Quantum coherence distribution and high-dimensional complementarity

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In the last few years, the l_1 norm has been used as a quantum coherence measure by many authors. In this work, we introduce a variable that mediates the complementarity relation between the l_1 norm of coherence, linear entropy, and the predictability of a quantum state of dimension greater than two. We show that this variable indicates the coherence distribution among the pairs of base states expanding the Hilbert space. We also show that a uniform quantum coherence distribution creates a direct complementarity between the l_1 norm, predictability, and linear entropy.

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I. INTRODUCTION

Complementarity relations have permeated theoretical descriptions and the modeling of quantum physical systems. These relations help determine some variable of interest from experimental data concerning a second variable complementary to the first one. The wave-particle duality, for example, corresponds numerically to the complementarity between the contrast and the which-way knowledge of an interferometer [1–3]. Establishing a quantitative complementarity between these effects gave rise to many works in the 1990s, and early 2000s [3–5].

Many authors explored the which-way information measure in the context of quantum states' discrimination of high-dimension states [1–3,6–8]. Concerning quantum coherence, Baumgratz *et al.* showed that the l_1 norm of coherence is a valid quantum coherence monotone against the l_2 norm, used by previous authors [1,3,9,10]. Currently, different authors have studied new interferometric complementarities involving the l_1 norm and the connection between quantum coherence and quantum entanglement [7,11,12]. In earlier works, authors showed that exact equality connects l_2 norm, linear entropy, and predictability [10,13]. In this article, we investigate the complementarity between the l_1 norm of quantum coherence, linear entropy, and predictability of any D-dimension discrete quantum state. We show that they are connected by a fourth variable quantity that we name T. We also investigate T's physical meaning and its rule in the complementarity upper bound.

II. BUILDING THE COMPLEMENTARITY

Streltsov *et al.* showed that quantum coherence is directly dependent on quantum state purity [12]. For a given degree of purity, quantum coherence is maximum for states formed by a superposition with equal coefficients. In parallel, for a given superposition of base states expanding the respective Hilbert space, quantum coherence is maximum if such a superposition is pure. An inequality was obtained relating the l_1 norm of coherence and the mixedness degree of a quantum state [14]. Here we analyze this inequality, including other variables, to get a relation as accurate as possible.

Considering a D-dimensional quantum state represented by the density operator

$$\hat{\rho} = \sum_{i=1}^{D}\sum_{j=1}^{D} \rho_{ij}\, |i\rangle\langle j|, \tag{1}$$

its normalized form of linear entropy $\mathbf{S}$, predictability $\mathbf{P}$, and l_1 norm of coherence $\mathbf{C}$ are defined, respectively, as [9,10,15]

$$\mathbf{S} = \frac{D}{(D-1)}[1-\mathrm{Tr}(\hat{\rho}^2)] = \frac{D}{(D-1)}\left[1-\sum_{i=1}^{D}\sum_{j=1}^{D}|\rho_{ij}|^2\right], \tag{2}$$

$$\mathbf{P} = \sqrt{\frac{D}{D-1}\left(\sum_{i=1}^{D}|\rho_{ii}|^2 - \frac{1}{D}\right)}, \tag{3}$$

$$\mathbf{C} = \frac{1}{D-1}\sum_{i=1}^{D}\sum_{\substack{j=1\\ j\neq i}}^{D}|\rho_{ij}| = \frac{1}{D-1}\|\vec{W}\|_1, \tag{4}$$

where $\|\vec{W}\|_1$ is the l_1 norm of the vector $\vec{W}$, composed by all the n nondiagonal coefficients of $\hat{\rho}$, with $n = D^2 - D$.

By calculating $\mathbf{S}+\mathbf{P}^2$ we obtain

$$\frac{D(D-1)}{2}\mathbf{C}^2 + \mathbf{S} + \mathbf{P}^2 = 1 + \frac{(D-2)(D+1)}{2}\mathbf{T}, \tag{5}$$

where $\mathbf{T}$ is a variable that is dependent on $\hat{\rho}$ nondiagonal coefficients, such that

$$\mathbf{T} = \frac{4D}{(D-2)(D^2-1)}\sum_{i=1}^{D}\sum_{\substack{j=1\\ j>i}}^{D}\sum_{k=1}^{D}\sum_{\substack{l=1\\ l>k\\ (i,j)\neq(k,l)}}^{D}|\rho_{ij}||\rho_{kl}|. \tag{6}$$

III. T AS A BOUNDING OF COMPLEMENTARITY

In Ref. [10], Jakob and Bergou obtained a complementarity relation between the linear entropy, predictability, and an assumed quantum coherence measure $\mathbf{V}$ (generalized visibility).

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In a normalized form, we can write the equation obtained by them as

$$\mathbf{P}^2 + \mathbf{V}^2 + \mathbf{S} = 1, \tag{7}$$

where

$$\mathbf{V} = \sqrt{\frac{D}{(D-1)} \sum_{i=1}^{D} \sum_{j=1_{(j\neq i)}}^{D} |\rho_{ij}|^2} = \sqrt{\frac{D}{(D-1)}} \|\vec{W}\|_2, \tag{8}$$

and $\|\vec{W}\|_2$ is the l_2 norm of the vector $\vec{W}$ composed by the n nondiagonal coefficients of $\hat{\rho}$.

The left side of Eqs. (7) and (5) differ from each other in the choice for quantum coherence measure. Using resource theory treatment, Baumgratz *et al.* showed that $\mathbf{V}$ is not a quantum coherence monotone. Consequently, we can consider it an invalid quantum coherence measure [9]. This finding invalidates Eq. (7) as a relation between linear entropy, predictability, and quantum coherence monotone for discrete states.

Considering the difference between the l_1 and l_2 norms, we could already expect that the sum of $\mathbf{P}^2$, $\mathbf{S}$, and $\mathbf{C}^2$ would not be equal to an exact value. Therefore, the establishment of a complementarity between $\mathbf{S}$, $\mathbf{P}$, and $\mathbf{C}$ evokes further discussions about the variable $\mathbf{T}$. In the first approach, we can use the relation between the l_1 and l_2 norms to establish an upper bound for $\mathbf{T}$ and the left side of Eq. (5).

Given l_1 and l_2 norms of a vector $\vec{W}$, the Cauchy-Schwarz inequality leads to [16]

$$\|\vec{W}\|_1^2 \leqslant n\|\vec{W}\|_2^2, \tag{9}$$

where n is the $\vec{W}$ dimension. In our context, $n = D^2 - D$ corresponds to the nondiagonal coefficients of $\hat{\rho}$.

From Eqs. (8) and (4) we can write

$$\mathbf{C}^2 = \frac{2}{D(D-1)}\mathbf{V}^2 + \frac{(D-2)(D+1)}{D(D-1)}\mathbf{T}. \tag{10}$$

Considering the definitions of Eqs. (4) and (8), from Eqs. (10) and (9) we obtain

$$\mathbf{T} \leqslant \mathbf{C}^2. \tag{11}$$

Both $\mathbf{C}$ and $\mathbf{T}$ are positive semi-defined. In particular, the variable $\mathbf{T}$ has the quantum coherence as upper bound. Taking Eq. (11) in Eq. (5), we obtain the complementarity

$$\mathbf{C}^2 + \mathbf{S} + \mathbf{P}^2 \leqslant 1. \tag{12}$$

Equation (12) achieves the equality when $\mathbf{T}$ achieves its upper bound in Eq. (11). Furthermore, we also can determine the $\mathbf{T}$ value that implies $\mathbf{C}^2 + \mathbf{S} + \mathbf{P}^2 = 1$ without resorting to Eq. (9), as we show next.

In Eq. (5) we observe that it is possible to obtain $\mathbf{C}^2 + \mathbf{S} + \mathbf{P}^2 = 1$ only in the case of

$$\frac{D(D-1)}{2}\mathbf{C}^2 - \frac{(D-2)(D+1)}{2}\mathbf{T} = \mathbf{C}^2,$$
$$\mathbf{T} = \mathbf{C}^2, \tag{13}$$

which matches the $\mathbf{T}$ upper bound in Eq. (11), and validates the construction of the inequality Eq. (12) from Eqs. (9) and (13).

IV. COMPLEMENTARITY REDUCTIONS

Another way to check our discussion is to analyze Eq. (5) for specific cases with one or more quantifiers equal to zero, resulting in reduced forms of complementarity.

$\mathbf{C} = 0$: If state $\hat{\rho}$ is completely mixed (incoherent), $\mathbf{C} = 0$ and $\mathbf{T} = 0$ also. In this case, from Eq. (5), we recover the known complementarity between predictability and linear entropy [10], namely,

$$\mathbf{S} + \mathbf{P}^2 = 1. \tag{14}$$

$\mathbf{T} = 0$: If state $\hat{\rho}$ has quantum coherence $\mathbf{C} \neq 0$, but only one nondiagonal coefficient different from 0 (and its complex conjugated), then $\mathbf{T} = 0$ even though $\mathbf{C} \neq 0$. In this case, from Eq. (5),

$$\frac{D(D-1)}{2}\mathbf{C}^2 + \mathbf{S} + \mathbf{P}^2 = 1, \tag{15}$$

which is also an exact complementarity relation between $\mathbf{C}$, $\mathbf{S}$, and $\mathbf{P}$.

The quantum coherence of a D-dimensionl state with $\mathbf{T} = 0$ is, at most, $\mathbf{C}_{\max} = 2/[D(D-1)]$, with the unique coefficient above diagonal equal to $1/D$. In this case, $\frac{D(D-1)}{2}\mathbf{C}_{\max}^2 = 2/[D(D-1)]$, and $\mathbf{S} + \mathbf{P}^2$ reaches the minimum. Further, in this situation the minimum of $\mathbf{P}$ and $\mathbf{S}$ are $\mathbf{P}_{\min} = 0$ and $\mathbf{S}_{\min} = 1 - 2/[D(D-1)]$, which agrees with Eq. (15).

$\mathbf{S} = 0$: If state $\hat{\rho}$ is pure, $\mathbf{S} = 0$ and from Eq. (5) we obtain

$$\frac{D(D-1)}{2}\mathbf{C}^2 + \mathbf{P}^2 = 1 + \frac{(D-2)(D+1)}{2}\mathbf{T}. \tag{16}$$

Pure states can be written as $\hat{\rho} = |\psi\rangle\langle\psi|$, such that $\mathbf{C}$ and $\mathbf{T}$ are always different from zero, unless $|\psi\rangle$ is a single element of the base states used for expanding the states in the D-dimensional Hilbert space. Only in this special case, $\mathbf{P} = 1$ and $\mathbf{C} = \mathbf{T} = 0$ for $\hat{\rho} = |\psi\rangle\langle\psi|$.

$\mathbf{P} = 0$: If state $\hat{\rho}$ has all diagonal coefficients $\{\rho_{ii}\}$ equal to $1/D$, $\mathbf{P} = 0$. In this case,

$$\frac{D(D-1)}{2}\mathbf{C}^2 + \mathbf{S} = 1 + \frac{(D-2)(D+1)}{2}\mathbf{T}, \tag{17}$$

with $\mathbf{S} = 1$ being only in the case of $\mathbf{C} = 0$ (so $\mathbf{T} = 0$).

Considering a general computational analysis of Eq. (5), Fig. 1 shows the plot of the numerical values of T and $D(D-1)\mathbf{C}^2/2 + \mathbf{S} + \mathbf{P}^2$, for different quantum states randomly generated by MATHEMATICA software. As expected, we observe that the left side of Eq. (5) is a first-order function of T, where the slope $(D-2)(D+1)/(2)$ increases as D increases. If we used nonnormalized definitions of the variables $\mathbf{S}$, $\mathbf{P}$, and $\mathbf{C}$, the straight lines shown in Fig. 1 would have the same slope and would cross the ordinate axis (vertical axis) at different points.

V. T PHYSICAL MEANING: QUANTUM COHERENCE DISTRIBUTION

It is interesting to observe that $\mathbf{T}$ [Eq. (6)] depends on the multiplication between absolute values of two nondiagonal coefficients. It is proportional to the sum of all possible outcomes, the sum of products of all pair combinations of nondiagonal coefficients. $\mathbf{T}$ is undefined for $D = 2$ since a density operator in this dimension has only one nondiagonal

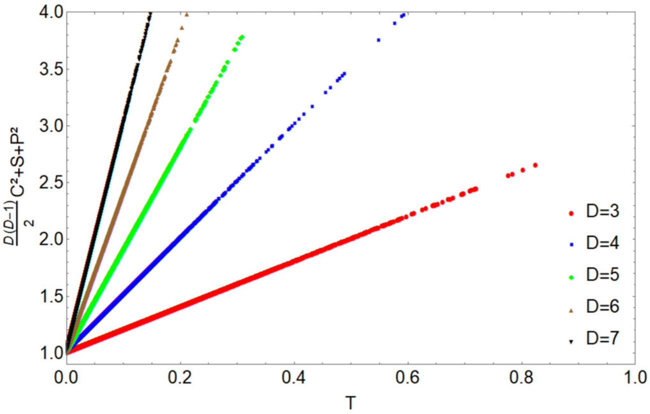

FIG. 1. Numerical plot of $D(D-1)\mathbf{C}^2/2+\mathbf{S}+\mathbf{P}^2$ as a function of T for random quantum states drawn. The states were generated by applying random unitary operations on diagonal states, initially generated by using the RANDOMREAL function in MATHEMATICA software. Each plotted marker corresponds to a specific quantum state and the points with the same color represent states with the same dimension D. For each value of D, 10 000 quantum states were generated.

element (and its complex conjugated). For other dimensions, $\mathbf{T}\neq 0$ if $\hat{\rho}$ has at least two coefficients $\rho_{ij}(i<j)$ different from zero.

In previous work, we treated each absolute value of the nondiagonal coefficient ($|\rho_{ij}|$, $i<j$) as a measure of the "partial coherence" between $|i\rangle$ and $|j\rangle$, belonging to the base states expanding $\hat{\rho}$ Hilbert space [13,17]. In that regard, a quantum state with all $|\rho_{ij}|$ $(i<j)$ equal to the same value presents the same "partial coherence" for all pairs $\{|i\rangle, |j\rangle\}$. In this case, we can say that quantum coherence $\mathbf{C}$ is uniformly distributed in the system. Moreover, if $\hat{\rho}$ has only one nondiagonal coefficient different from zero, its quantum coherence is concentrated in one pair belonging to the base states.

Concerning the variable $\mathbf{T}$, it reaches the minimum ($\mathbf{T}=0$) if $\hat{\rho}$ is incoherent, or if the quantum coherence of $\hat{\rho}$ is concentrated in one pair $(|i\rangle, |j\rangle)$. On the other hand, by replacing the definitions of $\mathbf{T}$ [Eq. (6)] and $\mathbf{C}$ [Eq. (4)] in Eq. (13), we obtain that $\mathbf{T}$ reaches the maximum if the state $\hat{\rho}$ is such a way that

$$\sum_{i=1}^{D}\sum_{\substack{j=1\\ j>i}}^{D}\sum_{k=1}^{D}\sum_{\substack{l=1\\ l>k\\ (i,j)\neq(k,l)}}^{D}|\rho_{ij}||\rho_{kl}| = \tfrac{(D+1)(D-2)}{4}\sum_{i=1}^{D}\sum_{\substack{j=1\\ j\neq i}}^{D}|\rho_{ij}|^2, \tag{18}$$

which is true only if absolute values of all coefficients ρ_{ij} are equal to the same value.

Figure 2 shows the behavior of $\mathbf{T}$ and the right side of Eq. (13) for dimensions $D=3$ and $D=5$, considering two situations: different numbers (n) of $|\rho_{ij}|\neq 0$ $(i<j)$ having the same fixed value (x) and a fixed number of $|\rho_{ij}|\neq 0$ $(i<j)$ but with variable values. We observe that $\mathbf{T}$ coincides with $(D^2-1)(D-2)\mathbf{C}^2/D$ and is equal to equal to $(D^2-D)/2$ if the number of $|\rho_{ij}|\neq 0$ $(i<j)$ is maximum, independently of the value of the nondiagonal coefficients of the density operator. For any fixed value of coherence $\mathbf{C}$, $\mathbf{T}$ reaches its maximum possible value in the case in which all $|\rho_{ij}|$ are equal. In other words, Eq. (13) is reached in a uniform distribution of coherence over all possible state pairs. Here, states are referred to as the states composing the base states used for defining the one-partite density operator $\hat{\rho}$. In this

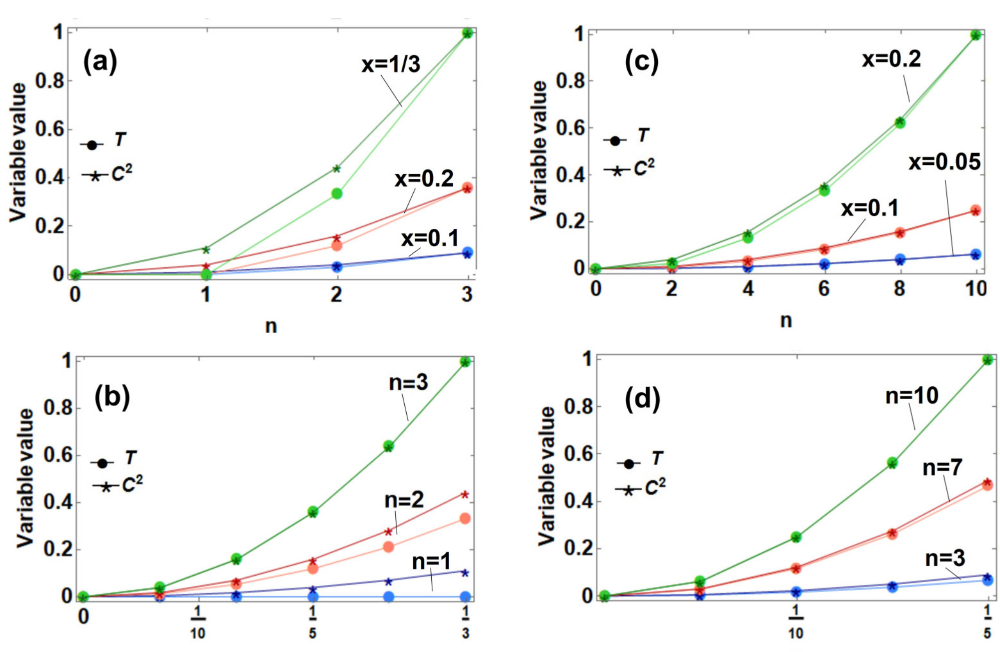

FIG. 2. $\mathbf{T}$ and $\mathbf{C}^2$ behavior for different quantities of (a), (c) nonnull nondiagonal density operator coefficients and (b), (d) different values of nonnull nondiagonal coefficient. In (a), (b) $D=3$, in (c), (d) $D=5$. n is the number of $|\rho_{ij}|\neq 0$ $(i<j)$ in the density operator. x is the value of each nondiagonal coefficient that is different from zero. In each graph the curves with the same color hue correspond to the same fixed values of (a), (c) x and (b), (d) n.

sense, **T** corresponds to a different kind of distribution than that presented in Refs. [18–20].

Here, states are referred to the states composing the base states used for defining the one-partite density operator $\hat{\rho}$. In this sense, **T** corresponds to a different kind of distribution than that presented in Refs. [18–20].

Without supposing an initial equality between the absolute values of the nonnull coefficients ρ_{ij} $(i < j)$, we can analyze equations Eqs. (13) and (18) for arbitrary coefficients in specific dimensions. Taking $D = 3$ as an example, we can write the general qutrit state as Eq. (13)

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} \\ \rho_{12}^* & \rho_{22} & \rho_{23} \\ \rho_{13}^* & \rho_{23}^* & \rho_{33} \end{pmatrix}. \tag{19}$$

Equation (13) is true if Eq. (18) holds, which for this case means that

$$|\rho_{12}||\rho_{13}| + |\rho_{12}||\rho_{23}| + |\rho_{13}||\rho_{23}| = |\rho_{12}|^2 + |\rho_{13}|^2 + |\rho_{23}|^2, \tag{20}$$

which is satisfied only if $|\rho_{12}| = |\rho_{13}| = |\rho_{23}|$, independently of their value ($\{|\rho_{12}|, |\rho_{13}|, |\rho_{23}|\} \in R$).

VI. CONCLUSION

We investigated the complementarity relation between linear entropy **S**, predictability **P**, and the l_1 norm of coherence **C**, for D-dimensional discrete quantum states. We showed that the complementarity relation is mediated by a fourth variable that is upper bounded by a multiple of $\mathbf{C}^2$ with the proportionality factor depending on D. The complementarity relation analyzed here is also helpful for quantum state characterization. If $\hat{\rho}$ corresponds to the density operator of one of the parts of a bipartite system, for example, the square root of its linear entropy, $\sqrt{\mathbf{S}}$, is proportional to the I-Concurrence [21] of the joint quantum state. We can determine the entanglement of a bipartite system by measuring the reduced state's quantum coherence, predictability, and the quantum coherence distribution **T**. **P** and **C** can be experimentally obtained by methods established in the literature [10,17,22,23], and **T** can be obtained by the pair-to-pair visibility measures, as proposed in Ref. [17]. Even if one cannot determine all the variables, it is possible to establish an upper bound for the I-Concurrence.

We demonstrated that **T** indicates the quantum coherence distribution among the levels of a one-partite system, being a nonentropic quantifier differing from previous works such as Refs. [18–20,24]. The distribution of coherence between pairs of base states in a one-partite system seem also to parallel the entanglement distribution in a multiqubit system. If all qubits pairs are entangled, the system presents genuine entanglement [25,26]. Concerning our discussions, if all level pairs are coherent, Eq. (12) reaches the upper bound. As genuine coherence has been used in another sense [27], we raise here the question of a parallel between genuine entanglement measures and the distribution of coherence among the levels of a unipartite system.

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