Reply to "Comment on 'Multiparty quantum mutual information: An alternative definition' "

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We reaffirm the claim of Lee *et al.* [preceding Comment, Phys. Rev. A **108**, 066401 (2023)] that the expression of quantum dual total correlation of a multipartite system in terms of quantum relative entropy as proposed in previous work [A. Kumar, Phys. Rev. A **96**, 012332 (2017)] is not correct.

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I. INTRODUCTION

In Ref. [1] two different expressions of quantum dual total correlation were obtained: one in terms of von Neumann entropy and the other in terms of quantum relative entropy. It was claimed that the two expressions are equivalent. In the preceding Comment [2] on Ref. [1], Lee *et al.* show that the quantum dual total correlation of an n-partite quantum state cannot be represented as the quantum relative entropy between $n-1$ copies of the quantum state and the product of n different reduced quantum states for $n \geqslant 3$. They arrive at this conclusion by considering explicitly the "support" condition of quantum relative entropy. Essentially, what Lee *et al.* show is that the following two expressions are not equal for $n \geqslant 3$:

$$I_n(\rho) := \sum_{k=1}^{n} S(\rho_{\bar{k}}) - (n-1)S(\rho), \tag{1}$$

where $\rho_{\bar{k}} = \mathrm{tr}_k(\rho)$ denotes the $(n-1)$-partite quantum state obtained by taking the partial trace on the kth party of ρ, and

$$J_n(\rho) := S\big(\rho^{\otimes(n-1)} \big|\big| \otimes_{k=1}^{n} \rho_{\bar{k}}\big), \tag{2}$$

where the quantum relative entropy is

$$S(\tau||\sigma) := \begin{cases} \mathrm{tr}(\tau \log_2 \tau) - \mathrm{tr}(\tau \log_2 \sigma) & \text{if } \mathrm{supp}(\tau) \subseteq \mathrm{supp}(\sigma) \\ \infty & \text{otherwise.} \end{cases}$$

To justify their claim, Lee *et al.* provide two examples which imply that the above two expressions of n-partite quantum mutual information are not equivalent: $I_n(\rho)$ in Eq. (1) is non-negative and nonincreasing under local completely positive and trace-preserving maps [3], and therefore is a suitable monotonic measure of multipartite correlations, while $J_n(\rho)$ in Eq. (2) is not.

II. REAFFIRMING THE CLAIM OF LEE *ET AL.*

The claim of Lee *et al.* is right. In this Reply we show analytically why the above two expressions are not equivalent. We begin with the expression of $I_n(\rho)$ [Eq. (1)] and proceed to show that this is not equal to $J_n(\rho)$ [Eq. (2)], as argued below,

$$\begin{aligned} I_n(\rho) &= \sum_{k=1}^{n} S(\rho_{\bar{k}}) - (n-1)S(\rho) \\ &= \sum_{k=1}^{n} [S(\rho_k) + S(\rho_{\bar{k}}) - S(\rho)] - \left(\sum_{k=1}^{n} S(\rho_k) - S(\rho)\right) \\ &= \sum_{k=1}^{n} S(\rho||\rho_k \otimes \rho_{\bar{k}}) - S\big(\rho\big|\big|\otimes_{k=1}^{n}\rho_k\big) \qquad (3) \\ &= S\big(\rho^{\otimes n}\big|\big|\otimes_{k=1}^{n}(\rho_k \otimes \rho_{\bar{k}})\big) - S\big(\rho\big|\big|\otimes_{k=1}^{n}\rho_k\big) \qquad (4) \\ &\overset{?}{=} S\big(\rho \otimes \rho^{\otimes(n-1)}\big|\big|\big(\otimes_{k=1}^{n}\rho_k\big) \otimes \big(\otimes_{k=1}^{n}\rho_{\bar{k}}\big)\big) \\ &\quad - S\big(\rho\big|\big|\otimes_{k=1}^{n}\rho_k\big) \qquad (5) \\ &= S\big(\rho\big|\big|\otimes_{k=1}^{n}\rho_k\big) + S\big(\rho^{\otimes(n-1)}\big|\big|\otimes_{k=1}^{n}\rho_{\bar{k}}\big) \\ &\quad - S\big(\rho\big|\big|\otimes_{k=1}^{n}\rho_k\big) \\ &= S\big(\rho^{\otimes(n-1)}\big|\big|\otimes_{k=1}^{n}\rho_{\bar{k}}\big) = J_n(\rho), \qquad (6) \end{aligned}$$

where the quantum relative entropy in Eqs. (3) and (4) is properly matched to satisfy the support condition in the sense that

$$\begin{aligned} S\big(\rho\big|\big|\otimes_{k=1}^{n}\rho_k\big) &\equiv S\big(\rho_{12\cdots n}\big|\big|\otimes_{k=1}^{n}\rho_k\big) \\ S(\rho||\rho_k \otimes \rho_{\bar{k}}) &\equiv S(\rho_{k\bar{k}}||\rho_k \otimes \rho_{\bar{k}}) \\ S\big(\rho^{\otimes n}\big|\big|\otimes_{k=1}^{n}(\rho_k \otimes \rho_{\bar{k}})\big) &\equiv S\big(\rho_{12\cdots n} \otimes \rho_{23\cdots n1} \otimes \cdots \\ &\quad \otimes \rho_{n1\cdots(n-1)}\big|\big|\otimes_{k=1}^{n}(\rho_k \otimes \rho_{\bar{k}})\big), \end{aligned}$$

where $\bar{1} = 23\cdots n$, $\bar{2} = 34\cdots n1$, and $\bar{k} = (k+1)\cdots n1\cdots(k-1)$. Equations (3) and (4) are alternate expressions equivalent to Eq. (1) in terms of quantum relative entropy. Equation (5) is not correct for two reasons: (i) noncommutativity of the tensor product and (ii) matching issue of subsystems. Therefore, we cannot arrive at Eq. (6).

III. SECOND REBUTTAL

Let us reconsider Eq. (2) for $n = 3$ explicitly:

$$\begin{aligned} J_3(\rho) &= S\big(\rho^{\otimes 2}\big|\big|\otimes_{k=1}^{3}\rho_{\bar{k}}\big) \\ &= S(\rho_{123} \otimes \rho_{123}||\rho_{23} \otimes \rho_{31} \otimes \rho_{12}). \end{aligned} \tag{7}$$

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Here we see that the subsystems in the first and second arguments of the quantum relative entropy are not properly matched. However, the subsystems in this expression could be matched up if we adopt the following conventions: (i) interpret the first argument in the usual way, with the subsystems in their standard order, and (ii) interpret the tensor product in the second argument with the values of k running from n to 1. Specifically, for $n = 3$, if we define

$$\tilde{J}_3(\rho) := S\big(\rho^{\otimes 2}\big|\big|\otimes_{k=3}^{1}\rho_{\bar{k}}\big) = S(\rho_{123}\otimes\rho_{123}||\rho_{12}\otimes\rho_{31}\otimes\rho_{23}), \tag{8}$$

then we see that the subsystems are properly matched. Now let us adopt the following notation:

$$\rho_{123}^{\otimes 2} = \rho_{123}\otimes\rho_{123} \equiv \rho_{A_1A_2A_3}\otimes\rho_{B_1B_2B_3} \equiv \rho_{123}\otimes\rho_{456}, \tag{9}$$

$$\otimes_{k=3}^{1}\rho_{\bar{k}} = \rho_{12}\otimes\rho_{31}\otimes\rho_{23} \equiv \rho_{A_1A_2}\otimes\rho_{A_3B_1}\otimes\rho_{B_2B_3} \equiv \rho_{12}\otimes\rho_{34}\otimes\rho_{56}. \tag{10}$$

Then, using the notation in Eqs. (9) and (10), one might attempt to show that Eq. (8) is equivalent to $I_3(\rho) = \sum_{k=1}^{3} S(\rho_{\bar{k}}) - 2S(\rho)$ as argued below:

$$\begin{aligned}
\tilde{J}_3(\rho) &:= S\big(\rho^{\otimes 2}\big|\big|\otimes_{k=3}^{1}\rho_{\bar{k}}\big)\\
&= S(\rho_{123}\otimes\rho_{123}||\rho_{12}\otimes\rho_{31}\otimes\rho_{23})\\
&\equiv S(\rho_{123}\otimes\rho_{456}||\rho_{12}\otimes\rho_{34}\otimes\rho_{56})\\
&= -\mathrm{tr}[\rho_{123}\otimes\rho_{456}\log_2(\rho_{12}\otimes\rho_{34}\otimes\rho_{56})]\\
&\quad +\mathrm{tr}[\rho_{123}\otimes\rho_{456}\log_2(\rho_{123}\otimes\rho_{456})]\\
&= -\mathrm{tr}[\rho_{123}\otimes\rho_{456}\log_2(\rho_{12}\otimes I_{34}\otimes I_{56})]\\
&\quad -\mathrm{tr}[\rho_{123}\otimes\rho_{456}\log_2(I_{12}\otimes\rho_{34}\otimes I_{56})]\\
&\quad -\mathrm{tr}[\rho_{123}\otimes\rho_{456}\log_2(I_{12}\otimes I_{34}\otimes\rho_{56})]\\
&\quad +\mathrm{tr}[\rho_{123}\otimes\rho_{456}\log_2(\rho_{123}\otimes I_{456})]\\
&\quad +\mathrm{tr}[\rho_{123}\otimes\rho_{456}\log_2(I_{123}\otimes\rho_{456})]\\
&= -\mathrm{tr}_{12}(\rho_{12}\log_2\rho_{12}) - \mathrm{tr}_{34}(\rho_3\otimes\rho_4\log_2\rho_{34})\\
&\quad -\mathrm{tr}_{56}(\rho_{56}\log_2\rho_{56}) + \mathrm{tr}_{123}(\rho_{123}\log_2\rho_{123})\\
&\quad +\mathrm{tr}_{456}(\rho_{456}\log_2\rho_{456})\\
&\overset{?}{=} S(\rho_{12}) + S(\rho_{34}) + S(\rho_{56}) - S(\rho_{123}) - S(\rho_{456})
\end{aligned} \tag{11}$$

$$\equiv S(\rho_{12}) + S(\rho_{31}) + S(\rho_{23}) - 2S(\rho_{123}) = I_3(\rho). \tag{12}$$

However, this is not correct because the second term in Eq. (11) cannot be obtained from the preceding equation.

Thus, even if we define the quantum dual total correlation of a multipartite system in terms of quantum relative entropy alternatively as

$$\tilde{J}_n(\rho) := S\big(\rho_{12\cdots n}^{\otimes(n-1)}\big|\big|\otimes_{k=n}^{1}\rho_{\bar{k}}\big) \tag{13}$$

and use the notation as discussed above, Eq. (13) is not equivalent to Eq. (1).

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[1] A. Kumar, Multiparty quantum mutual information: An alternative definition, Phys. Rev. A **96**, 012332 (2017).

[2] J. Lee, G. Noh, C. Noh, and J. Park, Comment on "Multiparty quantum mutual information: An alternative definition", Phys. Rev. A **108**, 066401 (2023).

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