the q-n-ME concurrence ($q\geqslant 2$) and global negativity of quantum state ρ is obtained as follows:

$$C_{q-n}(\rho)\geqslant\frac{\sum_{k=1}^{n}[(N^k+1)^{q-1}-1]^2}{n(2^{2q-2}-2^{q-1})}. \tag{18}$$

Here the inequality is saturated for n-qubit pure state when $q=2$. The proof is provided in Appendix C.

For higher-dimensional systems, the result is as follows.

Corollary 1. For any n-qudit quantum state $\rho\in\otimes_{i=1}^n\mathcal{H}_i$, $\dim\mathcal{H}_i=m$, $i=1,2,\dots,n$, q-n-ME concurrence ($q\geqslant 2$) satisfies the inequality

$$C_{q-n}(\rho)\geqslant\frac{\sum_{k=1}^{n}\{[(m-1)N^k+1]^{q-1}-1\}^2}{n(m^{2q-2}-m^{q-1})}. \tag{19}$$

In Ref. [59], Wei *et al*. provided lower bounds of q concurrence for any bipartite state ρ_{AB}, which are

$$C_q(\rho_{AB})\geqslant\frac{1-m^{1-q}}{(m-1)^2}\big\{\max[\|\rho^{T_A}\|_1,\|\mathcal{R}(\rho)\|_1]-1\big\}^2 \tag{20}$$

for either $q\geqslant 2$ with $m\geqslant 3$ or $q\geqslant 3$ with $m=2$, and

$$C_q(\rho_{AB})\geqslant\frac{1-2^{1-q}}{2-2^{2-s}}\big\{\max[\|\rho^{T_A}\|_1,\|\mathcal{R}(\rho)\|_1]-1\big\}^2 \tag{21}$$

for $2.4721=s\leqslant q<3$ with $m=2$. Here m is the smallest of the dimensions of the two systems.

Next, we improve the above results of Theorem 2 and Corollary 1 by utilizing the relation presented in formulas (20) and (21). The conclusion is shown in the following theorem.

Theorem 3. For any n-qubit quantum state ρ, one derives

$$C_{q-n}(\rho) \geqslant \frac{2^{q-1}-1}{2^{q-1}n}\sum_{k=1}^{n}(N^k)^2 \tag{22}$$

for $q \geqslant 3$, and

$$C_{q-n}(\rho) \geqslant \frac{1-2^{1-q}}{(2-2^{2-s})n}\sum_{k=1}^{n}(N^k)^2 \tag{23}$$

for $s \leqslant q < 3$. For any n-qudit quantum state $\rho \in \otimes_{i=1}^{n}\mathcal{H}_i$, $\dim\mathcal{H}_i = m$, $i = 1, 2, \ldots, n$, one obtains

$$C_{q-n}(\rho) \geqslant \frac{1-m^{1-q}}{n(m-1)^2}\sum_{k=1}^{n}(N^k)^2 \tag{24}$$

for $q \geqslant 2$ and $m \geqslant 3$.

For $0 \leqslant \alpha \leqslant \frac{1}{2}$, we have the following result.

Theorem 4. For any n-qubit quantum state ρ, the relation between the α-n-ME concurrence ($0 \leqslant \alpha \leqslant \frac{1}{2}$) and global negativity of quantum state ρ is

$$C_{\alpha-n}(\rho) \geqslant \frac{2^{1-\alpha}-1}{n}\sum_{k=1}^{n}N^k, \tag{25}$$

and for any n-qudit quantum state $\rho \in \otimes_{i=1}^{n}\mathcal{H}_i$, $\dim\mathcal{H}_i = m$, $i = 1, 2, \ldots, n$, we obtain

$$C_{\alpha-n}(\rho) \geqslant \frac{m^{1-\alpha}-1}{n(m-1)}\sum_{k=1}^{n}N^k. \tag{26}$$

Next we will use these bounds to detect the entangled states.

Example 1. Consider the mixture of the n-qubit W state and white noise

$$\rho = a|W\rangle\langle W| + \frac{1-a}{2^n}\mathbb{I},$$

where $|W\rangle = \frac{|0\cdots01\rangle+|0\cdots10\rangle+\cdots+|1\cdots00\rangle}{\sqrt{n}}$. By calculation, if $a \geqslant \frac{n}{n+2^n\sqrt{n-1}}$, then there is

$$N^1 = N^2 = \cdots = N^n = \frac{(2^n\sqrt{n-1}+n)a-n}{n2^{n-1}}.$$

Following from the relations presented in inequalities (22), (23), and (25), for $a \in [\frac{n}{n+2^n\sqrt{n-1}}, 1]$, one can obtain

$$C_{q-n}(\rho) \geqslant \frac{2^{q-1}-1}{2^{q-1}}\left(\frac{(2^n\sqrt{n-1}+n)a-n}{n2^{n-1}}\right)^2$$

for $q \geqslant 3$, and

$$C_{q-n}(\rho) \geqslant \frac{1-2^{1-q}}{2-2^{2-s}}\left(\frac{(2^n\sqrt{n-1}+n)a-n}{n2^{n-1}}\right)^2$$

for $s \leqslant q < 3$, and

$$C_{\alpha-n}(\rho) \geqslant (2^{1-\alpha}-1)\frac{(2^n\sqrt{n-1}+n)a-n}{n2^{n-1}}$$

for $0 \leqslant \alpha \leqslant \frac{1}{2}$.

When $a \in (\frac{n}{n+2^n\sqrt{n-1}}, 1]$, ρ is n nonseparable, that is, it is an entangled state. However, the result in Ref. [29] is that the state ρ is entangled if $a \in (\frac{n}{2^n+n}, 1]$. Due to $\frac{n}{n+2^n\sqrt{n-1}} < \frac{n}{2^n+n}$ when $n > 2$, the range of entanglement that can be detected using the negativity method is larger than that in Ref. [29].

V. DEGREE OF SEPARABILITY

In Ref. [30], Hong *et al*. gave two inequalities that can be used to determine whether a state is k nonseparable. Let $|\phi_1\rangle = |0\rangle^{\otimes n}$ and $|\phi_2\rangle = |1\rangle^{\otimes n}$ for Theorem 3 in Ref. [30], if an n-qubit quantum state ρ is k separable, then it fulfills the inequality

$$(2^k-2)A \leqslant B, \tag{27}$$

where

$$A = |\rho_{1,2^n}|,$$
$$B = \sum_{i=2}^{2^n-1}\sqrt{\rho_{i,i}\rho_{2^n-i+1,2^n-i+1}}.$$

Let $|\psi_i^s\rangle = |0\rangle^{\otimes(i-1)}|1\rangle|0\rangle^{\otimes(n-i)}$ for Theorem 4 in Ref. [30]. If an n-qubit quantum state ρ is k separable, then the result is accord with that of Ref. [37], which is

$$C \leqslant D + (n-k)E. \tag{28}$$

Here

$$C = \sum_{0\leqslant i\neq j\leqslant n-1}|\rho_{2^i+1,2^j+1}|,$$
$$D = \sum_{0\leqslant i\neq j\leqslant n-1}\sqrt{\rho_{1,1}\rho_{2^i+2^j+1,2^i+2^j+1}},$$
$$E = \sum_{i=0}^{n-1}|\rho_{2^i+1,2^i+1}|.$$

Violation of any of the above inequalities (27) and (28) implies that the state is k nonseparable.

Based on the relation of Eq. (27), an effective k_{eff}^1 can be defined

$$k_{\text{eff}}^1 = \log_2(2+\frac{B}{A}). \tag{29}$$

Note that, although Eq. (29) and Eq. (7) in Ref. [60] have the same form of expression, they represent completely different meanings. k_{eff}^1 quantifies the degree of separability, while Eq. (7) in Ref. [60] reflects the degree of entanglement. In addition, the effective k_{eff}^2 was presented in Ref. [37], which was obtained by inverting Eq. (28),

$$k_{\text{eff}}^2 = n - \frac{C-D}{E}. \tag{30}$$

So k_{eff}^2 can be also regarded as the degree of separability. If an n-partite quantum state is k separable, then we should have that $k_{\text{eff}}^i \geqslant k$ $(i = 1, 2)$. We can say that if $k_{\text{eff}}^1 < k$ or $k_{\text{eff}}^2 < k$, then the quantum state is k nonseparable. In particular, for a fully separable quantum state, one ought to obtain that both k_{eff}^1 and k_{eff}^2 are greater than or equal to n. If one of the k_{eff}^i $(i = 1, 2)$ is less than 2, then the quantum state is genuinely entangled.

To illustrate our results more clearly, we present two concrete examples.

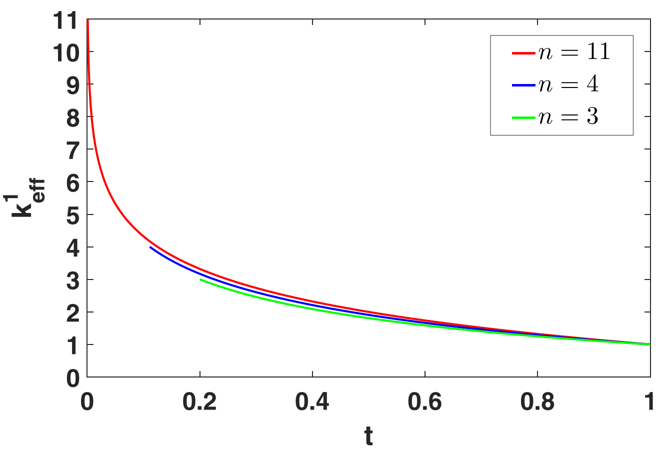

FIG. 1. Interpretation the degree of separability $k^1_{\mathrm{eff}}(t,n)$ of the state given in formula (31). The red (upper) line, blue (middle) line, and green (lower) line, respectively, correspond to the cases where n takes 11, 4, 3. Since $k^1_{\mathrm{eff}} > n$ is meaningless, it is omitted here.

Example 2. Consider an n-qubit quantum state

$$\rho = t|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}| + \frac{1-t}{2^n}\mathbb{I}, \tag{31}$$

where $|\mathrm{GHZ}\rangle = \frac{|0\rangle^{\otimes n}+|1\rangle^{\otimes n}}{\sqrt{2}}$. By calculation, when $t \in (\frac{2^n-2}{2^{n+k-1}-2}, 1]$, the quantum state ρ is k nonseparable. That is, if $t \in (\frac{2^n-2}{2^{n+k-1}-2}, 1]$, then $C_{q-k}(\rho) > 0$. Specifically, if $t \in (\frac{2^n-2}{2^{n+1}-2}, 1]$, ρ is not 2 separable, namely, the state ρ is genuinely multipartite entangled, which means $C_{q-2}(\rho) > 0$. The state ρ is fully separable ($k = n$) when $t \in [0, \frac{1}{2^{n-1}+1}]$. This range is consistent with the range given in Ref. [33]. Due to $\lim_{n\to+\infty} \frac{2^n-2}{2^{n+1}-2} = \frac{1}{2}$, when n is large enough, nearly half of the states are genuinely multipartite entangled. In addition, $\lim_{n\to+\infty} \frac{1}{2^{n-1}+1} = 0$, which means the quantum states are almost entangled when $n \to +\infty$. For this state, we can obtain

$$k^1_{\mathrm{eff}}(t,n) = \log_2\left(\frac{(2^n-2)(1-t)}{2^{n-1}t} + 2\right),$$

which is a function of t and n. Plotting $k^1_{\mathrm{eff}}(t,n)$, we take $n = 3, 4, 11$ here. As shown in Fig. 1, we observe that k^1_{eff} monotonically increases with the amount of decoherence.

Example 3. Consider an n-qubit quantum state ρ, which is a mixture of the Greenberger-Horne-Zeilinger (GHZ) state, W state, and white noise

$$\rho = a|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}| + b|W\rangle\langle W| + \frac{1-a-b}{2^n}\mathbb{I},$$

where $|W\rangle = \frac{|0\cdots01\rangle+|0\cdots10\rangle+\cdots+|1\cdots00\rangle}{\sqrt{n}}$. By simple calculation, one has $A = \frac{a}{2}$, $B = 2n\sqrt{(\frac{b}{n} + \frac{1-a-b}{2^n})\frac{1-a-b}{2^n}} + (2^{n-1}-n-1)\frac{1-a-b}{2^{n-1}}$, $C = (n-1)b$, $D = n(n-1)\sqrt{(\frac{a}{2} + \frac{1-a-b}{2^n})\frac{1-a-b}{2^n}}$, $E = n(\frac{b}{n} + \frac{1-a-b}{2^n})$. Set $n = 4$, when $k^1_{\mathrm{eff}} = \log_2[\frac{2}{a}(\frac{\sqrt{(1-a+3b)(1-a-b)}}{2} + \frac{3(1-a-b)}{8}) + 2] < k$ or $k^2_{\mathrm{eff}} = 4 - \frac{12b-3\sqrt{(1+7a-b)(1-a-b)}}{1-a+3b} < k$, the quantum state ρ is k nonseparable. To make it more intuitive, we plot

$$(2^k-2)a = \sqrt{(1-a+3b)(1-a-b)} + \frac{3(1-a-b)}{4} \tag{32}$$

and

$$4b = \sqrt{(1+7a-b)(1-a-b)} + (4-k)\frac{1-a+3b}{3} \tag{33}$$

for $k = 2$ and $k = 4$, respectively, in Fig. 2.

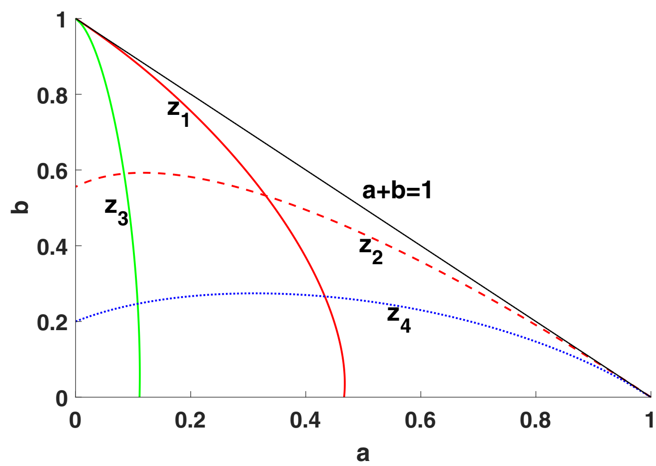

FIG. 2. The solid red line z_1 and the dashed red line z_2 represent the equality (32) and (33) for $k = 2$, respectively; the solid green line z_3 and the dotted blue line z_4 denote the equality (32) and (33) for $k = 4$, respectively. For case $a > b$, the inequality (27) is better at detecting entanglement, whereas for case $a < b$, the inequality (28) is better.

As we can see in Fig. 2, the states in the region bounded by line z_1, axis a, and line $a + b = 1$, and the region bounded by line z_2, axis b, and line $a + b = 1$ are genuinely four-partite entangled. The states in the region bounded by line z_1, z_2, and axis a can only be detected by the first form (27); the states in the region bounded by line z_1, z_2, and axis b can only be detected by the second form (28); and the states in the intersection region bounded by line z_1, line z_2, and line $a + b = 1$ are those where both inequalities can detect. Similarly, the states in the region bounded by line z_3, axis a, and line $a + b = 1$, and the region bounded by line z_4, axis b, and line $a + b = 1$ are not 4 separable, namely, they are entangled states.

Therefore, the combination of Eqs. (29) and (30) can be used to judge the separability of quantum states more effectively.

VI. COMPARING q-GME CONCURRENCE WITH OTHER GME MEASURES

We first introduce concurrence fill [61], which is defined based on the Heron formula of the triangle area

$$F_{A_1A_2A_3} = \Big[\tfrac{16}{3}P\big(P - C^2_{A_1|A_2A_3}\big)\big(P - C^2_{A_2|A_1A_3}\big) \times \big(P - C^2_{A_3|A_1A_2}\big)\Big]^{\frac{1}{4}},$$

where $P = \frac{1}{2}(C^2_{A_1|A_2A_3} + C^2_{A_2|A_1A_3} + C^2_{A_3|A_1A_2})$. $F_{A_1A_2A_3}$ perfectly characterizes the geometric meaning of three-qubit quantum states.

For any n-partite pure state $|\varphi\rangle$, the geometric mean of q concurrence (GqC) [41] is

$$\mathcal{G}_q(|\varphi\rangle) = [\mathcal{P}_q(|\varphi\rangle)]^{\frac{1}{c(\gamma)}},$$

where $\mathcal{P}_q(|\varphi\rangle) = \prod_{\gamma_i\in\gamma} C_{qA_{\gamma_i}|\overline{A}_{\gamma_i}}(|\varphi\rangle)$, $\gamma = \{\gamma_i\}$ denotes all of possible bipartitions, and $c(\gamma)$ represents the cardinality of the set γ.

Mathematically, GqC is defined based on the geometric mean of bipartite concurrence, however, the q-GME concurrence is a special case of the q-k-ME concurrence defined in terms of the minimum of all possible k partitions. Next we illustrate with a specific example, demonstrating that $C_{q-\mathrm{GME}}$ and $\mathcal{G}_q$ are distinct. In fact, the q-GME concurrence is consistent with GqC for three-partite completely symmetric pure states. For example, $|\mathrm{GHZ}_3\rangle = \frac{|000\rangle+|111\rangle}{\sqrt{2}}$ and $|W_3\rangle = \frac{|100\rangle+|010\rangle+|001\rangle}{\sqrt{3}}$, $C_{q-\mathrm{GME}}(|\mathrm{GHZ}_3\rangle) = \mathcal{G}_q(|\mathrm{GHZ}_3\rangle) = 1 - \frac{1}{2^{q-1}}$ and $C_{q-\mathrm{GME}}(|W_3\rangle) = \mathcal{G}_q(|W_3\rangle) = 1 - [(\frac{2}{3})^q + (\frac{1}{3})^q]$.

Now we compare these measures by a specific example. Theoretically, the q-GME concurrence defined by us may cause sharp peaks due to the minimization involved, but here we will present an example to show that the measure defined by us is sometimes smoother than GqC and concurrence fill.

Example 4. Considering a quantum state $|\phi_\theta\rangle = -\frac{1}{2}\cos\theta|010\rangle + \frac{\sqrt{3}}{2}\cos\theta|100\rangle + \sin\theta|011\rangle$, we obtain

$$C_{q-\mathrm{GME}}(|\phi_\theta\rangle) = \min\left\{1 - \left[\left(\frac{1}{4}+\frac{3}{4}\sin^2\theta\right)^q + \left(\frac{3}{4}\cos^2\theta\right)^q\right],\right.$$
$$\left. 1 - \left[\left(\frac{1+\sqrt{1-3\sin^2\theta\cos^2\theta}}{2}\right)^q + \left(\frac{1-\sqrt{1-3\sin^2\theta\cos^2\theta}}{2}\right)^q\right]\right\},$$

$$\mathcal{G}_q(|\phi_\theta\rangle) = \left\{1 - \left[\left(\frac{1}{4}+\frac{3}{4}\sin^2\theta\right)^q + \left(\frac{3}{4}\cos^2\theta\right)^q\right]\right\}^{\frac{2}{3}}$$
$$\times \left\{1 - \left[\left(\frac{1+\sqrt{1-3\sin^2\theta\cos^2\theta}}{2}\right)^q + \left(\frac{1-\sqrt{1-3\sin^2\theta\cos^2\theta}}{2}\right)^q\right]\right\}^{\frac{1}{3}},$$

$$F(|\phi_\theta\rangle) = \left[\frac{16}{3}P(P-\mathcal{C}_1)^2(P-\mathcal{C}_2)\right]^{1/4},$$

where

$$P = 3 - \left\{\left[\left(\frac{1+\sqrt{1-3\sin^2\theta\cos^2\theta}}{2}\right)^2 + \left(\frac{1-\sqrt{1-3\sin^2\theta\cos^2\theta}}{2}\right)^2\right] - 2\left[\left(\frac{1}{4}+\frac{3}{4}\sin^2\theta\right)^2 + \left(\frac{3}{4}\cos^2\theta\right)^2\right]\right\},$$

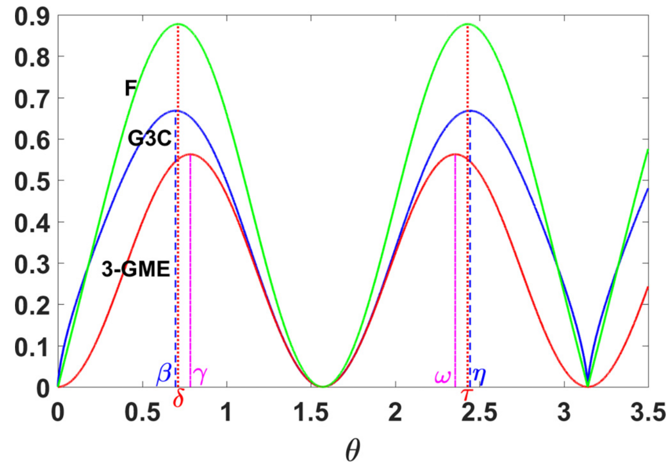

FIG. 3. Set $q = 3$. The red (lower) curve line expresses the $C_{3-\mathrm{GME}}(|\phi\rangle)$, the blue (middle) curve line is $\mathcal{G}_3(|\phi\rangle)$, the green (upper) curve line denotes concurrence fill of pure state $|\phi\rangle$.

$$\mathcal{C}_1 = 2 - 2\left[\left(\frac{1}{4}+\frac{3}{4}\sin^2\theta\right)^2 + \left(\frac{3}{4}\cos^2\theta\right)^2\right],$$
$$\mathcal{C}_2 = 2 - 2\left[\left(\frac{1+\sqrt{1-3\sin^2\theta\cos^2\theta}}{2}\right)^2 + \left(\frac{1-\sqrt{1-3\sin^2\theta\cos^2\theta}}{2}\right)^2\right],$$

when $q = 3$, $C_{3-\mathrm{GME}}(|\phi_\theta\rangle) = 1 - [(\frac{1+\sqrt{1-3\sin^2\theta\cos^2\theta}}{2})^3 + (\frac{1-\sqrt{1-3\sin^2\theta\cos^2\theta}}{2})^3]$. The comparison of the three measures is shown in Fig. 3.

Observing Fig. 3, we find the pure state $|\phi_\theta\rangle$ is not genuinely entangled when $\theta = 0, \frac{\pi}{2}, \pi$. When $\theta \in [\delta, \gamma] \cup [\omega, \tau]$, the entanglement order of $C_{3-\mathrm{GME}}$ is different from G3C and concurrence fill, that is, there exist $\vartheta_1,\ \vartheta_2 \in [\delta, \gamma]$ or $\vartheta_1,\ \vartheta_2 \in [\omega, \tau]$ such that $C_{3-\mathrm{GME}}(|\phi_{\vartheta_1}\rangle) \leqslant C_{3-\mathrm{GME}}(|\phi_{\vartheta_2}\rangle)$, while $\mathcal{G}_3(|\phi_{\vartheta_1}\rangle) \geqslant \mathcal{G}_3(|\phi_{\vartheta_2}\rangle)$ and $F(|\phi_{\vartheta_1}\rangle) \geqslant F(|\phi_{\vartheta_2}\rangle)$. In addition, when $\theta \in [\beta, \delta] \cup [\tau, \eta]$, the order of entanglement of $C_{3-\mathrm{GME}}$ and G3C is also different. When $\theta \in (0, \gamma)$, only the measure we defined corresponds to the unique quantum state. When $\theta = \pi$, the G3C and concurrence fill have a sharp peak, while our measure is smooth. Therefore, q-GME concurrence is advantageous in some cases.

To see whether $C_{q-\mathrm{GME}}(|\phi\rangle)$ and $C_{\alpha-\mathrm{GME}}(|\phi\rangle)$ have the same monotonicity about q and α, respectively, we plot two figures. In Fig. 4(a) we take $q \in [2, 12]$, the function $C_{q-\mathrm{GME}}$ is increasing of q. In Fig. 4(b) we take $\alpha \in [0, \frac{1}{2}]$, the function $C_{\alpha-\mathrm{GME}}$ is decreasing of α. This also reflects that C_{q-k} and $C_{\alpha-k}$ are different.

VII. CONCLUSION

In this work, we proposed two types of general parametrized entanglement measures, q-k-ME concurrence C_{q-k} $(q > 1,\ 2 \leqslant k \leqslant n)$ and α-k-ME concurrence $C_{\alpha-k}$ $(0 \leqslant \alpha < 1,\ 2 \leqslant k \leqslant n)$, in n-partite systems from the standpoint of k nonseparability, and shown that these measures C_{q-k} and $C_{\alpha-k}$ satisfy the requirements including

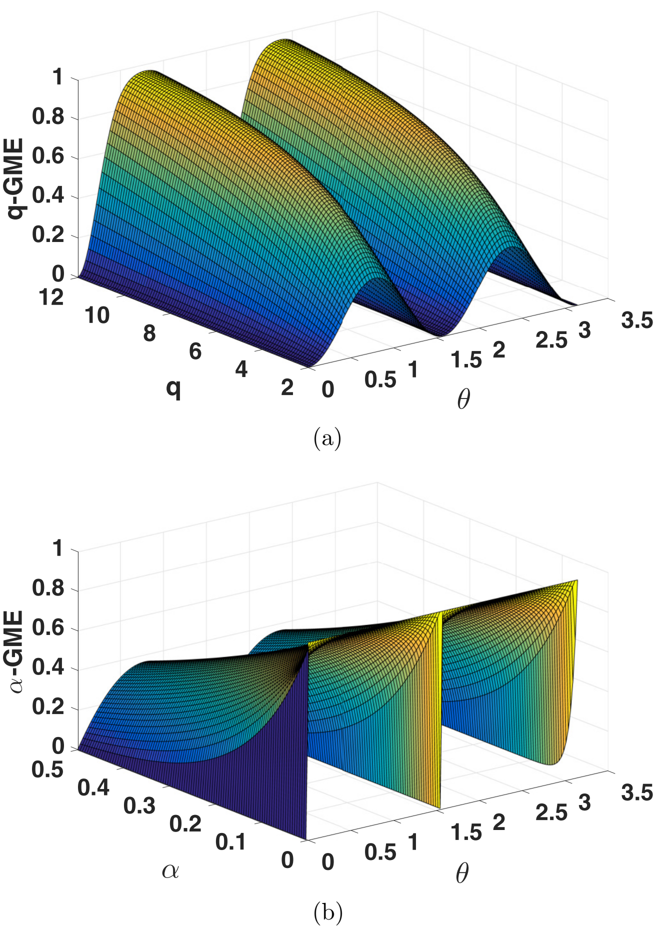

FIG. 4. For $q \in [2, 12]$, $C_{q-\mathrm{GME}}$ is an increasing function about q. For $\alpha \in [0, \frac{1}{2}]$, $C_{\alpha-\mathrm{GME}}$ is a decreasing function about α.

entanglement monotone, strong monotone, convexity, being zero for any k-separable state and strictly positive on any k-nonseparable state. In addition, C_{q-k} also satisfy subadditivity. It is evident that, as special cases of C_{q-k} and $C_{\alpha-k}$, parametrized GME measures $C_{q-\mathrm{GME}}$ and $C_{\alpha-\mathrm{GME}}$ can inherit their properties, respectively. The q-k-ME concurrence of ρ is lower bounded by the maximum of C_{q-k} of the PI part of ρ and so is $C_{\alpha-k}(\rho)$. Apart from that, we associated global negativity with q-n-ME concurrence (α-n-ME concurrence) and gave the lower bound of C_{q-n} ($C_{\alpha-n}$), which could be used to detect whether a quantum state is entangled. We presented an example which is a mixture of W state and white noise, and observed that when $a \in (\frac{n}{n+2^n\sqrt{n-1}}, \frac{n}{2^n+n}]$, our result can detect that these states are entangled, whereas the result in Ref. [29] cannot. What is more, we discussed the degree of entanglement of k-nonseparable states, where the violation of any of these inequalities (27) and (28) implies that the state is k nonseparable. The combination of these two formulas can detect entanglement more effectively. Comparing the q-GME concurrence with GqC and concurrence fill through a concrete example, we found that they generate different entanglement orders and that q-GME concurrence is sometimes smooth. The measures defined by us could be useful for further study of multipartite quantum entanglement.

ACKNOWLEDGMENTS

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APPENDIX A: PROOF OF PROPOSITION 1

First, we prove q-k-ME concurrence ($q > 1$) satisfies the conditions (M1) to (M6).

(M1) Since $\mathrm{Tr}\rho^q \leqslant 1$, we can know $C_{q-k}(\rho) \geqslant 0$. Given a k-separable pure state $|\varphi\rangle = \otimes_{t=1}^{k} |\varphi_t\rangle_{A_t}$, A_1 and $\overline{A}_1$, A_2 and $\overline{A}_2, \ldots,$ A_k and $\overline{A}_k$ do not exist correlation, where $\overline{A}_i$ is the complement of A_i, one obtains $C_q(|\varphi\rangle_{A_t|\overline{A}_t}) = 0$, $t = 1, 2, \ldots, k$. Then we can easily get $C_{q-k}(|\varphi\rangle) = 0$ for arbitrary k-separable pure states. For any k-separable mixed state ρ with pure-state ensemble decomposition $\{p_i, \rho_i\}$, $\rho_i = |\varphi_i\rangle\langle\varphi_i|$ and $|\varphi_i\rangle$ is k separable, then $C_{q-k}(\rho) \leqslant \sum_i p_i C_{q-k}(|\varphi_i\rangle) = 0$. Therefore, $C_{q-k}(\rho) = 0$ for any k-separable quantum state.

(M2) Suppose that the set $\{A_1|A_2|\cdots|A_k\}$ contains all of the k partitions ($2 \leqslant k \leqslant n$) of set $A = \{1, 2, \ldots, n\}$. For arbitrary k-nonseparable pure states, there exist $k' \in \{1, 2, \ldots, k\}$ such that subsystems $A_{k'}$ and $\overline{A}_{k'}$ are entangled, then $C_q(|\varphi\rangle_{A_{k'}|\overline{A}_{k'}}) = 1 - \mathrm{Tr}(\rho^q_{A_{k'}}) > 0$, thus we can easily derive $C_{q-k}(|\varphi\rangle) > 0$. For any k-nonseparable mixed state ρ, there is no convex combination of k-separable pure states. Hence $C_{q-k}(\rho) > 0$ for any k-nonseparable state.

(M3) By the property of trace, $C_{q-k}(\rho)$ is invariant under local unitary transformation.

(M4) We first demonstrate that C_{q-k} satisfies monotonicity.

Because q concurrence is nonincreasing under LOCC for arbitrary bipartite pure states [38], namely, the inequality $C_{qA_t|\overline{A}_t}[\Lambda_{\mathrm{LOCC}}(|\varphi\rangle)] \leqslant C_{qA_t|\overline{A}_t}(|\varphi\rangle)$ holds for any pure state $|\varphi\rangle$, then we can obtain

$$\begin{aligned} &C_{q-k}[\Lambda_{\mathrm{LOCC}}(|\varphi\rangle)] \\ &\leqslant \min \frac{\sum_{t=1}^{k} C_{qA_t|\overline{A}_t}[\Lambda_{\mathrm{LOCC}}(|\varphi\rangle)]}{k} \\ &\leqslant \min \frac{\sum_{t=1}^{k} C_{qA_t|\overline{A}_t}(|\varphi\rangle)}{k} \\ &= C_{q-k}(|\varphi\rangle). \end{aligned}$$

For any mixed state ρ with the optimal pure decomposition $\{p_i, \rho_i\}$, $\rho_i = |\varphi_i\rangle\langle\varphi_i|$, one has

$$\begin{aligned} &C_{q-k}[\Lambda_{\mathrm{LOCC}}(\rho)] \\ &\leqslant \sum_i p_i C_{q-k}[\Lambda_{\mathrm{LOCC}}(|\varphi_i\rangle)] \\ &\leqslant \sum_i p_i C_{q-k}(|\varphi_i\rangle) \\ &= C_{q-k}(\rho). \end{aligned}$$

Here the first inequality is due to the definition of $C_{q-k}(\rho)$, the second inequality holds because C_{q-k} is nonincreasing for any pure state under LOCC.

Next we show C_{q-k} conforms to strong monotonicity.

Owing to the fact that q concurrence ($q>1$) is entanglement monotone for arbitrary bipartite quantum states [38], that is, the inequality $C_{qA_t|\overline{A}_t}(\rho)\geqslant\sum_j p_jC_{qA_t|\overline{A}_t}(\sigma_j)$ holds, where an ensemble of state σ_j with the respective corresponding probability p_j is obtained by LOCC acting on ρ. We first consider the case that $\rho=|\varphi\rangle\langle\varphi|$ and σ_j are pure states. Assume that $A_1|A_2|\cdots|A_k$ is the optimal partition of ρ, then we get

$$\begin{aligned}
C_{q-k}(\rho)&=\frac{\sum_{t=1}^{k}\left(1-\mathrm{Tr}\rho_{A_t}^q\right)}{k}\\
&=\frac{\sum_{t=1}^{k}C_{qA_t|\overline{A}_t}(|\varphi\rangle)}{k}\\
&\geqslant\frac{\sum_{t=1}^{k}\sum_j p_jC_{qA_t|\overline{A}_t}(\sigma_j)}{k}\\
&=\sum_j p_j\frac{\sum_{t=1}^{k}C_{qA_t|\overline{A}_t}(\sigma_j)}{k}\\
&\geqslant\sum_j p_jC_{q-k}(\sigma_j),
\end{aligned}$$

where the last inequality holds according to Eq. (5).

For any mixed state ρ with the optimal pure decomposition $\{p_i,\rho_i\}$, $\rho_i=|\varphi_i\rangle\langle\varphi_i|$, one has

$$\begin{aligned}
C_{q-k}(\rho)&=\sum_i p_iC_{q-k}(|\varphi_i\rangle)\\
&\geqslant\sum_{ij}p_ip(j|i)C_{q-k}\left(|\varphi_i^j\rangle\right)\\
&=\sum_{ij}p_jp(i|j)C_{q-k}\left(|\varphi_i^j\rangle\right)\\
&=\sum_j p_j\left(\sum_i p(i|j)C_{q-k}\left(|\varphi_i^j\rangle\right)\right)\\
&\geqslant\sum_j p_jC_{q-k}(\sigma_j).
\end{aligned}$$

Here the state $|\varphi_i^j\rangle=\frac{\Lambda_j|\varphi_i\rangle}{\sqrt{\mathrm{Tr}(\Lambda_j|\varphi_i\rangle\langle\varphi_i|\Lambda_j^\dagger)}}$ is obtained with probability $p(j|i)=\mathrm{Tr}(\Lambda_j|\varphi_i\rangle\langle\varphi_i|\Lambda_j^\dagger)$ after performing stochastic LOCC on $|\varphi_i\rangle$, and $p_j=\mathrm{Tr}(\Lambda_j\rho\Lambda_j^\dagger)$ is the probability of the outcome j occurring with $\sigma_j=\sum_i p(i|j)|\varphi_i^j\rangle\langle\varphi_i^j|$, $p(i|j)=p_ip(j|i)/p_j$. The first inequality is true because C_{q-k} satisfies the strong monotonicity for any pure state, while the second inequality holds following from Eq. (7).

(M5) The convexity holds due to convex-roof extension.

(M6) Further we prove C_{q-k} fulfills the subadditivity. Let ρ and σ be two arbitrary pure states and $\rho=|\varphi\rangle\langle\varphi|$, $\sigma=|\phi\rangle\langle\phi|$. Suppose that there exist the optimal partitions $A_1|A_2|\cdots|A_k$ and $B_1|B_2|\cdots|B_k$ satisfying the condition of k partition such that $C_{q-k}(\rho)=\frac{\sum_{t=1}^k[1-\mathrm{Tr}(\rho_{A_t}^q)]}{k}$, $C_{q-k}(\sigma)=\frac{\sum_{t=1}^k[1-\mathrm{Tr}(\sigma_{B_t}^q)]}{k}$, we can get

$$\begin{aligned}
&C_{q-k}(\rho\otimes\sigma)-C_{q-k}(\rho)-C_{q-k}(\sigma)\\
&\quad\leqslant\frac{1}{k}\left\{\sum_{t=1}^{k}\left[1-\mathrm{Tr}\left(\rho_{A_t}^q\right)\mathrm{Tr}\left(\sigma_{B_t}^q\right)\right]-\sum_{t=1}^{k}\left[1-\mathrm{Tr}\left(\rho_{A_t}^q\right)\right]\right.\\
&\qquad\left.-\sum_{t=1}^{k}\left[1-\mathrm{Tr}\left(\sigma_{B_t}^q\right)\right]\right\}\\
&\quad=\frac{1}{k}\sum_{t=1}^{k}\left[-\mathrm{Tr}\left(\rho_{A_t}^q\right)\mathrm{Tr}\left(\sigma_{B_t}^q\right)+\mathrm{Tr}\left(\rho_{A_t}^q\right)+\mathrm{Tr}\left(\sigma_{B_t}^q\right)-1\right]\\
&\quad=-\frac{1}{k}\sum_{t=1}^{k}\left[1-\mathrm{Tr}\left(\rho_{A_t}^q\right)\right]\left[1-\mathrm{Tr}\left(\sigma_{B_t}^q\right)\right]\\
&\quad\leqslant 0.
\end{aligned}\tag{A1}$$

Suppose that ρ is any mixed state with optimal pure decomposition $\rho=\sum_i p_i\rho_i$ and $\rho_i=|\varphi_i\rangle\langle\varphi_i|$, $\sigma=|\phi\rangle\langle\phi|$ is any pure state, then we have

$$\begin{aligned}
C_{q-k}(\rho\otimes\sigma)&=C_{q-k}\left(\sum_i p_i\rho_i\otimes|\phi\rangle\langle\phi|\right)\\
&\leqslant\sum_i p_iC_{q-k}(|\varphi_i\rangle\langle\varphi_i|\otimes|\phi\rangle\langle\phi|)\\
&\leqslant\sum_i p_i[C_{q-k}(|\varphi_i\rangle)+C_{q-k}(|\phi\rangle)]\\
&=C_{q-k}(\rho)+C_{q-k}(\sigma),
\end{aligned}\tag{A2}$$

where the first inequality is owing to the convexity of C_{q-k} and the second inequality can be obtained from the result of inequality (A1).

By similar procedures, if ρ, σ are any two mixed states, and they have optimal pure decompositions $\rho=\sum_i p_i|\varphi_i\rangle\langle\varphi_i|$, $\sigma=\sum_j q_j|\phi_j\rangle\langle\phi_j|$, one has

$$\begin{aligned}
C_{q-k}(\rho\otimes\sigma)&=C_{q-k}\left(\sum_i p_i\rho_i\otimes\sum_j q_j\sigma_j\right)\\
&\leqslant\sum_j q_jC_{q-k}\left(\sum_i p_i|\varphi_i\rangle\langle\varphi_i|\otimes|\phi_j\rangle\langle\phi_j|\right)\\
&\leqslant\sum_j q_j[C_{q-k}(\rho)+C_{q-k}(\sigma_j)]\\
&=C_{q-k}(\rho)+C_{q-k}(\sigma).
\end{aligned}$$

Here the first inequality is due to the convexity of C_{q-k}, the second inequality holds because of the inequality (A2).

With similar methods, we can also prove α-k-ME concurrence ($0\leqslant\alpha<1$) meets the requirements of (M1) to (M5) for being a ME measure. It is not difficult to prove that $C_{\alpha-k}$ does not satisfy subadditivity.

APPENDIX B: PROOF OF THEOREM 1

Suppose that the set $\{1,2,\ldots,n\}$ is divided into $A_1|A_2|\cdots|A_k$ satisfying the condition (1), then $\Pi_j(A_1)|\Pi_j(A_2)|\cdots|\Pi_j(A_k)$ is still a k partition of the set $\{1,2,\ldots,n\}$. Let $|\varphi\rangle$ be any pure state, then $\Pi_j(|\varphi\rangle)$ is also a pure state, and we have

$$C_{q-k}(|\varphi\rangle)=C_{q-k}(\Pi_j|\varphi\rangle), \tag{B1}$$

where $\Pi_j\in S_n$, S_n is an n-order symmetric group.

By using the convexity of C_{q-k} and the relation shown in Eq. (B1), one gets

$$\begin{aligned}C_{q-k}(\rho^{\mathrm{PI}})&\leqslant\frac{1}{n!}\sum_{j=1}^{n!}C_{q-k}(\Pi_j|\varphi\rangle)\\&=\frac{1}{n!}\sum_{j=1}^{n!}C_{q-k}(|\varphi\rangle)=C_{q-k}(|\varphi\rangle).\end{aligned} \tag{B2}$$

Given a mixed state ρ, assume $\{p_i,\rho_i\}$ is the optimal pure decomposition of ρ, $\rho_i=|\varphi_i\rangle\langle\varphi_i|$, then we see

$$\begin{aligned}C_{q-k}(\rho)&=\sum_i p_iC_{q-k}(|\varphi_i\rangle)\geqslant\sum_i p_iC_{q-k}\big(\rho_i^{\mathrm{PI}}\big)\\&\geqslant C_{q-k}(\rho^{\mathrm{PI}}).\end{aligned}$$

Here the first inequality is based on inequality (B2) and the second inequality is due to the convexity of C_{q-k}.

Because the PI part depends on the choice of bases [37] and the relations listed above are true for ρ_U^{PI} obtained under any locally unitary transformation U, one derives

$$C_{q-k}(\rho)\geqslant\max_U C_{q-k}\big(\rho_U^{\mathrm{PI}}\big).$$

With similar procedures, the inequality (15) can be obtained.

APPENDIX C: PROOF OF THEOREM 2

For any n-qubit pure state $\rho=|\varphi\rangle\langle\varphi|$,

$$\begin{aligned}C_{q-n}(\rho)&=\frac{\sum_{k=1}^{n}\big[1-\mathrm{Tr}\big(\rho_k^q\big)\big]}{n}\\&=\frac{C_{q1|A^1}+C_{q2|A^2}+\cdots+C_{qn|A^n}}{n}\\&\geqslant\frac{[(N^1+1)^{q-1}-1]^2+\cdots+[(N^n+1)^{q-1}-1]^2}{n(2^{2q-2}-2^{q-1})}.\end{aligned}$$

Here $A^p=\{1,2,\ldots,n\}\setminus\{p\}$, $p=1,2,\ldots,n$. Note that when $q=2$, $C_{2p|A^p}(|\varphi\rangle)=\frac{[N(|\varphi\rangle)]^2}{2}$, then the inequality (18) holds with equality when $q=2$.

For any n-qubit mixed state ρ, suppose that $\{p_i,\rho_i\}$ is the optimal pure decomposition and $\rho_i=|\phi_i\rangle\langle\phi_i|$, then

$$\begin{aligned}C_{q-n}(\rho)&=\sum_i p_iC_{q-n}(|\phi_i\rangle)\\&\geqslant\sum_i p_i\frac{\sum_{k=1}^{n}[(N^k(|\phi_i\rangle)+1)^{q-1}-1]^2}{n(2^{2q-2}-2^{q-1})}\\&\geqslant\frac{\sum_{k=1}^{n}\big[\sum_i p_i(N^k(|\phi_i\rangle)+1)^{q-1}-1\big]^2}{n(2^{2q-2}-2^{q-1})}\\&\geqslant\frac{\sum_{k=1}^{n}\big[\big(\sum_i p_iN^k(|\phi_i\rangle)+1\big)^{q-1}-1\big]^2}{n(2^{2q-2}-2^{q-1})}\\&\geqslant\frac{\sum_{k=1}^{n}[(N^k(\rho)+1)^{q-1}-1]^2}{n(2^{2q-2}-2^{q-1})},\end{aligned}$$

where the second inequality holds because $y=x^2$ is a convex function, the third inequality is due to the fact that $y=x^{q-1}$ is convex for $q>2$, the third inequality is clearly true when $q=2$, and the last inequality holds from the convexity of N^k.

[1] A. K. Ekert, Quantum cryptography based on Bell's theorem, Phys. Rev. Lett. **67**, 661 (1991).
[2] C. H. Bennett, G. Brassard, and N. D. Mermin, Quantum cryptography without Bell's theorem, Phys. Rev. Lett. **68**, 557 (1992).
[3] B. A. Slutsky, R. Rao, P. C. Sun, and Y. Fainman, Security of quantum cryptography against individual attacks, Phys. Rev. A **57**, 2383 (1998).
[4] L. Masanes, Universally composable privacy amplification from causality constraints, Phys. Rev. Lett. **102**, 140501 (2009).
[5] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels, Phys. Rev. Lett. **70**, 1895 (1993).
[6] F. L. Yan and X. Q. Zhang, A scheme for secure direct communication using EPR pairs and teleportation, Eur. Phys. J. B **41**, 75 (2004).
[7] T. Gao, F. L. Yan, and Y. C. Li, Optimal controlled teleportation, Europhys. Lett. **84**, 50001 (2008).
[8] J. M. Renes and M. Grassl, Generalized decoding, effective channels, and simplified security proofs in quantum key distribution, Phys. Rev. A **74**, 022317 (2006).
[9] Y.-H. Zhou, Z. W. Yu, and X. B. Wang, Making the decoy-state measurement-device-independent quantum key distribution practically useful, Phys. Rev. A **93**, 042324 (2016).
[10] F. L. Yan, T. Gao, and E. Chitambar, Two local observables are sufficient to characterize maximally entangled states of N qubits, Phys. Rev. A **83**, 022319 (2011).
[11] M. Ikram, S. Y. Zhu, and M. S. Zubairy, Quantum teleportation of an entangled state, Phys. Rev. A **62**, 022307 (2000).
[12] H. W. Lee, Total teleportation of an entangled state, Phys. Rev. A **64**, 014302 (2001).
[13] H. J. Lee, D. Ahn, and S. W. Hwang, Dense coding in entangled states, Phys. Rev. A **66**, 024304 (2002).

[14] S. A. Hill and W. K. Wootters, Entanglement of a pair of quantum bits, Phys. Rev. Lett. **78**, 5022 (1997).

[15] W. K. Wootters, Entanglement of formation of an arbitrary state of two qubits, Phys. Rev. Lett. **80**, 2245 (1998).

[16] F. Mintert, M. Kuś, and A. Buchleitner, Concurrence of mixed multipartite quantum states, Phys. Rev. Lett. **95**, 260502 (2005).

[17] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Quantum entanglement, Rev. Mod. Phys. **81**, 865 (2009).

[18] K. Życzkowski, P. Horodecki, A. Sanpera, and M. Lewenstein, Volume of the set of separable states, Phys. Rev. A **58**, 883 (1998).

[19] G. Vidal and R. F. Werner, Computable measure of entanglement, Phys. Rev. A **65**, 032314 (2002).

[20] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, Mixed-state entanglement and quantum error correction, Phys. Rev. A **54**, 3824 (1996).

[21] P. W. Shor, Equivalence of additivity questions in quantum information theory, Commun. Math. Phys. **246**, 453 (2004).

[22] J. S. Kim, Tsallis entropy and entanglement constraints in multiqubit systems, Phys. Rev. A **81**, 062328 (2010).

[23] P. Krammer, H. Kampermann, D. Bruß, R. A. Bertlmann, L. C. Kwek, and C. Macchiavello, Multipartite entanglement detection via structure factors, Phys. Rev. Lett. **103**, 100502 (2009).

[24] O. Gühne and M. Seevinck, Separability criteria for genuine multiparticle entanglement, New J. Phys. **12**, 053002 (2010).

[25] O. Gittsovich, P. Hyllus, and O. Gühne, Multiparticle covariance matrices and the impossibility of detecting graph-state entanglement with two-particle correlations, Phys. Rev. A **82**, 032306 (2010).

[26] T. Gao and Y. Hong, Detection of genuinely entangled and nonseparable n-partite quantum states, Phys. Rev. A **82**, 062113 (2010).

[27] M. Huber, H. Schimpf, A. Gabriel, C. Spengler, D. Bruß, and B. C. Hiesmayr, Experimentally implementable criteria revealing substructures of genuine multipartite entanglement, Phys. Rev. A **83**, 022328 (2011).

[28] T. Gao and Y. Hong, Separability criteria for several classes of n-partite quantum states, Eur. Phys. J. D **61**, 765 (2011).

[29] T. Gao, Y. Hong, Y. Lu, and F. L. Yan, Efficient k-separability criteria for mixed multipartite quantum states, Europhys. Lett. **104**, 20007 (2013).

[30] Y. Hong, T. Gao, and F. L. Yan, Detection of k-partite entanglement and k-nonseparability of multipartite quantum states, Phys. Lett. A **401**, 127347 (2021).

[31] Y. Hong, X. F. Qi, T. Gao, and F. L. Yan, Detection of multipartite entanglement via quantum Fisher information, Europhys. Lett. **134**, 60006 (2021).

[32] Y. Hong, X. F. Qi, T. Gao, and F. L. Yan, Detection of the quantum states containing at most $k-1$ unentangled particles, Chin. Phys. B **30**, 100306 (2021).

[33] L. M. Zhang, T. Gao, and F. L. Yan, Relations among k-ME concurrence, negativity, polynomial invariants, and tangle, Quantum Inf. Process. **18**, 194 (2019).

[34] Z. H. Ma, Z. H. Chen, J. L. Chen, C. Spengler, A. Gabriel, and M. Huber, Measure of genuine multipartite entanglement with computable lower bounds, Phys. Rev. A **83**, 062325 (2011).

[35] Z. H. Chen, Z. H. Ma, J. L. Chen, and S. Severini, Improved lower bounds on genuine-multipartite-entanglement concurrence, Phys. Rev. A **85**, 062320 (2012).

[36] Y. Hong, T. Gao, and F. L. Yan, Measure of multipartite entanglement with computable lower bounds, Phys. Rev. A **86**, 062323 (2012).

[37] T. Gao, F. L. Yan, and S. J. van Enk, Permutationally invariant part of a density matrix and nonseparability of N-qubit states, Phys. Rev. Lett. **112**, 180501 (2014).

[38] X. Yang, M. X. Luo, Y. H. Yang, and S. M. Fei, Parametrized entanglement monotone, Phys. Rev. A **103**, 052423 (2021).

[39] G. Vidal, Entanglement monotones, J. Mod. Opt. **47**, 355 (2000).

[40] Z. W. Wei and S. M. Fei, Parameterized bipartite entanglement measure, J. Phys. A: Math. Theor. **55**, 275303 (2022).

[41] X. Shi and L. Chen, A genuine multipartite entanglement measure generated by the parametrized entanglement measure, Ann. Phys. **535**, 2300305 (2023).

[42] Y. F. Li and J. W. Shang, Geometric mean of bipartite concurrences as a genuine multipartite entanglement measure, Phys. Rev. Res. **4**, 023059 (2022).

[43] D. Bruß, N. Datta, A. Ekert, L. C. Kwek, and C. Macchiavello, Multipartite entanglement in quantum spin chains, Phys. Rev. A **72**, 014301 (2005).

[44] H. J. Briegel, D. E. Browne, W. Dür, R. Raussendorf, and M. Van den Nest, Measurement-based quantum computation, Nat. Phys. **5**, 19 (2009).

[45] R. Raussendorf and H. J. Briegel, A one-way quantum computer, Phys. Rev. Lett. **86**, 5188 (2001).

[46] S. Schauer, M. Huber, and B. C. Hiesmayr, Experimentally feasible security check for n-qubit quantum secret sharing, Phys. Rev. A **82**, 062311 (2010).

[47] G. Gour and B. C. Sanders, Remote preparation and distribution of bipartite entangled states, Phys. Rev. Lett. **93**, 260501 (2004).

[48] X. Y. Meng, J. X. Gao, and S. Havlin, Concurrence percolation in quantum networks, Phys. Rev. Lett. **126**, 170501 (2021).

[49] A. Banerjee and P. Bej, Braess paradox in a quantum network, Phys. Rev. A **104**, 052622 (2021).

[50] O. Malik, X. Y. Meng, S. Havlin, G. Korniss, B. K. Szymanski, and J. X. Gao, Concurrence percolation threshold of large-scale quantum networks, Commun. Phys. **5**, 193 (2022).

[51] S. Bose, V. Vedral, and P. L. Knight, Multiparticle generalization of entanglement swapping, Phys. Rev. A **57**, 822 (1998).

[52] S. Bose, V. Vedral, and P. L. Knight, Purification via entanglement swapping and conserved entanglement, Phys. Rev. A **60**, 194 (1999).

[53] C. Tsallis, Possible generalization of Boltzmann-Gibbs statistics, J. Stat. Phys. **52**, 479 (1988).

[54] E. Lutz, Anomalous diffusion and Tsallis statistics in an optical lattice, Phys. Rev. A **67**, 051402(R) (2003).

[55] P. Douglas, S. Bergamini, and F. Renzoni, Tunable Tsallis distributions in dissipative optical lattices, Phys. Rev. Lett. **96**, 110601 (2006).

[56] R. G. DeVoe, Power-law distributions for a trapped ion interacting with a classical buffer gas, Phys. Rev. Lett. **102**, 063001 (2009).
[57] R. M. Pickup, R. Cywinski, C. Pappas, B. Farago, and P. Fouquet, Generalized spin-glass relaxation, Phys. Rev. Lett. **102**, 097202 (2009).
[58] S. S. Sharma and N. K. Sharma, Quantum coherences, K-way negativities and multipartite entanglement, Phys. Rev. A **77**, 042117 (2008).
[59] Z. W. Wei, M. X. Luo, and S. M. Fei, Estimating parameterized entanglement measure, Quantum Inf. Process. **21**, 210 (2022).
[60] Y. Hong, X. F. Qi, T. Gao, and F. L. Yan, A $(k+1)$-partite entanglement measure of N-partite quantum states, Eur. Phys. J. Plus **138**, 1081 (2023).
[61] S. Xie and J. H. Eberly, Triangle measure of tripartite entanglement, Phys. Rev. Lett. **127**, 040403 (2021).