Comment on "Multiparty quantum mutual information: An alternative definition"

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We show that, contrary to the claim by Kumar [Phys. Rev. A **96**, 012332 (2017)], the quantum dual total correlation of an n-partite quantum state cannot be represented as the quantum relative entropy between $n-1$ copies of the quantum state and the product of n different reduced quantum states for $n \geqslant 3$. Specifically, we argue that the latter fails to yield a finite value for generalized n-partite Greenberger-Horne-Zeilinger states.

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In [1], a quantum version of the dual total correlation [2] for an n-partite quantum state ρ was proposed as

$$I_n(\rho) \equiv \sum_{k=1}^{n} S(\rho_{\bar{k}}) - (n-1)S(\rho), \tag{1}$$

where $S(\tau) = -\mathrm{tr}(\tau \log_2 \tau)$ is the von Neumann entropy of τ and $\rho_{\bar{k}} = \mathrm{tr}_k \rho$ denotes the $(n-1)$-partite quantum state obtained by taking the partial trace on the kth party of ρ. In addition, it was claimed that Eq. (1) can be represented as

$$I_n(\rho) = J_n(\rho) \equiv S\left(\rho^{\otimes(n-1)} \middle\| \bigotimes_{k=1}^{n} \rho_{\bar{k}}\right), \tag{2}$$

where $\rho^{\otimes j}$ represents j copies of ρ and $S(\tau||\sigma)$ is the quantum relative entropy of τ with respect to σ [3],

$$S(\tau||\sigma) = \begin{cases} \mathrm{tr}(\tau \log_2 \tau) - \mathrm{tr}(\tau \log_2 \sigma) & \text{if } \mathrm{supp}(\tau) \subseteq \mathrm{supp}(\sigma) \\ \infty & \text{otherwise,} \end{cases} \tag{3}$$

where the support of ω is the Hilbert space spanned by the eigenstates of ω with nonzero eigenvalue [4].

It is well known that $I_2(\rho)$ can be represented as the quantum relative entropy of a global quantum state ρ with respect to the product of two local quantum states ρ_1 and ρ_2, i.e., $I_2(\rho) = S(\rho||\rho_1 \otimes \rho_2)$ [5]. We emphasize that the parties between global and local quantum states must be properly matched to avoid the infinity in Eq. (3). While $\mathrm{supp}(\rho) \subseteq \mathrm{supp}(\rho_1 \otimes \rho_2)$ is always met, $\mathrm{supp}(\rho) \subseteq \mathrm{supp}(\rho_2 \otimes \rho_1)$ is not satisfied in general. It leads to a discrepancy between $I_2(\rho) = S(\rho||\rho_1 \otimes \rho_2)$ and $J_2(\rho) = S(\rho||\rho_2 \otimes \rho_1)$. For instance, we have $I_2(\rho) = 0$ and $J_2(\rho) = \infty$ for $\rho = |\psi_1\rangle\langle\psi_1| \otimes |\psi_2\rangle\langle\psi_2|$ satisfying $\langle\psi_1|\psi_2\rangle = 0$.

One can generalize this observation to the case of $n \geqslant 3$. Looking into the order of the parties in $\rho^{\otimes(n-1)}$ and $\bigotimes_{k=1}^{n} \rho_{\bar{k}}$, one immediately sees that they are mismatched for all n because the former and the latter start with the first and second parties of ρ, respectively. Similar to the case of $n = 2$, if we look into a product state $\rho = \bigotimes_{k=1}^{n} |\psi_k\rangle\langle\psi_k|$ satisfying $\langle\psi_i|\psi_j\rangle = \delta_{i,j}$, we obtain $I_n(\rho) = 0$ and $J_n(\rho) = \infty$.

One may think that rearranging the parties of quantum states can resolve the support condition problem. It is possible for $n = 2$ but impossible for $n \geqslant 3$. We show this by investigating the case of $\rho = |\phi\rangle\langle\phi|$ with $|\phi\rangle = \sqrt{p}|0\rangle^{\otimes n} + \sqrt{1-p}|1\rangle^{\otimes n}$. The state of $|\phi\rangle^{\otimes(n-1)}$ is represented by the superposition of the basis states having the multiples of n, i.e., $\{0, n, 2n, \ldots, n(n-1)\}$, copies of $|1\rangle$. On the other hand, the eigenstates of $\bigotimes_{k=1}^{n} \rho_{\bar{k}}$ have multiples of $n-1$, i.e., $\{0, (n-1), 2(n-1), \ldots, n(n-1)\}$, copies of $|1\rangle$. As n and $n-1$ are coprimes, $|\phi\rangle^{\otimes(n-1)}$ is orthogonal to the eigenstates of $\bigotimes_{k=1}^{n} \rho_{\bar{k}}$ except for $|0\rangle^{\otimes n(n-1)}$ and $|1\rangle^{\otimes n(n-1)}$, which means $\mathrm{supp}(\rho^{\otimes(n-1)}) \nsubseteq \mathrm{supp}(\bigotimes_{k=1}^{n} \rho_{\bar{k}})$ for $n \geqslant 3$. Importantly, the observation just described remains valid even if we rearrange the parties. Therefore, we have $I_n(\rho) = -n[p \log_2 p + (1-p)\log_2(1-p)]$ and $J_n(\rho) = \infty$ independent of the matching structure between $\rho^{\otimes(n-1)}$ and $\bigotimes_{k=1}^{n} \rho_{\bar{k}}$.

In [1], the equivalence between Eqs. (1) and (2), i.e., $I_n(\rho) = J_n(\rho)$, was invoked in order to prove the non-negativity and monotonicity of $I_n(\rho)$ under a partial trace and completely positive maps. In addition, the proof for non-negativity was reproduced in [6]. Our counterexample invalidates these proofs which rely on the facts that the relative entropy is non-negative and nonincreasing under completely positive and trace-preserving (CPTP) maps. Interestingly, however, one can find alternative proofs for the non-negativity and monotonicity in [7]. In Ref. [7], $I_n(\rho)$ was proposed as a quantum secrecy monotone and was shown to be non-negative and nonincreasing under local CPTP maps, owing to strong subadditivity of the conditional quantum mutual entropy. Therefore, $I_n(\rho)$ is a suitable monotonic measure of multipartite correlations, while $J_n(\rho)$ is not.

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