Self-testing of an unbounded number of mutually commuting local observables

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Based on the optimal quantum violation of the suitable Bell's inequality, device-independent self-testing of states and observables has been reported. It is well studied that locally commuting or compatible observables cannot be used to reveal quantum nonlocality. Therefore, self-testing of commuting local observables is not possible through the Bell test. In this work, we demonstrate the self-testing of a set of mutually commuting local observables. We show that the optimal quantum violations of suitably formulated bilocality and n-locality inequalities in networks uniquely fix the observables of one party to be mutually commuting. In particular, we first demonstrate that in a two-input arbitrary-party star network, two commuting local observables can be self-tested. Further, by considering an arbitrary-input three-party bilocal network scenario, we demonstrate the self-testing of an unbounded number of mutually commuting local observables.

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I. INTRODUCTION

The existence of noncommuting observables is a distinctive feature of quantum theory from its classical counterpart. Such a quantum feature plays a pivotal role in quantum information processing and cryptographic tasks. In quantum theory, the joint probability of two noncommuting observables does not exist, and any measurement of the prior observable influences the measurement of the posterior observable [1]. Incompatibility is a weaker notion of noncommutativity and is mainly motivated by the perspective of quantum measurement. Two noncommuting observables can be compatible, i.e., jointly measurable, depending on the suitably defined weakness of the measurement. However, two commuting observables are compatible as joint probability always exists, and the measurement of one does not disturb the other.

The demonstration of Bell's theorem [2] requires the relevant local observables to be noncommuting or, more generally, incompatible. The quantum violation of the two-input–two-output Clauser-Horne-Shimony-Holt (CHSH) inequality [3] in the simplest Bell scenario inevitably requires the local observables to be noncommuting. The optimal quantum violation is achieved when they are anticommuting. It was proved [4] that any pair of two-outcome incompatible measurements can violate the CHSH inequality. However, for more than two-input scenarios, this correspondence breaks down [5]. Given an arbitrary set of noncommuting observables, it is not *a priori* clear whether a suitable Bell's inequality can always be formulated to demonstrate the nonlocality by violating the said inequality.

Besides the immense conceptual insights Bell's theorem adds to researching quantum foundations, it provides a multitude of practical applications in quantum information processing (for extensive reviews, see [6–8]). Moreover, the nonlocal correlations are device independent; that is, no characterization of the devices needs to be assumed. Only the observed output statistics are enough to certify nonlocality. Device-independent nonlocal correlations are used as a resource for secure quantum key distribution [9–12], randomness certification [13–16], and witnessing the Hilbert space dimension [17–24] and to achieve advantages in communication-complexity tasks [25].

The maximum quantum value of a given Bell expression enables device-independent certification, commonly known as self-testing [26]. For a recent review of self-testing, we refer the reader to Ref. [27]. In its traditional form, self-testing is a device-independent protocol that aims to uniquely characterize the nature of the target quantum state and measurements solely from the input-output correlations. Essentially, this requires finding a suitable Bell's inequality whose maximum violation is achieved uniquely by the target state and measurements involved. In other words, the state and measurements are device independently certified with minimal assumptions; that is, the devices are uncharacterized (so-called black boxes), and the dimension of the system remains unspecified. Obtaining the maximum quantum value of a Bell inequality eventually guarantees the extremal points in a polytope caused by the behavior of the joint probabilities. For example, the optimal violation of the CHSH inequality self-tests the maximally entangled state and mutually anticommuting local observables. The self-testing scenario was first proposed by Mayers and Yao [26]. Later, McKague and Mosca used this isometric embedding to develop a generalized Mayers-Yao test [28]. Since then, a flurry of work on this topic has been reported [29–52].

Note that device-independent certification is quite challenging to implement experimentally. Semi-device-independent prepare-measure protocols with bounded dimensions are constructed which are experimentally less cumbersome. Self-testing quantum states and measurements in the prepare-measure scenario were demonstrated

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in [30,41]. Quite a number of works self-tested the nonprojective measurements in device-independent or semi-device-independent scenarios [42–47,53]. Semi-device-independent self-testing of an unsharp instrument through sequential measurements has also been reported [45,48–52]. Recently, device-independent certification of an unsharp instrument was also reported [54].

In network Bell tests [55,56], nontrivial forms of nonlocal correlations arise that cannot be traced back to the standard multipartite Bell scenario due to the independence condition of the sources. The set of quantum correlations in a network, in general, becomes nonconvex, thereby making the characterization of nonlocality and self-testing more complicated compared to the standard Bell test. The above issues were addressed in detail in [57–60], and the self-testing argument based on network nonlocality was discussed. Recently, considering the bilocal network scenario, how genuine nonlocal correlations enable self-testing of quantum state and observables was shown [61]. The self-testing of all entangled states was also demonstrated using quantum correlation in a network [62].

The aim of this paper is to certify a set of mutually commuting local observables. It is a common perception that commuting observables do not provide nonclassicality. As discussed above, no violation of Bell's inequality can be obtained if the local observables are mutually commuting. Against this backdrop, in this work, we propose self-testing schemes that certify an unbounded set of mutually commuting local observables. This is done through optimal quantum violations of the suitable network bilocal and n-local inequalities which are achieved when one observer performs the measurement of m mutually commuting local observables when both n and m is arbitrary. In other words, we show that the optimal quantum violation of certain network inequalities uniquely fixes a set of mutually commuting observables and thus the corresponding set of observables is self-tested, independent of the dimension.

We first consider the star network featuring arbitrary-n independent sources, n edge parties, and a central party. Each source distributes a physical state with an edge party and the central party, and each party performs two binary-outcome measurements. We demonstrate that the optimal quantum violation of a suitable n-locality inequality certifies that the two observables of the central party commute when n is even. Since our derivation of the optimal value is dimension independent, we demonstrate the self-testing of two commuting observables in arbitrary dimensions.

Further, we propose the self-testing of an arbitrary number of mutually commuting local observables using the simplest quantum network: the bilocality scenario involving two edge parties and a central party (Fig. 1). Each of the edge parties performs 2^{m-1} binary-outcome measurements, and the central party performs m binary-outcome measurements, where m is arbitrary. We propose a suitable bilocal inequality in an arbitrary-input scenario, and by using a dimension-independent approach, we derive the optimal quantum violation of a bilocality inequality. The optimal quantum value can be obtained only when the central party performs the measurements of m commuting observables.

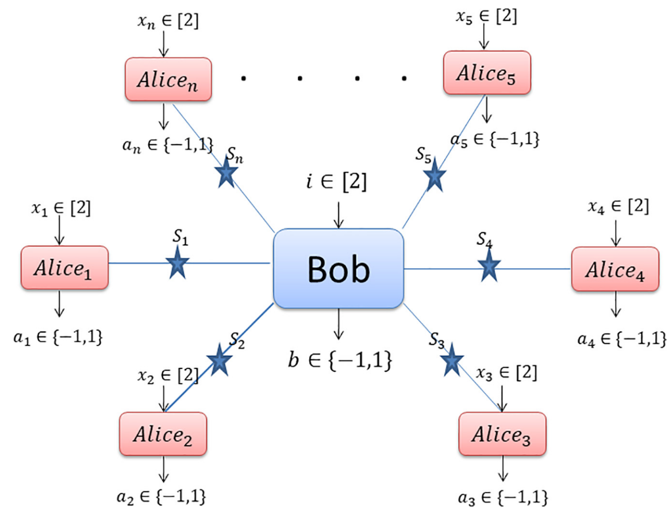

FIG. 1. Two-input arbitrary-party star network.

This then self-tests an unbounded number of mutually commuting local observables.

The plan of this paper is the following. In Sec. II, we consider the well-known star network and show that for an even number of edge parties, the optimal quantum violation self-tests a set of commuting local observables. In Sec. III, we consider the simplest bilocal network where the central party Bob performs three measurements (i.e., $m = 3$). We demonstrate the optimal quantum violation of the bilocality inequality self-tests the set of three mutually commuting local observables. In Sec. IV, we show that this feature is generic and valid for any arbitrary m-input case. Considering a similar bilocality scenario, we show that by increasing the number of inputs for each party, one can self-test a set of an arbitrary number of mutually commuting local observables. In Sec. V, we summarize the results and conclude by stating a few interesting open questions.

II. SELF-TESTING OF TWO COMMUTING LOCAL OBSERVABLES IN THE STAR NETWORK

Let us first consider the n-local configuration [63] featuring an arbitrary number n of edge party (Alices), say, Alice$_k$, $k \in [n]$, the central party Bob, and n independent sources S_k. Each Alice$_k$ measures two binary-outcome measurements, A_1^k and A_2^k, according to inputs $x_k = 1$ and 2, respectively, and gets outputs $a_k \in \{-1, 1\}$. Bob, upon receiving inputs $i = 1, 2$, performs two binary-outcome measurements B_1 and B_2 on the joint system he receives from n sources and obtains output $b \in \{-1, 1\}$. The complete independence of the resources S_k constitutes the assumption of n-locality which is the most crucial assumption in this context [63].

In an n-local model, we assume that the hidden variables λ_k, corresponding to the sources S_k distributed according to the probability density functions $\rho_k(\lambda_k)$, are independent of each other. Hence, the joint distribution $\rho(\lambda_1, \lambda_2, \ldots, \lambda_n)$ can be written in a factorized form as

$$\rho(\lambda_1, \lambda_2, \ldots, \lambda_n) = \prod_{k=1}^{n} \rho_k(\lambda_k), \tag{1}$$

which is the n-locality condition. Here, for each $k \in [n]$, $\rho_k(\lambda_k)$ satisfies the normalization condition $\int d\lambda_k \rho_k(\lambda_k) = 1$. Using the n-locality condition for a star-network scenario, the joint probability distribution can be written as

$$\begin{aligned}P(a_1, a_2, \dots, a_n, b, |x_1, x_2, \dots x_n, i) \\ = \int \left(\prod_{k=1}^{n} \rho_k(\lambda_k)\, d\lambda_k\, P(a_k|x_k, \lambda_k) \right) \\ \times P(b|i, \lambda_1, \lambda_2, \dots, \lambda_n). \end{aligned} \tag{2}$$

Clearly, Alice$_k$'s outcome solely depends on λ_k, but Bob's outcome depends on all of the λ_k, where $k \in [n]$. A suitable n-locality inequality was proposed in [64,65], which is given by

$$(\Delta_2^n)_{n-l} = |I_{2,1}^n|^{\frac{1}{n}} + |I_{2,2}^n|^{\frac{1}{n}} \leqslant 2, \tag{3}$$

where $n-l$ denotes the n-locality. Here, $I_{2,1}^n$ and $I_{2,2}^n$ are the linear combinations of suitably chosen correlations, defined as

$$\begin{aligned} I_{2,1}^n &= \langle (A_1^1 + A_2^1)(A_1^2 + A_2^2) \cdots (A_1^n + A_2^n) B_1 \rangle, \\ I_{2,2}^n &= \langle (A_1^1 - A_2^1)(A_1^2 - A_2^2) \cdots (A_1^n - A_2^n) B_2 \rangle, \end{aligned} \tag{4}$$

where A_1^k (A_2^k) denotes observables corresponding to input $x_k = 1$ (2) of the kth Alice and the correlations are defined as

$$\begin{aligned}&\langle A_{x_1}^1 \cdots A_{x_n}^n B_i \rangle \\ &= \sum_{a_1,\dots,a_n,b} (-1)^{\sum_{k=1}^n a_k + b} P(a_1, \dots, a_n, b|x_1, \dots, x_n, i).\end{aligned} \tag{5}$$

We define the expectation value of the observable of Alice$_k$ corresponding to the input x_k as

$$\langle A_{x_k}^k \rangle_{\lambda_k} = \sum_{a_k} (-1)^{a_k} P(a_k|x_k, \lambda_k), \tag{6}$$

where $k \in [n]$ and $x_k \in [2]$. Using the fact that $|\langle B_1 \rangle_{\lambda_1,\dots,\lambda_n}| \leqslant 1$ and the sources are independent, we can write

$$|I_{2,1}^n| \leqslant |\langle (A_1^1 + A_2^1)(A_1^2 + A_2^2) \cdots (A_1^n + A_2^n) \rangle|, \tag{7}$$

$$|I_{2,2}^n| \leqslant |\langle (A_1^1 - A_2^1)(A_1^2 - A_2^2) \cdots (A_1^n - A_2^n) \rangle|. \tag{8}$$

For simplicity let $|A_1^k + A_2^k| = z_k^1$, $|A_1^k - A_2^k| = z_k^2$. Now, using the inequality

$$\left(\prod_{k=1}^{n} z_k^1 \right)^{\frac{1}{n}} + \left(\prod_{k=1}^{n} z_k^2 \right)^{\frac{1}{n}} \leqslant \prod_{k=1}^{n} (z_k^1 + z_k^2)^{\frac{1}{n}}, \tag{9}$$

we find that

$$(\Delta_2^n)_{n-l} \leqslant \prod_{k=1}^{n} (|A_1^k + A_2^k| + |A_1^k - A_2^k|)^{\frac{1}{n}}. \tag{10}$$

Since each observable is dichotomic, clearly, we get $|A_1^k + A_2^k| + |A_1^k - A_2^k| \leqslant 2 \quad \forall k \in [n]$. Hence, we finally obtain $(\Delta_2^n)_{n-l} \leqslant 2$, as claimed in Eq. (3).

To derive the optimal quantum value of $(\Delta_2^n)_Q$, we use the following approach. Without loss of serious generality, we consider the state $|\psi\rangle = \otimes_{k=1}^n |\psi\rangle_{A_k B}$ and define two suitable vectors $M_{2,1}^n|\psi\rangle$ and $M_{2,2}^n|\psi\rangle$ as follows:

$$\begin{aligned} M_{2,1}^n|\psi\rangle &= \left[\bigotimes_{k=1}^{n} \left(\frac{A_1^k + A_2^k}{(\omega_{2,1}^n)_{A_k}} \right) \otimes B_1 \right] |\psi\rangle, \\ M_{2,2}^n|\psi\rangle &= \left[\bigotimes_{k=1}^{n} \left(\frac{A_1^k - A_2^k}{(\omega_{2,2}^n)_{A_k}} \right) \otimes B_2 \right] |\psi\rangle, \end{aligned} \tag{11}$$

where $|\psi\rangle_{A_k B}$ is the state shared between Alice$_k$ and Bob. Here, $(\omega_{2,i}^n)_{A_k}$ is the norm of the vector $[A_1^k - (-1)^i A_2^k]|\psi\rangle_{A_k B}$ such that each of the vectors $\frac{[A_1^k - (-1)^i A_2^k]|\psi\rangle_{A_k B}}{(\omega_{2,i}^n)_{A_k}}$ becomes normalized, which in turn ensures that $M_{2,i}^n|\psi\rangle$ is normalized. This implies that

$$I_{2,1}^n = \omega_{2,1}^n \langle M_{2,1}^n \rangle, \quad I_{2,2}^n = \omega_{2,2}^n \langle M_{2,2}^n \rangle, \tag{12}$$

where

$$\omega_{2,1}^n = \prod_{k=1}^{n} (\omega_{2,1}^n)_{A_k}, \quad \omega_{2,2}^n = \prod_{k=1}^{n} (\omega_{2,2}^n)_{A_k}. \tag{13}$$

Since $(\omega_{2,1}^n)$ and $(\omega_{2,2}^n)$ are products of norms, these are always positive. Hence, from Eq. (12), we can write

$$(\Delta_2^n)_Q = (\omega_{2,1}^n |\langle M_{2,1}^n \rangle|)^{1/n} + (\omega_{2,2}^n |\langle M_{2,2}^n \rangle|)^{1/n}. \tag{14}$$

From Eq. (14), it is straightforward to argue that the optimal value of $(\Delta_2^n)_Q$ is obtained when $\langle M_{2,1}^n \rangle = \pm 1$ and $\langle M_{2,2}^n \rangle = \pm 1$ hold. This ensures that the quantum state shared by each Alice and Bob has to be a pure state $|\psi\rangle$, which is the eigenvector of both of $M_{2,1}^n$ and $M_{2,2}^n$ corresponding to eigenvalues ± 1, i.e., $M_{2,1}^n|\psi\rangle = \pm|\psi\rangle$ and $M_{2,2}^n|\psi\rangle = \pm|\psi\rangle$. This implies that

$$(\Delta_2^n)_Q^{\text{opt}} = \max_{A_1^k, A_2^k} [(\omega_{2,1}^n)^{\frac{1}{n}} + (\omega_{2,2}^n)^{\frac{1}{n}}], \tag{15}$$

where $(\omega_{2,1}^n)_{A_k}$ and $(\omega_{2,2}^n)_{A_k}$ are given by

$$\begin{aligned} (\omega_{2,1}^n)_{A_k} &= ||(A_1^k + A_2^k)|\psi\rangle_{A_k B}||_2 = \sqrt{2 + \langle \{A_1^k, A_2^k\} \rangle}, \\ (\omega_{2,2}^n)_{A_k} &= ||(A_1^k - A_2^k)|\psi\rangle_{A_k B}||_2 = \sqrt{2 - \langle \{A_1^k, A_2^k\} \rangle}. \end{aligned} \tag{16}$$

By using inequality (9), we get the following:

$$\begin{aligned} (\omega_{2,1}^n)^{\frac{1}{n}} + (\omega_{2,2}^n)^{\frac{1}{n}} &\leqslant \prod_{k=1}^{n} ((\omega_{2,1}^n)_{A_k} + (\omega_{2,2}^n)_{A_k})^{\frac{1}{n}} \\ &= \prod_{k=1}^{n} \Big(\sqrt{2 + \langle \{A_1^k, A_2^k\} \rangle} \\ &\quad + \sqrt{2 - \langle \{A_1^k, A_2^k\} \rangle}\Big)^{\frac{1}{n}} \\ &= \prod_{k=1}^{n} \left(\sqrt{4 + 2\sqrt{4 - \langle \{A_1^k, A_2^k\} \rangle^2}}\right)^{\frac{1}{n}}. \end{aligned} \tag{17}$$

Hence, to obtain the optimal value of $(\Delta_2^n)_Q^{\text{opt}}$, observables of each Alice$_k$ have to be mutually anticommuting, i.e., $\{A_1^k, A_2^k\} = 0$. This provides an optimal quantum value of $(\Delta_2^n)_Q^{\text{opt}} = 2\sqrt{2}$. Considering the optimal scenario, from

Eq. (16), we then get $(\omega_{2,1}^n)_{A_k} = (\omega_{2,2}^n)_{A_k} = \sqrt{2}$. For convenience, we introduce the following notations:

$$\mathcal{A}_1^k = \frac{A_1^k + A_2^k}{\sqrt{2}}, \quad \mathcal{A}_2^k = \frac{A_1^k - A_2^k}{\sqrt{2}} \tag{18}$$

and $\mathcal{A}_1^n = \bigotimes_{k=1}^n \mathcal{A}_1^k$, $\mathcal{A}_2^n = \bigotimes_{k=1}^n \mathcal{A}_1^k$. It is then easy to check that the optimal quantum value $(\Delta_2^n)_Q^{\text{opt}} = 2\sqrt{2}$ is achieved when the following condition holds:

$$\mathcal{A}_1^n \mathcal{A}_2^n = (-1)^n \mathcal{A}_2^n \mathcal{A}_1^n. \tag{19}$$

We can then write $M_{2,1}^n = \mathcal{A}_1^n \otimes B_1$ and $M_{2,2}^n = \mathcal{A}_2^n \otimes B_2$. Since $M_{2,1}^n|\psi\rangle = \pm|\psi\rangle$ and $M_{2,2}^n|\psi\rangle = \pm|\psi\rangle$, the observables $M_{2,1}^n$ and $M_{2,2}^n$ commute, i.e., $[M_{2,1}^n, M_{2,2}^n] = 0$. Using Eq. (19), we get

$$\mathcal{A}_1^n \mathcal{A}_2^n \otimes (B_1 B_2 - (-1)^n B_2 B_1) = 0. \tag{20}$$

This implies that for even n, the observables B_1 and B_2 need to commute. In other words, the optimal quantum value $(\Delta_2^n)_Q^{\text{opt}}$ self-tests the commuting observables, an interesting self-testing that was not explored earlier.

The optimal quantum violation also self-tests the state $\rho = |\psi_{A^1B}\rangle\langle\psi_{A^1B}| \otimes |\psi_{A^2B}\rangle\langle\psi_{A^2B}| \otimes \cdots \otimes |\psi_{A^nB}\rangle\langle\psi_{A^nB}| \in (\otimes_{i=1}^n C^d)$. Optimal violation requires

$$\text{Tr}\big[\mathcal{A}_1^n \otimes \mathcal{A}_1^n \rho\big] = \text{Tr}\big[\mathcal{A}_2^n \otimes \mathcal{A}_2^n \rho\big] = 1, \tag{21}$$

which again confirms that ρ has to be a pure state. Let us introduce the state between Alice$_k$ and Bob in the Hilbert-Schmidt form as

$$\rho_{A^kB} = \frac{1}{d^2}\left[\mathbb{I} \otimes \mathbb{I} + \sum_{i=1}^{d^2-1} (-1)^\alpha \mathcal{A}_i^k \otimes \mathcal{A}_i^k\right], \tag{22}$$

where $\alpha = 0, 1$, $\{\mathcal{A}_i^k, \mathcal{A}_j^k\} = 0$, and, consequently, $[\mathcal{A}_i^k \otimes \mathcal{A}_j^k, \mathcal{A}_j^k \otimes \mathcal{A}_i^k] = 0$ for any arbitrary dimension d. For a density matrix ρ_{A^kB}, $\text{Tr}[\rho_{A^kB}] = 1$ has to be satisfied. This in turn leads to $\text{Tr}[\mathcal{A}_1^k] = \text{Tr}[\mathcal{A}_2^k] = 0$. Also, $\text{Tr}[\rho_{A^kB}^2] = 1$ ensures that the observables in the summation in Eq. (24) contains a full set of mutually commuting observables $\{\mathcal{A}_i^k \otimes \mathcal{A}_i^k\}$. Consequently, $\text{Tr}_{A^k}[\rho_{A^kB}] = \text{Tr}_B[\rho_{A^kB}] = \frac{\mathbb{I}}{d}$; that is, the partial trace of ρ_{A^kB} is a maximally mixed state for both Alice and Bob.

It is important to note that the above derivation is dimension independent, and hence, the conclusion holds for any dimensional system. However, a realization of such observables for each Alice$_k$ can be found even for the local qubit system as follows:

$$A_1^k = \frac{\sigma_z + \sigma_x}{\sqrt{2}}, \quad A_2^k = \frac{\sigma_z - \sigma_x}{\sqrt{2}} \;\; \forall k \in [n]. \tag{23}$$

Hence, Bob's observables B_1 and B_2 are

$$B_1 = \otimes^n \sigma_z, \quad B_2 = \otimes^n \sigma_x,$$

which in turn ensure that $B_1 B_2 = (-1)^n B_2 B_1$; that is, for an even value of n, the observables B_1 and B_2 commute. The optimal violation requires each source S_k to share a maximally entangled state ρ_{A^kB}, given by

$$\rho_{A^kB} = \tfrac{1}{4}[\mathbb{I} \otimes \mathbb{I} + \sigma_x \otimes \sigma_x - \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z]. \tag{24}$$

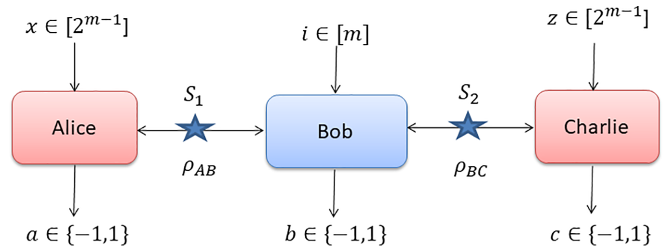

FIG. 2. Arbitrary-input scenario in a bilocal network.

Hence, the optimal quantum violation of the n-locality inequality (3) self-tests two mutually commuting observables if the star network features an even number of parties.

III. SELF-TESTING A SET OF THREE MUTUALLY COMMUTING LOCAL OBSERVABLES

We now focus on self-testing three mutually commuting local observables by considering the simplest bilocal scenario. Further, we generalize this result for an arbitrary-input bilocal scenario for self-testing an unbounded number of mutually commuting local observables.

As depicted in Fig. 2, we consider the bilocal scenario featuring two edge parties (Alice and Charlie) and a central party (Bob). Alice and Charlie each share a physical state with Bob generated from two independent sources, S_1 and S_2, respectively. The central party, Bob, performs three dichotomic measurements, B_1, B_2, and B_3, according to the inputs $i \in [3]$ and obtains the outcome $b \in \{-1, 1\}$. Alice (Charlie) receives inputs $x \in [4]$ $(z \in [4])$, performs four measurements A_x (C_z) accordingly, and obtains outputs $a(c) \in \{-1, 1\}$.

If the joint probability distribution $P(a, b, c|x, i, z)$ can be factorized as

$$P(a, b, c|x, i, z) = \iint \rho_1(\lambda_1)\rho_2(\lambda_2)P(a|x, \lambda_1)\, P(b|i, \lambda_1, \lambda_2) \times P(c|z, \lambda_2)\, d\lambda_1 d\lambda_2, \tag{25}$$

where λ_1 and λ_2 are the physical states generated from sources S_1 and S_2, respectively, then we propose that the inequality

$$\left(\Delta_3^2\right)_{b-l} = \sqrt{\left|I_{3,1}^2\right|} + \sqrt{\left|I_{3,2}^2\right|} + \sqrt{\left|I_{3,3}^2\right|} \leqslant 6 \tag{26}$$

is satisfied. Here $b - l$ denotes bilocality. Here, $I_{3,1}^2$ $I_{3,2}^2$, and $I_{3,3}^2$ are the linear combinations of suitably chosen correlations, defined as

$$\begin{aligned} I_{3,1}^2 &= \langle (A_1 + A_2 + A_3 - A_4)B_1(C_1 + C_2 + C_3 - C_4)\rangle, \\ I_{3,2}^2 &= \langle (A_1 + A_2 - A_3 + A_4)B_2(C_1 + C_2 - C_3 + C_4)\rangle, \\ I_{3,3}^2 &= \langle (A_1 - A_2 + A_3 + A_4)B_3(C_1 - C_2 + C_3 + C_4)\rangle, \end{aligned} \tag{27}$$

where the expectation value is defined as follows:

$$\langle A_x B_i C_z\rangle = \sum_{a,b,c} (-1)^{a+b+c} P(a, b, c|x, i, z)$$

and $\langle A_x\rangle_{\lambda_1} = \sum_a (-1)^a P(a|x, \lambda_1)$ $(x \in [4])$. Since $|\langle B_1\rangle_{\lambda_1,\lambda_2}| \leqslant 1$ and the sources are independent, we can write

$$\left|I_{3,1}^2\right| \leqslant |\langle (A_1 + A_2 + A_3 - A_4)\rangle||\langle (C_1 + C_2 + C_3 - C_4)\rangle|. \tag{28}$$

Similarly, we can factorize $|I^2_{3,2}|$ and $|I^2_{3,3}|$ like Eq. (28). Now, using the inequality

$$\forall\ z_k^i \geqslant 0, \quad \sum_{i=1}^{m}\left(\prod_{k=1}^{n} z_k^i\right)^{\frac{1}{n}} \leqslant \prod_{k=1}^{n}\left(\sum_{i=1}^{m} z_k^i\right)^{\frac{1}{n}} \tag{29}$$

for $m=3$ and $n=2$, we get the following:

$$\begin{aligned}(\Delta^2_3)_{b-l} &\leqslant [|\langle(A_1+A_2+A_3-A_4)\rangle||\langle(C_1+C_2+C_3-C_4)\rangle|]^{1/2}+[|\langle(A_1+A_2-A_3+A_4)\rangle||\langle(C_1+C_2-C_3+C_4)\rangle|]^{1/2}\\ &\quad+[|\langle(A_1-A_2+A_3+A_4)\rangle||\langle(C_1-C_2+C_3+C_4)\rangle|]^{1/2}\\ &\leqslant [|\langle(A_1+A_2+A_3-A_4)\rangle|+|\langle(A_1+A_2-A_3+A_4)\rangle|+|\langle(A_1-A_2+A_3+A_4)\rangle|]^{1/2}[|\langle(C_1+C_2+C_3-C_4)\rangle|\\ &\quad+|\langle(C_1+C_2-C_3+C_4)\rangle|+|\langle(C_1-C_2+C_3+C_4)\rangle|]^{1/2}.\end{aligned} \tag{30}$$

For notational convenience, let us consider the following: $|\langle(A_1+A_2+A_3-A_4)\rangle|+|\langle(A_1+A_2-A_3+A_4)\rangle|+|\langle(A_1-A_2+A_3+A_4)\rangle|=\eta_3^A$ and $|\langle(C_1+C_2+C_3-C_4)\rangle|+|\langle(C_1+C_2-C_3+C_4)\rangle|+|\langle(C_1-C_2+C_3+C_4)\rangle|=\eta_3^C$.

Since each observables of Alice and Charlie is dichotomic, we get $\eta_3^A=\eta_3^C\leqslant 6$. Substituting these values in Eq. (30), we finally obtain $(\Delta^2_3)_{b-l}\leqslant 6$, as claimed in Eq. (26).

To derive the optimal quantum value of $(\Delta^2_3)_Q$, we use the following approach. Without loss of generality, we consider the state $|\psi\rangle=|\psi\rangle_{AB}\otimes|\psi\rangle_{BC}$ and the suitable vectors $M^2_{3,1}|\psi\rangle$, $M^2_{3,2}|\psi\rangle$, and $M^2_{3,3}|\psi\rangle$ as follows:

$$\begin{aligned}M^2_{3,1}|\psi\rangle &= \left(\frac{A_1+A_2+A_3-A_4}{(\omega^2_{3,1})_A}\otimes\frac{C_1+C_2+C_3-C_4}{(\omega^2_{3,1})_C}\otimes B_1\right)|\psi\rangle,\\ M^2_{3,2}|\psi\rangle &= \left(\frac{A_1+A_2-A_3+A_4}{(\omega^2_{3,2})_A}\otimes\frac{C_1+C_2-C_3+C_4}{(\omega^2_{3,2})_C}\otimes B_2\right)|\psi\rangle,\\ M^2_{3,3}|\psi\rangle &= \left(\frac{A_1-A_2+A_3+A_4}{(\omega^2_{3,3})_A}\otimes\frac{C_1-C_2+C_3+C_4}{(\omega^2_{3,3})_C}\otimes B_3\right)|\psi\rangle.\end{aligned} \tag{31}$$

Here, $|\psi\rangle_{AB}$ ($|\psi\rangle_{BC}$) is the state shared between Alice (Charlie) and Bob. Also $(\omega^2_{3,1})_A$ is the norm of the vector $(A_1+A_2+A_3-A_4)|\psi\rangle_{AB}$ such that the vector $\frac{(A_1+A_2+A_3-A_4)|\psi\rangle_{AB}}{(\omega^2_{3,1})_A}$ becomes normalized. A similar argument holds for each $(\omega^2_{3,i})_{A/C}$, $i\in[3]$. This in turn ensures that the vectors $M^2_{3,i}|\psi\rangle$ are also normalized. Using the vectors in Eq. (31), we can write

$$I^2_{3,1}=\omega^2_{3,1}\langle M^2_{3,1}\rangle,\quad I^2_{3,2}=\omega^2_{3,2}\langle M^2_{3,2}\rangle,\quad I^2_{3,3}=\omega^2_{3,3}\langle M^2_{3,3}\rangle, \tag{32}$$

where $\omega^2_{3,i}$ is defined as $\omega^2_{3,i}=(\omega^2_{3,i})_A(\omega^2_{3,i})_C$. Since $(\omega^2_{3,i})$ are products of norms, they are always positive. Hence, from Eq. (32), we can write the following:

$$(\Delta^2_3)_Q=\sqrt{\omega^2_{3,1}|\langle M^2_{3,1}\rangle|}+\sqrt{\omega^2_{3,2}|\langle M^2_{3,2}\rangle|}+\sqrt{\omega^2_{3,3}|\langle M^2_{3,3}\rangle|}.$$

According to the construction, the vectors $M^2_{3,1}|\psi\rangle$, $M^2_{3,2}|\psi\rangle$, and $M^2_{3,3}|\psi\rangle$ are normalized, and hence, the optimal value of $(\Delta^2_3)_Q$ is obtained when the conditions $\langle M^2_{3,1}\rangle=\pm1$, $\langle M^2_{3,2}\rangle=\pm1$, and $\langle M^2_{3,3}\rangle=\pm1$ hold simultaneously. This ensures that the quantum state $|\psi\rangle$ has to be a pure state, and it is the eigenvector of each of the observables $M^2_{3,1}$, $M^2_{3,2}$, and $M^2_{3,3}$ corresponding to eigenvalues ±1, i.e., $M^2_{3,1}|\psi\rangle=\pm|\psi\rangle$, $M^2_{3,2}|\psi\rangle=\pm|\psi\rangle$, and $M^2_{3,3}|\psi\rangle=\pm|\psi\rangle$. This implies that

$$(\Delta^2_3)^{\text{opt}}_Q=\max_{A_x,C_z}\left(\sqrt{\omega^2_{3,1}}+\sqrt{\omega^2_{3,2}}+\sqrt{\omega^2_{3,3}}\right). \tag{33}$$

The norm $(\omega^2_{3,1})_A$ is given by

$$\begin{aligned}(\omega^2_{3,1})_A &= ||(A_1+A_2+A_3-A_4)|\psi\rangle_{AB}||_2\\ &= [4+\langle\{A_1,(A_2+A_3-A_4)\}+\{A_2,(A_3-A_4)\}\\ &\quad-\{A_3,A_4\}\rangle]^{1/2}.\end{aligned} \tag{34}$$

We can similarly write the expressions for $(\omega^2_{3,2})_A$ $[(\omega^2_{3,2})_C]$ and $(\omega^2_{3,3})_A$ $[(\omega^2_{3,3})_C]$. Since we have defined $\omega^2_{3,i}=(\omega^2_{3,i})_A(\omega^2_{3,i})_C$ $\forall i\in[3]$ by using the inequality in Eq. (29), we find that

$$\sum_{i=1}^{3}\sqrt{\omega^2_{3,i}}\leqslant\sqrt{\sum_{i=1}^{3}(\omega^2_{3,i})_A}\sqrt{\sum_{i=1}^{3}(\omega^2_{3,i})_C}. \tag{35}$$

Further using the convex inequality, we have

$$\sum_{i=1}^{3}(\omega^2_{3,i})_A\leqslant\sqrt{3\sum_{i=1}^{3}\left[(\omega^2_{3,i})_A\right]^2}. \tag{36}$$

The equality holds when each of $(\omega^n_{3,i})_A$, $i\in[3]$, is equal. We can then write the following:

$$\begin{aligned}\sum_{i=1}^{3}\left[(\omega^2_{3,i})_A\right]^2 &= \langle\psi|12+[\{A_1,(A_2+A_3+A_4)\}\\ &\quad-\{A_2,(A_3+A_4)\}-\{A_3,A_4\}]|\psi\rangle\\ &= \langle\psi(12+\delta_3)|\psi\rangle,\end{aligned} \tag{37}$$

where δ_3 is defined as follows:

$$\delta_3 = [\{A_1, (A_2 + A_3 + A_4)\} - \{A_2, (A_3 + A_4)\} - \{A_3, A_4\}]. \tag{38}$$

Let $|\psi'\rangle = (A_1 - A_2 - A_3 - A_4)|\psi\rangle$ such that $|\psi\rangle \neq 0$. Then we get $\langle\psi'|\psi'\rangle = \langle\psi|(4 - \delta_3)|\psi\rangle$, which implies that $\langle\delta_3\rangle = 4 - \langle\psi'|\psi'\rangle$. Evidently, $\langle\delta_3\rangle_{\max}$ is obtained only when $\langle\psi'|\psi'\rangle = 0$. Since $|\psi\rangle \neq 0$, we have the following condition:

$$A_1 - A_2 - A_3 - A_4 = 0. \tag{39}$$

Hence, to obtain the optimal value of $(\Delta_3^2)_Q^{\text{opt}}$, Alice's observables must satisfy the linear condition of Eq. (39). In turn $\langle\delta_3\rangle_{\max} = 4$ provides

$$\left[(\omega_{3,1}^2)_A\right]^2 + \left[(\omega_{3,2}^2)_A\right]^2 + \left[(\omega_{3,3}^2)_A\right]^2 = 16. \tag{40}$$

Plugging it in inequality (36), we get $\sum_{i=1}^{3}(\omega_{3,i}^2)_A \leqslant 4\sqrt{3}$. Since each observable A_x is dichotomic, premultiplying and postmultiplying Eq. (39) by A_1 and adding the results, we get

$$\{A_1, A_2\} + \{A_1, A_3\} + \{A_1, A_4\} = 2. \tag{41}$$

Similarly, we can find three more such relations as follows:

$$\begin{aligned} \{A_1, A_2\} - \{A_2, A_3\} - \{A_2, A_4\} &= 2, \\ \{A_1, A_3\} - \{A_2, A_3\} - \{A_3, A_4\} &= 2, \\ \{A_1, A_4\} - \{A_2, A_4\} - \{A_3, A_4\} &= 2. \end{aligned} \tag{42}$$

Solving Eqs. (41) and (42), we get

$$\begin{aligned} \{A_1, A_2\} = \{A_1, A_3\} = \{A_1, A_4\} &= \tfrac{2}{3}, \\ \{A_2, A_3\} = \{A_2, A_4\} = \{A_3, A_4\} &= -\tfrac{2}{3}. \end{aligned} \tag{43}$$

We thus obtain the relations between the observables for each Alice. Also, for the optimal quantum violation, we check that $(\omega_{3,1}^2)_A = (\omega_{3,2}^2)_A = (\omega_{3,3}^2)_A = 4/\sqrt{3}$. Following a similar calculation for the other party, Charlie, we get

$$\begin{aligned} \{C_1, C_2\} = \{C_1, C_3\} = \{C_1, C_4\} &= \tfrac{2}{3}, \\ \{C_2, C_3\} = \{C_2, C_4\} = \{C_3, C_4\} &= -\tfrac{2}{3}. \end{aligned} \tag{44}$$

Again, the optimal quantum violation gives $(\omega_{3,1}^2)_C = (\omega_{3,2}^2)_C = (\omega_{3,3}^2)_C = 4/\sqrt{3}$. This in turn provides the optimal quantum value

$$(\Delta_3^2)_Q^{\text{opt}} = 4\sqrt{3}. \tag{45}$$

Considering the optimal scenario, let us denote

$$\begin{aligned} \mathcal{A}_1^{2,3} &= \left(\frac{A_1 + A_2 + A_3 - A_4}{4/\sqrt{3}}\right) \otimes \left(\frac{C_1 + C_2 + C_3 - C_4}{4/\sqrt{3}}\right), \\ \mathcal{A}_2^{2,3} &= \left(\frac{A_1 + A_2 - A_3 + A_4}{4/\sqrt{3}}\right) \otimes \left(\frac{C_1 + C_2 - C_3 + C_4}{4/\sqrt{3}}\right), \\ \mathcal{A}_3^{2,3} &= \left(\frac{A_1 - A_2 + A_3 + A_4}{4/\sqrt{3}}\right) \otimes \left(\frac{C_1 - C_2 + C_3 + C_4}{4/\sqrt{3}}\right). \end{aligned} \tag{46}$$

Since the optimal quantum violation certifies relations (43) and (44), we obtain

$$\mathcal{A}_i^{2,3}\mathcal{A}_j^{2,3} = \mathcal{A}_j^{2,3}\mathcal{A}_i^{2,3} \quad \forall\, i \neq j \in [3]. \tag{47}$$

We can then write $M_{3,1}^2 = \mathcal{A}_1^{2,3} \otimes B_1, M_{3,2}^2 = \mathcal{A}_2^{2,3} \otimes B_2$, and $M_{3,3}^2 = \mathcal{A}_3^{2,3} \otimes B_3$. Since $M_{3,i}^2|\psi\rangle = \pm|\psi\rangle\ \forall\, i \in [3]$, the observables $M_{3,1}^2$, $M_{3,2}^2$, and $M_{3,3}^2$ are mutually commuting, i.e., $[M_{3,1}^2, M_{3,2}^2] = 0$. Using Eq. (47), we get

$$\mathcal{A}_1^{2,3}\mathcal{A}_2^{2,3} \otimes (B_1B_2 - B_2B_1) = 0, \tag{48}$$

which implies that B_1 and B_2 commute. Following a similar argument, it is straightforward to show that $[B_2, B_3] = [B_1, B_3] = 0$. Thus, the optimal quantum violation of the bilocality inequality (26) uniquely fixes Bob's observables, and simultaneously, a set of three mutually commuting local observables has been self-tested. Following an approach similar to that mentioned in Sec. II, the required maximally entangled state can be straightforwardly derived.

In the next section, we show that this feature is generic and valid for any arbitrary m-input case. In other words, we demonstrate the self-testing of an unbounded number of mutually commuting local observables.

IV. SELF-TESTING A SET OF ARBITRARY *m* MUTUALLY COMMUTING LOCAL OBSERVABLES

In order to self-test a set of unbounded m mutually commuting local observables, we consider the bilocality scenario with arbitrary-m inputs and derive the bilocality inequality. Here, the central party, Bob, receives arbitrary-m inputs, and each of the edge parties, Alice and Charlie, receives 2^{m-1} inputs. In this tripartite scenario, the two edge parties, Alice and Charlie, receive inputs $x, z \in \{1, 2, 3, \ldots, 2^{m-1}\}$ producing outputs $a, c \in \{-1, 1\}$, respectively. Bob's inputs are denoted as $i \in \{1, 2, \ldots, m\}$ and produce output $b \in \{-1, 1\}$. It is assumed that there are two independent sources, S_1 and S_2, and each of them distributes a state with the central party Bob.

We propose the generalized bilocality inequality for arbitrary m as

$$\left(\Delta_m^2\right)_{b-l} = \sum_{i=1}^{m} \sqrt{|J_{m,i}|} \leqslant m\binom{m-1}{\lfloor\frac{m-1}{2}\rfloor}, \tag{49}$$

where $J_{m,i}$ is the linear combination of suitable correlations, defined as follows:

$$J_{m,i} = \left\langle \left(\sum_{x=1}^{2^{m-1}}(-1)^{y_i^x}A_x\right)B_i\left(\sum_{z=1}^{2^{m-1}}(-1)^{y_i^z}C_z\right)\right\rangle. \tag{50}$$

Here, $y_i^{x(z)}$ takes a value of either 0 or 1. For our purpose, we fix the values of y_i^x and y_i^z by using the encoding scheme used in random access codes [24,66,67] as a tool. This will fix 1 or -1 values of $(-1)^{y_i^x}$ in Eq. (50) for a given i. Let us consider a random variable $y^\alpha \in \{0, 1\}^m$ with $\alpha \in \{1, 2, \ldots, 2^m\}$. Each element of the bit string can be written as $y^\alpha = y_{i=1}^\alpha y_{i=2}^\alpha y_{i=3}^\alpha \cdots\ y_{i=m}^\alpha$. For example, if $y^\alpha = 011\cdots00$, then $y_{i=1}^\alpha = 0$, $y_{i=2}^\alpha = 1$, $y_{i=3}^\alpha = 1$, and so on. Here, we denote the length-m binary strings as y^x. Now we consider the bit strings such that for any two x and x', $y^x \oplus_2 y^{x'} = 11\cdots1$. Clearly, we have $x \in \{1, 2, \ldots, 2^{m-1}\}$ constituting the inputs for Alice. If $i = 1$, we get all the first bits of each bit string y^x for every $x \in \{1, 2, \ldots, 2^m\}$. Similar encoding holds for the other party, Charlie.

An example for $m = 2$ could be useful here. In this case, we have $y^\delta \in \{00, 01, 10, 11\}$, with $\delta = 1, 2, 3, 4$. We then denote $y^x \equiv \{y^1, y^2\} \in \{00, 01\}$, with $y^1 = 00$ and $y^2 = 01$. This also means $y^1_{i=1} = 0$, $y^1_{i=2} = 0$, $y^2_{i=1} = 0$, and $y^2_{i=2} = 1$.

Here, using the fact that the observable $|B_i|_{\lambda_1,\lambda_2} \leqslant 1$, we get

$$|J_{m,i}| \leqslant \left|\left\langle \sum_{x=1}^{2^{m-1}} (-1)^{y^x_i} A_x \right\rangle\right| \left|\left\langle \sum_{z=1}^{2^{m-1}} (-1)^{y^z_i} C_z \right\rangle\right|. \tag{51}$$

This implies that

$$\sum_{i=1}^{m} \sqrt{|J_{m,i}|} \leqslant \sum_{i=1}^{m} \sqrt{\left|\left\langle \sum_{x=1}^{2^{m-1}} (-1)^{y^x_i} A_x \right\rangle\right| \left|\left\langle \sum_{z=1}^{2^{m-1}} (-1)^{y^z_i} C_z \right\rangle\right|}. \tag{52}$$

Using the inequality

$$\sum_{i=1}^{m} \sqrt{z^i_1 z^i_2} \leqslant \sqrt{\left(\sum_{i=1}^{m} z^i_1\right)\left(\sum_{i=1}^{m} z^i_2\right)}, \tag{53}$$

we get the following:

$$\begin{aligned}(\Delta^2_m)_{b-l} &\leqslant \sqrt{\left(\sum_{i=1}^{m}\left|\left\langle \sum_{x=1}^{2^{m-1}} (-1)^{y^x_i} A_x \right\rangle\right|\right)\left(\sum_{i=1}^{m}\left|\left\langle \sum_{z=1}^{2^{m-1}} (-1)^{y^z_i} C_z \right\rangle\right|\right)} \\ &\leqslant \sqrt{\eta^A_m \times \eta^C_m}.\end{aligned} \tag{54}$$

Here, we denote η^A_m and η^C_m as follows:

$$\eta^A_m = \sum_{i=1}^{m}\left|\left\langle \sum_{x=1}^{2^{m-1}} (-1)^{y^x_i} A_x \right\rangle\right|, \quad \eta^C_m = \sum_{i=1}^{m}\left|\left\langle \sum_{z=1}^{2^{m-1}} (-1)^{y^z_i} C_z \right\rangle\right|. \tag{55}$$

Since the encoding schemes used for Alice and Charlie are identical, clearly,

$$(\eta^A_m)_{\max} = (\eta^C_m)_{\max}, \tag{56}$$

which is the optimal bilocal bound of Δ^2_m. Since each observable of Alice and Charlie is dichotomic, we obtain

$$(\eta^A_m)_{\max} = m\binom{m-1}{\lfloor \frac{m-1}{2} \rfloor}. \tag{57}$$

A detailed derivation is provided in Appendix A. Substituting this value in Eq. (54), we get the bilocality inequality as defined in Eq. (49).

In order to obtain the quantum upper bound of the expression (Δ^2_m), we use the following approach. Without loss of serious generality, we consider the state $|\psi\rangle = |\psi\rangle_{AB} \otimes |\psi\rangle_{BC}$ and the suitable vectors $M^2_{m,i}|\psi\rangle$ as follows:

$$\begin{aligned}M^2_{m,i}|\psi\rangle = \Bigg[&\left(\frac{1}{(\omega^2_{m,i})_A} \sum_{x=1}^{2^{m-1}} (-1)^{y^x_i} A_x\right) \otimes B_i \\ &\otimes \left(\frac{1}{(\omega^2_{m,i})_C} \sum_{z=1}^{2^{m-1}} (-1)^{y^z_i} C_z\right)\Bigg]|\psi\rangle,\end{aligned} \tag{58}$$

where $(\omega^2_{m,i})_A$ is the norm of the vector $[\sum_{x=1}^{2^m-1} (-1)^{y^x_i} A_x]|\psi\rangle_{AB}$ such that the vector $[\sum_{x=1}^{2^m-1} (-1)^{y^x_i} A_x]|\psi\rangle_{AB}/(\omega^2_{m,i})_A$ becomes normalized. A similar argument holds for each $(\omega^2_{m,i})_C$, $i \in [m]$. This in turn ensures that the vectors $M^2_{m,i}|\psi\rangle_{AB}$ are also normalized. Here, $|\psi\rangle_{AB}(|\psi\rangle_{BC})$ is the state shared between Alice (Charlie) and Bob. Using the vectors in Eq. (58), we can write

$$J_{m,i} = \omega^2_{m,i}\langle M^2_{m,i}\rangle. \tag{59}$$

Here, $|\psi\rangle_{AB}$ $(|\psi\rangle_{BC})$ is the state shared between Alice (Charlie) and Bob. Also, $\omega^2_{m,i}$ is defined as $\omega^2_{m,i} = (\omega^2_{m,i})_A(\omega^2_{m,i})_C$. Since $(\omega^2_{m,i})$ are products of norms, $(\omega^2_{m,i}) \geqslant 0 \; \forall\, i \in [m]$. Hence, from Eq. (59), we can write

$$(\Delta^2_m)_Q = \sum_{i=1}^{m} \sqrt{\omega^2_{m,i}|\langle M^2_{m,i}\rangle|}.$$

As we have defined, the vectors $M^2_{m,i}|\psi\rangle$ are normalized for each $i \in [m]$. Hence, the optimal value of $(\Delta^2_m)_Q$ is obtained when $\langle M^2_{m,i}\rangle = \pm 1$ for each $i \in [m]$. This ensures that the state $|\psi\rangle$ has to be a pure state and it is the eigenvector of each of the observables $M^2_{m,i}$ corresponding to eigenvalues ± 1, i.e., $M^2_{m,i}|\psi\rangle = \pm|\psi\rangle \; \forall\, i \in [m]$. This implies that

$$(\Delta^2_m)^{\text{opt}}_Q = \max_{A_x, C_z}\left(\sum_{i=1}^{m} \sqrt{\omega^2_{m,i}}\right). \tag{60}$$

The norms $(\omega^2_{m,i})_A$ and $(\omega^2_{m,i})_C$ are given by $(\omega^2_{m,i})_A = ||[\sum_{x=1}^{2^m-1} (-1)^{y^x_i} A_x]|\psi\rangle_{AB}||_2$ and $(\omega^2_{m,i})_C = ||[\sum_{z=1}^{2^m-1} (-1)^{y^z_i} C_z]|\psi\rangle_{BC}||_2$.

Since $\omega^2_{3,i} = (\omega^2_{3,i})_A(\omega^2_{3,i})_C \; \forall\, i \in [m]$, by using inequality (29), we get

$$\sum_{i=1}^{m} \sqrt{\omega^2_{m,i}} \leqslant \sqrt{\sum_{i=1}^{m} (\omega^2_{m,i})_A} \sqrt{\sum_{i=1}^{m} (\omega^2_{m,i})_C}. \tag{61}$$

Further using the convex inequality, we get the following:

$$\sum_{i=1}^{m} (\omega^2_{m,i})_A \leqslant \sqrt{m \sum_{i=1}^{m} \left[(\omega^2_{m,i})_A\right]^2}. \tag{62}$$

We have found that

$$\max_{A_x, x}\left(\sum_{i=1}^{m} (\omega^2_{m,i})_A\right) = \max_{C_z, z}\left(\sum_{i=1}^{m} (\omega^2_{m,i})_C\right) = 2^{m-1}\sqrt{m}, \tag{63}$$

which implies that $(\Delta^2_m)^{\text{opt}}_Q = 2^{m-1}\sqrt{m}$. Also, we check that for the optimal value the following condition holds:

$$(\omega^2_{m,i})_A = (\omega^2_{m,i})_C = \frac{2^{m-1}}{\sqrt{m}}, \quad i \in [m]. \tag{64}$$

The detailed derivation of this optimal quantum bound is provided in Appendix B.

Considering the optimal scenario, let us introduce the following notations:

$$\mathcal{A}_i^{2,m} = \left(\frac{\sqrt{m}}{2^{m-1}}\sum_{x=1}^{2^{m-1}}(-1)^{y_i^x}A_x\right)\otimes\left(\frac{\sqrt{m}}{2^{m-1}}\sum_{z=1}^{2^{m-1}}(-1)^{y_i^z}C_z\right). \tag{65}$$

Since the optimal quantum violation certifies a certain number of relations of the observables of Alice and Charlie, using relations (B7) and (B8), we obtain

$$\mathcal{A}_i^{2,m}\mathcal{A}_j^{2,m} = \mathcal{A}_j^{2,m}\mathcal{A}_i^{2,m} \quad \forall\, i \neq j \in [m]. \tag{66}$$

We can then write $M_{m,i}^2 = \mathcal{A}_i^{2,m}\otimes B_i$. Since $M_{m,i}^2|\psi\rangle = \pm|\psi\rangle\ \forall\, i \in [m]$, the observables $M_{m,i}^2$ and $M_{m,j}^2$ are mutually commuting, i.e., $[M_{m,i}^2, M_{m,j}^2] = 0\ \forall\, i \neq j \in [m]$. Using Eq. (66), we get

$$\mathcal{A}_i^{2,m}\mathcal{A}_j^{2,m}\otimes(B_iB_j - B_jB_i) = 0. \tag{67}$$

That is, Bob's observables have to be mutually commuting to obtain the optimal quantum value. Since m is arbitrary, the optimal quantum violation of the bilocality inequality self-tests a set of an unbounded number of mutually commuting local observables.

However, the number of mutually commuting observables is restricted by the dimension. For example, for a two-qubit system, at most three observables can be mutually commuting. We find that to obtain the optimal value for arbitrary m, the local dimension of every Alice (Charlie) has to be at least $d = 2^{\lfloor m/2\rfloor}$. In other words, Alice (Charlie) shares at least $\lfloor m/2\rfloor$ copies of a two-qubit maximally entangled state with Bob. The total state can be written as

$$|\psi_{ABC}\rangle = |\phi_{AB}^+\rangle^{\otimes\lfloor\frac{m}{2}\rfloor}\otimes|\phi_{BC}^+\rangle^{\otimes\lfloor\frac{m}{2}\rfloor}. \tag{68}$$

Thus, the optimal quantum violation of arbitrary-input bilocality inequality (49) uniquely fixes Bob's observables, and eventually, a set of an unbounded number of mutually commuting local observables is self-tested. Following an approach similar to that mentioned in Sec. II, the required maximally entangled state can be found.

V. SUMMARY AND DISCUSSION

In summary, we provided schemes for self-testing an unbounded number of mutually commuting local observables. It is a common perception that commuting observables cannot lead to nonclassicality because they have common eigenstates and hence joint probability exists. It is also well known that the demonstration of Bell's theorem requires incompatible observables. Since commuting observables are compatible, they cannot reveal quantum nonlocality. Against this backdrop, in this work, we demonstrated that the optimal quantum violation of network inequalities can be obtained only when the observables of one party are mutually commuting. Therefore, we showed that the optimal quantum violation self-tests a set of mutually commuting local observables.

To demonstrate this, we first considered a star network involving an arbitrary number n of edge parties and a central party. Each party, including the central party, receives two inputs $m = 2$ and performs the measurement of two dichotomic observables accordingly. We showed that the optimal quantum violation of the n-locality inequality can be obtained only when Bob's two observables are mutually commuting for even n. Importantly, we invoked an elegant approach that enabled us to derive the optimal quantum violation without any reference to the dimension of the system. In other words, the dimension of Bob's commuting observables remains unspecified.

Further, we demonstrated the self-testing of an unbounded number of mutually commuting local observables. This feature is also generic and valid for any arbitrary dimensional system. To demonstrate this, we considered a bilocal scenario $n = 2$ in which the central party Bob performs an arbitrary number m of dichotomic measurements and each of the two edge parties performs 2^{m-1} dichotomic measurements. We showed that optimal quantum violation of a suitably formulated bilocality inequality can be obtained only when Bob's observables are mutually commuting. Therefore, the optimal quantum value self-tests an unbounded number of mutually commuting local observables because m is arbitrary.

The significance of this work is that it challenges the usual perception of commuting observables in quantum theory. Optimal quantum violation of Bell's inequality commonly certifies the anticommuting observables. Based on the optimal quantum violation of suitably formulated network inequalities, we demonstrated the self-testing of an unbounded number of mutually commuting local observables.

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APPENDIX A: DERIVATION OF THE BILOCAL BOUND OF $(\eta_m^A)_{\max}$ IN EQUATION (57)

Let us consider an expression $\mathcal{B} = \sum_{i=1}^{m}\sum_{x=1}^{2^{m-1}}(-1)^{y_i^x}A_xB_i$, where A_x and B_i are all dichotomic and the encoding scheme y_i^x is the same as depicted in Sec. III. Since each of $A_x, B_i \in \{-1, 1\}$, the observables A_x and B_i are basically equivalent, and the functional $\mathcal{B}$ is invariant under the interchange of indices x and i. We can then write $\mathcal{B} = \sum_{i=1}^{2^{m-1}}\sum_{x=1}^{m}(-1)^{y_x^i}A_xB_i$. Hence, using the fact that $|B_i| \leqslant 1\ \forall\, i$, we get

$$|\mathcal{B}| \leqslant \sum_{i=1}^{m}\left|\sum_{x=1}^{2^{m-1}}(-1)^{y_i^x}A_x\right|, \quad |\mathcal{B}| \leqslant \sum_{i=1}^{2^{m-1}}\left|\sum_{x=1}^{m}(-1)^{y_x^i}A_x\right|. \tag{A1}$$

Considering the tightness of the bound and the uniqueness of the supremum property, we can write

$$\sum_{i=1}^{m}\left|\sum_{x=1}^{2^{m-1}}(-1)^{y_i^x}A_x\right| = \sum_{i=1}^{2^{m-1}}\left|\sum_{x=1}^{m}(-1)^{y_x^i}A_x\right|. \tag{A2}$$

Using the augmented Hadamard code [68], Refs. [64,65] already derived that $(\sum_{i=1}^{2^{m-1}}|\sum_{x=1}^{m}(-1)^{y_x^i}A_x|)_{\max} = m\binom{m-1}{\lfloor\frac{m-1}{2}\rfloor}$, which implies that $(\sum_{i=1}^{m} | \sum_{x=1}^{2^{m-1}} (-1)^{y_i^x} A_x)_{\max} = m\binom{m-1}{\lfloor \frac{m-1}{2} \rfloor}$. Since we considered $\sum_{i=1}^{m} | \sum_{x=1}^{2^{m-1}} (-1)^{y_i^x} A_x| = \eta_m^A$, clearly,

$$\left(\eta_m^A\right)_{\max} = m\binom{m-1}{\lfloor \frac{m-1}{2} \rfloor}. \tag{A3}$$

APPENDIX B: DETAILED DERIVATION OF THE OPTIMAL QUANTUM BOUND OF $\sum_{i=1}^{m}(\omega_{m,i}^2)_A(\sum_{i=1}^{m}(\omega_{m,i}^2)_C)$ IN EQUATION (63)

Since, from Eq. (61), we can write

$$\left(\Delta_m^2\right)_Q^{\text{opt}} \leqslant \sqrt{\left(\sum_{i=1}^{m}\left(\omega_{m,i}^2\right)_A\right)}\sqrt{\left(\sum_{i=1}^{m}\left(\omega_{m,i}^2\right)_C\right)}, \tag{B1}$$

here, we optimize $(\sum_{i=1}^{m}(\omega_{m,i}^2)_A)$. Using the convex inequality, we can write

$$\sum_{i=1}^{m}\left(\omega_{m,i}^2\right)_A \leqslant \sqrt{m\sum_{i=1}^{m}\left(\left(\omega_{m,i}^2\right)_A\right)^2}. \tag{B2}$$

Using the definition of $(\omega_{m,i}^2)_A$, we can write $\sum_{i=1}^{m}((\omega_{m,i}^2)_A)^2 = \langle\psi|m2^{m-1} + \delta_m|\psi\rangle$, where

$$\delta_m = \sum_{l=1}^{2^{m-1}-m}(\delta_m)_l \tag{B3}$$

$$\begin{aligned} &= (m-2)\sum_{j'=2}^{1+\binom{m}{1}}\{A_1, A_{j'}\} + (m-4) \\ &\times \sum_{j'=2+\binom{m}{1}}^{1+\binom{m}{1}+\binom{m}{2}}\{A_1, A_{j'}\} + \cdots + \left(m - 2\left\lfloor\frac{m}{2}\right\rfloor\right) \\ &\times \sum_{j'=2+\binom{m}{1}+\binom{m}{2}+\cdots+\binom{m}{\lfloor \frac{m}{2}\rfloor - 1}}^{1+\binom{m}{1}+\binom{m}{2}+\cdots\binom{m}{\lfloor \frac{m}{2}\rfloor}}\{A_1, A_{j'}\} + (m-4) \\ &\times \sum_{j,j'=2 j\neq j'}^{1+\binom{m}{1}}\{A_j, A_{j'}\} + \cdots + (m-4)\{A_{2^{m-1}-1}, A_{2^{m-1}}\}, \end{aligned} \tag{B4}$$

such that $(\delta_m)_l = 2^{m-1} - \langle\psi_l|\psi_l\rangle$. Hence,

$$\delta_m = (2^{m-1} - m)2^{m-1} - \sum_{l=1}^{2^{m-1}-m}\langle\psi_l|\psi_l\rangle, \tag{B5}$$

where we define

$$|\psi_l\rangle = \sum_{x=1}^{2^{m-1}}(-1)^{s_l \cdot y^x}A_x|\psi\rangle. \tag{B6}$$

To find the element s_l, we consider a set $\mathcal{L}_m = \{s|s \in \{0, 1\}^m, \sum_r s_r \geqslant 2\}$, $r \in \{1, 2, \ldots, m\}$. The element $s_l \in \mathcal{L}_m$ is such that $\sum_r (s_l)_r \neq 2u$ for some $u \in \mathbb{N}$. We then find a number $(2^{m-1} - m)$ of s_l, where $l \in [2^{m-1} - m]$. Clearly, $(\delta_m)_{\max} = (2^{m-1} - m)2^{m-1}$, and it holds only when, for each $l \in [2^{m-1} - m]$, $|\psi_l\rangle = 0$. Since $|\psi\rangle \neq 0$, for optimization, the observables for each Alice must satisfy the conditions $\sum_{x=1}^{2^{m-1}}(-1)^{s_l \cdot y^x}A_x = 0$ for each $l \in [2^{m-1} - m]$. Finally, we get $\sum_{i=1}^{m}((\omega_{m,i}^2)_A)_{\text{opt}}^2 = 2^{2(m-1)}$, which in turn provides $\sum_{i=1}^{m}[(\omega_{m,i}^2)_A]_{\text{opt}} = 2^{m-1}\sqrt{m}$. The observables of each Alice satisfy the condition

$$\{A_j, A_{j'}\} = 2 - \frac{4p}{m}\ \forall\, j, j' = x \in [2^{m-1}]. \tag{B7}$$

Clearly, there exists a jth (j'th) bit string denoted by y^j ($y^{j'}$) from the set of 2^{m-1} bit strings as defined earlier. Let the set $\{y^j\}$ contain all the elements (0 or 1) of that corresponding bit string. Hence, for $x = j\ (j') \in [2^{m-1}]$, we can consider the set $\{y^j\} \cup \{y^{j'}\}$ as the collection of those elements corresponding to bit strings $\{y^j\}$ and $\{y^{j'}\}$. Without loss of generality, let us assume $\{y^j\} \cup \{y^{j'}\}$ contains q ones in it. Clearly, from the construction of the bit strings, here, $0 \leqslant q \leqslant m$.

Now we divide the bit strings y^j and $y^{j'}$ into $(\lfloor \frac{m}{2} \rfloor + 1)$ classes according to the number of ones in them. Let $y^j \in C^\nu$ if the corresponding bit string of y^j contains ν ones in it. Let there be two classes C^ν and $C^{\nu'}$ such that $y^j \in C^\nu$ and $y^{j'} \in C^{\nu'}$ and $\nu + \nu' = q$ $(0 \leqslant \nu, \nu' \leqslant \lfloor \frac{m}{2} \rfloor)$. For a given pair of (j, j'), a $t \in \mathbb{T} \subseteq [m]$ exists such that $y_t^j = y_t^{j'} = 1$. Let the cardinality of the set be $\mathbb{T}$, i.e., $|\mathbb{T}| = d$. Then a number p exists such that $p = q - 2d$. Using it in Eq. (B7), we get the observables for each Alice$_k$. Similarly, we can find

$$\{C_j, C_{j'}\} = 2 - \frac{4p}{m}\ \ \forall\, j\ (j') = z \in [2^{m-1}]. \tag{B8}$$

For example, we can consider the $m = 3$ scenario. Considering Eq. (B3), here, we can write

$$\delta_3 = \{A_1, (A_2 + A_3 + A_4)\} - \{A_2, (A_3 + A_4)\} - \{A_3, A_4\}, \tag{B9}$$

as derived in Eq. (38). Here, the set $\mathcal{L}_3$ is defined as $\mathcal{L}_3 = \{s|s \in \{0, 1\}^3, \sum_r s_r \geqslant 2\}$, $r \in \{1, 2, 3\}$. Hence, here, $\mathcal{L}_3$ contains only the elements $\{110, 101, 011, 111\}$. We denote the element $s_l \in \mathcal{L}_3$ such that $\sum_r (s_l)_r \neq 2u$ for some $u \in \mathbb{N}$. Hence, here, we then find $(2^{3-1} - 3) = 1$ element s_1, which is $s_1 = 111$. Clearly, $(\delta_3)_{\max} = (2^{3-1} - 3)2^{3-1} = 4$, and it holds only when $|\psi_1\rangle = \sum_{x=1}^{4}(-1)^{s_1 \cdot y^x}A_x|\psi\rangle = 0$. Since $|\psi\rangle \neq 0$, for optimization, the observables for each Alice must satisfy the conditions $\sum_{x=1}^{4}(-1)^{s_1 \cdot y^x}A_x = 0$. Since, here, $s_1 = 111$, $\sum_{x=1}^{4}(-1)^{s_1 \cdot y^x}A_x = 0 \Rightarrow A_1 - A_2 - A_3 - A_4 = 0$, which is derived in Sec III. Finally, we get $\sum_{i=1}^{3}[(\omega_{3,i}^2)_A]_{\text{opt}}^2 = 2^{2(3-1)} = 16$, which in turn provides $(\Delta_3^2)_Q^{\text{opt}} = 4\sqrt{3}$.

The observables of each Alice satisfy the condition

$$\{A_j, A_{j'}\} = 2 - \frac{4p}{3}, \tag{B10}$$

where $j, j' = x \in [4]$. Clearly, the jth (j'th) bit string denoted by y^j ($y^{j'}$) from the set of four bit strings defined earlier exists. Let the set $\{y^j\}$ contain all the elements (0 or 1) of that corresponding bit string. Let us consider that $j = 1$, $j' = 2$. Hence, the corresponding three-length bit strings are $y^1 = 000$ and $y^2 = 001$. Hence, we can consider the set $\{y^1\} \cup \{y^2\}$ to be the collection of those elements corresponding to bit strings y^1

and y^2. Clearly, we can see that $\{y^1\} \cup \{y^2\} = \{0, 0, 0, 0, 0, 1\}$ contains a single 1 in it. Hence, in this case $q = 1$.

Since we divide the bit strings y^j and $y^{j'}$ into $(\lfloor \frac{3}{2} \rfloor + 1) = 2$ classes according to the number of ones in it, here, we can see that $y^1 \in C^{\nu=0}$ and $y^2 \in C^{\nu'=1}$, and here, $\nu + \nu' = q = 1$. For a given pair of (1,2), a $t \in \mathbb{T} \subseteq [3]$ exists such that $y^1_t = y^2_t = 1$. Since $y^1 = 000$ and $y^2 = 001$, there is no t such that $y^1_t = y^2_t = 1$. Hence, the cardinality of the set $\mathbb{T}$ is zero, i.e., $|\mathbb{T}| = d = 0$. Then we get $p = q - 2d = 1$. Using this in Eq. (B10), we get

$$\{A_1, A_2\} = 2 - \tfrac{4}{3} = \tfrac{2}{3}, \tag{B11}$$

as derived in Sec III. Similarly, we can find all the anticommutation relations of both Alice and Charlie.

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