Letter

Information propagation in long-range quantum many-body systems

Marius Lemm[*] and Carla Rubiliani[†]
Department of Mathematics, University of Tübingen, 72076 Tübingen, Germany

Israel Michael Sigal[‡]
Department of Mathematics, University of Toronto, Toronto, Canada ON M5S 2E4

Jingxuan Zhang (张景宣)[§]
Yau Mathematical Sciences Center, Tsinghua University, Beijing 100084, China
and Department of Mathematical Sciences, University of Copenhagen, Copenhagen 2100, Denmark

(Received 4 July 2023; accepted 16 October 2023; published 13 December 2023)

We study general lattice bosons with long-range hopping and long-range interactions decaying as $|x-y|^{-\alpha}$ with $\alpha \in (d+2, 2d+1)$. We find a linear light cone for the information propagation starting from suitable initial states. We apply these bounds to estimate the minimal time needed for quantum messaging, for the propagation of quantum correlations, and for quantum state control. The proofs are based on the ASTLO method (adiabatic spacetime localization observables). Our results pose previously unforeseen limitations on the applicability of fast-transfer and entanglement-generation protocols developed for breaking linear light cones in long-range and/or bosonic systems.

DOI: 10.1103/PhysRevA.108.L060401

Introduction. The finite speed of information propagation is an empirical fact of Nature. In relativistic theory, the existence of the light cone is a fundamental requirement with wide-ranging consequences. It is remarkable that similar effective "light" cones also constrain the nonrelativistic quantum theory that governs condensed-matter physics. The existence of such an effective "light" cone was discovered by Lieb and Robinson [1] 50 years ago in quantum spin systems.

In the early 2000s, starting with the work of Hastings [2] on the Lieb-Schultz-Mattis theorem, several works leveraged Lieb-Robinson bounds (LRBs) to great effect to control ground-state correlation properties and study topological quantum phase transitions [3–11]. These important developments revolutionized our understanding of the information content of quantum matter at zero temperature. Since then, inspired by these works, an active research area dealing with the dynamics of quantum information has sprung to life. A variety of improvements and extensions of the original LRB have been found, e.g., to long-range spin interactions [12–15], lattice fermions [16,17], open quantum lattice systems [18,19], and anomalous transport [20,21]. Moreover, the applications of LRBs and related propagation bounds have been expanded and deepened to include, e.g., quantum state transport [22,23] and error bounds on quantum simulation algorithms, e.g., [13,14,24,25], equilibration times [26] in condensed-matter physics, and scrambling times relevant to high-energy physics [27–29]. See the survey papers [30–33] for further background on LRBs, and [34,35] for their experimental observation.

The present paper focuses on lattice bosons, for which it had been a long-standing problem to derive useful propagation bounds for general initial states because interactions are effectively unbounded. This problem has recently seen rapid progress [17,23,36–39] especially for nearest-neighbor boson hopping [38]. Nonetheless, some problems have remained, particularly concerning propagation in systems of bosons with long-range hopping and long-range interactions. Long-range interactions are subtle in quantum spin systems—see [13] for the phase diagram for LRBs in long-range quantum spin systems, and [40,41] for reviews of the effect of the long-range interactions on the transmission of quantum information—and the relation to the bosonic case is also subtle [42]. For bosons, the long-range hopping and long-range interactions are effectively unbounded, and the challenge is to derive LRBs with performance guarantees that are independent of ad-hoc truncation of the local particle number.

To tackle this problem, we consider a broad class of many-boson Hamiltonians with *long-range hopping and long-range interactions* on lattices. We present results on the minimal time for quantum messaging and the propagation of quantum correlations (including nonexistence of fast scrambling; cf. [27,28]) and control of states. Concretely, fixing a finite lattice $\Lambda \subset \mathbb{R}^d$, $d \geqslant 1$, we consider many-boson Hamiltonians of the form

$$H_\Lambda = \sum_{x,y\in\Lambda} h_{xy} a_x^* a_y + \frac{1}{2} \sum_{x,y\in\Lambda} a_x^* a_y^* w_{xy} a_y a_x, \qquad (1)$$

[*]marius.lemm@uni-tuebingen.de
[†]carla.rubiliani@uni-tuebingen.de
[‡]im.sigal@utoronto.ca
[§]jingxuan@tsinghua.edu.cn

acting on the bosonic Fock spaces $\mathcal{F}_\Lambda$ over the one-particle space $\ell^2(\Lambda)$. Here a_x and a_x^* are bosonic annihilation and creation operators, respectively, h_{xy} is a Hermitian $|\Lambda| \times |\Lambda|$ matrix representing a one-particle Hamiltonian h (where $|\Lambda|$ denotes the number of vertices of Λ), and w_{xy} is a real-valued $|\Lambda| \times |\Lambda|$ matrix representing a two-particle pair potential. The results we present here extend to fermionic lattice systems.

The class of Hamiltonians (1) goes beyond Bose-Hubbard-type Hamiltonians because the hopping matrix h_{xy} and the two-particle interaction w_{xy} can be of infinite range. Our propagation estimates take into account the decay properties of the hopping matrix and the interaction through moments of the position operator. Namely, we assume that there exists some integer $p \geqslant 1$ such that

$$\kappa_p = \|h\|_p + \|w\|_p < \infty,$$
$$\text{where} \quad \|u\|_p = \sup_{x\in\Lambda} \sum_{y\in\Lambda} |u_{xy}||x-y|^{p+1}. \tag{2}$$

For example, in the short-range (e.g., finite-range or subexponentially decaying) regime, condition (2) holds for arbitrarily large $p \geqslant 1$. Importantly, condition (2) holds also for long-range interactions $|h_{xy}|, |w_{xy}| \leqslant C(1+|x-y|)^{-\alpha}$ with decay rate $\alpha > d+2$ for any $p < \alpha - d - 1$.

Our first main result is the maximal velocity bound (MVB), Theorem 1 (i), which holds for completely general initial states. It essentially says that even bosons with long-range hopping and long-range interactions cannot propagate particles superballistically. We also extend an idea of [23] to bound the speed of propagation of macroscopic particle clouds in Theorem 2.

Next, we use the MVB to derive Theorem 3 on the light-cone approximation of quantum dynamics, which, in turn, yields the weak LRB, Theorem 4, for localized initial states. It is remarkable that a linear light cone for information propagation can be proved for a subclass of initial states for such general long-range bosonic Hamiltonians because such a linear light cone is expected to break for bosonic Hamiltonians with only nearest-neighbor hopping in general [38]. Another reason the result is surprising is that for long-range spin interactions with decay rates $\alpha \in (d+2, 2d+1)$, there exist fast-transfer and entanglement-generation protocols [15,43–45] that break any linear light cone. Nonetheless, our results yield a linear light cone for such decay rates $\alpha \in (d+2, 2d+1)$. Hence, our Theorem 4 places unforeseen constraints on the applicability of these fast-transfer and entanglement-generation protocols for localized initial states. We emphasize that our results are not in contradiction to these fast-transfer protocols because the Hamiltonians we treat in (1) only involve boson density-density interactions, and we require the initial state to be suitably localized.

We then turn to applications in Theorems 5–8. These results provide general constraints on propagation/creation of correlation, quantum messaging, state control times, and the relation between a spectral gap and the decay of correlations for the broad class of lattice boson Hamiltonians we consider. These physically meaningful consequences can be derived from the main results Theorems 3 and 4 through what are by now standard arguments [2,4,6,10,11,46] highlighting the "unreasonable effectiveness of Lieb-Robinson bounds"; see also [30–33]. Of specific interest is Theorem 5 for pure states, which bounds the minimal time for *creation of quantum entanglement* between different regions.

Our work builds on a completely different approach to propagation bounds originally introduced in [47] for few-body quantum mechanics in continuous space and further developed in [48–53]. The method was recently extended to Bose-Hubbard Hamiltonians with long-range hopping in [23,36], and we draw on the insights developed therein.

There has recently been intensive research activity concerning propagation bounds for bosonic lattice systems. Results similar to Theorems 1, 3, and 4 have been obtained in [37–39,54] for the case of nearest-neighbor hopping. Earlier influential works derived Lieb-Robinson bounds for systems of harmonic oscillators [55], which can be coupled to a finite-dimensional quantum system [56] or perturbed by an anharmonic interaction [57]; see also [24,25].

Of particular interest is [38], which considers nearest-neighbor hopping and derives superlinear light cones $|x| \sim t \log t$ for particle propagation, and $|x| \sim t^d \operatorname{polylog} t$ for information propagation, where d is the lattice dimension. Regarding particle propagation, Theorem 1, part (ii), extends the first-moment bound in [38] from nearest-neighbor interactions to any finite range, and it improves the light cone to exactly linear $|x| \sim t$ under a mild initial-density assumption. Considering information propagation, we obtain much stronger linear light cones $|x| \sim t$ (and various information-theoretic consequences) for general long-range interactions under the assumption of initially localized states. The results thus point to a certain dichotomy in the information-propagation capabilities of even long-range lattice bosons depending on the localization of the initial state, an effect first found in [54].

Setup and main results. When testing observables against quantum states, we identify the density matrix ρ with the linear functional $\omega(A) \equiv \omega_\rho(A) = \operatorname{Tr}(A\rho)$ on observables A. We consider the time evolution of observables in the Heisenberg picture,

$$\alpha_t(A) = e^{itH} A e^{-itH} \text{ so that } \omega_t(A) = \omega(\alpha_t(A)). \tag{3}$$

We consider initial quantum states satisfying

$$\omega \in \mathcal{D}, \quad \omega(N^2) < \infty, \tag{4}$$

where $\mathcal{D}$ is the domain of the commutator with H [58]. Examples include initial states of fixed particle number.

For any region $X \subset \Lambda$, we write $d_X(x) = \inf_{y\in X} |x-y|$ for the associated distance function, and $X_\eta = \{x : d_X(x) \leqslant \eta\}$ for the η-enlarged set with $\eta > 0$.

A key role is played by the first moment of the hopping matrix,

$$\kappa = \sup_{x\in\Lambda} \sum_{y\in\Lambda} |h_{xy}||x-y|, \tag{5}$$

which, as we will see, gives an explicit, calculable bound on the maximal velocity (i.e., the light cone slope). We recall that our standing assumption is that (2) holds for some $p \geqslant 1$.

Our first result is a general maximal velocity bound (MVB) for particle transport. Here and below we say that the interaction is of finite range $R > 0$ if $h_{xy} = 0$ for all $x, y \in \Lambda$ with $|x-y| > R$.

Theorem 1 (MVB). For every $v > \kappa$, the following holds:

(i) There exists $C = C(p, \|h\|_p, v) > 0$ such that for all $\eta \geqslant 1$, $X \subset \Lambda$, and any initial state ω,

$$\sup_{|t|<\eta/v} \omega_t(N_X) \leqslant C(\omega(N_{X_\eta}) + \eta^{-p}\omega(N)). \tag{6}$$

(ii) Suppose that the interaction is of finite range R. Then there exists $C = C(p, \|h\|_p, v, R) > 0$ such that for any $X \subset \Lambda$, there exists $\eta_0 = \eta_0(X) \geqslant 1$ such that for any $\eta \geqslant \eta_0$ and initial state ω of controlled density [i.e., $0 < a \leqslant \omega(n_x) \leqslant b < \infty$ for all x], we have

$$\sup_{|t|<\eta/v} \omega_t(N_X) \leqslant C\frac{b^2}{a^2}\omega(N_{X_\eta}). \tag{7}$$

The key steps in the proof of Theorem 1 are given below with details relegated to [59,60].

Theorem 1, part (i), says that even bosons with long-range hopping cannot propagate superballistically. The error term $\eta^{-p}N$ grows in the thermodynamic limit, and so the propagation is only controlled on macroscopic lengthscales $\eta \gg N^{-1/p}$. This restriction is removed in part (ii) under additional assumptions.

The N-dependence of the remainder can also be removed if we focus on macroscopic particle transport as follows. For a given $S \subset \Lambda$, we define the (macroscopic) local relative particle numbers as $\bar{N}_S = \frac{N_S}{N_\Lambda}$. For $0 \leqslant \nu \leqslant 1$, we write $P_{\bar{N}_S \leqslant \nu}$, $P_{\bar{N}_{S^c} \geqslant \nu}$ for spectral projections associated with $\bar{N}_S$.

Theorem 2 (MVB for macroscopic particle transport). Suppose the initial state ω satisfies $\omega(P_{\bar{N}_{X_\eta} \geqslant \nu}) = 0$ with some $\eta \geqslant 1$, $\nu \geqslant 0$, $X \subset \Lambda$. Then, for all $\nu' > \nu$, $v > \kappa$, there exists $C = C(p, \kappa_p, v, \nu' - \nu) > 0$ such that $|t| < \eta/v$:

$$\omega_t(P_{\bar{N}_X \geqslant \nu'}) \leqslant C\eta^{-p}. \tag{8}$$

In other words, this result (8) says that macroscopic particle clouds of $(\nu' - \nu)N$ particles travel at most with speed κ. Theorems 1 and 2 hold without assumption (2) on w.

Let us now suppose that all the bosons are initially localized in a region X, i.e., the initial condition ω satisfies $\omega(N_{X^c}) = 0$. Then, for all $|t| < \eta/v$ we can apply Theorem 1 to complements to obtain

$$\omega_t\big(N_{X_\eta^c}\big) = \omega\big(\alpha_t\big(N_{X_\eta^c}\big)\big) \leqslant C\eta^{-p}\omega(N_X). \tag{9}$$

Here we write $X^c = \Lambda \setminus X$ and set $X_\eta^c \equiv (X_\eta)^c$.

It is this setup that we will use for proving the light cone approximation and LRBs next. We say that an operator A acting on $\mathcal{F}$ is *localized* in $X \subset \Lambda$ (in symbols, $\mathrm{supp}A \subset X$) if $[A, a_x^\sharp] = 0$ for all $x \in X^c$, where $a_x^\#$ stands for either a_x or a_x^*. In Theorem 3 below, we show that the evolution of initially localized observables under (3) is approximated by a family of observables localized within the LC of the initial support.

For any subset $S \subset \Lambda$, we define the *localized evolution* of observables as $\alpha_t^S(A) = e^{itH_S}Ae^{-itH_S}$, where H_S is defined in (1) with S in place of Λ, and

$$\mathcal{B}_S = \{A \in \mathcal{B}(\mathcal{F}) : [A, N] = 0,\ \mathrm{supp}A \subset S\} \tag{10}$$

denotes bounded particle number conserving operators localized in S.

Theorem 3 (LC approximation of quantum evolution). Suppose that the initial state ω satisfies (4) and

$$\omega(N_{X^c}) = 0 \tag{11}$$

for some $X \subset \Lambda$. Then, for every $v > 2\kappa$, there exists $C = C(p, \kappa_p, v) > 0$ such that for all $\xi \geqslant 1$ and $A \in \mathcal{B}_X$, the full evolution $\alpha_t(A)$ is approximated by the local evolution $\alpha_t^{X_\xi}(A)$, for all $|t| < \xi/v$, as

$$\big|\omega\big(\alpha_t(A) - \alpha_t^{X_\xi}(A)\big)\big| \leqslant C|t|\xi^{-p}\|A\|\omega\big(N_X^2\big). \tag{12}$$

We remark that the term $\omega(N_X^2)$ which appears here and in the following is controlled by the state's local energy density and particle density near X [61].

Theorem 4 (weak Lieb-Robinson bound). Suppose the assumptions of Theorem 3 hold with $n \geqslant 1$, $X \subset \Lambda$. Then, for every $v > 2\kappa$, there exists $C = C(p, \kappa_p, v) > 0$ such that for all $\xi \geqslant 1$, $Y \subset \Lambda$ with $\mathrm{dist}(X, Y) \geqslant 2\xi$, and operators $A \in \mathcal{B}_X$, $B \in \mathcal{B}_Y$, we have, for all $|t| < \xi/v$,

$$|\omega([\alpha_t(A), B])| \leqslant C|t|\|A\|\|B\|\xi^{-p}\omega\big(N_X^2\big). \tag{13}$$

We call a bound of the form (13) the *weak Lieb-Robinson bound (LRB)* because unlike the classical LRB, estimate (13) depends on a subclass of states.

Applications.

Propagation/creation of correlations. Assuming a state ω is weakly correlated in a domain $Z^c \subset \Lambda$, how long does it take to create substantial correlations in Z^c? For subsets $X, Y, Z \subset \Lambda$, let $d_{XY} = \mathrm{dist}(X, Y)$ and $d_{XY}^{Z^c} = \min(d_{XY}, d_{XZ^c}, d_{YZ^c})$.

Definition 1 (weakly correlated state). We say a state ω is weakly correlated inside a subset Z on the lengthscale $\lambda > 0$ if there are $C > 0$, $p \geqslant 1$ such that for all X, $Y \subset Z$ with $d_{XY}^{Z^c} > 0$ and operators $A \in \mathcal{B}_X$, $B \in \mathcal{B}_Y$ [see (10)], we have

$$|\omega^c(A, B)| \leqslant C\big(d_{XY}^{Z^c}/\lambda\big)^{1-p}\|A\|\|B\|, \tag{14}$$

where $\omega^c(A, B) = \omega(AB) - \omega(A)\omega(B)$. In this case, we write $\mathrm{WC}(Z, \lambda, C, p)$.

Theorem 5 (propagation/creation of correlation). Let $Z \subset \Lambda$ and suppose the initial state ω satisfies (4), $\omega(N_Z) = 0$, and is $\mathrm{WC}(Z, \lambda, C, p)$.

Then ω_t is $\mathrm{WC}(Z, 3\lambda, C\omega(N_{Z^c}^2), p)$ for all $|t| < \lambda/3\kappa$.

Constraint on the propagation of quantum signals. The weak LRB (13) imposes a direct constraint on the speed of quantum messaging (cf. [4,19,23]). Assume that Bob at a location Y is in possession of a state ρ and an observable B and would like to send a signal through the quantum channel α_t' to Alice who is at X and who possesses the same state ρ and an observable A. To send a message, Bob sends Alice the state $\rho_r = e^{-iBr}\rho e^{iBr}$. To see whether Bob sent his message, Alice computes the difference between the expectations of A in the states $\alpha_t'(\rho_r)$ and $\alpha_t'(\rho)$, which we call the *signal detector*: $\mathrm{SD}(t, r) = \mathrm{Tr}[A\alpha_t'(\rho_r) - A\alpha_t'(\rho)]$.

Theorem 6 (bound on messaging time). Under the assumptions of Theorem 3, for every $v > 4\kappa$, there exists $C = C(p, \kappa_p, v) > 0$ such that for all $\xi \geqslant 2$, X, $Y \subset \Lambda$ with $\mathrm{dist}(X, Y) \geqslant 2\xi$, and operators $A \in \mathcal{B}_X$, $B \in \mathcal{B}_Y$ with $\|B\|_n < \infty$ [see after (2)], we have, for all r, $|t| < \xi/v$:

$$|\mathrm{SD}(t, r)| \leqslant Cr|t|\xi^{-p}\|A\|\|B\|\mathrm{Tr}\big(N_X^2\rho\big). \tag{15}$$

Bound on quantum state control. For any $S \subset \Lambda$, we denote by $\mathcal{F}_S$ the bosonic Fock space $\mathcal{F}_S = \oplus_{n=0}^{\infty} \otimes_{\mathrm{S}}^n \ell^2(S)$, where $\otimes_{\mathrm{S}}$ stands for the symmetric tensor product. Due to the tensorial structure $\mathcal{F}_\Lambda \simeq \mathcal{F}_Y \otimes_{\mathrm{S}} \mathcal{F}_{Y^c}$ (see [59], Appendix A), we can define the partial trace $\mathrm{Tr}_{\mathcal{F}_{Y^c}}$ over $\mathcal{F}_{Y^c}$. We define the *restriction* of a state ρ to the density operators on the local Fock space $\mathcal{F}_Y$, $Y \subset \Lambda$, by $[\rho]_Y = \mathrm{Tr}_{\mathcal{F}_{Y^c}} \rho$.

Let τ be a quantum map (or *state control map*) supported in X. Given a density operator ρ, our task is to design τ so that at some time t, the evolution $\rho_t^\tau = \alpha_t(\rho^\tau)$ of the density operator $\rho^\tau = \tau(\rho)$ has the restriction $[\rho_t^\tau]_Y$ to $S(\mathcal{F}_Y)$, which is close to a desired state, say σ. To measure the success of the transfer operation, one can maximize the figure of merit $F([\rho_t^\tau]_Y, \sigma)$, where $F(\rho, \sigma) = \|\sqrt{\rho}\sqrt{\sigma}\|_1$ is the fidelity.

To show that the state transfer is impossible in a given time interval, we compare ρ_t^τ and $\rho_t = \alpha_t(\rho)$ by using $F([\rho_t^\tau]_Y, [\rho_t]_Y)$ as a figure of merit (cf. [22,23]), and we try to show that it is close to 1 for $t \leqslant t_*$ and for all state preparation (unitary) maps τ localized in X. If this is true, then using τ's localized in X does not affect states in Y. Let $\tau(\rho) = U\rho U^* \equiv \rho^U$, where U is a unitary.

Theorem 7 (quantum control bound). Let ω be a pure state. Under the assumptions of Theorem 3, for every $v > 8\kappa$, there exists $C = C(p, \kappa_p, v) > 0$ such that for all $\xi \geqslant 4$, $Y \subset \Lambda$ with $\mathrm{dist}(X, Y) \geqslant 2\xi$, and unitary operator $U \in \mathcal{B}_X$ [see (10)], we have, for all $|t| < \xi/v$:

$$F([\alpha_t'(\rho)]_Y, [\alpha_t'(\rho^U)]_Y) \geqslant 1 - C|t|\xi^{-p}\mathrm{Tr}\big(N_X^2\rho\big).$$

The estimate above imposes a lower bound on the time for the best-possible quantum control protocols for quantum many-body dynamics and bounds quantum state transfer as in [22,23].

Spectral gap and decay of correlation.

Theorem 8 (gap implies decay of correlations). Suppose H in (1) has a spectral gap of size $\gamma > 0$ at the ground-state energy. Suppose the assumptions of Theorem 3 hold with $n \geqslant 1$, $X \subset \Lambda$, and ω is the ground state. Then, there exists $C = C(n, \kappa_n) > 0$ such that for all $\xi \geqslant 1$, $Y \subset \Lambda$ with $\mathrm{dist}(X, Y) \geqslant 2\xi$, and operators $A \in \mathcal{B}_X$, $B \in \mathcal{B}_Y$, we have

$$|\omega(BA)| \leqslant C\|A\|\|B\|\big[\gamma^{-1}\xi^{-2} + \xi^{1-p}\omega\big(N_X^2\big)\big]. \tag{16}$$

Key steps in the proofs. We sketch the key ideas involved in the proofs and refer to [60] and [59] for the full details. The method is an adaptation of the ASTLO (adiabatic spacetime localization observables) approach developed in [23,36].

Proof idea for Theorem 1. For a function $f : \Lambda \to \mathbb{C}$, we define its second quantization $\hat{f} = \mathrm{d}\Gamma(f) = \sum_{x\in\Lambda} f(x)a_x^* a_x$. As in [23,36], we control the time evolution associated with (1) by recursive monotonicity estimates for the *ASTLOs*:

$$\hat{\chi}_{ts} = \mathrm{d}\Gamma(\chi_{ts}), \quad \chi_{ts} = \chi\left(\frac{d_X - v't}{s}\right), \tag{17}$$

where $s > t \geqslant 0$, d_X is the distance function to X, $v' = \frac{v+\kappa}{2}$, and χ belongs to the set $\mathcal{X}$ of smooth monotonic cutoff functions which interpolate between 0 and 1 and satisfy $\sqrt{\chi'} \in C^\infty$ and $\mathrm{supp}\,\chi' \subset (0, v - v')$.

For a differentiable path of observables, define the Heisenberg derivative $DA(t) = \frac{\partial}{\partial t}A(t) + i[H, A(t)]$, with

$$\partial_t\alpha_t(A(t)) = \alpha_t(DA(t)). \tag{18}$$

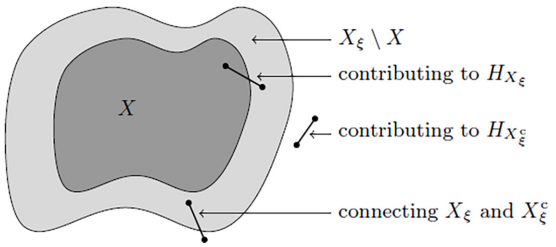

FIG. 1. Set X_ξ and splitting of H.

Proposition 9 (RME). Suppose the assumptions of Theorem 1 hold. Then, for every $\chi \in \mathcal{X}$, there exist $C = C(p, \kappa_p, \chi) > 0$ and functions $\xi^k = \xi^k(\chi) \in \mathcal{X}$, $k = 2, \ldots, p$, such that for all $s, t > 0$, we have the operator inequality

$$D\hat{\chi}_{ts} \leqslant -\frac{(v' - \kappa)\hat{\chi}_{ts}'}{s} + C\sum_{k=2}^{p}\frac{\widehat{(\xi^k)'}_{ts}}{s^k} + \frac{CN}{s^{p+1}}. \tag{19}$$

Since the second term on the r.h.s. is of the same form as the leading, negative term, estimate (19) can be bootstrapped to obtain an integral inequality with remainder $O(s^{-p})$, which gives Theorem 1, part (i). For details, see [59], Sec. 3. The proof of Theorem 1, part (ii) uses novel techniques (which we call band-limited ASTLOs) and is fully contained in [60]. The proof of Theorem 2 is similar to the proof of Theorem 1, part (i) except that we introduce a "second-order" ASTLO by using spectral calculus to approximate the spectral projector as in [36]. See [59], Sec. 4, for details.

Proof idea for Theorem 3. Let $A_t = \alpha_t(A)$ and $A_t^\xi \equiv \alpha_t^{X_\xi}(A)$. By the fundamental theorem of calculus, we have $A_t - A_t^\xi = \int_0^t \partial_r\alpha_r(\alpha_{t-r}^{X_\xi}(A))\,dr$. Using identity (18) for α_r and $\alpha_{t-r}^{X_\xi}$, we find

$$A_t - A_t^\xi = i\int_0^t \alpha_r([R', A_{t-r}^\xi])\,dr, \tag{20}$$

where $R' = H - H_{X_\xi}$. Since A_s^ξ is localized in X_ξ, only terms in R' which connect X_ξ and X_ξ^c contribute to $[R', A_{t-r}^\xi]$ (see Fig. 1). Then we apply the MVB (6) to estimate $[R', A_{t-r}^\xi]$ similar to [23]. We defer further details to [59], Secs. 5 and 6, and Appendix D.

As mentioned before, Theorems 5–8 are by now relatively standard consequences of Lieb-Robinson bounds with some modifications required due to the bosonic nature of the Hamiltonian. In [59], these results are stated as Theorems 2.4–2.7 and complete proofs are given in [59], Secs. 7–10, respectively.

Conclusions. We have presented the existence of a linear light cone for a many-boson system with long-range hopping and long-range interactions. The results provide practical constraints on sending quantum information in such systems. Our results identify good classes of initial states in which even bosons with long-range hopping and long-range interactions cannot propagate information superballistically. Our results thus complement the existence of fast-transfer protocols for nearest-neighbor Bose-Hubbard Hamiltonians [38] and quantum spin systems with long-range interactions [15,37,43–45].

The results can be extended to time-dependent and few-body interactions, quantum spin systems, and lattice fermions. Extensions to open systems of lattice bosons with applications to estimating the decoherence and thermalization times are also in reach using ideas from [52,53].

Acknowledgments. The authors are grateful to Jérémy Faupin and Avy Soffer for enjoyable and fruitful collaborations. The research of M.L. and C.R. is supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) through Grant No. TRR 352–470903074. The research of I.M.S. is supported in part by NSERC Grant No. NA7901. J.Z. is supported by DNRF Grant CPH-GEOTOP-DNRF151, DAHES Fellowship Grant No. 2076-00006B, DFF Grant No. 7027-00110B, and the Carlsberg Foundation Grant No. CF21-0680. His research was also supported in part by NSERC Grant No. NA7901. Parts of this work were done while J.Z. was visiting MIT. We thank the referees for their helpful comments.

[1] E. H. Lieb and D. W. Robinson, The finite group velocity of quantum spin systems, Commun. Math. Phys. **28**, 251 (1972).
[2] M. B. Hastings, Lieb-Schultz-Mattis in higher dimensions, Phys. Rev. B **69**, 104431 (2004).
[3] S. Bravyi and M. B. Hastings, Topological quantum order: Stability under local perturbations, J. Math. Phys. **51**, 093512 (2010).
[4] S. Bravyi, M. B. Hastings, and F. Verstraete, Lieb-Robinson bounds and the generation of correlations and topological quantum order, Phys. Rev. Lett. **97**, 050401 (2006).
[5] J. Eisert and T. J. Osborne, General entanglement scaling laws from time evolution, Phys. Rev. Lett. **97**, 150404 (2006).
[6] M. B. Hastings and T. Koma, Spectral gap and exponential decay of correlations, Commun. Math. Phys. **265**, 781 (2006).
[7] M. B. Hastings, An area law for one-dimensional quantum systems, J. Stat. Mech. Theor. Exp. (2007) P08024.
[8] M. B. Hastings, Quantum belief propagation, an algorithm for thermal quantum systems, Phys. Rev. B **76**, 201102(R) (2007).
[9] M. B. Hastings and X.-G. Wen, Quasiadiabatic continuation of quantum states: The stability of topological ground-state degeneracy and emergent gauge invariance, Phys. Rev. B **72**, 045141 (2005).
[10] B. Nachtergaele, Y. Ogata, and R. Sims, Propagation of correlations in quantum lattice systems, J. Stat. Phys. **124**, 1 (2006).
[11] B. Nachtergaele and R. Sims, Lieb-Robinson bounds and the exponential clustering theorem, Commun. Math. Phys. **265**, 119 (2006).
[12] M. Foss-Feig, Z.-X. Gong, C. W. Clark, and A. V. Gorshkov, Nearly-linear light cones in long-range interacting quantum systems, Phys. Rev. Lett. **114**, 157201 (2015).
[13] M. C. Tran, A. Y. Guo, C. L. Baldwin, A. Ehrenberg, A. V. Gorshkov, and A. Lucas, Lieb-Robinson light cone for power-law interactions, Phys. Rev. Lett. **127**, 160401 (2021).
[14] M. C. Tran, Y. Su, D. Carney, and J. M. Taylor, Faster digital quantum simulation by symmetry protection, PRX Quantum **2**, 010323 (2021).
[15] M. C. Tran, C.-F. Chen, A. Ehrenberg, A. Y. Guo, A. Deshpande, Y. Hong, Z.-X. Gong, A. V. Gorshkov, and A. Lucas, Hierarchy of linear light cones with long-range interactions, Phys. Rev. X **10**, 031009 (2020).
[16] B. Nachtergaele, R. Sims, and A. Young, Lieb-Robinson bounds, the spectral flow, and stability of the spectral gap for lattice fermion systems, in *Mathematical Problems in Quantum Physics*, Contemporary Mathematics Vol. 717 (American Mathematical Society, Providence, RI, 2018), pp. 93–115.
[17] Z. Wang and K. R. A. Hazzard, Tightening the Lieb-Robinson bound in locally interacting systems, PRX Quantum **1**, 010303 (2020).
[18] B. Nachtergaele, A. Vershynina, and V. Zagrebnov, Lieb-Robinson bounds and existence of the thermodynamic limit for a class of irreversible quantum dynamics, Contemp. Math. **552**, 161 (2011).
[19] D. Poulin, Lieb-Robinson bound and locality for general Markovian quantum dynamics, Phys. Rev. Lett. **104**, 190401 (2010).
[20] D. Damanik, M. Lemm, M. Lukic, and W. Yessen, New anomalous Lieb-Robinson bounds in quasiperiodic XY chains, Phys. Rev. Lett. **113**, 127202 (2014).
[21] D. Damanik, M. Lemm, M. Lukic, and W. Yessen, On anomalous Lieb–Robinson bounds for the fibonacci XY chain, J. Spectral Theor. **6**, 601 (2016).
[22] J. M. Epstein and K. B. Whaley, Quantum speed limits for quantum-information-processing task, Phys. Rev. A **95**, 042314 (2017).
[23] J. Faupin, M. Lemm, and I. M. Sigal, On Lieb-Robinson bounds for the Bose-Hubbard model, Commun. Math. Phys. **394**, 1011 (2022).
[24] M. P. Woods, M. Cramer, and M. B. Plenio, Simulating bosonic baths with error bars, Phys. Rev. Lett. **115**, 130401 (2015).
[25] M. P. Woods and M. B. Plenio, Dynamical error bounds for continuum discretisation via Gauss quadrature rules—a Lieb-Robinson bound approach, J. Math. Phys. **57**, 022105 (2016).
[26] C. Gogolin and J. Eisert, Equilibration, thermalisation, and the emergence of statistical mechanics in closed quantum systems, Rep. Prog. Phys. **79**, 056001 (2016).
[27] C.-F. Chen and A. Lucas, Finite speed of quantum scrambling with long range interactions, Phys. Rev. Lett. **123**, 250605 (2019).
[28] T. Kuwahara and K. Saito, Absence of fast scrambling in thermodynamically stable long-range interacting systems, Phys. Rev. Lett. **126**, 030604 (2021).
[29] D. A. Roberts and B. Swingle, Lieb-Robinson bound and the butterfly effect in quantum field theories, Phys. Rev. Lett. **117**, 091602 (2016).
[30] C.-F. Chen, A. Lucas, and C. Yin, Speed limits and locality in many-body quantum dynamics, Rep. Prog. Phys. **86**, 116001 (2023).
[31] M. Kliesch, C. Gogolin, and J. Eisert, Lieb-Robinson bounds and the simulation of time-evolution of local observables in lattice systems, *in Many-Electron Approaches in Physics, Chemistry and Mathematics* (Springer, Cham, 2014), pp. 301–318.
[32] B. Nachtergaele and R. Sims, Lieb-Robinson bounds in quantum many-body physics, Contemp. Math. **529**, 141 (2010).

[33] B. Nachtergaele, R. Sims, and A. Young, Quasi-locality bounds for quantum lattice systems. I. Lieb-Robinson bounds, quasi-local maps, and spectral flow automorphisms, J. Math. Phys. **60**, 061101 (2019).

[34] M. Cheneau, P. Barmettler, D. Poletti, M. Endres, P. Schauß, T. Fukuhara, C. Gross, I. Bloch, C. Kollath, and S. Kuhr, Light-cone-like spreading of correlations in a quantum many-body system, Nature (London) **481**, 484 (2012).

[35] M. Cheneau, Experimental tests of Lieb-Robinson bounds, *The Physics and Mathematics of Elliott Lieb: The 90th Anniversary* (EMS Press, Berlin, 2022), Vol. I, Chap. 10, pp. 225–245.

[36] J. Faupin, M. Lemm, and I. M. Sigal, Maximal speed for macroscopic particle transport in the Bose-Hubbard model, Phys. Rev. Lett. **128**, 150602 (2022).

[37] T. Kuwahara and K. Saito, Lieb-Robinson bound and almost-linear light cone in interacting boson systems, Phys. Rev. Lett. **127**, 070403 (2021).

[38] T. Kuwahara, T. V. Vu, and K. Saito, Optimal light cone and digital quantum simulation of interacting bosons, arXiv:2206.14736.

[39] C. Yin and A. Lucas, Finite speed of quantum information in models of interacting bosons at finite density, Phys. Rev. X **12**, 021039 (2022).

[40] S. Bose, Quantum communication through spin chain dynamics: An introductory overview, Contemp. Phys. **48**, 13 (2007).

[41] N. Defenu, T. Donner, T. Macrì, G. Pagano, S. Ruffo, and A. Trombettoni, Long-range interacting quantum systems, Rev. Mod. Phys. **95**, 035002 (2023).

[42] L. Cevolani, G. Carleo, and L. Sanchez-Palencia, Protected quasilocality in quantum systems with long-range interactions, Phys. Rev. A **92**, 041603(R) (2015).

[43] T. Kuwahara and K. Saito, Strictly linear light cones in long-range interacting systems of arbitrary dimensions, Phys. Rev. X **10**, 031010 (2020).

[44] Z. Eldredge, Z.-X. Gong, J. T. Young, A. H. Moosavian, M. Foss-Feig, and A. V. Gorshkov, Fast quantum state transfer and entanglement renormalization using long-range interactions, Phys. Rev. Lett. **119**, 170503 (2017).

[45] M. C. Tran, A. Y. Guo, A. Deshpande, A. Lucas, and A. V. Gorshkov, Optimal state transfer and entanglement generation in power-law interacting systems, Phys. Rev. X **11**, 031016 (2021).

[46] M. B. Hastings, Locality in quantum and Markov dynamics on lattices and networks, Phys. Rev. Lett. **93**, 140402 (2004).

[47] I. M. Sigal and A. Soffer, Long-range many-body scattering, Invent. Math **99**, 115 (1990).

[48] E. Skibsted, Propagation estimates for N-body Schrödinger operators, Commun. Math. Phys. **142**, 67 (1991).

[49] I. Herbst and E. Skibsted, Free channel Fourier transform in the long-range N-body problem, J. Anal. Math. **65**, 297 (1995).

[50] J.-F. Bony, J. Faupin, and I. M. Sigal, Maximal velocity of photons in non-relativistic QED, Adv. Math. **231**, 3054 (2012).

[51] J. Arbunich, F. Pusateri, I. M. Sigal, and A. Soffer, Maximal speed of quantum propagation, Lett. Math. Phys. **111**, 62 (2021).

[52] S. Breteaux, J. Faupin, M. Lemm, and I. M. Sigal, Maximal speed of propagation in open quantum systems, in *The Physics and Mathematics of Elliott Lieb—The 90th Anniversary* (EMS, Berlin, 2022), Vol. I, pp. 109–130.

[53] S. Breteaux, J. Faupin, M. Lemm, D. H. Ou Yang, I. M. Sigal, and J. Zhang, Light cones for open quantum systems, arXiv:2303.08921.

[54] N. Schuch, S. K. Harrison, T. J. Osborne, and J. Eisert, Information propagation for interacting-particle systems, Phys. Rev. A **84**, 032309 (2011).

[55] J. Jünemann, A. Cadarso, D. Pérez-García, A. Bermudez, and J. J. García-Ripoll, Lieb-robinson bounds for spin-boson lattice models and trapped ions, Phys. Rev. Lett. **111**, 230404 (2013).

[56] M. Cramer, A. Serafini, and J. Eisert, Locality of dynamics in general harmonic quantum systems, in *Quantum Information and Many Body Quantum Systems*, CRM Series (Edizioni della Normale Pisa, 2008), pp. 51–73.

[57] B. Nachtergaele, H. Raz, B. Schlein, and R. Sims, Lieb-Robinson bounds for harmonic and anharmonic lattice systems, Commun. Math. Phys. **286**, 1073 (2009).

[58] Here is the rigorous definition of D. Let $S(\mathcal{F})$ be the Schatten class of quantum states, i.e. positive operators ρ on the bosonic Fock space $\mathcal{F}$ with $\mathrm{Tr}(\rho) = 1$. We denote by $\mathcal{D}$ the domain of $A \mapsto [A, H]$, given by $\mathcal{D} = \{\rho \in S(\mathcal{F}) \mid \rho\mathcal{D}(H) \subset \mathcal{D}(H)$ and $[H, \rho] \in S(\mathcal{F})\}$, and we write $\omega \in \mathcal{D}$ if $\omega = \omega_\rho$ for some $\rho \in \mathcal{D}$.

[59] I. M. Sigal and J. Zhang, On propagation of information in quantum many-body systems, arXiv:2212.14472.

[60] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevA.108.L060401 for the proof of Theorem 1, part (ii).

[61] Fix a region X. We can apply Cauchy-Schwarz twice (once to the hopping term and once to $n_x n_y \leqslant \frac{n_x^2+n_y^2}{2}$) to obtain the operator inequality $N_X^2 \leqslant C|X|(H_X + N_{X_1})$. This shows that for finite regions X and for states of locally bounded energy and particle density, the expectation of N_X^2 is well-controlled.