Proposals for ruling out real quantum theories in an entanglement-swapping quantum network with causally independent sources

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The question of whether complex numbers play a fundamental role in quantum theory has been debated since the inception of quantum mechanics. Recently, a feasible proposal to differentiate between real and complex quantum theories based on the technique of testing Bell nonlocalities has emerged [Renou *et al.*, Nature (London) **600**, 625 (2021)]. Based on this method, the real quantum theory has been falsified experimentally in both photonic and superconducting quantum systems [Li *et al.*, Phys. Rev. Lett. **128**, 040402 (2022); Chen, *et al.*, Phys. Rev. Lett. **128**, 040403 (2022)]. The quantum networks with multiple independent sources which are not causally connected have gained significant interest as they offer a new perspective on studying the nonlocalities. The independence of these sources imposes additional constraints on observable covariances and leads to new bounds for classical and quantum correlations. In this study, we examine the discrimination between the real and complex quantum theories with an entanglement swapping scenario under a stronger assumption that the two sources are causally independent, which was not made in previous works. Using a revised Navascúes-Pironio-Acín method and Bayesian optimization, we find a proposal with optimal coefficients of the correlation function which could give a larger discrimination between the real and quantum theories compared with the existing proposals. This work opens up avenues for further exploration of the discrimination between real and complex quantum theories within intricate quantum networks featuring causally independent parties.

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I. INTRODUCTION

Quantum mechanics, which was established nearly 100 years ago, has achieved numerous significant accomplishments that have had profound impacts on science and technology [1,2]. Especially in the past two decades, quantum information science, based on the principles of quantum mechanics and information science, has activated a series of advanced technologies such as quantum computing [3–8], quantum communication [9,10], and quantum metrology [11,12]. However, the fundamental role of complex numbers in quantum mechanics has long puzzled its founders and subsequent researchers [13]. Despite this, a group of dedicated scientists has persistently pursued a quantum theory that relies on only real numbers in its mathematical formulation, running parallel to the development of the standard quantum theory [14–21]. With a fixed Hilbert space dimension, the real and complex quantum theories can be discriminated by a single-site experiment using local tomography [22]. However, without bounding the dimension, one should consider experiments involving several distant labs, such as the Bell nonlocality test [23].

Bell nonlocality was introduced by Bell in 1964, with studying quantum correlations in a groundbreaking two-party model through the analysis of outcome statistics in experiments [23]. Bell's model has made significant contributions to our understanding of quantum phenomena and has been instrumental in experimental tests that successfully ruled out local hidden variable theories [24–30]. In past decades, the studies of nonlocalities have been extended to the scenarios of quantum networks [31], such as the nonbilocalities in an entanglement-swapping quantum network [32,33]. These studies offer new opportunities and perspectives for studying quantum nonlocalities [33–54]. In addition, quantum networks also play important roles in ruling out quantum theories based on only real numbers. It turns out that without bounding the dimension of Hilbert space, the real and complex quantum theories cannot be discriminated without a network-based scenario, even in a conventional Bell scenario with more than two separate parties [17,55,56]. Recently, Renou *et al.*, with the entanglement-swapping scenario [57], have given an interesting proposal to falsify real quantum theories. Their model involves three observers, namely, Alice, Bob, and Charlie, and two independent entangled pairs without quantum correlations but allowing sharing a global hidden variable λ as shown in Fig. 1(a). In their protocol, Bob performs a single joint measurement with four possible outcomes recorded as b, while Alice and Charlie perform three and six measurements

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with two outcomes with x and z as the input and a and c as the outcome, respectively. We refer to such a protocol as the (3, 6) scenario in this paper. By defining a correlation function based on the outcome probability distributions from experimental data, Renou *et al.* [57] have demonstrated that this function can assume different values under the constraints of real and complex quantum theories. These values can be calculated numerically using semidefinite programming (SDP) optimization techniques. This meticulously designed experiment has been conducted in both photonic and superconducting qubit systems and has successfully falsified the quantum theory based on only real numbers with compelling experimental evidence [58–60]. Following the work of Renou *et al.* [57], Bednorz and Batle reduced the number of Charlie's measurement settings from six to four and three constructing the (3,3) and (3,4) scenarios and proved that a (2,2) scenario does not exist [61]. In these models, a potential global hidden variable is taken into consideration, allowing for prior classical correlations among the parties. In quantum networks that incorporate multiple independent sources [31,32,34], the causal structure becomes increasingly complex. The independence of these sources in various network structures introduces additional constraints on classical and quantum correlations. For the simplest entanglement-swapping quantum network with two casually independent sources involving two independent hidden variables λ_1 and λ_2 as shown in Fig. 1(b), there are constraints on the observers' measurement results coming from the independence of two sources [32,33]:

$$\sum_b P(a, b, c|x, z) = P(a|x)P(c|z) = \sum_b P(a, b|x) \sum_b P(b, c|z), \quad (1)$$

where x and z are the input of Alice and Charlie, while a, b, and c are the outcomes of Alice, Bob, and Charlie. These new constraints bring a new maximum bound for numerically calculating a Bell type function under classical and standard quantum theories with a revised Navascúes-Pironio-Acín (NPA) method [38] which introduces a scalar extension of the moment matrices.

In this work, we extend the study on the discrimination between real and complex quantum theories to the scenarios of quantum networks with multiple independent sources. Specifically, we focus on the simplest entanglement-swapping model involving two independent hidden variables. We introduce a revised NPA method, building upon the technologies used in previous works [38,57] for numerical calculation of the maximum bounds of the correlation function under the quantum theories that rely on only real numbers. Furthermore, we employ Bayesian optimization techniques to effectively search the optimal correlation function for discriminating the real and quantum theories, and we successfully find a correlation function with a group of coefficients that exhibits superior robustness compared to existing proposals.

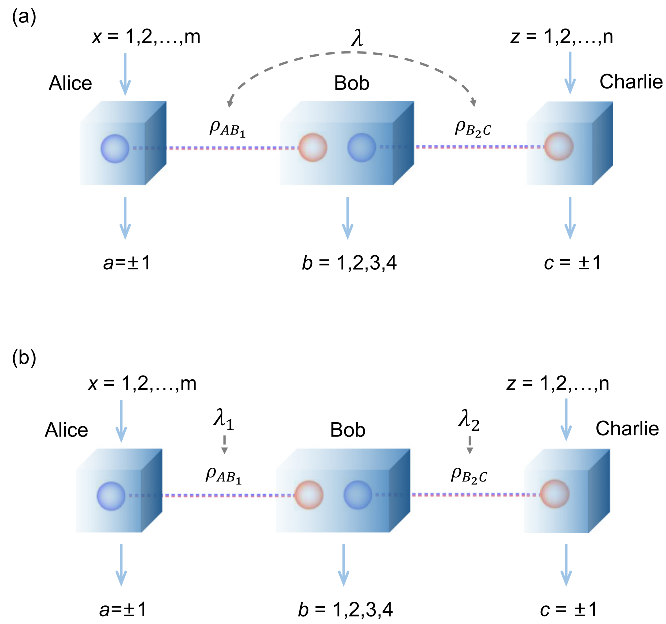

FIG. 1. Scenarios to discriminate complex and real quantum theories. (a) The model used in Refs. [57,61]. (b) The model used in this work. Two independent hidden variables emphasize the causal independence of two sources. x and z represent the input values of Alice and Charlie, while a, b, and c represent the outcomes of Alice, Bob, and Charlie, respectively.

II. THE ENTANGLEMENT-SWAPPING SCENARIO

The entanglement-swapping scenario has been used to discriminate the real and complex quantum theories as discussed in Ref. [57] with the assumption that the two sources are allowed to share a classical hidden variable. In this section we briefly elaborate the entanglement swapping model with causally independent sources. As shown in Fig. 1(b), such a scenario involves three observers, Alice, Bob, and Charlie, as well as two independent entanglement sources, ρ_{AB_1} and ρ_{B_2C}. The entanglement source ρ_{AB_1} is shared between Alice and Bob, while the entanglement ρ_{B_2C} is shared between Bob and Charlie. Unlike the previous model used in Ref. [57] [Fig. 1(a)], these two sources are entirely causally independent, devoid of both quantum and classical correlations. They are characterized by two potential hidden variables with independent origin, denoted as λ_1 and λ_2. In this scenario, Bob conducts a single joint measurement on the two particles he received from the two entanglement sources, obtaining four different outcomes recorded as $b = \{1, 2, 3, 4\}$, while Alice and Charlie randomly perform m and n dichotomic measurements on their received particles, obtaining the outcomes $a = \{1, -1\}$ and $c = \{1, -1\}$, respectively. We refer to such a protocol as the (m, n) scenario in this paper. We use a group of conditional probabilities to denote the experimental results. For example $P(a, b, c|x, z)$ represents the probability of outcome results a, b, and c when Alice, Bob, and Charlie's measurement settings A_x, B_b, and C_z, respectively, $x \in \{1, 2, \dots, m\}$ and $z \in \{1, 2, \dots, n\}$.

In this work, we define an open correlation function with the outcome probability distribution and an $m \times n$ free coefficient matrix $\mathcal{E}$ in the form of.

$$F=\sum_{x,z,b} g(b,x) e_{xz} S^b_{x,z},$$
$$\text{where} \quad S^b_{x,z}=\sum_{a,c\in\{\pm 1\}} acP(a,b,c|x,z),$$
$$g(b,x)=\begin{cases}1, & \text{if } b=x \text{ or } b=4,\\ -1, & \text{otherwise.}\end{cases} \tag{2}$$

Here, e_{xz} is the element of the coefficient matrix $\mathcal{E}$. Similar to the Bell correlation function, the function F may take different values depending on whether it is evaluated in the context of classical theory, real quantum theory, or complex quantum theory, and the maximum bound in each respective condition can be recorded as F_c, F_r, and F_q. If we observe a value of F which exceeds F_c and F_r but does not reach F_q, we can falsify quantum theory based on only real numbers. The gap of F_q and F_r represents the discrimination between real and complex quantum theories. We define the value of $R=F_r/F_q$ as representing the ratio of the maximum bound of Eq. (2) under real and complex quantum theory, respectively. A smaller R indicates a larger discrimination between real and complex quantum theories and may reduce the requirements for the fidelities of the entanglement sources and measurements in practical experiments. For example, we discuss the case that three are white noises in two entanglement sources and the Bell state measurements Bob performed on the two particles he received. Then the entanglement source and the Bell state measurement in complex quantum mechanics can be rewritten as

$$\begin{aligned}\rho_{AB_1}=\rho_{B_2C}&=v_E|\Phi^+\rangle\langle\Phi^+|+(1-v_E)I,\\ B_1&=v_I|\Psi^+\rangle\langle\Psi^+|+(1-v_I)I/4,\\ B_2&=v_I|\Psi^-\rangle\langle\Psi^-|+(1-v_I)I/4,\\ B_3&=v_I|\Phi^+\rangle\langle\Phi^+|+(1-v_I)I/4,\\ B_4&=v_I|\Phi^-\rangle\langle\Phi^-|+(1-v_I)I/4,\end{aligned} \tag{3}$$

where v_E and v_I are the visibility of the entanglement source and Bell Standard Measurement, respectively, and I is the identity matrix. The noise value $S'^b_{x,z}$ can be calculated as

$$S'^b_{x,z}=\text{tr}\,(\rho_{AB_1}\otimes\rho_{B_2C})(A_x\otimes B_b\otimes C_z)=v_E^2 v_I S^b_{x,z}. \tag{4}$$

Then the noise correlation function $F'_q=v_E^2 v_I F_q$. To experimentally falsify real quantum mechanics, it is necessary for us to satisfy the condition $v_E^2 v_I > F_r/F_q$. It is clear that a lower value of R requires lower visibility in experimental realization. To optimize the coefficient matrix $\mathcal{E}$ and attain a lower value of $R=F_r/F_q$, we employ Bayesian optimization, as outlined in Sec. IV.

III. CALCULATING THE BOUND OF REAL QUANTUM THEORIES IN THE MODEL INVOLVING TWO INDEPENDENT HIDDEN VARIABLES

In this section, we initially outline the process of formulating an SDP optimization problem for the computation of F_r and F_q, primarily in accordance with the approach detailed in Ref. [57]. Subsequently, for the model used in this work, we demonstrate the incorporation of supplementary constraints from causal independence, as specified in Eq. (1), into the SDP optimization problem.

In the swapping scenario shown in Fig. 1(b), the probability distribution of measurement outputs can be represented by

$$P(a,b,c|x,z)=\text{tr}[(\rho_{ABC})(A_{a|x}\otimes B_b\otimes C_{c|z})], \tag{5}$$

where $\rho_{ABC}=\rho_{AB_1}\otimes\rho_{B_2C}$, and $A_{a|x}$ denotes the measurement operator we use when Alice chooses x from possible settings and gets output a; notice that measurement operators and density operators here can live in either complex Hilbert space or real Hilbert space.

To establish the constraints for Eq. (5) when computing the upper bound of Eq. (2), we employ methodologies previously employed in earlier works [57,58,61], building upon Moroder *et al.*'s extension [55] of the NPA hierarchy [62–64]. Our approach begins with the creation of two sets, $\mathcal{A}$ and $\mathcal{C}$, derived from the settings of Alice and Charlie. Specifically, $\mathcal{A}$ encompasses all monomials of I, $A_{1|1}$, $A_{1|2}$, and $A_{1|3}$ with degrees of n_A or less (for the definition of monomial degree, refer to Ref. [62]). Similarly, $\mathcal{C}$ is constructed following the same principle. Each monomial within $\mathcal{A}$ is linked to a ket denoted as $|\alpha\rangle$, with an associated property $\langle\alpha|\alpha'\rangle=\delta_{\alpha,\alpha'}$. Analogously, for monomials within $\mathcal{C}$, an orthonormal set $|\gamma\rangle$ can also be associated. Then we define two completely positive maps,

$$\begin{aligned}\Omega_A(\eta)&=\sum_{\alpha,\alpha'}\text{tr}(A^\dagger_\alpha\eta A_{\alpha'})|\alpha\rangle\langle\alpha'|,\\ \Omega_C(\eta)&=\sum_{\gamma,\gamma'}\text{tr}(C^\dagger_\gamma\eta C_{\gamma'})|\gamma\rangle\langle\gamma'|,\end{aligned} \tag{6}$$

where A_α denotes the monomial that $|\alpha\rangle$ is associated to, and then we define the matrix

$$\Gamma^b=(\Omega_A\otimes\Omega_C)(\rho_{AC|b}), \tag{7}$$

where $\rho_{AC|b}=\text{tr}_B\{(\rho_{AB_1}\otimes\rho_{B_2C})(I\otimes B_b\otimes I)\}$ is the reduced state of systems A and C after Bob conducts the measurement, obtaining output b.

Since Ω_A and Ω_C are completely positive, Γ^b is positive semidefinite, and it has other properties due to orthogonality of $\{|\alpha\rangle\}$ and $\{|\gamma\rangle\}$; that is, if $\alpha_2\alpha_1^\dagger=\alpha_4\alpha_3^\dagger=\alpha$ and $\gamma_2\gamma_1^\dagger=\gamma_4\gamma_3^\dagger=\gamma$, we have

$$\langle\alpha_1\gamma_1|\Gamma^b|\alpha_2\gamma_2\rangle=\langle\alpha_3\gamma_3|\Gamma^b|\alpha_4\gamma_4\rangle=\text{tr}\{\rho_{AC|b}(A_\alpha\otimes C_\gamma)\}, \tag{8}$$

which allows us to write Γ^b in a more simple way,

$$\Gamma^b=\sum_{\alpha\in\mathcal{A}\cdot\mathcal{A},\gamma\in\mathcal{C}\cdot\mathcal{C}} d^b_{\alpha,\gamma}M^\alpha\otimes N^\gamma, \tag{9}$$

where

$$M^\alpha_{a,a'}=\delta_{\alpha,a'a^\dagger},\quad N^\gamma_{c,c'}=\delta_{\gamma,c'c^\dagger}, \tag{10}$$

and the real coefficients $\{d^b_{\alpha,\gamma}\}$ are the variable set of the optimization problem, it follows

$$\begin{aligned}d^b_{A_{1|x},C_{1|z}}&=P(1,b,1|x,z),\quad d^b_{A_{1|x},I}=P(1,b|x),\\ d^b_{I,I}&=P(b),\quad d^b_{\alpha,\gamma}=\text{tr}\{(A_\alpha\otimes B_b\otimes C_\gamma)(\rho_{ABC})\},\end{aligned} \tag{11}$$

and the normalization constraint is $\sum_b d^b_{I,I}=1$.

Now we consider $\Gamma = \sum_b \Gamma^b$, and by the independence of ρ_{AB_1} and ρ_{B_2C}, we have

$$\Gamma = \Omega_A(\rho_A) \otimes \Omega_C(\rho_C). \tag{12}$$

Separability leads to different constraints in complex and real quantum theories [65,66]:

$$\begin{aligned} &\Gamma^{T_A} \geqslant 0, \text{ for complex quantum theory,} \\ &\Gamma^{T_A} = \Gamma, \text{ for real quantum theory.} \end{aligned} \tag{13}$$

the latter constraint in Eq. (13) is stronger, leading to a lower upper bound of F.

As we mentioned in Sec. I, in our model involving λ_1 and λ_2 shown in Fig. 1(b), the independence of ρ_{AB_1} and ρ_{B_2C} brings new constraints on probability distribution [notice that this formula is the same as Eq. (1)]:

$$\sum_b P(a, b|x) \sum_b P(b, c|z) = \sum_b P(a, b, c|x, z); \tag{14}$$

however, due to their nonlinear nature, direct inclusion of the constraints from Eq. (14) into an SDP optimization problem is not feasible. To address this, we expand upon the technique presented in Ref. [38], adapting it to a bipartite form. This adaptation enhances compatibility with the aforementioned numerical approach, rendering it more suitable for implementation.

We modify the set $\mathcal{C}$ constructed from Charlie's settings as [take (3, 3) for instance]

$$\mathcal{C} = \{I, C_{1|1}, C_{1|2}, C_{1|3}, \ldots, C_{1|2}C_{1|3}, c_1 I, c_2 I, c_3 I\}, \tag{15}$$

where $c_i = P(c = 1|z = i)$, and $\mathcal{A}$ does not change.

$\{d^b_{\alpha,\gamma}\}$ are constructed by the same process as Eqs. (6), (7), (8), and (9), with more constraints coming from Eq. (14):

$$\begin{aligned} \sum_b d^b_{\alpha,c_i I} &= \sum_b d^b_{\alpha,C_{1|i}}, \quad \text{for } \forall\alpha,\ \forall i \in \{1, 2, 3\}, \\ \sum_b d^b_{I,c_i C_{1|j}} &= \sum_b d^b_{I,c_i c_j I}, \quad \text{for } \forall i,\ j \in \{1, 2, 3\}, \end{aligned} \tag{16}$$

which can be derived from Eqs. (11) and (14), specifically,

$$\begin{aligned} \sum_b d^b_{\alpha,c_i I} &= \sum_b \operatorname{tr}\{(\alpha \otimes c_i I)\rho_{AC|b}\} \\ &= c_i \sum_b \operatorname{tr}\{(\alpha \otimes I)\rho_{AC|b}\} \\ &= c_i \sum_b d^b_{\alpha,I} = \sum_b d^b_{\alpha,C_{1|i}}; \end{aligned} \tag{17}$$

the second formula in Eq. (16) can be derived in the same way.

The new constraints in Eq. (16) should also be satisfied under complex quantum theory. Hence, to calculate the maximum bound of F [Eq. (2)] under real and complex quantum theories, we can construct and solve an SDP optimization problem as follows:

$$\begin{aligned} &\max\ F \\ &s.t.\ \ \Gamma^b = \sum_{\alpha \in \mathcal{A}\cdot\mathcal{A}, \gamma \in \mathcal{C}\cdot\mathcal{C}} d^b_{\alpha,\gamma} M^\alpha \otimes N^\gamma \geqslant 0, \\ &\qquad \sum_b P(b) = 1, \\ &\qquad \sum_b d^b_{\alpha,c_i I} = \sum_b d^b_{\alpha,C_{1|i}}, \text{ for } \forall\alpha,\ \forall i \in \{1, 2, 3\}, \\ &\qquad \sum_b d^b_{I,c_i C_{1|j}} = \sum_b d^b_{I,c_i c_j I}, \text{ for } \forall i,\ j \in \{1, 2, 3\}, \\ &\qquad \times \left[\sum_b \Gamma^b = \left(\sum_b \Gamma^b \right)^{T_A} \text{(if real)} \right]. \end{aligned} \tag{18}$$

The corresponding relationship between $\{P(a, b, c|x, z)\}$ and $\{d^b_{\alpha,\gamma}\}$ is illustrated by Eq. (11).

IV. SEARCHING OPTIMAL CORRELATION FUNCTION WITH BAYESIAN OPTIMIZATION

Bayesian optimization is a powerful technique for hyperparameter tuning, which involves finding the optimal values of a set of parameters for a given objective function. One popular implementation of Bayesian optimization is sequential model-based optimization (SMBO), the detailed principles of which are discussed in Ref. [67]. Here we briefly introduce SMBO and illustrate how to use it in the discrimination of complex and real theories.

SMBO is an iterative algorithm that generates a model of the objective function in each iteration to identify the next set of parameters to test, ultimately finding the optimal set of parameters. To construct the objective function model, SMBO utilizes the parameter tuning history $H = [x_{1:i}, f(x_{1:i})]$, where $x_{1:i}$ and $f(x_{1:i})$ denote parameters and corresponding function values obtained in i times iterations. In the $(i+1)^{\text{th}}$ iteration, an acquisition function is employed to determine the next parameter set value x_{i+1}. This acquisition function balances exploitation, which involves selecting values close to the current most optimal parameter set, and exploration, which employs randomness to prevent falling into local optimal solutions. SMBO is terminated when the predefined number of iterations is reached. This algorithm is particularly beneficial when the objective function is a black-box function that is challenging to evaluate and does not have well-defined derivatives.

In the discrimination between complex and real quantum theories, the parameter set to optimize is $\{e_{xz}\}$ in Eq. (2), and the objective function is $R(\mathcal{E}) = F_r/F_q$, where F_r and F_q are maximum bounds of Eq. (2) under real quantum theory and complex quantum theory, respectively. Both F_r and F_q can be calculated by the SDP optimization problem constructed in Sec. III through MATLAB packages MOSEK [68] and YALMIP [69], sometimes F_q can be calculated analytically under certain assumptions. We use the MATLAB Bayesian optimization package [70] to realize SMBO on the objective function $R(\mathcal{E})$.

In the (3, 3) scenario, the optimal values for $\{e_{xz}\}$ we get are (permutation has been done to make results symmetric)

$$\mathcal{E}=\begin{pmatrix}0.31993 & 0.5 & -0.5\\ 0.5 & 0.31933 & 0.5\\ -0.5 & 0.5 & 0.31933\end{pmatrix}, \tag{19}$$

with $F_r = 2.1134$, $F_q = 2.3283$, and $R(\mathcal{E}) = 0.9077$, The specific states and measurements to achieve F_q are [61]

$$\{A_x\} = \{\sigma_i\},$$

$$\{C_z\} = \left\{\sum_i c_z^i \sigma_i\right\}, \quad c_z^i = -\frac{e_{iz}}{\sqrt{e_{1z}^2 + e_{2z}^2 + e_{3z}^2}},$$

$$B_1 = (I\otimes I - \sigma_X\otimes\sigma_X - \sigma_Y\otimes\sigma_Y - \sigma_Z\otimes\sigma_Z)/4,$$

$$B_2 = (I\otimes I - \sigma_X\otimes\sigma_X + \sigma_Y\otimes\sigma_Y + \sigma_Z\otimes\sigma_Z)/4,$$

$$B_3 = (I\otimes I + \sigma_X\otimes\sigma_X - \sigma_Y\otimes\sigma_Y + \sigma_Z\otimes\sigma_Z)/4,$$

$$B_4 = (I\otimes I + \sigma_X\otimes\sigma_X + \sigma_Y\otimes\sigma_Y - \sigma_Z\otimes\sigma_Z)/4,$$

$$\rho_{AB_1} = \rho_{B_2C} = |\Phi^+\rangle\langle\Phi^+|, |\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle),$$

where $\{\sigma_i\}$ are Pauli operators.

Similarly, in the (3,4) scenario, the optimal set of parameters we obtained is

$$\mathcal{E}=\begin{pmatrix}-0.19883 & 0.1996 & 0.20026 & 0.19944\\ 0.20094 & -0.19971 & 0.20083 & 0.1987\\ 0.2006 & 0.19961 & -0.2 & 0.19971\end{pmatrix}. \tag{20}$$

We can infer that absolute values of all coefficients are the same, and hence the optimal set is (with every coefficient timing 5)

$$\mathcal{E}=\begin{pmatrix}-1 & 1 & 1 & 1\\ 1 & -1 & 1 & 1\\ 1 & 1 & -1 & 1\end{pmatrix}, \tag{21}$$

with $F_q = 6.9282$, $F_r = 6.4722$, and $R(\mathcal{E}) = 0.8847$, which is lower than $R(\mathcal{E})$ of a (3,6) scenario in Ref. [57]. To achieve F_q, $\{A_a\}$ and $\{B_b\}$ used are the same as what the (3,3) scenario uses, and (C_z) are

$$\begin{pmatrix}C_1\\ C_2\\ C_3\\ C_4\end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix}1 & -1 & -1\\ -1 & 1 & -1\\ -1 & -1 & 1\\ -1 & -1 & -1\end{pmatrix}\begin{pmatrix}\sigma_X\\ \sigma_Y\\ \sigma_Z\end{pmatrix}. \tag{22}$$

In the swapping-entanglement model involving λ_1 and λ_2, the new constraints (16) are included, so we obtain tighter upper bounds for F_r. Since F_q at least has the same forms as previous work [57,61], we achieve lower values of $R(\mathcal{E})$. We compare the results presented in this work with those from previous research, as illustrated in Table I. The findings indicate that

TABLE I. Comparison between this work and previous works [57,58,61].

Scenario	$R(\mathcal{E})$	Causal constraint
(3, 3) [61]	0.9381	No
(3, 3) [This work]	0.9077	Yes
(3, 4) [61]	0.9341	No
(3, 4) [This work]	0.8847	Yes
(3, 6) [57]	0.9028	No

we have achieved lower values of $R(\mathcal{E})$, implying reduced visibility requirements for experimental implementation and hence exhibiting superior robustness.

V. SUMMARY AND DISCUSSIONS

In summary, our work offers proposals to discriminate real and complex quantum theories in an entanglement-swapping model involving two independent hidden variables, which emphasize the causally independent nature of sources. To address new constraints of the causal independence on the experimental outcome probability distribution, we have developed a numerical method based on the NPA technologies, enabling discrimination between real and complex quantum theories in quantum networks with causally independent parties. In this work, we further employ Bayesian optimization to search for the optimal coefficient matrix of the correlation function. As a result, we obtain a more experimentally feasible scenario that allows discriminating real and quantum theories with lower visibility of the sources and measurements, and we compare this scenario to existing proposals.

Finally, we discuss the reasonableness of the independence assumption in our model. Renou *et al.*'s and other previous models [57,58,61] consider a global hidden variable shared between the two sources in the entanglement-swapping scenario. This is because the two entangled sources may be produced in the same factory or may be operated using the same power socket. The causal network with an additional source should be considered more general than the one without such a source. However, a very similar assumption is actually needed in Renou *et al.*'s and other previous models, which is the independence of randomness sources for choosing measurements, which also corresponds to the "free choice" or "measurement independence" assumption in standard Bell tests [32,33]. The stronger assumption that the entanglement source is absolutely independent brings new constraints on the probability distribution of the input-output experiment as the form of Eq. (1). These constraints are not only added to the experimental results predicted by complex quantum theory but are also added to the cases predicted by real quantum theories and classical theory. These constraints may change the effective probability distributions for all three cases and help us find a more experimentally feasible scenario with lower demand of visibility of the sources and measurements. These considerations should be extended to exploring discrimination between real and complex quantum theories in more complicated quantum networks with causally independent parties which will introduce more additional constraints [31]. We believe further advancements for the discrimination of real and complex quantum theories will be achieved, ultimately making experiments more accessible in the near future.

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[1] P. Dirac, *The Principles of Quantum Mechanics* (Clarendon, Oxford, UK, 1981).

[2] J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton University, Princeton, NJ, 1955).

[3] T. D. Ladd, F. Jelezko, R. Laflamme, Y. Nakamura, C. Monroe, and J. L. O'Brien, Quantum computers, Nature (London) **464**, 45 (2010).

[4] F. Arute, K. Arya, R. Babbush, D. Bacon, J. C. Bardin, R. Barends, R. Biswas, S. Boixo, F. G. Brandao, D. A. Buell *et al.*, Quantum supremacy using a programmable superconducting processor, Nature (London) **574**, 505 (2019).

[5] H.-S. Zhong, H. Wang, Y.-H. Deng, M.-C. Chen, L.-C. Peng, Y.-H. Luo, J. Qin, D. Wu, X. Ding, Y. Hu *et al.*, Quantum computational advantage using photons, Science **370**, 1460 (2020).

[6] Y. Wu, W.-S. Bao, S. Cao, F. Chen, M.-C. Chen, X. Chen, T.-H. Chung, H. Deng, Y. Du, D. Fan, M. Gong, C. Guo, C. Guo, S. Guo, L. Han, L. Hong, H.-L. Huang, Y.-H. Huo, L. Li, N. Li *et al.*, Strong quantum computational advantage using a superconducting quantum processor, Phys. Rev. Lett. **127**, 180501 (2021).

[7] Q. Zhu, S. Cao, F. Chen, M.-C. Chen, X. Chen, T.-H. Chung, H. Deng, Y. Du, D. Fan, M. Gong *et al.*, Quantum computational advantage via 60-qubit 24-cycle random circuit sampling, Sci. Bull. **67**, 240 (2022).

[8] L. S. Madsen, F. Laudenbach, M. F. Askarani, F. Rortais, T. Vincent, J. F. Bulmer, F. M. Miatto, L. Neuhaus, L. G. Helt, M. J. Collins *et al.*, Quantum computational advantage with a programmable photonic processor, Nature (London) **606**, 75 (2022).

[9] N. Gisin and R. Thew, Quantum communication, Nat. Photon. **1**, 165 (2007).

[10] F. Xu, X. Ma, Q. Zhang, H.-K. Lo, and J.-W. Pan, Secure quantum key distribution with realistic devices, Rev. Mod. Phys. **92**, 025002 (2020).

[11] V. Giovannetti, S. Lloyd, and L. Maccone, Advances in quantum metrology, Nat. Photon. **5**, 222 (2011).

[12] E. Polino, M. Valeri, N. Spagnolo, and F. Sciarrino, Photonic quantum metrology, AVS Quantum Sci. **2**, 024703 (2020).

[13] A. Einstein, K. Przibram, and M. Klein, *Letters on Wave Mechanics: Correspondence with H. A. Lorentz, Max Planck, and Erwin Schrödinger* (Philosophical Library/Open Road, 2011).

[14] E. C. G. Stueckelberg, Quantum theory in real Hilbert-space, Helv. Phys. Acta **33**, 727 (1960).

[15] W. K. Wootters, Statistical distance and Hilbert space, Phys. Rev. D **23**, 357 (1981).

[16] W. H. Zurek, *Complexity, Entropy, and the Physics of Information*, edited by W. H. Zurek (Addison-Wesley, Boston, 1990).

[17] M. McKague, M. Mosca, and N. Gisin, Simulating quantum systems using real Hilbert spaces, Phys. Rev. Lett. **102**, 020505 (2009).

[18] L. Hardy and W. K. Wootters, Limited holism and real-vector-space quantum theory, Found. Phys. **42**, 454 (2012).

[19] A. Aleksandrova, V. Borish, and W. K. Wootters, Real-vector-space quantum theory with a universal quantum bit, Phys. Rev. A **87**, 052106 (2013).

[20] V. Moretti and M. Oppio, Quantum theory in real Hilbert space: How the complex Hilbert space structure emerges from Poincaré symmetry, Rev. Math. Phys. **29**, 1750021 (2017).

[21] P. Drechsel, Foundation of quantum mechanics: Once again, Found. Sci. **24**, 375 (2019).

[22] K.-D. Wu, T. V. Kondra, S. Rana, C. M. Scandolo, G.-Y. Xiang, C.-F. Li, G.-C. Guo, and A. Streltsov, Operational resource theory of imaginarity, Phys. Rev. Lett. **126**, 090401 (2021).

[23] J. S. Bell, On the Einstein Podolsky Rosen paradox, Phys. Phys. Fiz. **1**, 195 (1964).

[24] S. J. Freedman and J. F. Clauser, Experimental test of local hidden-variable theories, Phys. Rev. Lett. **28**, 938 (1972).

[25] A. Aspect, P. Grangier, and G. Roger, Experimental tests of realistic local theories via Bell's theorem, Phys. Rev. Lett. **47**, 460 (1981).

[26] M. Giustina, M. A. M. Versteegh, S. Wengerowsky, J. Handsteiner, A. Hochrainer, K. Phelan, F. Steinlechner, J. Kofler, J.-A. Larsson, C. Abellán, W. Amaya, V. Pruneri, M. W. Mitchell, J. Beyer, T. Gerrits, A. E. Lita, L. K. Shalm, S. W. Nam, T. Scheidl, R. Ursin *et al.*, Significant-loophole-free test of Bell's theorem with entangled photons, Phys. Rev. Lett. **115**, 250401 (2015).

[27] B. Hensen, H. Bernien, A. E. Dréau, A. Reiserer, N. Kalb, M. S. Blok, J. Ruitenberg, R. F. L. Vermeulen, R. N. Schouten, C. Abellán, W. Amaya, V. Pruneri, M. W. Mitchell, M. Markham, D. J. Twitchen, D. Elkouss, S. Wehner, T. H. Taminiau, and R. Hanson, Loophole-free Bell inequality violation using electron spins separated by 1.3 kilometres, Nature (London) **526**, 682 (2015).

[28] L. K. Shalm, E. Meyer-Scott, B. G. Christensen, P. Bierhorst, M. A. Wayne, M. J. Stevens, T. Gerrits, S. Glancy, D. R. Hamel, M. S. Allman, K. J. Coakley, S. D. Dyer, C. Hodge, A. E. Lita, V. B. Verma, C. Lambrocco, E. Tortorici, A. L. Migdall, Y. Zhang, D. R. Kumor *et al.*, Strong loophole-free test of local realism, Phys. Rev. Lett. **115**, 250402 (2015).

[29] W. Rosenfeld, D. Burchardt, R. Garthoff, K. Redeker, N. Ortegel, M. Rau, and H. Weinfurter, Event-ready Bell test using entangled atoms simultaneously closing detection and locality loopholes, Phys. Rev. Lett. **119**, 010402 (2017).

[30] M.-H. Li, C. Wu, Y. Zhang, W.-Z. Liu, B. Bai, Y. Liu, W. Zhang, Q. Zhao, H. Li, Z. Wang, L. You, W. J. Munro, J. Yin, J. Zhang, C.-Z. Peng, X. Ma, Q. Zhang, J. Fan, and J.-W. Pan, Test of local realism into the past without detection and locality loopholes, Phys. Rev. Lett. **121**, 080404 (2018).

[31] A. Tavakoli, A. Pozas-Kerstjens, M.-X. Luo, and M.-O. Renou, Bell nonlocality in networks, Rep. Prog. Phys. **85**, 056001 (2022).

[32] C. Branciard, N. Gisin, and S. Pironio, Characterizing the nonlocal correlations created via entanglement swapping, Phys. Rev. Lett. **104**, 170401 (2010).

[33] C. Branciard, D. Rosset, N. Gisin, and S. Pironio, Bilocal versus nonbilocal correlations in entanglement-swapping experiments, Phys. Rev. A **85**, 032119 (2012).

[34] T. Fritz, Beyond Bell's theorem: Correlation scenarios, New J. Phys. **14**, 103001 (2012).

[35] T. Fritz, Beyond Bell's theorem II: Scenarios with arbitrary causal structure, Commun. Math. Phys. **341**, 391 (2016).

[36] G. Carvacho, F. Andreoli, L. Santodonato, M. Bentivegna, R. Chaves, and F. Sciarrino, Experimental violation of local causality in a quantum network, Nat. Commun. **8**, 14775 (2017).

[37] D. J. Saunders, A. J. Bennet, C. Branciard, and G. J. Pryde, Experimental demonstration of nonbilocal quantum correlations, Sci. Adv. **3**, e1602743 (2017).

[38] A. Pozas-Kerstjens, R. Rabelo, L. Rudnicki, R. Chaves, D. Cavalcanti, M. Navascués, and A. Acín, Bounding the sets of classical and quantum correlations in networks, Phys. Rev. Lett. **123**, 140503 (2019).

[39] Q.-C. Sun, Y.-F. Jiang, B. Bai, W. Zhang, H. Li, X. Jiang, J. Zhang, L. You, X. Chen, Z. Wang, Q. Zhang, J. Fan, and J.-W. Pan, Experimental demonstration of non-bilocality with truly independent sources and strict locality constraints, Nat. Photon. **13**, 687 (2019).

[40] M.-O. Renou, E. Bäumer, S. Boreiri, N. Brunner, N. Gisin, and S. Beigi, Genuine quantum nonlocality in the triangle network, Phys. Rev. Lett. **123**, 140401 (2019).

[41] D. Poderini, I. Agresti, G. Marchese, E. Polino, T. Giordani, A. Suprano, M. Valeri, G. Milani, N. Spagnolo, G. Carvacho, R. Chaves, and F. Sciarrino, Experimental violation of n-locality in a star quantum network, Nat. Commun. **11**, 2467 (2020).

[42] I. Agresti, B. Polacchi, D. Poderini, E. Polino, A. Suprano, I. Šupić, J. Bowles, G. Carvacho, D. Cavalcanti, and F. Sciarrino, Experimental robust self-testing of the state generated by a quantum network, PRX Quantum **2**, 020346 (2021).

[43] E. Bäumer, N. Gisin, and A. Tavakoli, Demonstrating the power of quantum computers, certification of highly entangled measurements and scalable quantum nonlocality, npj Quantum Inf. **7**, 117 (2021).

[44] A. Tavakoli, N. Gisin, and C. Branciard, Bilocal Bell inequalities violated by the quantum elegant joint measurement, Phys. Rev. Lett. **126**, 220401 (2021).

[45] C.-X. Huang, X.-M. Hu, Y. Guo, C. Zhang, B.-H. Liu, Y.-F. Huang, C.-F. Li, G.-C. Guo, N. Gisin, C. Branciard, and A. Tavakoli, Entanglement swapping and quantum correlations via symmetric joint measurements, Phys. Rev. Lett. **129**, 030502 (2022).

[46] A. Pozas-Kerstjens, N. Gisin, and A. Tavakoli, Full network nonlocality, Phys. Rev. Lett. **128**, 010403 (2022).

[47] E. Håkansson, A. Piveteau, S. Muhammad, and M. Bourennane, Experimental demonstration of full network nonlocality in the bilocal scenario, arXiv:2201.06361.

[48] X.-M. Gu, L. Huang, A. Pozas-Kerstjens, Y.-F. Jiang, D. Wu, B. Bai, Q.-C. Sun, M.-C. Chen, J. Zhang, S. Yu, Q. Zhang, C.-Y. Lu, and J.-W. Pan, Experimental full network nonlocality with independent sources and strict locality constraints, Phys. Rev. Lett. **130**, 190201 (2023).

[49] N.-N. Wang, A. Pozas-Kerstjens, C. Zhang, B.-H. Liu, Y.-F. Huang, C.-F. Li, G.-C. Guo, N. Gisin, and A. Tavakoli, Certification of non-classicality in all links of a photonic star network without assuming quantum mechanics, Nat. Commun. **14**, 2153 (2023).

[50] Y.-L. Mao, Z.-D. Li, A. Steffinlongo, B. Guo, B. Liu, S. Xu, N. Gisin, A. Tavakoli, and J. Fan, Recycling nonlocality in quantum star networks, Phys. Rev. Res. **5**, 013104 (2023).

[51] L. T. Ligthart and D. Gross, The inflation hierarchy and the polarization hierarchy are complete for the quantum bilocal scenario, J. Math. Phys. **64**, 072201 (2023).

[52] A. Tavakoli, A. Pozas-Kerstjens, P. Brown, and M. Araújo, Semidefinite programming relaxations for quantum correlations, arXiv:2307.02551.

[53] B. Doolittle and E. Chitambar, Maximal qubit violations of n-locality in star and chain networks, Phys. Rev. A **108**, 042409 (2023).

[54] E. Polino, D. Poderini, G. Rodari, I. Agresti, A. Suprano, G. Carvacho, E. Wolfe, A. Canabarro, G. Moreno, G. Milani, R. W. Spekkens, R. Chaves, and F. Sciarrino, Experimental nonclassicality in a causal network without assuming freedom of choice, Nat. Commun. **14**, 909 (2023).

[55] T. Moroder, J.-D. Bancal, Y.-C. Liang, M. Hofmann, and O. Gühne, Device-independent entanglement quantification and related applications, Phys. Rev. Lett. **111**, 030501 (2013).

[56] K. F. Pál and T. Vértesi, Efficiency of higher-dimensional Hilbert spaces for the violation of Bell inequalities, Phys. Rev. A **77**, 042105 (2008).

[57] M.-O. Renou, D. Trillo, M. Weilenmann, T. P. Le, A. Tavakoli, N. Gisin, A. Acín, and M. Navascués, Quantum theory based on real numbers can be experimentally falsified, Nature (London) **600**, 625 (2021).

[58] Z.-D. Li, Y.-L. Mao, M. Weilenmann, A. Tavakoli, H. Chen, L. Feng, S.-J. Yang, M.-O. Renou, D. Trillo, T. P. Le, N. Gisin, A. Acín, M. Navascués, Z. Wang, and J. Fan, Testing real quantum theory in an optical quantum network, Phys. Rev. Lett. **128**, 040402 (2022).

[59] M.-C. Chen, C. Wang, F.-M. Liu, J.-W. Wang, C. Ying, Z.-X. Shang, Y. Wu, M. Gong, H. Deng, F.-T. Liang, Q. Zhang, C.-Z. Peng, X. Zhu, A. Cabello, C.-Y. Lu, and J.-W. Pan, Ruling out real-valued standard formalism of quantum theory, Phys. Rev. Lett. **128**, 040403 (2022).

[60] D. Wu, Y.-F. Jiang, X.-M. Gu, L. Huang, B. Bai, Q.-C. Sun, X. Zhang, S.-Q. Gong, Y. Mao, H.-S. Zhong, M.-C. Chen, J. Zhang, Q. Zhang, C.-Y. Lu, and J.-W. Pan, Experimental refutation of real-valued quantum mechanics under strict locality conditions, Phys. Rev. Lett. **129**, 140401 (2022).

[61] A. Bednorz and J. Batle, Optimal discrimination between real and complex quantum theories, Phys. Rev. A **106**, 042207 (2022).

[62] M. Navascués, S. Pironio, and A. Acín, Bounding the set of quantum correlations, Phys. Rev. Lett. **98**, 010401 (2007).
[63] M. Navascués, S. Pironio, and A. Acín, A convergent hierarchy of semidefinite programs characterizing the set of quantum correlations, New J. Phys. **10**, 073013 (2008).
[64] S. Pironio, M. Navascués, and A. Acín, Convergent relaxations of polynomial optimization problems with noncommuting variables, SIAM J. Optim. **20**, 2157 (2010).
[65] M. Horodecki, P. Horodecki, and R. Horodecki, Separability of mixed states: necessary and sufficient conditions, Phys. Lett. A **223**, 1 (1996).
[66] C. M. Caves, C. A. Fuchs, and P. Rungta, Entanglement of formation of an arbitrary state of two rebits, Found. Phys. Lett. **14**, 199 (2001).
[67] J. Bergstra, R. Bardenet, Y. Bengio, and B. Kégl, Algorithms for hyper-parameter optimization, in *Proceedings of the 24th International Conference on Neural Information Processing Systems, NIPS'11* (Curran Associates, Red Hook, NY, 2011), pp. 2546–2554.
[68] L. Vandenberghe and S. Boyd, The MOSEK Optimization Toolbox for MATLAB Manual, Version 7.0 (Revision 140), https://www.mosek.com/.
[69] J. Lofberg, Yalmip: a toolbox for modeling and optimization in MATLAB, in *2004 IEEE International Conference on Robotics and Automation (IEEE Cat. No.04CH37508)* (IEEE, New York, 2004), pp. 284–289.
[70] MathWorks, Bayesian Optimization Algorithm, https://www.mathworks.com/help/stats/bayesian-optimization-workflow.html.