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Letter

Universal Landauer-like inequality from the first law of thermodynamics

Junjie Liu[1,*] and Hanlin Nie[2]
[1]*Department of Physics, International Center of Quantum and Molecular Structures, Shanghai University, Shanghai 200444, China*
[2]*Centre for Quantum Technologies, National University of Singapore, Block S15, 3 Science Drive 2, 117543, Singapore*

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We demonstrate that the equality-based first law of thermodynamics inherently implies a universal Landauer-like inequality, connecting variations in system entropy and energy. This Landauer-like inequality is shown to depend solely on system information and is highly applicable in scenarios where the implementation of the conventional Landauer principle becomes challenging. Moreover, we unveil that this Landauer-like inequality complements the Landauer principle by establishing an alternative upper bound on heat dissipation. To underscore its practicality, we illustrate the utility of the Landauer-like inequality in contexts such as dissipative quantum state preparation and quantum information erasure applications. Our findings offer insights into identifying thermodynamic constraints, with particular relevance to the domains of quantum thermodynamics and the energetics of quantum information processing. Additionally, this approach paves the way for investigating systems coupled to nonthermal baths or those characterized by limited access to bath information.

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Introduction. To ensure the sustainability of quantum technologies, conducting a comprehensive assessment of their energetic footprints is of paramount importance [1]. In this context, the Landauer principle (LP) [2] marks the pioneering effort, offering an inequality that establishes a lower bound on the dissipated heat $Q(t)$ by the entropy change $\Delta S(t)$ of information-bearing degrees of freedom during the time interval $[0,t]$ (setting $\hbar=1$ and $k_B=1$):

$$Q(t)\geqslant -T\Delta S(t). \tag{1}$$

Here, T represents the temperature of the thermal bath. Within the framework of information thermodynamics [3,4], it is now recognized that Eq. (1) precisely corresponds to the Clausius inequality for the total entropy production in a scenario with a single thermal bath [5–7], with $Q(t)$ and $\Delta S(t)$ identified as the averaged energy change of the thermal bath $\Delta E_B(t)$ and the system's von Neumann entropy production, respectively [5–7]. Here, we define $\Delta A(t)\equiv A(t)-A(0)$ for an arbitrary quantity A. Nevertheless, this thermodynamic interpretation of Eq. (1) necessitates energy measurements of a thermal bath, which are often challenging to implement due to limited control or access.

One established strategy for circumventing the bath energy measurement issue is to employ a dynamical map that converges to the system Gibbs state. A widely accepted implementation employs a quantum Lindblad master equation with Lindblad operators satisfying the detailed balance condition [8]. By adopting this approach, a Clausius inequality is obtained, where the dissipated heat $Q(t)=-\Delta E_S(t)$ is directly linked to the system's averaged energy change $\Delta E_S(t)$, which is experimentally friendly [9]. This favorable result, however, comes at the expense of requiring weak system-bath couplings [10].

The LP has been refined [5,11–16] and generalized [6,17–27] in response to its fundamental and conceptual significance. Recent experimental verifications [28–30] have further advanced its validation. However, challenges persist when implementing the LP equation (1) and its generalizations in specific scenarios. Nonthermal baths frequently encountered at the nanoscale [31–35] lead to an ambiguous definition of thermodynamic temperature. Additionally, calculating the total entropy production can be challenging when the final state is pure [36,37], as is the case in quantum state preparation tasks [38–40], or when systems undergo unidirectional transitions [41], thereby limiting the universality of the LP. It is also worth noting that the existing LPs are typically derived for setups immersed in a single bath [2,5–7,14,22–25,33,42], hindering their applicability in systems coupled to multiple baths. Even within the validity regime of the LP, the existence of a dual upper bound for the dissipated heat remains largely elusive.

Exerting additional refinements to overcome the limitations of the LP represents a formidable task. In this Research Letter, we aim for alternatives and leverage the equality-based first law of thermodynamics to derive universal Landauer-like inequalities for undriven and driven quantum systems [Eqs. (4) and (8), respectively]. These inequalities, originating from a distinct framework, offer applicability in scenarios where the LP fails, thereby enabling investigations into trade-off relations between heat dissipation and system entropy production in generic open quantum systems, including those coupled to nonthermal baths or lacking access to bath information—a scenario frequently encountered in today's nanoscale experiments [43,44]. Moreover, they can complement the LP in its validity regime by providing dual upper bounds on dissipated heat, effectively constraining it from

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[*] jjliu.fd@gmail.com

both sides. This provides a comprehensive assessment of the potential range of dissipated heat during quantum processes, a property with applications not only in computation [42] but also beyond [1]. We showcase the practical utility of these Landauer-like inequalities in a dissipative quantum Bell state preparation process [45,46], where the LP is inapplicable due to ill-defined quantities involved in the second law of thermodynamics, and a quantum information erasure task on a driven qubit [16,22,24], where the LP remains valid and can be combined.

Landauer-like inequality for undriven systems. We first consider a generic undriven quantum system with a Hamiltonian H_S and a time-dependent density matrix $\rho_S(t)$, allowing for nonequilibrium conditions. Introducing a reference Gibbsian state $\rho_{\rm th}=e^{-\beta_R H_S}/Z_S$, where $\beta_R=T_R^{-1}$ is an inverse reference parameter and $Z_S\equiv{\rm Tr}[e^{-\beta_R H_S}]$ is the partition function, we can obtain a universal form of the first law of thermodynamics which reduces to that in Ref. [47] for isothermal processes,

$$E_S(t) = T_R S(t)+\mathcal{F}(t). \tag{2}$$

Here, $E_S(t)\equiv{\rm Tr}[H_S\rho_S(t)]$ denotes the average system energy, $S(t)=-{\rm Tr}[\rho_S(t)\ln\rho_S(t)]$, and $\mathcal{F}$ represents the nonequilibrium information free energy [3,13,47,48], $\mathcal{F}(t)\equiv F+T_R D[\rho_S(t)||\rho_{\rm th}]$, where $F\equiv -T_R\ln Z_S$ and $D[\rho_1||\rho_2]\equiv{\rm Tr}[\rho_1(\ln\rho_1-\ln\rho_2)]$ is the relative entropy of states $\rho_{1,2}$.

From Eq. (2), we find that $\Delta S(t)=\beta_R[\Delta E_S(t)-T_R D[\rho_S(t)||\rho_{\rm th}]+T_R D[\rho_S(0)||\rho_{\rm th}]]$. To obtain general bounds enabling direct evaluations, we fix the reference state $\rho_{\rm th}$ by ensuring that it possesses the same entropy as the actual initial state $\rho_S(0)$ [32,34,35,49,50],

$${\rm Tr}[\rho_S(0)\ln\rho_S(0)] = {\rm Tr}[\rho_{\rm th}\ln\rho_{\rm th}]. \tag{3}$$

Since ${\rm Tr}[\rho_{\rm th}\ln\rho_{\rm th}]$ is monotonically related to β_R within the interval $0\leqslant S(t)\leqslant\ln d_S$, where d_S represents the dimension of H_S, a unique solution for β_R can be found from Eq. (3) once $\rho_S(0)$ is specified [35]. It is important to emphasize that β_R is a parameter determined solely by the system information. Only when the initial state is thermal does T_R acquire a meaningful thermodynamic interpretation.

With Eq. (3), one finds $\Delta E_S(t)+T_R D(\rho_S(0)||\rho_{\rm th})={\rm Tr}\{[\rho_S(t)-\rho_{\rm th}]H_S\}\equiv\Delta E_S^R(t)$; namely, one can map the initial quantum relative entropy, a measure characterizing distance between states in the system Hilbert space, onto the energy contrast with respect to the reference state. Inserting it into the expression for $\Delta S(t)$ above Eq. (3), we attain an inequality

$$\beta_R\Delta E_S^R(t)-\Delta S(t) \geqslant 0. \tag{4}$$

The universality of inequality (4) arises from its derivation without any approximations. Noting that the left-hand side of Eq. (4) denoted as $\mathcal{P}\equiv\beta_R\Delta E_S^R(t)-\Delta S(t)$ equals the relative entropy $D(\rho_S(t)||\rho_{\rm th})$, the equality condition of Eq. (4) is satisfied when the system approaches the reference state within finite times, i.e., $\rho_S(t)=\rho_{\rm th}$. Thus Eq. (4) emerges as a consequence of the non-negative distance of $\rho_S(t)$ from an initially determined reference state $\rho_{\rm th}$. Adopting the quantum coherence definition ${\rm Coh}(t)\equiv S'(t)-S(t)$ with diagonal entropy $S'(t)\equiv-{\rm Tr}(\Pi[\rho_S(t)]\ln\Pi[\rho_S(t)])$ and $\Pi[\rho_S(t)]\equiv\sum_n|E_n\rangle\langle E_n|\rho_S(t)|E_n\rangle\langle E_n|$ in the energy basis $\{|E_n\rangle\}$ of H_S [51], we have $\Delta S(t)=\Delta S'(t)-\Delta{\rm Coh}(t)$, which allows us to identify the quantum coherence contribution to the bound; we relegate details to the Supplemental Material (SM) [52].

We term the inequality (4) a Landauer-like one since it similarly imposes constraints on energy change using system entropy production. However, the inequality (4) distinguishes itself from the LP equation (1) in two key aspects. Firstly, Eq. (4) operates within a broader framework, relying solely on system information. This feature enables its applications in scenarios where the LP may fail due to limited prior knowledge of bath information or ill-defined quantities associated with the second law of thermodynamics [36,37,46]. Moreover, unlike the LP and its generalizations [2,5–7,14,22–25,33,42] relying on a single-bath assumption, Eq. (4) can be applied to systems coupled to multiple baths.

Secondly, we derive the inequality (4) *exclusively* from the first-law-of-thermodynamics equation (2). This unique foundation enables Eq. (4) to offer *complementary* constraints to the LP equation (1). To clarify, we decompose $\Delta E_S^R(t)=\Delta E_S(t)+\Delta E_S^{\rm in}$, where $\Delta E_S(t)\equiv{\rm Tr}\{[\rho_S(t)-\rho_S(0)]H_S\}$ represents the actual system energy change and $\Delta E_S^{\rm in}\equiv{\rm Tr}\{[\rho_S(0)-\rho_{\rm th}]H_S\}$ accounts for the possible initial energy contrast. Inserting this decomposition into Eq. (4), a universal upper bound on $-\Delta E_S(t)$ is obtained,

$$-\Delta E_S(t) \leqslant \mathcal{Q}_u(t). \tag{5}$$

Here, we have defined $\mathcal{Q}_u(t)\equiv\Delta E_S^{\rm in}-T_R\Delta S(t)$. Equation (5) estimates the amount of energy that the system *at most* dissipates. Alternatively, Eq. (5) can be interpreted as an upper bound on the extractable work $-\Delta E_S(t)$ (defined, for instance, in Ref. [53]) from a quantum system, utilizing protocols that extend beyond mere unitary transformations. In arriving at the upper bound in Eq. (5), we have assumed a non-negative reference parameter T_R which can cover typical initial states. For initial states which correspond to a negative reference parameter T_R, we will receive instead a lower bound.

At weak system-bath couplings, $Q(t)=-\Delta E_S(t)$. Combining Eq. (5) with the LP equation (1), we obtain dual constraints on the dissipated heat from both sides:

$$-T\Delta S(t) \leqslant Q(t) \leqslant \mathcal{Q}_u(t). \tag{6}$$

Equations (4)–(6) constitute our first main results. It is worth noting that the whole equation (6) applies to systems weakly coupled to a thermal bath at temperature T. Interestingly, when we set $\beta_R=\beta$ and $\rho_S(0)=\rho_{\rm th}$, we find $\Delta E_S^{\rm in}=0$ and $\mathcal{Q}_u(t)=-T\Delta S(t)$. Consequently, the upper and lower bounds in Eq. (6) coincide, indicating a vanishing dissipated heat [54]. Importantly, the left-hand side of Eq. (6) is derived using the original LP equation (1), and one can employ its generalizations (for instance, the one in Ref. [24]) to tighten the lower bound while maintaining the validity of Eq. (6).

Landauer-like inequality for driven systems. We now generalize Eqs. (4)–(6) to account for driven systems with a time-dependent Hamiltonian $H_S(t)$. With $H_S(t)$ and noting the fact that the system entropy can change during nonunitary evolution, we consider an instantaneous reference Gibbsian state $\rho_{\rm th}(t)=e^{-\beta_R(t)H_S(t)}/Z_S(t)$ with $\beta_R(t)=1/T_R(t)$ being a time-dependent inverse reference parameter and $Z_S(t)={\rm Tr}[e^{-\beta_R(t)H_S(t)}]$. We find that the first law of thermodynamics still takes a similar form to that of Eq. (2), $E_S(t) = T_R(t)S(t) + \widetilde{\mathcal{F}}(t)$, but with $E_S(t) \equiv \mathrm{Tr}[H_S(t)\rho_S(t)]$ and $\widetilde{\mathcal{F}}(t) \equiv F(t) + T_R(t)D[\rho_S(t)||\rho_{\mathrm{th}}(t)]$; $F(t) = -T_R(t)\ln Z_S(t)$. We remark that $\beta_R(t)$ can gain a meaningful thermodynamic interpretation when the system can stay in an instantaneous thermal state [18]. With the generalized first law of thermodynamics, the change in the system's von Neumann entropy can be expressed as $\Delta S(t) = \beta_R(t)[E_S(t) - \widetilde{\mathcal{F}}(t)] - \beta_R(0)[E_S(0) - \widetilde{\mathcal{F}}(0)]$.

In this scenario, we fix the instantaneous reference state by requiring an instantaneous equivalence of entropy, allowing for a unique solution for $\beta_R(t)$ [35]:

$$\mathrm{Tr}[\rho_S(t)\ln\rho_S(t)] \;=\; \mathrm{Tr}[\rho_{\mathrm{th}}(t)\ln\rho_{\mathrm{th}}(t)]. \tag{7}$$

By exploiting the above condition at $t = 0$, we can transfer $D[\rho_S(0)||\rho_{\mathrm{th}}(0)] = \mathrm{Tr}\{[\rho_{\mathrm{th}}(0) - \rho_S(0)]\ln\rho_{\mathrm{th}}(0)\} = -\beta_R(0)\mathrm{Tr}\{[\rho_{\mathrm{th}}(0) - \rho_S(0)]H_S(0)\}$. Inserting this expression into the above equation for $\Delta S(t)$ and rearranging terms, we obtain a generalized Landauer-like inequality that extends Eq. (4),

$$\beta_R(0)\Delta\widetilde{E}_S^R(t) - \Delta S(t) + \mathcal{C}(t) \;\geqslant\; 0. \tag{8}$$

Here, we define $\Delta\widetilde{E}_S^R(t) \equiv E_S(t) - E_S^{\mathrm{th}}(0)$ with $E_S^{\mathrm{th}}(0) = \mathrm{Tr}[H_S(0)\rho_{\mathrm{th}}(0)]$, and $\mathcal{C}(t) \equiv \Delta\beta_R(t)E_S(t) + \ln\frac{Z_S(t)}{Z_S(0)}$ with $\Delta\beta_R(t) = \beta_R(t) - \beta_R(0)$. $\mathcal{C}(t)$ vanishes when H_S becomes time independent, as then β_R has no time dependence. The equality condition of Eq. (8) is satisfied when $\rho_S(t) = \rho_{\mathrm{th}}(t)$, as the left-hand side of Eq. (8) precisely corresponds to $D[\rho_S(t)||\rho_{\mathrm{th}}(t)]$. Similar to Eq. (4), one can still decompose $\Delta S(t) = \Delta S'(t) - \Delta\mathrm{Coh}(t)$ with $S'(t)$ expressed in terms of the instantaneous energy basis [51] and identify the quantum coherence contribution to the bound as shown in the SM [52].

Equation (8) leads to generalizations of Eqs. (5) and (6), forming the second main results of this study,

$$-\Delta E_S(t) \;\leqslant\; \widetilde{\mathcal{Q}}_u(t), \tag{9}$$

$$-T\Delta S(t) \;\leqslant\; Q(t) \;\leqslant\; \widetilde{\mathcal{Q}}_u(t) + W(t). \tag{10}$$

Here, $\Delta E_S(t) = \mathrm{Tr}[\rho_S(t)H_S(t) - \rho_S(0)H_S(0)]$. We define $\widetilde{\mathcal{Q}}_u(t) \equiv \Delta\widetilde{E}_S^{\mathrm{in}} - T_R(0)\Delta S(t) + T_R(0)\mathcal{C}(t)$ with $\Delta\widetilde{E}_S^{\mathrm{in}} \equiv \mathrm{Tr}\{[\rho_S(0) - \rho_{\mathrm{th}}(0)]H_S(0)\}$, and $W(t) = \int_0^t \mathrm{Tr}[\dot{H}_S(t')\rho_S(t')]dt'$ denotes the work performed on the system with $\dot{A}(t) \equiv dA(t)/dt$ for an arbitrary $A(t)$. We remark that Eq. (9) holds under the assumption of a non-negative initial reference parameter $T_R(0)$ and Eq. (10) applies to driven systems weakly coupled to a thermal bath at the temperature T. The usual LP holds in driven systems with dissipated heat $Q(t) = -\int_0^t \mathrm{Tr}[H_S(t')\dot{\rho}_S(t')]dt'$ [24] and $\Delta E_S(t) = -Q(t) + W(t)$. Equations (5) and (6) are recovered when H_S becomes time independent. Notably, the upper and lower bounds in Eq. (10) differ, even when $\rho_S(0) = \rho_{\mathrm{th}}(0)$ and $T_R(0) = T$, as the upper bound includes a nonzero work contribution.

Application 1: Dissipative quantum state preparation. To validate the framework, we first consider the dissipative quantum state preparation (DQSP) which harnesses dissipative processes to prepare useful quantum pure states. One generally utilizes a Markovian quantum Lindblad master equation to this end [38–40,55],

$$\frac{d}{dt}\rho_S(t) \;=\; -i[H_S,\rho_S(t)] + \sum_{\mu=1}\gamma_\mu\mathcal{D}[L_\mu]\rho_S(t). \tag{11}$$

Here, $\gamma_\mu \geqslant 0$ is the damping coefficient of channel μ, and $\mathcal{D}[L_\mu]\rho = L_\mu\rho L_\mu^\dagger - \frac{1}{2}\{L_\mu^\dagger L_\mu, \rho\}$ is the Lindblad superoperator with L_μ being the Lindblad jump operator and $\{A, B\} = AB + BA$. The final stationary state would be a pure one $|\Phi\rangle$ when it fulfills the conditions $H_S|\Phi\rangle = E_n|\Phi\rangle$ and $L_\mu|\Phi\rangle = 0$, $\forall\mu$ [38].

The DQSP process is typically accomplished through dissipation engineering [43] involving nonthermal baths [44], wherein the engineered Lindblad jump operators do not adhere to the detailed balance condition [38]. Furthermore, the final pure state challenges the conventional definition of total entropy production, rendering it ill defined [36,37]. As a result, the LP equation (1) cannot quantify the thermodynamic cost of DQSP processes [46]. In contrast, the inequality (4) remains applicable.

In this application, we validate the upper bound in Eq. (6), noting that $Q(t) = -\Delta E_S(t)$ for the description equation (11) [9,24]. We consider a concrete DQSP setup proposed in Ref. [45] consisting of two Λ-type three-level Rydberg atoms, each one containing two ground states $|0\rangle$ and $|1\rangle$, and one Rydberg state $|r\rangle$. Combining an unconventional Rydberg pumping mechanism with the spontaneous emission of two atoms, Ref. [45] showed that one can dissipatively generate the Bell state $|\Phi\rangle = (|00\rangle - |11\rangle)/\sqrt{2}$ with $|00(11)\rangle$ being understood as $|0(1)\rangle \otimes |0(1)\rangle$. The elements in Eq. (11) are [45] $H_S = \Omega_2(|10\rangle\langle r0| + |01\rangle\langle 0r|) + \omega[(|11\rangle + |00\rangle) \otimes (\langle 01| + \langle 10|)] + \mathrm{H.c.}$ (H.c. denotes the Hermitian conjugate), $\gamma_\mu = \gamma/2$, and four Lindblad jump operators describing spontaneous emission: $L_1 = |01\rangle\langle 0r|$, $L_2 = |00\rangle\langle 0r|$, $L_3 = |10\rangle\langle r0|$, and $L_4 = |00\rangle\langle r0|$. It is evident that the Bell state satisfies conditions $H_S|\Phi\rangle = 0$ and $L_{1,2,3,4}|\Phi\rangle = 0$ as required by the DQSP scheme. Nevertheless, one should bear in mind that the adopted model overlooks other decaying channels such as the dephasing one.

In Fig. 1, we depict a set of results for both $Q(t)$ and its upper bound $\mathcal{Q}_u(t)$ for the aforementioned model. In the main plot, we take $\rho_S(0) = \rho_{\mathrm{th}}$ with $\beta_R = 30$, while in the inset, we consider a nonthermal-form initial state obtained by sorting the same diagonal elements of ρ_{th} in an increasing order with respect to an ordered energy basis with increasing eigenenergies. As can be seen from the figure, $\mathcal{Q}_u(t)$ indeed bounds the dissipated heat $Q(t)$ from above. We also note that the contribution from quantum coherence, $T_R\Delta\mathrm{Coh}(t)$, is only impactful at short times and gradually diminishes as time progresses. Consequently, the dominant factor governing the upper bound at extended times is the reduction in diagonal entropy, $-T_R\Delta S'(t)$. The finite distance $\mathcal{Q}_u(t) - Q(t) = T_R\mathcal{P}$ [see definition below Eq. (4)] becomes maximum at long times as $|\Phi\rangle$ significantly deviates from a full-rank reference state. One can potentially reduce the distance by strategically adjusting the initial conditions to mitigate the contribution $-T_R\Delta S'(t)$ [57]. Notably, the initial state ρ_{th} (more generally, passive states [46]) leads to a negative dissipated heat as can be seen from Fig. 1: This occurs because the final Bell state is the third excited state of the system and the system gains

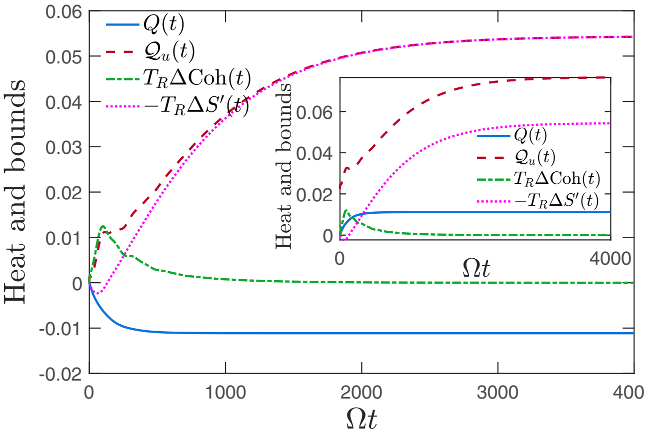

FIG. 1. Results for the time-dependent dissipated heat $Q(t)$ (blue solid curve), the upper bound $\mathcal{Q}_u(t)$ given by Eq. (6) (red dashed curve), the quantum coherence contribution $T_R\Delta\mathrm{Coh}(t)$ (green dash-dotted curve), and the diagonal entropy contribution $-T_R\Delta S'(t)$ (magenta dotted curve). Main panel: $\rho_S(0) = \rho_{\mathrm{th}}$ with $\beta_R = 30$. Inset: $\rho_S(0)$ has the same diagonal elements with ρ_{th} but sorted in an increasing order with respect to an ordered energy basis with increasing eigenenergies. We set $\Omega = 2\pi$ MHz as the unit and adopt experimental values $(\Omega_2, \omega, \gamma) = 2\pi \times (0.02, 0.01, 0.03)$ MHz [56].

energy from the environment to complete the preparation process [46]. With the LP equation (1), one can just deduce $\beta Q(t) \geqslant -\Delta S(t) \geqslant 0$, given that $\Delta S(t) \leqslant 0$ in simulations. However, due to the undefined nature of β, extracting information about $Q(t)$ alone is impossible.

Application 2: Information erasure. We then turn to an information erasure model in which a driven qubit is coupled to a thermal bath at the temperature $T = \beta^{-1}$ [16,22,24]. The system is described by a time-dependent Hamiltonian

$$H_S(t) = \frac{\varepsilon(t)}{2}(\cos[\theta(t)]\sigma_z + \sin[\theta(t)]\sigma_x). \tag{12}$$

Here, $\sigma_{x,z}$ are the Pauli matrices, and $\varepsilon(t)$ and $\theta(t)$ are time-dependent control parameters. We adopt the control protocols $\varepsilon(t) = \varepsilon_0 + (\varepsilon_\tau - \varepsilon_0)\sin(\pi t/2\tau)^2$ and $\theta(t) = \pi(t/\tau - 1)$ [22]. The evolution of $\rho_S(t)$ is still governed by the quantum Lindblad master equation (11) but with the time-dependent Hamiltonian in Eq. (12) and two time-dependent jump operators, $L_1(t) = \sqrt{\varepsilon(t)[N_B(t)+1]}|0_t\rangle\langle 1_t|$ and $L_2(t) = \sqrt{\varepsilon(t)N_B(t)}|1_t\rangle\langle 0_t|$ [24]. Here, $|0_t\rangle$ ($|1_t\rangle$) is the instantaneous ground (excited) state of $H_S(t)$, $N_B(t) = 1/(e^{\beta\varepsilon(t)} - 1)$, and $\gamma_{1,2} = \gamma$.

For this driven setup, we consider validating Eq. (10). In the case of a maximally mixed initial state $\rho_S(0) = \mathrm{I}/2$ with "I" being a 2×2 identity matrix [24], the upper bound in Eq. (10) is trivially divergent due to the divergent initial reference parameter. For demonstration, we consider an easily prepared initial thermal state $\rho_S(0) = e^{-\beta H_S(0)}/\mathrm{Tr}[e^{-\beta H_S(0)}]$ [58]. In this case, the upper bound in Eq. (10) becomes finite and nontrivial. A set of numerical results are depicted in Fig. 2.

In the upper panel of Fig. 2, we observe that the dissipated heat $Q(t)$ is indeed bounded by both the derived upper bound $\widetilde{\mathcal{Q}}_u(t) + W(t)$ and the LP lower bound $-T\Delta S(t)$. Opting for $\beta_R(0) = \beta$, both bounds in Eq. (10) incorporate an identical quantum coherence term $T_R(0)\Delta\mathrm{Coh}(t)$. This term exhibits oscillations but does not significantly contribute to the bound. However, decreasing τ amplifies the oscillation magnitude of $T_R(0)\Delta\mathrm{Coh}(t)$; see SM [52] for more details. In the lower panel of Fig. 2, we notice the monotonic increase of the inverse reference parameter $\beta_R(t)$ from the initial actual inverse temperature $\beta_R(0) = \beta = 1$ (black dash-dotted line). This behavior allows us to utilize $\beta_R(t)$ as a monitoring tool for the effectiveness of the erasure process, driving the qubit towards the ground state with a divergent effective inverse temperature. Interestingly, we also observe a negative $W(t)$ in the lower panel of Fig. 2, implying a work output from the driven qubit. Given analogous heat engine setups involving a single bath and two driving fields [59–61], there is a compelling question about adapting the information erasure model for heat engine design—a topic for future exploration.

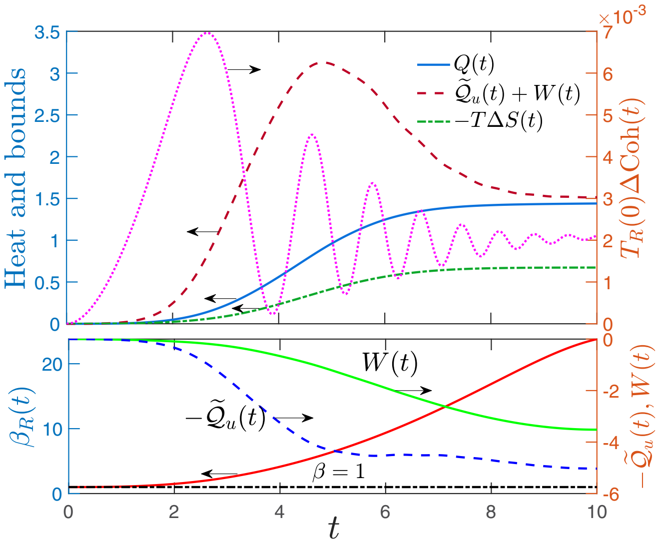

FIG. 2. Validating Eq. (10) with an initial thermal state $\rho_S(0) = e^{-\beta H_S(0)}/\mathrm{Tr}[e^{-\beta H_S(0)}]$. Upper panel: The dissipated heat $Q(t)$ (blue solid curve), the derived upper bound $\widetilde{\mathcal{Q}}_u(t) + W(t)$ (red dashed curve), the LP bound $-T\Delta S(t)$ (green dash-dotted curve), and the quantum coherence contribution $T_R(0)\Delta\mathrm{Coh}(t)$ (right axis, magenta dotted curve). Lower panel: The time-dependent inverse reference temperature $\beta_R(t)$ (left axis), and the work $W(t)$ and $-\widetilde{\mathcal{Q}}_u(t)$ (right axis). Parameters are $\gamma = 0.2$, $\beta = T^{-1} = 1$, $\varepsilon_0 = 0.4$, $\varepsilon_\tau = 10$, and $\tau = 10$.

Discussion and conclusion. Our Landauer-like inequalities are compatible with current experimental capacities. By utilizing quantum state tomography, the reduced system state and all relevant quantities in the inequalities, including the reference parameter, can be fully determined. While we focused on applications related to dissipated heat, these inequalities possess a broader scope. For instance, they can be applied to investigate irreversibility in thermal relaxation processes, especially those where entropy production plays a significant role [62]. In this regard, we remark that both Eqs. (4) and (8) can be transformed into universal upper bounds on the system entropy production.

We note that Ref. [18] revealed a nonequilibrium Landauer principle (NLP) using the non-negativity of quantum relative entropy as well. While the NLP gives rise to inequalities resembling Eqs. (4) and (8), a meticulous analysis confirms their distinction from Eqs. (4) and (8), with our results being more general; detailed insights are available in the SM [52].

In conclusion, we offer a unique perspective for exploring thermodynamic trade-off relations, departing from the traditional focus on the second law. Grounding our investigation in the first law, we introduce an analytical framework to establish Landauer-like inequalities, thereby contributing to the ongoing discourse on thermodynamic principles. Notably, our work brings us closer to a comprehensive understanding of energy dynamics in quantum systems interfacing with diverse bath types, extending beyond the confines of thermal baths alone. Importantly, the applicability of our Landauer-like inequalities does not hinge on specific knowledge of the bath or the degree of control over it. These alignments with contemporary requirements in quantum thermodynamics [35,63] and the burgeoning field of quantum information experiments underscore the relevance of our study to both quantum thermodynamics and quantum information science.

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