Nonclassical quantifier based on skewed information

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(Received 22 August 2023; accepted 27 November 2023; published 5 January 2024)

We propose a convenient and easily computable nonclassicality quantifier for bosonic field states based on the Wigner-Yanase skew information. The proposed nonclassicality quantifier is reflected by the quantum interaction between the maximum phase angle of the homodyne rotated quadrature operator and the bosonic field states. If the value of the nonclassical quantifier is greater than one-half, the state is nonclassical, and the quantifier is one-half for the pure classical. It is worth mentioning that an increase in the strength of nonclassicality inducing operations, such as squeezing and photon addition, leads to an enhancement of the nonclassicality in the quantum state. By computing some well-known nonclassical states and summarizing their existence features, we have confirmed the validity of our proposed nonclassicality quantifier. We have shown that in a range of values of the nonclassicality of the Gaussian state, revealing the sufficiency of the quantifier. We also compare it with two common nonclassical quantifiers to illustrate its advantages.

DOI: 10.1103/PhysRevA.109.012203

I. INTRODUCTION

Nonclassicality is not only a fundamental characteristic of quantum mechanics, but also an indispensable resource in quantum optics and quantum information processing. The study of quantum optics in the classical photonic field is closely linked to many fundamental problems of quantum mechanics. In recent years, significant progress has been made in constructing a large number of nonclassical states in quantum optics, which have practical applications [1–9] in quantum information schemes. These nonclassical states include photon number states, Glauber coherent states, multiphoton coherent states, even coherent states, odd coherent states, squeezed states, Schrödinger cat states, phase states, and more. The nonclassical properties of the optical field are of great interest, and are reflected in specific quantum statistical characteristics such as antibunching [10–13], sub-Poisson photon statistics [14,15], quadrature squeezing [16,17], and the partially negative distribution of the Wigner function [18,19]. A quantitative description of nonclassicality, which is crucial for the formation of nonclassical states, is provided by these nonclassical states.

Various methods have been proposed to quantify the nonclassicality of quantum states, with the aim of better understanding this nonclassical nature as a quantum resource. These methods include Mandel's Q parameter [20], nonclassical depth [21–26], distance-based measures [27–34], and so on. In the case of single-mode optical fields, the Glauber-Sudarshan P function [35–37] can be used to distinguish between classical and nonclassical states, but it is not able to accurately quantify the amount of nonclassicality. Recently, a nonclassicality measure based on the volume of the negative part of the Wigner function in phase space was described [38]. However, none of these measures can fully capture the degree of nonclassicality in quantum states. For instance, Mandel's Q parameter only reflects nonclassical behavior up to photon number unification, nonclassical distance is challenging to determine the classical state closest to the quantum state in practice, and nonclassical depth lacks continuity. Therefore, exploring new approaches for quantifying quantum nonclassicality is an important and intriguing topic in resource theory.

In this paper, we introduce an approach to distinguish between classical and nonclassical states in quantum theory based on the Wigner-Yanase skew information [39]. A quantifier for nonclassicality is proposed, which is based on the quantum interaction between the maximum phase angle of the homodyne rotated quadrature operator and the bosonic field states [40]. This quantifier not only provides a quantitative method to study nonclassical properties in quantum theory, but also contributes to a deeper understanding of quantum properties from multiple perspectives. The quantifier is demonstrated using classical states in the optical field, which highlights its theoretical significance and practical value for the study of quantum phase-space theory in experiment.

The structure of this paper is as follows. In Sec. II, we present the nonclassicality quantifier and outline its associated properties, providing a straightforward proof. In Sec. III, we examine common quantum states in the light field and analyze the nonclassical differences between them using this quantifier. In Sec. IV, we compare our quantifier with other methods for quantifying nonclassicality and discuss its advantages. Finally, in Sec. V, we give the discussion and conclusions.

II. NONCLASSICAL QUANTIFIER

A. Preparation of basic knowledge

To begin with, we have a brief introduction to the light field. Light is an electromagnetic wave, and its classical

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properties in free space can be described by Maxwell's equations [41–43]. However, in the quantum physics, the nonclassical properties of light are closely related to its quantum state. To describe the quantum state of light, we need to introduce the creation operator $a^\dagger$ and the annihilation operator a. These operators are non-Hermitian and cannot be directly observed, but they satisfy the canonical commutation relation

$$[a, a^\dagger] = 1. \tag{1}$$

At this point, we introduce the Wigner-Yanase skew information [39,44]

$$I(\rho, X) = -\tfrac{1}{2}\mathrm{tr}[(\sqrt{\rho}, X)^2], \tag{2}$$

which reflects the fact how the quantum state ρ of an optical field does not commute with the conserved observable X [45–48]. This concept is a valuable tool for characterizing the features of the quantum state of the optical field, can also be interpreted as a variation of Fisher information [49,50]. $I(\rho, X)$ possesses the following properties.

(i) *Non-negativity*. For any quantum state ρ, it holds that

$$I(\rho, X) \geqslant 0. \tag{3}$$

Consequently, if ρ and X are exchangeable, then $I(\rho, X) = 0$.

(ii) *Convexity*. For any distribution $\{p_i\}$, it holds that

$$I\left(\sum_i p_i\rho_i, X\right) \leqslant \sum_i p_i I(\rho_i, X), \tag{4}$$

where p_i satisfies $\sum_i p_i = 1$, $p_i \geqslant 0$. This property indicates that mixing different quantum states does not increase the skew information.

(iii) *Unitary invariance*. For any arbitrary quantum state, performing the unitary operation U does not result in a change in the skew information, denoted as

$$I(U\rho U^\dagger, X) = I(\rho, UXU^\dagger). \tag{5}$$

However, the squeezing operator is a special type of unitary operation that can change the photon number distribution and phase distribution of the state, which can lead to a change in the skew information.

(iv) *Relationship with variance*. For any arbitrary quantum state, the variance serves as an upper bound on the skew information:

$$I(\rho, X) \leqslant V(\rho, X) = \mathrm{tr}\rho \mathrm{X}^2 - (\mathrm{tr}\rho \mathrm{X})^2, \tag{6}$$

and the equality sign holds for any quantum pure state $|\Psi\rangle$, which holds that

$$I(|\Psi\rangle, X) = V(|\Psi\rangle, X) = \langle\Psi|X^2|\Psi\rangle - (\langle\Psi|X|\Psi\rangle)^2. \tag{7}$$

B. Definitions and properties

For the bosonic field state ρ, inspired by the definition and properties of the skew information, we define

$$N(\rho) = \max_\theta I(\rho, H_\theta) \tag{8}$$

as a quantifier for nonclassicality of ρ, where $H_\theta = \frac{ae^{i\theta} + a^\dagger e^{-i\theta}}{\sqrt{2}}$ is defined the homodyne rotated quadrature operator and θ represents the phase of the local oscillator in homodyne detection arrangement ($\theta \in [0, 2\pi]$).

Performing a simple manipulation of Eq. (8), the quantifier $N(\rho)$ can be given by

$$N(\rho) = \max_\theta \left[\mathrm{tr}\rho \mathrm{H}_\theta^2 - \mathrm{tr}\sqrt{\rho}\mathrm{H}_\theta\sqrt{\rho}\mathrm{H}_\theta\right], \tag{9}$$

for any pure state ρ, it turns out that $N(\rho)$ can also be expressed as

$$N(\rho) = \max_\theta \left[\langle H_\theta^2\rangle - \langle H_\theta\rangle^2\right]. \tag{10}$$

By considering the characteristics of skew information, we can list the following properties of a nonclassical quantifier $N(\rho)$:

(B_1) *Non-negativity.* For any quantum state ρ, it holds that

$$N(\rho) \geqslant 0, \tag{11}$$

and $N(\rho) = 0$ when the quantum state is a free state.

(B_2) *Convexity*. For a set of probability distributions $\{p_i|\rho_i \geqslant 0, \sum_i p_i = 1\}$, it holds that

$$N\left(\sum_i p_i\rho_i\right) \leqslant \sum_i p_i N(\rho_i). \tag{12}$$

From a mathematical perspective, this property is advantageous for calculating the resources associated with a given quantum state.

(B_3) *Rotation invariance*. For the single-mode phase-space rotation operator

$$R(\theta) = e^{i\theta a^\dagger a}, \qquad \theta \in R \tag{13}$$

it holds that

$$N[R(\theta)\rho R^\dagger(\theta), H_\theta] = N(\rho, H_\theta). \tag{14}$$

(B_4) *Displacement invariance*. For the single-mode phase-space displacement operator

$$D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}, \qquad \alpha \in C, \tag{15}$$

it holds that

$$N(D(\alpha)\rho D^\dagger(\alpha), H_\theta) = N(\rho, H_\theta). \tag{16}$$

(B_5) *Monotonicity*. For any quantum state ρ performing a measurement operation $\Pi(\rho)$, which involves observing a quantum system to obtain its properties or states, it holds that

$$N(\Pi(\rho)) \leqslant N(\rho), \tag{17}$$

which shows that the value of the quantifier $N(\rho)$ cannot be increased.

(B_6) For any coherent state $|\alpha\rangle$, it holds that

$$N(|\alpha\rangle\langle\alpha|) = \tfrac{1}{2}. \tag{18}$$

Furthermore,

$$N(|\Psi\rangle\langle\Psi|) \geqslant \tfrac{1}{2}, \tag{19}$$

for any pure state $|\Psi\rangle$. This implies that among all pure states, any coherent state exhibits the minimum value of nonclassicality.

For a classical state, it holds that

$$N(\rho) \leqslant \tfrac{1}{2}. \tag{20}$$

For a nonclassical state, it holds that

$$N(\rho) > \tfrac{1}{2}. \tag{21}$$

It is important to note that this condition is only sufficient, but not necessary, for ρ to be considered nonclassical.

Now we proceed to establish the above properties. Item (B_1) follows readily from the non-negativity of Wigner-Yanase skew information.

For item (B_2), assuming H'_θ as the optimal observable and according to the convexity of Wigner-Yanase skew information, the Eq. (12) follows from

$$\begin{aligned} N\left(\sum_i p_i\rho_i\right) &= \max_\theta I\left(\sum_i p_i\rho_i, H_\theta\right) \\ &= I\left(\sum_i p_i\rho_i, H_{\theta'}\right) \\ &\leqslant \sum_i p_i I(\rho_i, H_{\theta'}) \\ &\leqslant \sum_i p_i \max_\theta I(\rho_i, H_\theta) \\ &\leqslant \sum_i p_i N(\rho_i). \end{aligned} \tag{22}$$

This is consistent with the intuition that classical mixing does not increase nonclassicality.

Item (B_3) and item (B_4) follow from the unitary invariance of Wigner-Yanase skew information.

For item (B_5), considering the phase-space measurement in the measurement operation

$$\Pi(\rho) = \frac{1}{2\pi}\int_0^{2\pi} e^{i\theta a^\dagger a}\rho e^{-i\theta a^\dagger a} d\theta. \tag{23}$$

Equation (17) follows from the convexity [item (B_2)] and the rotation invariance [item (B_3)] as

$$\begin{aligned} N(\Pi(\rho)) &= N\left(\frac{1}{2\pi}\int_0^{2\pi} e^{i\theta a^\dagger a}\rho e^{-i\theta a^\dagger a} d\theta\right) \\ &\leqslant \frac{1}{2\pi}\int_0^{2\pi} N(e^{i\theta a^\dagger a}\rho e^{-i\theta a^\dagger a}) d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} N(\rho) d\theta \\ &= N(\rho). \end{aligned} \tag{24}$$

For item (B_6), we have now

$$a|\alpha\rangle = \alpha|\alpha\rangle, \tag{25}$$

where $\alpha = |\alpha|e^{i\phi}$ is the eigenvalue of the annihilation operator. Equation (18) follows from

$$\begin{aligned} &N(|\alpha\rangle\langle\alpha|) \\ &= \max_\theta \left[\langle\alpha|H_\theta^2|\alpha\rangle - \langle\alpha|H_\theta|\alpha\rangle^2\right] \\ &= \max_\theta \left[\tfrac{1}{2}\langle\alpha|a^2e^{2i\theta} + 1 + 2a^\dagger a + a^{\dagger 2}e^{-2i\theta}|\alpha\rangle \right. \\ &\quad \left. - \tfrac{1}{2}\langle\alpha|ae^{i\theta} + a^\dagger e^{-i\theta}|\alpha\rangle^2\right] \\ &= \max_\theta \left[\tfrac{1}{2}(\alpha^2e^{2i\theta} + 1 + 2|\alpha|^2 + \alpha^{*2}e^{-2i\theta}) \right. \\ &\quad \left. - \tfrac{1}{2}(\alpha^2e^{2i\theta} + 2|\alpha|^2 + \alpha^{*2}e^{-2i\theta})\right] \\ &= \tfrac{1}{2}. \end{aligned} \tag{26}$$

For any pure state $|\Psi\rangle$, Eq. (19) follows from

$$\begin{aligned} &N(|\Psi\rangle\langle\Psi|) \\ &= \max_\theta \left[\langle\Psi|H_\theta^2|\Psi\rangle - \langle\Psi|H_\theta|\Psi\rangle^2\right] \\ &= \max_\theta \tfrac{1}{2}[e^{2i\theta}(\langle\Psi|a^2|\Psi\rangle - \langle\Psi|a|\Psi\rangle^2) \\ &\quad + e^{-2i\theta}(\langle\Psi|a^{\dagger 2}|\Psi\rangle - \langle\Psi|a^\dagger|\Psi\rangle^2) + 1 \\ &\quad + 2\langle\Psi|a^\dagger a|\Psi\rangle - 2\langle\Psi|a^\dagger|\Psi\rangle\langle\Psi|a|\Psi\rangle] \\ &\geqslant \tfrac{1}{2}[2|\langle\Psi|a^2|\Psi\rangle - \langle\Psi|a|\Psi\rangle^2| + 1 \\ &\quad + 2\langle\Psi|a^\dagger a|\Psi\rangle - 2\langle\Psi|a^\dagger|\Psi\rangle\langle\Psi|a|\Psi\rangle] \\ &\geqslant \tfrac{1}{2} + |\langle a^2\rangle - \langle a\rangle^2| + \langle a^\dagger a\rangle - \langle a\rangle\langle a^\dagger\rangle, \end{aligned} \tag{27}$$

where $\langle a^\dagger a\rangle = \langle\Psi|a^\dagger a|\Psi\rangle$, $\langle a^2\rangle = \langle\Psi|a^2|\Psi\rangle$, and $\langle a^\dagger\rangle = \langle\Psi|a^\dagger|\Psi\rangle$.

In conventional treatments of quantum optics, classical states typically refer to states that possess well-defined probability distributions (denoted as $\rho_i = \sum_n p_i|\alpha_i\rangle\langle\alpha_i|$), similar to coherent states in classical optics. Based on item (B_2), we can conclude that any classical state satisfies the following relation:

$$N\left(\sum_i p_i|\alpha_i\rangle\langle\alpha_i|\right) \leqslant \sum_i p_i N(|\alpha_i\rangle\langle\alpha_i|) = \frac{1}{2}. \tag{28}$$

Thus for a nonclassical state, it holds that

$$N(\rho) > \tfrac{1}{2}. \tag{29}$$

We emphasize that the advantage of using skew information as a measure of nonclassicality is its ability to quantify the degree of nonclassicality in a state and provide a unified standard for comparing nonclassicality between different states. Furthermore, the boundary of 1/2 for skew information does not imply arbitrariness. It is a threshold chosen based on experience, which offers practicality and convenience in practical applications.

III. EXAMPLES OF TYPICAL LIGHT FIELD STATES

In the preceding sections, we have presented the general form of our nonclassicality quantifier. We now illustrate that this quantifier $N(\cdot)$ reflects nonclassicality by typical examples in the single-mode bosonic field.

Example 1. For the coherent states $\rho = |\alpha\rangle\langle\alpha|$, we have now

$$N(|\alpha\rangle\langle\alpha|) = \tfrac{1}{2}. \tag{30}$$

This observation aligns with the widely recognized interpretation of the coherent state as the most classical pure state [27,37,51].

Example 2. For the squeezed coherent states

$$\rho_{sc} = S(z)|\alpha\rangle\langle\alpha|S^\dagger(z), \tag{31}$$

where $S(z) = e^{(za^{\dagger 2} - z^*a^2)/2}$ is the squeezing operator, $z = |z|e^{i\varphi}(0 \leqslant |z| < \infty)$ is called the compression amplitude, and

$\varphi(0 \leqslant \varphi \leqslant 2\pi)$ is the compression angle. Since $S^\dagger a S = a\cosh|z| - a^\dagger e^{i\varphi}\sinh|z|$, $S^\dagger a^\dagger S = a^\dagger\cosh|z| - ae^{-i\varphi}\sinh|z|$, we have

$$N(\rho_{\rm sc}) = \max_\theta \left[\langle\alpha|S^\dagger H_\theta^2 S|\alpha\rangle - \langle\alpha|S^\dagger H_\theta S|\alpha\rangle^2\right] \quad (32)$$

with

$$\begin{aligned}
&\langle\alpha|S^\dagger H_\theta^2 S|\alpha\rangle \\
&= \tfrac{1}{2}\langle\alpha|\cosh^2|z|(a^2 e^{2i\theta} + a^{\dagger 2}e^{-2i\theta}) + 2(1 + a^\dagger a)\sinh^2|z| \\
&\quad + \sinh^2|z|(a^{\dagger 2}e^{2i\varphi} + a^2 e^{-2i\varphi}) + 2a^\dagger a\cosh^2|z| + 1 \\
&\quad - (1 + 2a^\dagger a)\cosh|z|\sinh|z|(e^{2i\theta + i\varphi} + e^{-2i\theta - i\varphi}) \\
&\quad - 2(a^{\dagger 2}e^{2i\varphi} + a^2 e^{-2i\varphi})\cosh|z|\sinh|z||\alpha\rangle \\
&= \tfrac{1}{2}[\cosh^2|z|(\alpha^2 e^{2i\theta} + \alpha^{*2}e^{-2i\theta}) + 2(1 + |\alpha|^2)\sinh^2|z| \\
&\quad + \sinh^2|z|(\alpha^{*2}e^{2i\varphi} + \alpha^2 e^{-2i\varphi}) + 2|\alpha|^2\cosh^2|z| + 1 \\
&\quad - (1 + 2|\alpha|^2)\cosh|z|\sinh|z|(e^{2i\theta + i\varphi} + e^{-2i\theta - i\varphi}) \\
&\quad - 2(\alpha^{*2}e^{2i\varphi} + \alpha^2 e^{-2i\varphi})\cosh|z|\sinh|z|],
\end{aligned} \quad (33)$$

and

$$\begin{aligned}
&\langle\alpha|S^\dagger H_\theta S|\alpha\rangle^2 \\
&= \tfrac{1}{2}\langle\alpha|\cosh|z|(a^\dagger e^{-i\theta} + ae^{i\theta}) - \sinh|z|(a^\dagger e^{i\varphi} + ae^{-i\varphi})|\alpha\rangle^2 \\
&= \tfrac{1}{2}[\cosh^2|z|(\alpha^2 e^{2i\theta} + \alpha^{*2}e^{-2i\theta}) + 2|\alpha|^2(\sinh^2|z| \\
&\quad + \cosh^2|z|) - 2|\alpha|^2\cosh|z|\sinh|z|(e^{2i\theta + i\varphi} + e^{-2i\theta - i\varphi}) \\
&\quad - 2(\alpha^{*2}e^{2i\varphi} + \alpha^2 e^{-2i\varphi})\cosh|z|\sinh|z| \\
&\quad + \sinh^2|z|(\alpha^{*2}e^{2i\varphi} + \alpha^2 e^{-2i\varphi})].
\end{aligned} \quad (34)$$

Now, the quantifier of the squeezed coherent state $\rho_{\rm sc}$ is given by

$$\begin{aligned}
&N(\rho_{\rm sc}) \\
&= \max_\theta \tfrac{1}{2}[\cosh^2|z| + \sinh^2|z| \\
&\quad + \cosh|z|\sinh|z|\cosh|z|(2\theta + \varphi)] \\
&= \tfrac{1}{2}[\cosh^2|z| + \sinh^2|z| + 2\cosh|z|\sinh|z|] \\
&= \tfrac{1}{2}[\cosh|z| + \sinh|z|]^2 \\
&= \tfrac{1}{2}e^{2|z|}(|z| \neq 0).
\end{aligned} \quad (35)$$

It is obvious that the squeezed coherent states are nonclassical, and that nonclassicality increases with the compressiveness.

Example 3. The optical Schrödinger cat states can be defined as the superposition of two coherent states $|\alpha\rangle$ and $|-\alpha\rangle$ with the same amplitude and opposite phase. They are valuable resources in quantum computing, quantum teleportation, and quantum precision measurements.

For the even cat states

$$|\alpha_+\rangle = \frac{1}{(2 + 2e^{-2|\alpha|^2})^{1/2}}(|\alpha\rangle + |-\alpha\rangle), \quad (36)$$

we have

$$N(|\alpha_+\rangle\langle\alpha_+|) = \max_\theta \left[\langle\alpha_+|H_\theta^2|\alpha_+\rangle - \langle\alpha_+|H_\theta|\alpha_+\rangle^2\right], \quad (37)$$

with

$$\begin{aligned}
&\langle\alpha_+|H_\theta^2|\alpha_+\rangle \\
&= \tfrac{1}{2}\langle\alpha_+|a^2 e^{2i\theta} + 1 + 2a^\dagger a + a^{\dagger 2}e^{-2i\theta}|\alpha_+\rangle \\
&= \tfrac{1}{2}[\alpha^2 e^{2i\theta} + 1 + 2|\alpha|^2\tanh|\alpha|^2 + \alpha^{*2}e^{-2i\theta}].
\end{aligned} \quad (38)$$

Here, we take advantage of the interconversion effect of annihilation operators $a|\alpha_+\rangle = \alpha\sqrt{\tanh|\alpha|^2}|\alpha_-\rangle$ and $a|\alpha_-\rangle = \alpha\sqrt{\coth|\alpha|^2}|\alpha_+\rangle$. In addition, the even cat states and odd cat states are orthogonal and satisfy $\langle\alpha_+|\alpha_-\rangle = 0$, so we can get

$$\begin{aligned}
&\langle\alpha_+|H_\theta|\alpha_+\rangle^2 \\
&= \langle\alpha_+|\frac{ae^{i\theta} + a^\dagger e^{-i\theta}}{\sqrt{2}}|\alpha_+\rangle^2 \\
&= \tfrac{1}{2}[e^{i\theta}\langle\alpha_+|a|\alpha_+\rangle + e^{-i\theta}\langle\alpha_+|a^\dagger|\alpha_+\rangle]^2 \\
&= 0.
\end{aligned} \quad (39)$$

It follows that

$$\begin{aligned}
&N(|\alpha_+\rangle\langle\alpha_+|) \\
&= \max_\theta \tfrac{1}{2}[\alpha^2 e^{2i\theta} + 1 + 2|\alpha|^2\tanh|\alpha|^2 + \alpha^{*2}e^{-2i\theta}] \\
&= \tfrac{1}{2}[1 + 2|\alpha|^2 + 2|\alpha|^2\tanh|\alpha|^2] \\
&= \tfrac{1}{2} + |\alpha|^2\tanh|\alpha|^2 + |\alpha|^2,
\end{aligned} \quad (40)$$

which is an increasing function of $|\alpha|^2$.

For the odd cat states

$$|\alpha_-\rangle = \frac{1}{(2 - 2e^{-2|\alpha|^2})^{1/2}}(|\alpha\rangle - |-\alpha\rangle), \quad (41)$$

we can use the same formula as before to obtain

$$N(|\alpha_-\rangle\langle\alpha_-|) = \tfrac{1}{2} + |\alpha|^2\coth|\alpha|^2 + |\alpha|^2. \quad (42)$$

From the above calculation results, we can see that $|\alpha|^2$ is also an increasing function, which reflects the good nonclassicality of both even cat states and odd cat states [52]. In contrast, the odd cat states are more nonclassical, meaning that $N(|\alpha_-\rangle\langle\alpha_-|) > N(|\alpha_+\rangle\langle\alpha_+|)$, which may be related to its compression effect.

For the photon-added even cat states $\rho_e \sim a^\dagger|\alpha_+\rangle\langle\alpha_+|a$, using the wave function's normalizing condition, we obtain

$$\langle\alpha_+|aa^\dagger|\alpha_+\rangle = \langle\alpha_+|1 + aa^\dagger|\alpha_+\rangle = 1 + |\alpha|^2\tanh|\alpha|^2, \quad (43)$$

then we can rewrite $N(\rho_e)$ as

$$N(\rho_e) = \frac{\max_\theta\left[\langle\alpha_+|aH_\theta^2 a^\dagger|\alpha_+\rangle - \langle\alpha_+|aH_\theta a^\dagger|\alpha_+\rangle^2\right]}{1 + |\alpha|^2\tanh|\alpha|^2}, \quad (44)$$

where

$$\begin{aligned}
&\langle\alpha_+|aH_\theta^2 a^\dagger|\alpha_+\rangle \\
&= \tfrac{1}{2}\langle\alpha_+|a(a^2 e^{2i\theta} + 1 + 2aa^\dagger + a^{\dagger 2}e^{-2i\theta})a^\dagger|\alpha_+\rangle \\
&= \tfrac{1}{2}[(3 + |\alpha|^2\tanh|\alpha|^2)(\alpha^2 e^{2i\theta} + \alpha^{*2}e^{-2i\theta} + 1) \\
&\quad + 6|\alpha|^2\tanh|\alpha|^2 + 2|\alpha|^4],
\end{aligned} \quad (45)$$

with

$$\langle\alpha_+|aH_\theta a^\dagger|\alpha_+\rangle^2 = 0. \quad (46)$$

It follows that

$$
\begin{aligned}
N(\rho_e) &= \frac{\max_\theta \left[\langle\alpha_+|aH_\theta^2a^\dagger|\alpha_+\rangle - \langle\alpha_+|aH_\theta a^\dagger|\alpha_+\rangle^2\right]}{1+|\alpha|^2\tanh|\alpha|^2}\\
&= \frac{\frac{1}{2}[(2|\alpha|^2+7)(1+|\alpha|^2\tanh|\alpha|^2)+2(|\alpha|^2+1)^2-6]}{1+|\alpha|^2\tanh|\alpha|^2}\\
&= |\alpha|^2+\frac{7}{2}+\frac{(|\alpha|^2+1)^2-3}{1+|\alpha|^2\tanh|\alpha|^2}. \qquad (47)
\end{aligned}
$$

We can obtain that when the external environment changes (the number of photons increases), the properties of the superposition state change and its nonclassicality is significantly enhanced.

For the photon-added odd cat states $\rho_o \sim a^\dagger|\alpha_-\rangle\langle\alpha_-|a$, the final calculation result is given by

$$N(\rho_o) = |\alpha|^2+\frac{7}{2}+\frac{(|\alpha|^2+1)^2-3}{1+|\alpha|^2\coth|\alpha|^2}, \qquad (48)$$

which also reveals $N(\rho_o) > N(|\alpha_-\rangle\langle\alpha_-|)$, adding a photon increases the nonclassicality.

Example 4. For the Fock (number) state $|n\rangle\langle n|$ [53], we have

$$
\begin{aligned}
N(|n\rangle\langle n|) &= \max_\theta\left[\langle n|H_\theta^2|n\rangle - \langle n|H_\theta|n\rangle^2\right]\\
&= \max_\theta \tfrac{1}{2}[\langle n|1+2a^\dagger a|n\rangle] = \tfrac{1}{2}+n, \qquad (49)
\end{aligned}
$$

which is the general most common representation of the state, since the Hamiltonian has eigenvalues of $\hbar\omega(n+\frac{1}{2})$, where n is the photon number. In this regard, we can see that $N(\cdot)$ is the sum of the nonclassicality of the vacuum state and the number of photons, which suggests that in addition to the vacuum state, all other Fock states are nonclassical.

Example 5. For the thermal state

$$\rho_{\text{th}} = \frac{e^{-(\hbar\omega/k_BT)a^\dagger a}}{\text{tr}e^{-(\hbar\omega/k_BT)a^\dagger a}} = (1-\gamma)\sum_{n=0}^{\infty}\gamma^n|n\rangle\langle n|, \qquad (50)$$

with

$$\gamma = e^{-\hbar\omega/k_BT}, \qquad (51)$$

we have

$$N(\rho_{\text{th}}) = \max_\theta\left[\text{tr}\rho_{\text{th}}H_\theta^2 - \text{tr}\sqrt{\rho}_{\text{th}}H_\theta\sqrt{\rho}_{\text{th}}H_\theta\right], \qquad (52)$$

where

$$
\begin{aligned}
\text{tr}\rho_{\text{th}}H_\theta^2 &= (1-\gamma)\sum_{m,n}\langle m|\gamma^n|n\rangle\langle n|H_\theta^2|m\rangle\\
&= \frac{1}{2}(1-\gamma)\sum_n\gamma^n\langle n|1+2a^\dagger a|n\rangle\\
&= \frac{1}{2}(1-\gamma)\sum_n\gamma^n(1+2n)\\
&= \frac{1}{2}(1-\gamma)\left[\frac{1}{1-\gamma}+\frac{2\gamma}{(1-\gamma)^2}\right]\\
&= \frac{1}{2}\frac{1+\gamma}{1-\gamma}, \qquad (53)
\end{aligned}
$$

and

$$
\begin{aligned}
&\text{tr}\sqrt{\rho}_{\text{th}}H_\theta\sqrt{\rho}_{\text{th}}H_\theta\\
&= (1-\gamma)\sum_{m,n,q}\gamma^{\frac{n+q}{2}}\langle m|n\rangle\langle n|H_\theta|q\rangle\langle q|H_\theta|m\rangle\\
&= (1-\gamma)\sum_{n,q}\gamma^{\frac{n+q}{2}}\langle n|H_\theta|q\rangle\langle q|H_\theta|n\rangle\\
&= \frac{1}{2}(1-\gamma)\sum_{n,q}\gamma^{\frac{n+q}{2}}(e^{i\theta}\langle n|a|q\rangle+e^{-i\theta}\langle n|a^\dagger|q\rangle)\\
&\quad\times(e^{i\theta}\langle q|a|n\rangle+e^{-i\theta}\langle q|a^\dagger|n\rangle)\\
&= \frac{1}{2}(1-\gamma)\sum_{n,q}\gamma^{\frac{n+q}{2}}(\sqrt{q}\sqrt{n+1}\langle n|q-1\rangle\langle q|n+1\rangle\\
&\quad+(\sqrt{q+1}\sqrt{n}\langle n|q+1\rangle\langle q|n-1\rangle)\\
&= \frac{1}{2}(1-\gamma)\sum_{n,q}\gamma^{\frac{n+q}{2}}(q\delta_{n,q-1}+n\delta_{q,n-1})\\
&= (1-\gamma)\sum_{n=0}^{\infty}n\gamma^{n-\frac{1}{2}}\\
&= \frac{\sqrt{\gamma}}{1-\gamma}. \qquad (54)
\end{aligned}
$$

It follows that

$$
\begin{aligned}
N(\rho_{\text{th}}) &= \frac{1}{2}\frac{1+\gamma}{1-\gamma}-\frac{\sqrt{\gamma}}{1-\gamma}\\
&= \frac{1}{2}\frac{1-\sqrt{\gamma}}{1+\sqrt{\gamma}}. \qquad (55)
\end{aligned}
$$

From this we can see that this is a decreasing function on γ, which is $N(\rho_{\text{th}}) < \frac{1}{2}$. In other words, the higher the temperature, and the less nonclassicality is exhibited. This is also indicative of a little worse nonclassicality than the coherent states and Fock states, probably due to the influence of thermal noise.

For the truncated thermal state

$$\rho_{\text{tth}} = \frac{1-\gamma}{\gamma}\sum_{n=1}^{\infty}\gamma^n|n\rangle\langle n|,\ \ 0<\gamma<1 \qquad (56)$$

we have

$$N(\rho_{\text{tth}}) = \max_\theta\left[\text{tr}\rho_{\text{tth}}H_\theta^2 - \text{tr}\sqrt{\rho}_{\text{tth}}H_\theta\sqrt{\rho}_{\text{tth}}H_\theta\right], \qquad (57)$$

where

$$
\begin{aligned}
\text{tr}\rho_{\text{tth}}H_\theta^2 &= \frac{1-\gamma}{\gamma}\text{tr}\left(\sum_{n=0}^{\infty}\gamma^n|n\rangle\langle n|-|0\rangle\langle 0|\right)H_\theta^2\\
&= \frac{1-\gamma}{\gamma}\left(\sum_{n=0}^{\infty}\gamma^n\langle n|H_\theta^2|n\rangle-\langle 0|H_\theta^2|0\rangle\right)\\
&= \frac{1-\gamma}{2\gamma}\sum_{n=0}^{\infty}\gamma^n(1+2n)-\frac{1-\gamma}{2\gamma}\\
&= \frac{1}{2}+\frac{1}{1-\gamma}, \qquad (58)
\end{aligned}
$$

and

$$\begin{aligned}
&\mathrm{tr}\sqrt{\rho}_{\mathrm{tth}}H_\theta\sqrt{\rho}_{\mathrm{tth}}H_\theta\\
&=\frac{1-\gamma}{\gamma}\mathrm{tr}\left(\sum_{n=0}^{\infty}\gamma^{\frac{n}{2}}|n\rangle\langle n|-|0\rangle\langle 0|\right)H_\theta\\
&\times\left(\sum_{m=0}^{\infty}\gamma^{\frac{m}{2}}|m\rangle\langle m|-|0\rangle\langle 0|\right)H_\theta\\
&=\frac{1-\gamma}{2\gamma}\left[\sum_{n=0}^{\infty}\gamma^{\frac{m+n}{2}}\sqrt{n+1}\sqrt{m}\langle n|m-1\rangle\langle m|n+1\rangle\right.\\
&\left.+\sum_{m=0}^{\infty}\gamma^{\frac{m+n}{2}}\sqrt{n}\sqrt{m+1}\langle n|m+1\rangle\langle m|n-1\rangle-2\sqrt{\gamma}\right]\\
&=\frac{1-\gamma}{2\gamma}\left[2\sum_{n}^{\infty}nr^{n-\frac{1}{2}}-2\sqrt{\gamma}\right]\\
&=\frac{\sqrt{\gamma}(2-\gamma)}{1-\gamma},
\end{aligned}\tag{59}$$

then we obtain

$$\begin{aligned}
N(\rho_{\mathrm{tth}})&=\frac{1}{2}+\frac{1}{1-\gamma}-\frac{\sqrt{\gamma}(2-\gamma)}{1-\gamma}\\
&=\frac{1}{2}\frac{1-\sqrt{\gamma}}{1+\sqrt{\gamma}}+(1-\sqrt{\gamma}).
\end{aligned}\tag{60}$$

Similar to the thermal state, the truncated thermal state is also a decreasing function with respect to γ, but the nonclassical nature of the truncated thermal state is better than the thermal state in terms of the value of $N(\cdot)$. Indeed, $N(\rho_{\mathrm{tth}})>\frac{1}{2}$ for which signals nonclassicality.

For the photon-added thermal state

$$\rho_{\mathrm{pth}}=\frac{a^\dagger\rho_{\mathrm{th}}a}{\mathrm{tr}a^\dagger\rho_{\mathrm{th}}a}=\frac{(1-\gamma)^2}{\gamma}\sum_{n=1}^{\infty}\gamma^n|n\rangle\langle n|,\quad 0<\gamma<1$$

we have

$$N(\rho_{\mathrm{pth}})=\max_\theta\left[\mathrm{tr}\rho_{\mathrm{pth}}H_\theta^2-\mathrm{tr}\sqrt{\rho}_{\mathrm{pth}}H_\theta\sqrt{\rho}_{\mathrm{pth}}H_\theta\right],\tag{61}$$

where

$$\begin{aligned}
\mathrm{tr}\rho_{\mathrm{pth}}H_\theta^2&=\frac{(1-\gamma)^2}{\gamma}\mathrm{tr}\left(\sum_{n}^{\infty}n\gamma^2|n\rangle\langle n|H_\theta^2\right)\\
&=\frac{(1-\gamma)^2}{2\gamma}\sum_{m,n}^{\infty}n\gamma^n\langle m|n\rangle\langle n|1+2a^\dagger a|m\rangle\\
&=\frac{(1-\gamma)^2}{2\gamma}\sum_{n=1}^{\infty}n\gamma^n(1+2n)\\
&=\frac{1}{2}+\frac{1+\gamma}{1-\gamma},
\end{aligned}\tag{62}$$

and

$$\begin{aligned}
&\mathrm{tr}\sqrt{\rho}_{\mathrm{pth}}H_\theta\sqrt{\rho}_{\mathrm{pth}}H_\theta\\
&=\frac{(1-\gamma)^2}{\gamma}\sum_{m,n,q}^{\infty}\gamma^{\frac{n+q}{2}}\sqrt{n}\sqrt{q}\langle m|n\rangle\langle n|H_\theta|q\rangle\langle q|H_\theta|m\rangle\\
&=\frac{(1-\gamma)^2}{2\gamma}\sum_{n,q}^{\infty}\gamma^{\frac{n+q}{2}}\sqrt{n}\sqrt{q}(\langle n|a|q\rangle\langle q|a^\dagger|n\rangle\\
&+\langle n|a^\dagger|q\rangle\langle q|a|n\rangle)\\
&=\frac{(1-\gamma)^2}{2\gamma}\sum_{n,q}^{\infty}\gamma^{\frac{n+q}{2}}\sqrt{n}\sqrt{q}(q\delta_{n,q-1}+n\delta_{q,n-1})\\
&=\frac{(1-\gamma)^2}{\gamma^{3/2}}\sum_{n=1}^{\infty}n^{3/2}(n-1)^{1/2}\gamma^n.
\end{aligned}\tag{63}$$

It follows that

$$\begin{aligned}
&N(\rho_{\mathrm{pth}})\\
&=\frac{1}{2}+\frac{1+\gamma}{1-\gamma}-\frac{(1-\gamma)^2}{\gamma^{3/2}}\sum_{n=1}^{\infty}n^{3/2}(n-1)^{1/2}\gamma^n,
\end{aligned}\tag{64}$$

which is a decreasing function of γ. And when at $\gamma<\gamma_c\approx 0.223$, $N(\rho_{\mathrm{pth}})>\frac{1}{2}$, showing its nonclassicality [54,55].

Example 6. For the Gaussian state [56]

$$\rho_{\mathrm{g}}=D(\alpha)S(z)\rho_{\mathrm{th}}S^\dagger(z)D^\dagger(\alpha),$$

where $D(\alpha)$ refers to the displacement operator mentioned previously [57–59], $S(z)$ is the squeezing operator mentioned above and ρ_{th} is a thermal state. From the criterion in item (B_4) of the properties of $N(.)$, we obtain

$$N(D\rho'_{\mathrm{th}}D^\dagger)=N(\rho'_{\mathrm{th}}),\tag{65}$$

with

$$\rho'_{\mathrm{th}}=S(z)\rho_{\mathrm{th}}S^\dagger(z),\tag{66}$$

then we have

$$N(\rho_{\mathrm{g}})=\max_\theta\left[\mathrm{tr}\rho'_{\mathrm{th}}H_\theta^2-\mathrm{tr}\sqrt{\rho'_{\mathrm{th}}}H_\theta\sqrt{\rho'_{\mathrm{th}}}H_\theta\right],\tag{67}$$

where

$$\begin{aligned}
&\mathrm{tr}\rho'_{\mathrm{th}}H_\theta^2\\
&=\mathrm{tr}(1-\gamma)\sum_{n=1}^{\infty}\gamma^n|n\rangle\langle n|S^\dagger H_\theta^2S\\
&=(1-\gamma)\sum_{n=1}^{\infty}\gamma^n\langle n|S^\dagger H_\theta^2S|n\rangle\\
&=\frac{1-\gamma}{2}\sum_{n=1}^{\infty}\gamma^n\langle n|S^\dagger(a^2e^{2i\theta}+1+2a^\dagger a+a^{\dagger 2}e^{-2i\theta})S|n\rangle\\
&=\frac{1-\gamma}{2}\sum_{n=1}^{\infty}\gamma^n\langle n|-e^{i(2\theta+\varphi)}\cosh|z|\sinh|z|(1+2n)\\
&-e^{-i(2\theta+\varphi)}\cosh|z|\sinh|z|(1+2n)+2\sinh^2|z|+1\\
&+2n(\sinh^2|z|+\cosh^2|z|)|n\rangle\\
&=\frac{1-\gamma}{2}\sum_{n=1}^{\infty}\gamma^n[-2\cosh(2\theta+\varphi)\cosh|z|\sinh|z|(1+2n)\\
&+2\sinh^2|z|(1+2n)+2n\cosh^2|z|+\cosh^2|z|-\sinh^2|z|]
\end{aligned}$$

$$= \frac{1-\gamma}{2}\sum_{n=1}^{\infty}(1+2n)\gamma^n[\cosh^2|z| + \sinh^2|z| - 2\cosh(2\theta+\varphi)\cosh|z|\sinh|z|], \quad (68)$$

and

$$\mathrm{tr}\sqrt{\rho'_{\mathrm{th}}}H_\theta\sqrt{\rho'_{\mathrm{th}}}H_\theta = (1-\gamma)\sum_{m,n}^{\infty}\gamma^{\frac{m+n}{2}}\langle m|S^\dagger H_\theta S|n\rangle\langle n|S^\dagger H_\theta S|m\rangle$$
$$= \frac{1-\gamma}{2}\sum_{m,n}^{\infty}\gamma^{\frac{m+n}{2}}[\langle m|S^\dagger(ae^{i\theta}+a^\dagger e^{-i\theta})S|n\rangle \times \langle n|S^\dagger(ae^{i\theta}+a^\dagger e^{-i\theta})S|m\rangle]$$
$$= \frac{1-\gamma}{2}\sum_{m,n}^{\infty}\gamma^{\frac{m+n}{2}}2n\delta_{m,n-1}[\cosh^2|z| + \sinh^2|z| - 2\cosh(2\theta+\varphi)\cosh|z|\sinh|z|], \quad (69)$$

then we have

$$N(\rho_g) = \max_\theta\left[(1-\gamma)\sum_{n=0}^{\infty}(1+2n)\gamma^n - (1-\gamma)\sum_{n=0}^{\infty}2n\gamma^{n-\frac{1}{2}}\right] \times[\cosh^2|z| + \sinh^2|z| - 2\cosh(2\theta+\varphi)\cosh|z|\sinh|z|]$$
$$= \frac{1}{2}\frac{1-\sqrt{\gamma}}{1+\sqrt{\gamma}}e^{2|z|}. \quad (70)$$

We specifically draw the conclusion that Gaussian states with $|z| > \frac{1}{2}\ln\frac{1+\sqrt{\gamma}}{1-\sqrt{\gamma}}$ are nonclassical based on the item (B_6) of the properties of $N(\cdot)$. However, from the phase-space analysis in Refs. [60–62], it has been clearly stated that the Gaussian states are nonclassical if and only if $|z| > \frac{1}{2}\ln\frac{1+\gamma}{1-\gamma}$. Consequently, Gaussian state $N(\rho_g)$ with $\frac{1}{2}\ln\frac{1+\gamma}{1-\gamma} < |z| < \frac{1}{2}\ln\frac{1+\sqrt{\gamma}}{1-\sqrt{\gamma}}$ are considered nonclassical, despite having $N(\rho_g) < \frac{1}{2}$ in this particular case. Compared to the nonclassicality range of $\frac{1}{2}\ln\frac{1+\gamma}{1-\gamma} < |z| < \frac{1}{2}\ln\frac{1+\gamma^{1/4}}{1-\gamma^{1/4}}$ given in Ref. [63] for Gaussian states, it can be observed that our quantifier provides a significantly improved range. To further verify the progressiveness of our given quantifier, we plot the three functions $z_1 = \frac{1}{2}\ln\frac{1+\gamma}{1-\gamma}$, $z_2 = \frac{1}{2}\ln\frac{1+\sqrt{\gamma}}{1-\sqrt{\gamma}}$, and $z_3 = \frac{1}{2}\ln\frac{1+\gamma^{1/4}}{1-\gamma^{1/4}}$ relative to the variable γ in Fig. 1. From Fig. 1, we can get $z_1 < z_2 < z_3$. And the difference between the function z_2 and z_3 also shows the superiority of our given quantifier in Eq. (8). This observation suggests that the quantifier we use to measure nonclassicality is only sufficient.

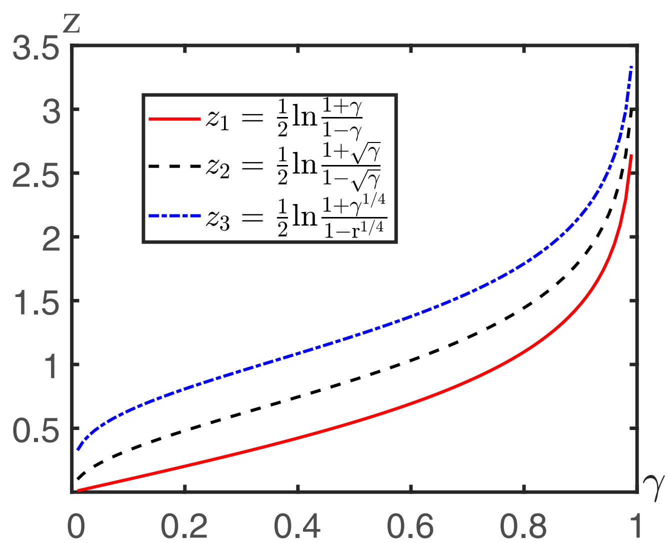

FIG. 1. Comparison of the three z curves by variation of the γ parameter.

IV. COMPARISON

We will list two representative nonclassical metrics that reveal the advantages of our proposed nonclassical quantifier. Mandel introduced the Mandel's Q parameter [20] to describe the difference between the photon number statistics and the Poisson statistical distribution of the optical field, with the expression

$$Q = \frac{\langle(\Delta N)^2\rangle}{\langle N\rangle},$$

where $(\Delta N)^2$ denotes the particle number quantum square fluctuation value, defining as $\langle(\Delta N)^2\rangle = \langle N^2\rangle - \langle N\rangle^2$, $N = a^\dagger a$ is the particle number operator.

For a particular light field state, if $Q = 0$, it means that the light field is coherent light and the photon number distribution of this light field state is Poisson distribution; if $Q > 0$, it is said that the distribution of photon number of this light field is wider than Poisson distribution and it is called super-Poisson distribution (understood from the classical statistical point of view, the mean-squared deviation of particle number of classical light field cannot be smaller than the average photon number); if $Q < 0$, that is, the mean-squared deviation of its photon number measurement is smaller than the average photon number. This means that the light field is noncoherent light and the distribution of the photon number of the light field is narrower than the Poisson distribution, which is called the sub-Poisson distribution. The sub-Poisson distribution is purely a quantum effect of the light field. However, the Mandel's Q parameter does not reflect the nonclassical behavior beyond the photon number statistics, which is one of its shortcomings. Our nonclassicality quantifier enjoys such a desirable property since $N(|n\rangle\langle n|) = \frac{1}{2} + n$.

The Wigner function $W(q, p)$ is a quasiprobability distribution in the phase space, which is in one-to-one correspondence with the quantum state. The q and p are the real and imaginary parts of the complex amplitude α in the phase space. The expression of the Wigner function for any quantum state ρ is shown as follows [19]:

$$W(q,p) = \frac{1}{\pi}\int_{-\infty}^{\dagger\infty}\exp(2ipy)\langle q-y|\rho|q+y\rangle dy.$$

Having obtained the expression of the Wigner function for any quantum state, we can then integrate the negative part in the phase space to obtain its negative volume of the Wigner function. The absolute value of the negative volume of the Wigner function $P_{\rm NW}$ is

$$P_{\rm NW} = |\int_{\Omega} W(q, p) dq dp|,$$

where Ω is the region of negative distribution of the Wigner function. $P_{\rm NW}$ is a nonnegative real number, and the larger $P_{\rm NW}$ the stronger we consider the nonclassicality of the corresponding quantum state [38]. For the Fock state, $P_{\rm NW}$ increases monotonically as the quantum number n increases, which is consistent with the Fock state, which becomes more nonclassical when n is larger. The negative values of the Wigner function are considered to be a representation of nonclassicality, but not all nonclassical states of the Wigner function have a negative probability distribution. For example, the Glauber-Sudarshan P function of the Fock state has singular values and the Wigner function has negative values, yet it has no quadrature component compression [1]; the quadrature amplitude (potential phase) of the squeezed state is smaller than that of the vacuum state, which is a typical nonclassical state, yet it has a positive Wigner function. Our nonclassicality quantifier gives a quantitative description of the nonclassicality of a given quantum state, thus avoiding this situation.

V. DISCUSSION AND CONCLUSIONS

In conclusion, with the help of the Wigner-Yanase skew information, we have presented an approach to quantifying quantum nonclassicality. Our measure of quantum nonclassicality quantization can be understood as the quantum interaction between the maximum phase angle of the homodyne rotated quadrature operator and the bosonic field states. In particular, we have shown that classical and nonclassical states can be distinguished by this quantifier. By listing the quantum states that are common to calculations, we can show that the property of this quantifier is applicable and reflects that the nonclassicality of different states is different. It is worth mentioning that in the calculation of thermal states associated with single-mode optical field, by comparing thermal states, truncated thermal states, and photon addition thermal states, we are surprised to find that the truncated thermal states should always be more nonclassical than the corresponding thermal states with the same parameters γ or equivalently the same temperature, and the photon addition thermal states are better nonclassical.

We also compute in detail the expression for the nonclassical measure of the Gaussian state and give the range of values for which it is in nonclassicality, again proving that our nonclassical measure is only a sufficient, not necessary, condition for the computation of certain states. This plays a crucial role in our study of the nonclassicality of mixed states.

The nonclassical effects of quantum states of optical fields are closely related to various research fields such as quantum measurements, quantum computing and quantum confidential communication. We hope this work will be useful in the exploration of this boundary from a quantitative perspective.

ACKNOWLEDGMENT

This work is supported by the Fundamental Research Program of Shanxi Province under Grant No. 202203021222225.

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